BLOW-UP OF SOLUTIONS
OF NONLINEAR PARABOLIC INEQUALITIES

STEVEN D. TALIAFERRO

ABSTRACT. We study nonnegative solutions \( u(x,t) \) of the nonlinear parabolic inequalities

\[
a u^\lambda \leq u_t - \Delta u \leq u^\lambda
\]

in various subsets of \( \mathbb{R}^n \times \mathbb{R} \), where \( \lambda > \frac{n+2}{n} \) and \( a \in (0,1) \) are constants. We show that changing the value of \( a \) in the open interval \((0,1)\) can dramatically affect the blow-up of these solutions.

1. INTRODUCTION

In this paper, we study nonnegative solutions \( u(x,t) \) of the nonlinear parabolic inequalities

\[
a u^\lambda \leq u_t - \Delta u \leq u^\lambda
\]

in various subsets of \( \mathbb{R}^n \times \mathbb{R} \), where \( a \in (0,1) \) is a constant and \( n \geq 1 \) is an integer.

In order to state our results, we define \(|(x,t)|\) for \((x,t) \in \mathbb{R}^n \times \mathbb{R}\) by

\[
|(x,t)| = \max\{|x|,|t|^{1/2}\},
\]

where \(|x|\) is the usual Euclidean norm of \( x \) in \( \mathbb{R}^n \), and we define

\[
\lambda_B = \begin{cases} 
\frac{n+2}{n} \left(\frac{n}{n-1}\right)^2 & \text{if } n \geq 2, \\
\infty & \text{if } n = 1.
\end{cases}
\]

Note that \( \lambda_B > \frac{n+2}{n} \).

Our result on the blow-up at the origin of nonnegative solutions of (1.1) is

**Theorem 1.** Suppose \( \lambda > \frac{n+2}{n} \). Then there exists \( a = a(n,\lambda) \in (0,1) \) and \( C = C(n,\lambda) \in (1,\infty) \) such that for each continuous function

\( \varphi: (0,1) \rightarrow (0,\infty) \) (resp. \( \varphi: (-1,0) \rightarrow (0,\infty) \))

there exists a \( C^\infty \) positive solution \( u(x,t) \) of (1.1) in \((\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\}\) such that

\[
 u(0,t) \neq O(\varphi(t)) \quad \text{as } t \rightarrow 0^+ \quad (\text{resp. } t \rightarrow 0^-)
\]

and \(|(x,t)|^{\lambda_B} u(x,t)\) is bounded between \(1/C\) and \( C \) in the region

\( (\mathbb{R}^n \times \mathbb{R}) - \{(x,t): |x|^2 \leq t \leq 1\} \) (resp. \( (\mathbb{R}^n \times \mathbb{R}) - \{(x,t): -1 \leq t \leq -|x|^2\} \)).

**Theorem 1** is in strong contrast to the following result of Poláčik, Quittner, and Souplet [11, 15].
Theorem 2. Suppose $1 < \lambda < \lambda_B$. Then there exists $a = a(n, \lambda) \in (0, 1)$ and $C = C(n, \lambda) \in (1, \infty)$ such that if $u(x, t)$ is a $C^{2,1}$ nonnegative solution of \eqref{1.1} in
\[ B_2(0) \times (0, 2) \quad \text{(resp. } B_2(0) \times (-2, 0)), \]
then $u(x, t) \leq C|t|^\frac{n+2}{n-2}$ for
\[(x, t) \in B_1(0) \times (0, 1) \quad \text{(resp. } (x, t) \in B_1(0) \times (-1, 0)). \]

Our result on the blow-up at $t = \pm \infty$ of nonnegative solutions of \eqref{1.1} is

Theorem 3. Suppose $\lambda > \frac{n+2}{n}$. Then there exists $a = a(n, \lambda) \in (0, 1)$ and $C = C(n, \lambda) \in (1, \infty)$ such that for each continuous function
\[ \varphi: (1, \infty) \to (0, \infty) \quad \text{(resp. } \varphi: (-\infty, -1) \to (0, \infty)) \]
there exists a $C^\infty$ positive solution $u(x, t)$ of \eqref{1.1} in $\mathbb{R}^n \times \mathbb{R}$ such that
\[ u(0, t) \neq O(\varphi(t)) \quad \text{as } t \to \infty \quad \text{(resp. } t \to -\infty) \]
and
\[ (1 + |(x, t)|)^n u(x, t) \]
is bounded between $1/C$ and $C$ in the region
\[ \{(x, t) : t < |x|^2\} \quad \text{(resp. } \{(x, t) : t > -|x|^2\}). \]

Theorem 3 is in strong contrast to the following result of Poláčik, Quittner, and Souplet [11, 15].

Theorem 4. Suppose $1 < \lambda < \lambda_B$. Then there exists $a = a(n, \lambda) \in (0, 1)$ and $C = C(n, \lambda) \in (1, \infty)$ such that if $u(x, t)$ is a $C^{2,1}$ nonnegative solution of \eqref{1.1} in
\[ \{(x, t) : t > |x|^2\} \quad \text{(resp. } \{(x, t) : t < -|x|^2\}), \]
then $u(x, t) \leq C|t|^\frac{1}{n-2}$ for
\[(x, t) \in B_1(0) \times (2, \infty) \quad \text{(resp. } (x, t) \in B_1(0) \times (-\infty, -2)). \]

When $\frac{n+2}{n} < \lambda < \lambda_B$, these four theorems show that changing the value of $a$ in the open interval $(0, 1)$ can dramatically affect the blow-up of positive solutions of \eqref{1.1}.

Theorem 1 is not true when $\lambda \leq \frac{n+2}{n}$. In fact, we prove in [17] that if $u(x, t)$ is a $C^{2,1}$ nonnegative solution of the parabolic inequalities
\[ 0 \leq u_t - \Delta u \leq u^{\frac{n+2}{n}} + 1 \]
in a punctured neighborhood of the origin in $\mathbb{R}^n \times [0, \infty)$, then
\[ u(x, t) = O(t^{-n/2}) \quad \text{as } (x, t) \to (0, 0), \ t > 0. \]

If $\lambda > \frac{n+2}{n}$, then by Theorem 1 there exists $a \in (0, 1)$ such that \eqref{1.1} has $C^{2,1}$ positive solutions in $B_1(0) \times (0, 1)$ which are arbitrarily large as $(x, t)$ approaches $(0, 0)$ along the positive $t$-axis. Let $I_1 = I_1(n, \lambda)$ be the set of all such $a$.

If $1 < \lambda < \lambda_B$, then by Theorem 2 there exists $a \in (0, 1)$ such that every $C^{2,1}$ positive solution $u(x, t)$ of \eqref{1.1} in $B_1(0) \times (0, 1)$ satisfies
\[ u(0, t) = O(t^{\frac{1}{n-2}}) \quad \text{as } t \to 0^+. \]

Let $I_2 = I_2(n, \lambda)$ be the set of all such $a$. 
An interesting open question is whether
\[ I_1(n, \lambda) \cup I_2(n, \lambda) = (0, 1) \quad \text{for all} \quad \lambda \in \left( \frac{n + 2}{n}, \lambda_B \right) \quad \text{and} \quad n \geq 1. \]

If not, how do the \( C^{2,1} \) positive solutions of \((1.1)\) in \( B_1(0) \times (0, 1) \) behave as \((x, t)\) approaches the origin along the positive \( t \)-axis when \( a \in (0, 1) - (I_1 \cup I_2) \)? A similar question can be asked about Theorems \( 3 \) and \( 4 \). These questions seem to be very difficult.

The blow-up of solutions of the equation
\[ u_t - \Delta u = u^\lambda \]
has been extensively studied in \([1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 18]\) and elsewhere. See \([13]\) and \([5]\) for a summary of many of these results. However, other than \([15]\), we know of no previous results for the inequalities \((1.1)\). When \( \frac{n + 2}{n} < \lambda < \lambda_B \), our results show that it is more appropriate to study the inequalities \((1.1)\) rather than the equation \((1.3)\).

An elliptic analog of the results in this paper can be found in \([16]\).

2. Preliminary results

In this section, we introduce some notation and obtain some results that will be used in Sections \(3\) and \(4\) to prove Theorems \(1\) and \(3\) respectively.

**Lemma 1.** Let \( f \) be a \( C^\infty \) nonnegative function in an open subset \( \Omega \) of \( \mathbb{R}^n \times \mathbb{R} \) and define
\[ u(x, t) := \int_{\Omega} \Phi(x - y, t - s) f(y, s) \, dy \, ds \quad \text{for} \quad (x, t) \in \Omega, \]
where
\[ \Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0 \end{cases} \]
is the heat kernel. If \( u \in L_1^1(\Omega) \), then \( u \) is \( C^\infty \) in \( \Omega \) and \( Hu = f \) in \( \Omega \), where \( Hu = u_t - \Delta u \) is the heat operator.

**Proof:** Let \( \psi \in C_0^\infty(\Omega) \). Multiplying \((2.1)\) by \( H^* \psi := \psi_t + \Delta \psi \), integrating the resulting equation over \( \Omega \), and using Fubini’s theorem and the fact that \( H\Phi = \delta \), we see that \( Hu = f \) in \( D'(\Omega) \). Thus by standard parabolic regularity theory, \( u \in C^\infty(\Omega) \). \( \square \)

If \((x, t), (y, s) \in \mathbb{R}^n \times \mathbb{R} \) and \( c \in \mathbb{R} \), then it follows from \((1.2)\) that
\[ |(x, t) + (y, s)| \leq |(x, t)| + |(y, s)| \]
and \( |(cx, c^2t)| = |c| \cdot |(x, t)| \).

Throughout this section we assume \( \lambda > \frac{n + 2}{n} \), which implies
\[ n > \frac{2}{\lambda - 1} \quad \text{and} \quad 2 < \frac{2\lambda}{\lambda - 1} < n + 2. \]

Define \( W: (\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\} \to \mathbb{R} \) by
\[ W(y, s) = (|y|^4 + s^2)^{-\frac{1}{4n-4}}. \]
Then $W$ is $C^\infty$ on $(\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\}$ and
\begin{equation}
W(y, s) \sim |(y, s)|^{-\frac{1}{n-1}} \text{ for } 0 < |(y, s)| < \infty.
\end{equation}
(Here and later the notation $X \sim Y$ (resp. $X \lesssim Y$) means $\frac{1}{2}Y \leq X \leq CY$ (resp. $X \leq CY$) for some positive constant $C$ which depends only on $n$ and $\lambda$.)

Define $W_0: (\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\} \to \mathbb{R}$ by
\begin{equation}
W_0 = \varphi W,
\end{equation}
where $\varphi: \mathbb{R}^n \times \mathbb{R} \to [0,1]$ is a $C^\infty$ function satisfying $\varphi(y, s) = 1$ for $|(y, s)| \leq 1$ and
\begin{equation}
\varphi(y, s) = 0 \text{ for } |(y, s)| \geq \sqrt{\frac{3}{2}}.
\end{equation}

Define $w, w_0: (\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\} \to \mathbb{R}$ by
\begin{equation}
w(x, t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x-y, t-s)W(y, s)^\lambda \, dy \, ds
\end{equation}
and
\begin{equation}w_0(x, t) = \iint_{\mathbb{R}^n \times \mathbb{R}} \Phi(x-y, t-s)W_0(y, s)^\lambda \, dy \, ds.
\end{equation}

It follows from (2.3), (2.4), and (2.5) that $w$ and $w_0$ are locally bounded in $(\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\}$. Thus by Lemma 1, $w$ and $w_0$ are $C^\infty$ in $(\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\}$, $Hw = W_0^\lambda$ and $Hw_0 = W_0^\lambda$ in $(\mathbb{R}^n \times \mathbb{R}) - \{(0,0)\}$, and
\begin{equation}
0 \leq w(x, t) - w_0(x, t) \lesssim \int_{|(y, s)| \geq 1} \Phi(x-y, t-s)\varphi(y, s)|y, s|^{-\frac{2\lambda}{n-1}} \, dy \, ds
\end{equation}
\begin{equation}
\lesssim 1 \text{ for } 0 < |(x, t)| \leq 1.
\end{equation}

**Lemma 2.** The functions $w, W, w_0,$ and $W_0$ satisfy
\begin{equation}
\frac{w(x, t)}{W(x, t)} \sim 1 \text{ for } 0 < |(x, t)| < \infty,
\end{equation}
\begin{equation}
\frac{w_0(x, t)}{W_0(x, t)} \sim 1 \text{ for } 0 < |(x, t)| \leq 1,
\end{equation}
and
\begin{equation}
\frac{w_0(x, t)}{|(x, t)|^{-\frac{n}{n-1}}} \lesssim 1 \text{ and } \frac{W_0(x, t)}{|(x, t)|^{-\frac{n}{n-1}}} \lesssim 1 \text{ for } 1 \leq |(x, t)| < \infty.
\end{equation}

**Proof:** Making in (2.7) the change of variables
\begin{equation}
x = c\xi \quad t = c^2\tau \quad y = c\eta \quad s = c^2\zeta
\end{equation}
where $c$ is a positive constant, we get
\begin{equation}w(x, t) = c^{-\frac{1}{n-1}}w(\xi, \tau).
\end{equation}
Taking $c = |(x, t)|$ in (2.12) we find
\begin{equation}|(\xi, \tau)| = \left|\left(\frac{1}{c}x, \frac{1}{c^2}t\right)\right| = \frac{1}{c}|(x, t)| = 1
\end{equation}
and hence (2.9) follows from (2.13) and (2.4).
It follows from (2.4), (2.5), (2.15), (2.16), and (2.17) that for 0 < |(x, t)| ≤ 1 we have
\[
\frac{|w_0(x, t) - w(x, t)|}{W_0(x, t) - W(x, t)} = \frac{|w_0(x, t) - w(x, t)|}{W(x, t)} \lesssim |(x, t)|^{\frac{2}{n-1}} \to 0 \quad \text{as} \quad |(x, t)| \to 0.
\]
Thus (2.10) follows from (2.9) and from the continuity and positivity of \(w_0\) and \(W_0\) on \(0 < |(x, t)| \leq 1\).

Taking \(c = |(x, t)| \geq 4\) in (2.12) we have that (2.14) holds, \(\frac{2}{|(x, t)|} \leq \frac{1}{2}\), and
\[
\frac{w_0(x, t)}{|(x, t)|^{n-\frac{2}{n}}} \lesssim |(x, t)|^{n} \int_{|y,s| \leq 2} \Phi(x - y, t - s)(y, s)|^{-\frac{2\lambda}{n}} dy ds
= |(x, t)|^{n-\frac{2}{n}} \int_{|\eta,\zeta| \leq 2/(x, t)} \Phi(\xi - \eta, \tau - \zeta)(\eta, \zeta)|^{-\frac{2\lambda}{n}} d\eta d\zeta
\lesssim |(x, t)|^{n-\frac{2}{n}} \int_{|\eta,\zeta| \leq 2/(x, t)} |(\eta, \zeta)|^{-\frac{2\lambda}{n}} d\eta d\zeta \sim 1.
\]
Thus the first inequality of (2.11) follows from the continuity of \(w_0(x, t)\) for \(1 \leq |(x, t)| \leq 4\). The second inequality of (2.11) follows from (2.5) and (2.4). \(\square\)

For \(0 < r \leq \frac{1}{2}\), define \(W_r : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) by
\[
W_r(y, s) = (|y|^4 + s^2 + r^4)^{-\frac{2}{n-1}} \varphi(y, s),
\]
where \(\varphi\) is the function in (2.5). Then
\[
(2.15) \quad W_r(y, s) \sim r^{-\frac{2}{n-1}} \quad \text{for} \quad 0 \leq |(y, s)| \leq r,
(2.16) \quad W_r(y, s) \sim W_0(y, s) \quad \text{for} \quad r \leq |(y, s)| < \infty.
\]

Recall that according to our definition of \(X \sim Y\) after equation (2.4), the constants \(C\) for the relations (2.15) and (2.16) above and the relations (2.18) and (2.19) below do not depend on \(r\).

For \(0 < r \leq \frac{1}{2}\), define \(w_r : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) by
\[
(2.17) \quad w_r(x, t) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x - y, t - s)W_r(y, s)^\lambda dy ds.
\]
It follows from Lemma 1 that \(w_r\) is \(C^\infty\) in \(\mathbb{R}^n \times \mathbb{R}\) and \(Hw_r = W_r^\lambda\).

**Lemma 3.** For \(0 < r \leq \frac{1}{2}\) we have
\[
(2.18) \quad \frac{w_r(x, t)}{W_r(x, t)} \sim 1 \quad \text{for} \quad 0 \leq |(x, t)| \leq 1,
\]
and
\[
(2.19) \quad \frac{w_r(x, t) + W_r(x, t)}{|(x, t)|^{-n}} \lesssim 1 \quad \text{for} \quad 1 \leq |(x, t)| < \infty.
\]

**Proof.** It follows from (2.3), (2.5), (2.15), (2.16), and (2.17) that
\[
w_r(x, t) \sim I_r(x, t) + J_r(x, t) + K_r(x, t) \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},
\]
where
\[ I_r(x, t) = r^{-\frac{2\lambda}{\kappa}} \int_{|(y,s)|<r} \Phi(x - y, t - s) \, dy \, ds, \]
\[ J_r(x, t) = \int_{r<|(y,s)|<1} \Phi(x - y, t - s)|/(y,s)|^{-\frac{2\lambda}{\kappa}} \, dy \, ds, \]
\[ K_r(x, t) = \int_{1<|(y,s)|<2} \Phi(x - y, t - s) \varphi(y, s)|/(y,s)|^{-\frac{2\lambda}{\kappa}} \, dy \, ds. \]

For \( |(x, t)| \leq r \in (0, \frac{1}{2}] \) we have
\[ \frac{I_r(x, t)}{r^{-\frac{2\lambda}{\kappa}}} = r^{-2} \int_{|(y,s)|<r} \Phi(x - y, t - s) \, dy \, ds \leq r^{-2} \int_{r \in \mathbb{R}^n} \Phi(x - y, t - s) \, dy \, ds \]
\[ \leq r^{-2} 2r^2 = 2. \]

Making in (2.20) the change of variables (2.12) with \( c = r \in (0, \frac{1}{2}] \), we get
\[ \frac{J_r(x, t)}{r^{-\frac{2\lambda}{\kappa}}} = \tilde{J}_r \left( \frac{x}{r^2}, \frac{t}{r^2} \right), \]
where
\[ \tilde{J}_r(\xi, \tau) = \int_{1 \leq |(\eta,\zeta)| \leq \frac{\lambda}{\kappa}} \Phi(\xi - \eta, \tau - \zeta)|/(\eta,\zeta)|^{-\frac{2\lambda}{\kappa}} \, d\eta \, d\zeta. \]
For \( |(x, t)| \leq r \in (0, \frac{1}{2}] \) we have \(|(\xi, \tau)| \leq 1\). Also
\[ \sup_{|(\xi, \tau)| \leq 1} \tilde{J}_r(\xi, \tau) \leq \sup_{|(\xi, \tau)| \leq 1} \int_{1 \leq |(\eta,\zeta)| \leq \infty} \Phi(\xi - \eta, \tau - \zeta)|/(\eta,\zeta)|^{-\frac{2\lambda}{\kappa}} \, d\eta \, d\zeta \]
\[ = C(n, \lambda) < \infty \]
and
\[ \inf_{|(\xi, \tau)| \leq 1} \tilde{J}_r(\xi, \tau) \geq \inf_{|(\xi, \tau)| \leq 1} \int_{1 < |(\eta,\zeta)| < 2} \Phi(\xi - \eta, \tau - \zeta)|/(\eta,\zeta)|^{-\frac{2\lambda}{\kappa}} \, d\eta \, d\zeta \]
\[ \geq C(n, \lambda) > 0. \]

For \( r \in (0, \frac{1}{2}] \) and \((x, t) \in \mathbb{R}^n \times \mathbb{R}\), we have
\[ \frac{K_r(x, t)}{r^{-\frac{2\lambda}{\kappa}}} \leq \int_{1<|(y,s)|<2} \Phi(x - y, t - s) \, dy \, ds \leq 8. \]

Combining (2.21)–(2.25) and using (2.15) we obtain for \( r \in (0, \frac{1}{2}] \) that
\[ \frac{w_r(x, t)}{W_r(x, t)} \sim \frac{w_r(x, t)}{r^{-\frac{2\lambda}{\kappa}}} \sim 1 \quad \text{for} \quad |(x, t)| < r. \]
Since for \(|(x, t)| \geq r\), \(W_r(x, t) \sim W_0(x, t)\) and \(w_r(x, t) \leq w_0(x, t)\), it follows from Lemma 2 that to complete the proof of Lemma 3 we only need to show
\[
(2.26) \quad \frac{w_r(x, t)}{W_r(x, t)} \geq 1 \quad \text{for} \quad r \leq |(x, t)| \leq 1.
\]

To do this, we make in (2.20) the change of variables (2.12) with \(c = |(x, t)| \in [r, \frac{1}{2}]\) to get (2.14) and
\[
(2.27) \quad \frac{w_r(x, t)}{W_r(x, t)} \sim \frac{J_r(x, t)}{c^{-}\frac{n}{-1}} = \int \int_{\xi \leq (n, \zeta) \leq \frac{1}{2}} \Phi(\xi - \eta, \tau - \zeta)(\eta, \zeta)|^{-\frac{2\lambda}{2}} \ d\eta \ d\zeta
\]
\[
(2.28) \quad \frac{w_r(x, t)}{W_r(x, t)} \sim w_r(x, t) \geq K_r(x, t) \geq C(n, \lambda) > 0.
\]
Relation (2.28) follows from (2.27) and (2.28). \(\square\)

For \(0 < r \leq \frac{1}{2}\) and \(h > 0\), define \(W_{h,r}: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\) by
\[
(2.29) \quad W_{h,r}(y, s) = h^{-\frac{n}{-2}} W_r\left(\frac{y}{h}, \frac{s}{h^2}\right).
\]

It follows from (2.5), (2.15), (2.16), and (2.6) that
\[
(2.30) \quad W_{h,r}(y, s) \sim (hr)^{-\frac{n}{2}} \quad \text{for} \quad 0 \leq |(y, s)| \leq hr,
\]
\[
(2.31) \quad W_{h,r}(y, s) \sim |(y, s)|^{-\frac{n}{2}} \quad \text{for} \quad hr \leq |(y, s)| \leq h,
\]
\[
(2.32) \quad W_{h,r}(y, s) \sim |(y, s)|^{-\frac{n}{2}} \varphi\left(\frac{y}{h}, \frac{s}{h^2}\right) \quad \text{for} \quad h \leq |(y, s)| \leq \sqrt{3} h / 2,
\]
\[
(2.33) \quad W_{h,r}(y, s) = 0 \quad \text{for} \quad |(y, s)| \geq \sqrt{3} h / 2.
\]

For \(0 < r \leq \frac{1}{2}\) and \(h > 0\), define \(w_{h,r}: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\) by
\[
(2.34) \quad w_{h,r}(x, t) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x - y, t - s) W_{h,r}(y, s) dy ds.
\]

Making in (2.34) the change of variables (2.12) with \(c = h\) and using (2.17) and (2.29), we get
\[
(2.35) \quad w_{h,r}(x, t) = h^{-\frac{n}{2} - \frac{n}{2}} w_r\left(\frac{x}{h}, \frac{t}{h^2}\right).
\]

The following lemma follows immediately from Lemma 3 and equations (2.29) and (2.35).

**Lemma 4.** For \(0 < r \leq \frac{1}{2}\) and \(h > 0\) we have
\[
(2.36) \quad \frac{w_{h,r}(x, t)}{W_{h,r}(x, t)} \sim 1 \quad \text{for} \quad 0 \leq |(x, t)| \leq h,
\]
\[
(2.37) \quad \frac{w_{h,r}(x, t) + W_{h,r}(x, t)}{h^n - \frac{n}{2} |(x, t)|^{-n}} \lesssim 1 \quad \text{for} \quad h \leq |(x, t)| < \infty.
\]
Define $\widehat{W}$, $\widehat{w} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by
\[
\widehat{W}(y, s) = (|y|^4 + s^2 + 1)^{-\frac{1}{2}}.
\]
(2.36)
\[
\widehat{w}(x, t) = \int \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x - y, t - s)\widehat{W}(y, s)\lambda dy ds.
\]
Then
\[
\widehat{W}(y, s) \sim (1 + |(y, s)|) - \frac{\lambda}{2} \quad \text{for} \quad (y, s) \in \mathbb{R}^n \times \mathbb{R},
\]
which implies
\[
\widehat{W}(y, s) \sim 1 \quad \text{for} \quad 0 \leq |(y, s)| \leq 1
\]
and
\[
\widehat{W}(y, s) \sim |(y, s)|^{-\frac{\lambda}{2}} \quad \text{for} \quad |(y, s)| \geq 1
\]
and it follows from (2.36) and Lemma 1 that $\widehat{w}$ is $C^\infty$ in $\mathbb{R}^n \times \mathbb{R}$ and $H\widehat{w} = \widehat{W}^\lambda$.

For $|(x, t)| \geq 1$, we obtain from (2.4) and (2.9) that
\[
|(x, t)| \frac{2}{\lambda} \widehat{w}(x, t) \leq |(x, t)| \frac{2}{\lambda} w(x, t) \sim \frac{w(x, t)}{W(x, t)} \sim 1
\]
and making the change of variables (2.12) with $c = |(x, t)| \geq 1$ and using (2.14) and (2.38) we get
\[
|(x, t)| \frac{2}{\lambda} \widehat{w}(x, t) \gtrsim |(x, t)| \frac{2}{\lambda} \int \int_{|(y, s)| \geq 2} \Phi(x - y, t - s) \frac{1}{|(y, s)|^\frac{1}{\lambda}} dy ds
\]
\[
= \int \int_{|(\eta, \zeta)| \geq 2} \Phi(\xi - \eta, \tau - \zeta) \frac{1}{|(\eta, \zeta)|^\frac{1}{\lambda}} d\eta d\zeta
\]
\[
\geq \min_{2 \leq |(\eta, \zeta)| \leq 3} \int \int_{|(\eta, \zeta)| \geq 2} \Phi(\xi - \eta, \tau - \zeta) \frac{1}{|(\eta, \zeta)|^\frac{1}{\lambda}} d\eta d\zeta
\]
\[
= C(n, \lambda) > 0.
\]
So $\widehat{w}(x, t) \sim |(x, t)| - \frac{\lambda}{2}$ for $|(x, t)| \geq 1$, and thus by (2.38),
\[
(\frac{\widehat{w}^\lambda}{H\widehat{w}})^{\frac{1}{\lambda}} = \frac{\widehat{w}}{W} \sim 1 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.
\]
(2.39)

For $0 < r \leq \frac{1}{2}$ and $h > 0$, define $V_{h, r}^+, V_{h, r}^- : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by
\[
V_{h, r}^+(x, t) = W_{h, r}((x, t) - (0, 2h^2)),
\]
\[
V_{h, r}^-(x, t) = W_{h, r}((x, t) + (0, 2h^2)).
\]
We abbreviate these last two equations by writing
\[
V_{h, r}^\pm(x, t) = W_{h, r}((x, t) \mp (0, 2h^2))
\]
and in what follows we abbreviate other pairs of equations in a similar way.

For $0 < r \leq \frac{1}{2}$ and $h > 0$, define $v_{h, r}^\pm : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by
\[
v_{h, r}^\pm(x, t) = \int \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x - y, t - s)V_{h, r}^\pm(y, s)\lambda dy ds.
\]
(2.40)
Then
\[(2.41) \quad Hv_{h,r}^\pm = (V_{h,r}^\pm)^\lambda\]
and by \((2.34)\)
\[v_{h,r}^\pm(x, t) = w_{h,r}(x, t \mp (0, 2h^2)).\]

Thus the following lemma follows directly from Lemma 4.

**Lemma 5.** For \(0 < r \leq \frac{1}{2}\) and \(h > 0\) we have

\[
\begin{align*}
\frac{v_{h,r}^\pm(x, t)}{V_{h,r}^\pm(x, t)} &\sim 1 \quad \text{for } |(x, t) \mp (0, 2h^2)| \leq h, \\
\frac{v_{h,r}^\pm(x, t) + V_{h,r}^\pm(x, t)}{h^{-\frac{2}{n-1}} |(x, t) \mp (0, 2h^2)|^{-\frac{n}{n-1}}} &\lesssim 1 \quad \text{for } |(x, t) \mp (0, 2h^2)| \geq h.
\end{align*}
\]

**Lemma 5.** and equations \((2.30), (2.31), (2.4),\) and \((2.9)\) imply
\[(2.42) \quad v_{h,r}^\pm \sim V_{h,r}^\pm \geq h^{-\frac{2}{n}} \sim |(x, t)|^{-\frac{2}{n}} \sim W \sim w \quad \text{for } |(x, t) \mp (0, 2h^2)| \leq h.
\]

Since for \(|(x, t) \mp (0, 2h^2)| \geq h,
\[
\frac{h^{-\frac{2}{n}} |(x, t) \mp (0, 2h^2)|^{-\frac{n}{n-1}}}{|x, t|^{-\frac{2}{n}}} \lesssim \min \left\{ \left( \frac{h}{|(x, t)|} \right)^{\frac{2}{n-1}}, \left( \frac{|(x, t)|}{h} \right)^{\frac{2}{n}} \right\},
\]

it follows from Lemma 5 that
\[(2.43) \quad \frac{v_{h,r}^\pm(x, t) + V_{h,r}^\pm(x, t)}{|(x, t)|^{-\frac{2}{n}}} \lesssim \min \left\{ \left( \frac{h}{|(x, t)|} \right)^{\frac{2}{n-1}}, \left( \frac{|(x, t)|}{h} \right)^{\frac{2}{n}} \right\}
\]
for \(|(x, t) \mp (0, 2h^2)| \geq h.

Let \(h_j = 3^j\) for \(j \in \mathbb{Z}\). Let \(\varphi: (-\infty, 0) \cup (0, \infty) \to (0, \infty)\) be a continuous function. (There is no loss of generality in assuming the functions \(\varphi\) in Theorems 1 and 3 are all positive and continuous on the larger set \((-\infty, 0) \cup (0, \infty).\) Choose \(r_j \in (0, \frac{1}{2}]\) such that
\[(2.44) \quad \frac{(h_j r_j)^{-\frac{2}{n}}}{\varphi(\pm 2h_j^2)} \to \infty \quad \text{as } |j| \to \infty.
\]

Let \(V_j^\pm = V_{h_j, r_j}^\pm\) and \(v_j^\pm = v_{h_j, r_j}^\pm\). Since by \((2.33)\) the support of \(V_j^\pm\) is contained in \(R \sqrt{2h_j}(0, \pm 2h_j^2)\), where
\[R_h(x, y) := \{(y, s) \in \mathbb{R}^n \times \mathbb{R} : |(y, s) - (x, t)| \leq h\},
\]
we see that the functions \(V_j^\pm, j \in \mathbb{Z}\), have disjoint supports. By Lemma 5 and equation \((2.30),\)
\[(2.45) \quad v_j^\pm(0, \pm 2h_j^2) \sim V_j^\pm(0, \pm 2h_j^2) = W_{h_j, r_j}(0, 0) \sim (h_j r_j)^{-\frac{2}{n}}
\]
for \(j \in \mathbb{Z}.
\]
If $A$ is any subset of $\mathbb{Z}$ and $R_j \pm = R_h \pm (0, \pm 2h_j)$ it follows from (2.43) that for $(x, t) \notin \bigcup_{j \in A} R_j \pm$ and $(x, t) \neq (0, 0)$ we have

$$
\sum_{j \in A} v_j^\pm (x, t) + V_j^\pm (x, t) \lesssim \sum_{h_j \leq |(x, t)|} \left( \frac{h_j}{|x, t|} \right)^{n-\frac{2}{s+1}} + \sum_{h_j \geq |(x, t)|} \left( \frac{|(x, t)|}{h_j} \right)^{n-\frac{2}{s+1}}
$$

(2.46)

and thus, since the functions $V_j^\pm$ have disjoint supports,

$$
\sum_{j \in A} V_j^\pm (x, t)^\lambda \lesssim \left( \frac{\sum_{j \in A} V_j^\pm (x, t)}{|(x, t)|^{\frac{2}{s+1}}} \right)^\lambda \lesssim 1
$$

(2.47)

for $(x, t) \notin \bigcup_{j \in A} R_j \pm$ and $(x, t) \neq (0, 0)$.

3. Proof of Theorem 1

In this section, we use the notation and results in Section 2 to prove Theorem 1.

Proof of Theorem 1. Since the functions $V_j^\pm$, $j \in \mathbb{Z}$, are $C^\infty$ and have disjoint support, $\sum_{j \leq -1} (V_j^\pm)^\lambda$ converges in $(\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\}$ to a $C^\infty$ function. It follows from the monotone convergence theorem and (2.40) that

$$
v^\pm(x, t) := \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x - y, t - s) \left( \sum_{j \leq -1} V_j^\pm(y, s)^\lambda \right) dy \, ds
$$

$$
= \sum_{j \leq -1} v_j^\pm (x, t)
$$

(3.1)

by (2.46). Thus $v^\pm$ is bounded on compact subsets of $(\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\}$, and so by Lemma 1, $v^\pm$ is $C^\infty$ on $(\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\}$ and

$$
Hv^\pm = \sum_{j \leq -1} (V_j^\pm)^\lambda = \sum_{j \leq -1} Hv_j^\pm
$$

by (2.41).

Define $u^\pm: (\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\} \to \mathbb{R}$ by $u^\pm = w + v^\pm$, where $w$ is given by (2.7). Then $u^\pm$ is $C^\infty$ and

$$
Hu^\pm = W^\lambda + \sum_{j \leq -1} (V_j^\pm)^\lambda.
$$

(3.2)

We now show

$$
Hu^\pm \sim (u^\pm)^\lambda \quad \text{in} \quad (\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\},
$$

(3.3)

which, after scaling $u^\pm$ if necessary, implies $u^\pm$ satisfies (1.1) in $(\mathbb{R}^n \times \mathbb{R}) - \{(0, 0)\}$. 


If \((x, t) \notin \bigcup_{j \leq -1} R_j\) and \((x, t) \neq (0, 0)\), then by \((3.2)\), \((2.4)\), \((2.9)\), \((2.47)\), and \((2.46)\),

\[
Hu^{\pm} = W^{\lambda} \left[ 1 + \sum_{j \leq -1} \left( \frac{V^\pm_j}{W} \right)^{\lambda} \right] \\
\sim w^{\lambda} \left[ 1 + \sum_{j \leq -1} \frac{(V^\pm_j)^{\lambda}}{|(x, t)|^{\frac{2}{\lambda}}} \right] \\
\sim w^{\lambda} \sim w^{\lambda} \left( 1 + \left( \frac{v^\pm_j}{w} \right)^{\lambda} \right) = (u^\pm)^{\lambda}.
\]

If \((x, t) \in R_{j_0}\) for some \(j_0 \leq -1\), then by \((3.2)\), \((2.42)\), \((2.46)\), and Lemma 5,

\[
Hu^{\pm} = (V^\pm_{j_0})^{\lambda} + W^{\lambda} \\
= (V^\pm_{j_0})^{\lambda} \left[ 1 + \left( \frac{W}{V^\pm_{j_0}} \right)^{\lambda} \right] \\
\sim (V^\pm_{j_0})^{\lambda} \sim (v^\pm_{j_0})^{\lambda} \\
\sim (v^\pm_{j_0})^{\lambda} \left[ 1 + \sum_{j \leq -1, j \neq j_0} \frac{v^\pm_j}{v^\pm_{j_0}} + \frac{w}{v^\pm_{j_0}} \right]^{\lambda} \\
= (u^\pm)^{\lambda},
\]

which proves \((3.3)\).

It follows from \((2.41)\) and \((2.43)\) that \(u^\pm(0, t) \neq O(\varphi(t))\) as \(t \to 0^\pm\).

By \((2.41)\), \((2.40)\), and \((3.1)\) we see that \(|(x, t)|^{\frac{2}{\lambda}} u^\pm(x, t) \sim 1\) in the regions stated in Theorem 1.

Taking \(u = u^+\) (resp. \(u = u^-\)), we obtain Theorem 1. \(\Box\)

4. PROOF OF THEOREM 3

In this section, we use the notation and results in Section 2 to prove Theorem 3.

**Proof of Theorem 3.** Since the functions \(V^\pm_j, j \in \mathbb{Z}\), are \(C^\infty\) and have disjoint support, \(\sum_{j \geq 1} (V^\pm_j)^{\lambda}\) converges on \(\mathbb{R}^n \times \mathbb{R}\) to a \(C^\infty\) function.

Let \(B\) be a subset of \(\mathbb{N}\). If \(|(x, t)| < 1\), then \((x, t) \notin \bigcup_{j \in B} R^\pm_j\) and it therefore follows from \((2.43)\) that

\[
\sum_{j \in B} v^\pm_j(x, t) + V^\pm_j(x, t) \lesssim \sum_{j \geq 1} h^\pm_j \sim 1.
\]

Thus, by \((2.40)\), we have for \((x, t) \notin \bigcup_{j \in B} R^\pm_j\) that

\[
\sum_{j \in B} v^\pm_j(x, t) + V^\pm_j(x, t) \lesssim (1 + |(x, t)|)^{-\frac{2}{\lambda}}.
\]
Hence, since the functions $V_j^\pm$ have disjoint support,

$$
\sum_{j \in B} V_j^\pm(x, t)^\lambda \lesssim (1 + |(x, t)|)^{-\frac{2\lambda}{n}}
$$

for $(x, t) \notin \bigcup_{j \in B} R_j^\pm$.

It follows from the monotone convergence theorem and (2.40) that $v^\pm := \int X R^n \times \int X R^n \Phi(x - y, t - s) \sum_{j \geq 1} V_j^\pm(y, s)^\lambda dy ds$

$$
= \sum_{j \geq 1} v_j^\pm(x, t)
$$

(4.3)

\begin{align*}
\lesssim \begin{cases}
(1 + |(x, t)|)^{-\frac{2\lambda}{n}} & \text{if } (x, t) \notin \bigcup_{j \geq 1} R_j^\pm, \\
v_{j_0}(x, t) + (1 + |(x, t)|)^{-\frac{2\lambda}{n}} & \text{if } (x, t) \in R_j^\pm \text{ for some } j_0 \geq 1
\end{cases}
\end{align*}

by (4.1). Thus $v^\pm$ is bounded on compact subsets of $R^n \times R$ and so by Lemma 1, $v^\pm$ is $C^\infty$ in $R^n \times R$ and

$$
 Hv^\pm = \sum_{j \geq 1} (V_j^\pm)^\lambda = \sum_{j \geq 1} Hv_j^\pm
$$

by (2.41).

Define $u^\pm : R^n \times R \rightarrow R$ by $u^\pm = \hat{u} + v^\pm$, where $\hat{u}$ is given by (2.36). Then $u^\pm$ is $C^\infty$ and

$$
 Hu^\pm = \hat{W}^\lambda + \sum_{j \geq 1} (V_j^\pm)^\lambda.
$$

(4.4)

We now show

$$
 Hu^\pm \sim (u^\pm)^\lambda \quad \text{in } R^n \times R,
$$

which, after scaling $u^\pm$ if necessary, implies $u^\pm$ satisfies (1.1) in $R^n \times R$.

If $(x, t) \notin R_j^\pm$, then by (4.1), (4.2), (2.37), (2.39), and (4.1), we have

$$
 Hu^\pm = \hat{W}^\lambda \left[ 1 + \sum_{j \geq 1} \left( \frac{V_j^\pm}{\hat{W}} \right)^\lambda \right] 
\sim \hat{u}^\lambda \sim \hat{u}^\lambda \left[ 1 + \left( \frac{u^\pm}{\hat{u}} \right) \right]^\lambda = (u^\pm)^\lambda.
$$

If $(x, t) \in R_j^\pm$ for some $j_0 \geq 1$, then $|(x, t)| \geq 1$ and so (2.42), (2.37), and (2.39) imply

$$
v_{j_0}^\pm \sim V_{j_0}^\pm \geq h_{j_0}^{-\frac{n}{2\lambda}} \sim |(x, t)|^{-\frac{n}{2\lambda}} \sim (1 + |(x, t)|)^{-\frac{n}{2\lambda}} \sim \hat{W} \sim \hat{w}.
$$
Hence, if \((x, t) \in R^+_{j_0}\) for some \(j_0 \geq 1\), then by (4.4), (4.1) and Lemma 5, we have
\[
Hu^\pm = (V_{j_0}^\pm)^\lambda + \hat{W}^\lambda = (V_{j_0}^\pm)^\lambda \left[1 + \left(\frac{\hat{W}}{V_{j_0}^\pm}\right)^\lambda\right]
\]
\[
\sim (V_{j_0}^\pm)^\lambda \sim (v_{j_0}^\pm)^\lambda \sim (u_{j_0}^\pm)^\lambda
\]
\[
= (u^\pm)^\lambda,
\]
which proves (4.5).

It follows from (2.44) and (2.45) that \(u^\pm(0, t) \neq O(\varphi(t))\) as \(t \to \pm \infty\). By (2.37), (2.39), and (4.3) we see that \((1 + |(x, t)|)^{-\lambda} u^\pm(x, t) \sim 1\) in the regions stated in Theorem 3.

Taking \(u = u^+\) (resp. \(u = u^-\)), we obtain Theorem 5. □

5. PROOFS OF THEOREMS 2 AND 4

Souplet [15] showed that the proof of Theorem 3.1 in [11] can be very slightly modified to prove the following theorem.

**Theorem 5.** Suppose \(1 < \lambda < \lambda_B\) and \(D\) is a proper open subset of \(R^n \times R\). Then there exists \(a = a(n, \lambda) \in (0, 1)\) and \(C = C(n, \lambda) \in (1, \infty)\) such that if \(u(x, t)\) is a \(C^{2,1}\) nonnegative solution of (1.1) in \(D\), then
\[
u(x, t) \leq C \left(\inf_{(y, s) \in \partial D} |(y, s) - (x, t)|\right)^{-\lambda} \quad \text{for all } (x, t) \in D.
\]

Theorems 2 and 4 are immediate consequences of Theorem 5.

The proofs of Theorem 5 and [11, Theorem 3.1] rely heavily on the following Liouville-type result of Bidaut-Véron [3].

**Theorem 6.** Suppose \(1 < \lambda < \lambda_B\). Then the only \(C^{2,1}\) nonnegative solution \(u(x, t)\) of
\[
u_t - \Delta u = u^\lambda \quad \text{in } R^n \times R
\]
is \(u \equiv 0\).

**REFERENCES**


Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368

E-mail address: stalia@math.tamu.edu