BOUNDEDNESS OF PROJECTION OPERATORS AND CESÁRO MEANS IN WEIGHTED $L^p$ SPACE ON THE UNIT SPHERE

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Abstract. For the weight function \( \prod_{i=1}^{d+1} |x_i|^{2\kappa_i} \) on the unit sphere, sharp local estimates of the orthogonal projection operators are obtained and used to prove the convergence of the Cesàro \((C, \delta)\) means in the weighted \(L^p\) space for \( \delta \) above the critical index. Similar results are also proved for corresponding weight functions on the unit ball and on the simplex.

1. Introduction

For spherical harmonic expansions on the unit sphere \( S^d := \{(x_1, \cdots, x_{d+1}) : (x_1^2 + \cdots + x_{d+1}^2)^{1/2} = 1 \} \) of \( \mathbb{R}^{d+1} \), it is well known that their Cesàro \((C, \delta)\) means are uniformly bounded, in terms of degrees, in the \(L^p\) norm for all \( 1 \leq p \leq \infty \) if and only if \( \delta \geq \frac{d-1}{2} \) \([3]\). For \( \delta \) below the critical index \( \frac{d-1}{2} \), C. Sogge \([14]\) proved a much deeper result that the \((C, \delta)\) means are uniformly bounded on \(L^p(S^d)\) if
\[
\left| \frac{1}{2} - \frac{1}{p} \right| \geq \frac{1}{d+1} \quad \text{and} \quad \delta > \delta(p):= \max\{d\left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0\}
\]
for \( d \geq 2 \) and, moreover, the condition \( |1/2 - 1/p| \geq 1/(d + 1) \) is not needed in the case of \( d = 2 \). The condition \( \delta > \delta(p) \) is also known to be necessary \([3]\).

Later in \([15]\), Sogge proved that the condition (1.1) ensures the boundedness of the Riesz means of eigenfunction expansions associated to the second-order elliptic differential operators on compact connected \(C^\infty\) manifolds of dimension \( d \).

The purpose of the present paper is to establish analogous results for the Cesàro means of orthogonal expansions associated with the weight function \( h_\kappa^2(x) \), where
\[
h_\kappa(x) := \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \quad \kappa := (\kappa_1, \cdots, \kappa_{d+1}), \quad \min_{1 \leq i \leq d+1} \kappa_i \geq 0,
\]
on the unit sphere \( S^d \), as well as for orthogonal expansions for related weight functions (see (2.16) and (2.12) below) on the unit ball and on the simplex.

The function \( h_\kappa \) in (1.2) is invariant under the group \( \mathbb{Z}_d^2 \) and it is the simplest example of weight functions invariant under reflection groups studied first by Dunkl \([9]\). Homogeneous polynomials that are orthogonal with respect to \( h_\kappa^2 \) on \( S^d \) are called \( h \)-harmonics and their restrictions on \( S^d \) are eigenfunctions of a second-order
that for means (21). Our main result in this paper (see Theorem 3.1 in Section 3) shows the theory of differential-difference operator, which plays the role of the ordinary Laplacian. For \( h \) the critical index for the \((C, \delta)\) means in the uniform norm turned out to be (11)

\[
\delta > \sigma_\kappa := \frac{d-1}{2} + |\kappa| - \min_{1 \leq i \leq d+1} \kappa_i.
\]

This condition is also necessary for the almost everywhere convergence of the \((C, \delta)\) means (21)). Our main result in this paper (see Theorem 3.1 in Section 3) shows that for \( h_\kappa \) in (1.2), the \((C, \delta)\) means of \( h_\kappa \)-harmonic expansions converge in the \( L^p(h_\kappa^2; S^d) \) norm if

\[
|\frac{1}{2} - \frac{1}{p}| \geq \frac{1}{2\sigma_\kappa - 1} \quad \text{and} \quad \delta > \delta_\kappa(p) := \max\{(2\sigma_\kappa + 1)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}
\]

and that the condition \( \delta > \delta_\kappa(p) \) is also necessary. Note that (1.5) agrees with (1.1) when \( \kappa = 0 \), while \( \delta_\kappa(p) > \delta(p) \) when \( \kappa > 0 \).

The reason that these sharp results can be established for \( h_\kappa \) in (1.2) lies in an explicit formula for the kernel \( P_\kappa(h_\kappa^2; \cdot, \cdot) \) of the orthogonal projection operator (definition in the next section), while no explicit formula for the kernel is known for other reflection invariant weight functions. For (1.2), we have

\[
P_\kappa(h_\kappa^2; x, y) = c_\kappa \frac{n + \lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^{d+1}} C_\lambda^{\lambda_\kappa}(u(x, y, t)) \prod_{i=1}^{d+1} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt,
\]

where \( C_\lambda^{\lambda_\kappa} \) is the Gegenbauer polynomial of degree \( n \),

\[
\lambda_\kappa := \frac{d-1}{2} + |\kappa|, \quad \text{and} \quad u(x, y, t) = x_1 y_1 t_1 + \ldots + x_{d+1} y_{d+1} t_{d+1},
\]

and \( c_\kappa \) is the normalization constant of the weight function \( \prod_{i=1}^{d+1} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} \). If some \( \kappa_i = 0 \), then the formula holds under the limit relation

\[
\lim_{\lambda \to 0} c_\lambda \int_{-1}^{1} g(t)(1 + t)(1 - t)^{\lambda - 1} dt = g(1).
\]

For the spherical harmonic expansions, this kernel is the familiar \( P_\kappa(x, y) := \frac{d+1}{2\lambda} C_\lambda^{\lambda_\kappa}(\langle x, y \rangle) \) with \( \lambda = \frac{d+1}{2} \) (cf. [10]), which is also called a zonal harmonic.

The simple structure of the zonal harmonics means that one can derive various properties and estimates relatively easily. The structure of the kernel \( P_\kappa(h_\kappa^2; x, y) \) in (1.5) is far more complicated, making the derivation of any information from it more difficult. There is, however, a deeper reason that the study of \( h \)-harmonic expansion is more difficult than that of ordinary spherical harmonics. The zonal harmonics are invariant under the rotation group \( O(d+1) \) in the sense that \( P_\kappa(x, y) = P_\kappa(xy, yy) \) for all \( g \in O(d+1) \), which reflects the fact that the sphere is a homogeneous space. The \( P_\kappa(h_\kappa^2; x, y) \) in (1.5) is invariant under \( \mathbb{Z}^{d+1}_2 \), a subgroup of \( O(d+1) \), and we are in fact working with a weighted sphere that has singularity on the largest circles of the coordinate planes. In particular, we can no longer treat the sphere as a homogeneous space and many of our estimates of various kernels have to be local, depending on the location of the points.
The difficulty manifests itself acutely in the study of the $L^p$ boundedness of Cesàro means of $h$-harmonic expansions. For the ordinary spherical harmonics, the proof of Sogge \cite{14} relies on the sharp asymptotic bounds for the $(L^p, L^2)$ norms of the orthogonal projection operators, and a result of Bonami-Clerc \cite{3}, which says that the sharp results for Cesàro summation on $L^p$ can be deduced from these asymptotic estimates of orthogonal projections. (See also Sogge \cite{15} for the case of general compact manifolds.) For our study, while we can obtain global sharp asymptotic bounds for the $L^p(h^2_{κ}; S^d) \to L^2(h^2_{κ}; S^d)$ norms of the orthogonal projection operators of $h$-harmonic expansions (see Theorem 3.3 in Section 3), which are in full analogy with those of Sogge \cite{14} for ordinary spherical harmonics, seemingly, these global estimates are not enough for the proof of the uniform boundedness of Cesàro means on weighted $L^p$. (See Section 3.1 for the precise definition of $L^p(h^2_{κ}; S^d).$) In order to obtain our main result on the boundedness of Cesàro means, we have to replace the norm of the orthogonal projection operator by a local estimate of the projection operator over a spherical cap. (See Theorem 3.4 in Section 3.) The latter local result is substantially more difficult to establish, since only a part of the proof can follow Sogge’s strategy based on Stein’s theorem on analytic interpolation and the rest has to rely on a sharp pointwise local estimate of the kernels.

Analogues of our main results also hold for orthogonal expansions on the unit ball and on the simplex for weight functions related to $h^2_{κ}$, including in particular the Lebesgue measure (see Section 2). In fact, they follow more or less from the results for $h$-harmonics. In particular, the same condition (1.3) guarantees the convergence of the Cesàro means in the corresponding weighted $L^p$ space.

The paper is organized as follows: The next section contains preliminary results, whereas the main results are stated and discussed in Section 3. The local estimate of the projection operator is studied in Section 4. The proof of the main result for the projection operators on the sphere is given in Section 5, while the proof of the main result for the Cesàro means on the sphere is presented in Section 6. Finally, the results on the ball and on the simplex are proved in Section 7.

2. Preliminary results

2.1. $h$-spherical harmonics. We restrict our discussion to $h_{κ}$ in (1.2). Unless otherwise stated, the main reference for the material in this section is \cite{10}. An $h$-harmonic is a homogeneous polynomial $P$ that satisfies the equation $Δ_{h}P = 0$, where $Δ_{h} := D_{1}^2 + \cdots + D_{d+1}^2$ and

$$D_{i}f(x) := \partial_{i}f(x) + κ_{i} \frac{f(x) - f(x - 2x_{i}e_{i})}{x_{i}}, \quad 1 \leq i \leq d + 1,$$

$e_{1}, \ldots, e_{d+1}$ denoting the usual coordinate vectors in $\mathbb{R}^{d+1}$. The differential-difference operators $D_{1}, \ldots, D_{d+1}$ are the Dunkl operators, which commute. An $h$-harmonic is an orthogonal polynomial with respect to the weight function $h^2_{κ}(x)$ on $S^d$. Its restriction on the sphere is called a spherical $h$-harmonic. Let $\mathcal{H}^d_n(h^2_{κ})$ denote the space of spherical $h$-harmonics of degree $n$ on $S^d$. It is known that $\dim \mathcal{H}^d_n(h^2_{κ}) = (n+d+1) - (n+2d-1)$. Let $L^2(h^2_{κ}; S^d)$ denote the $L^2$ space with respect to $h^2_{κ}$ on $S^d$; see Section 3.1 below for the precise definition. The Hilbert space theory shows that

$$L^2(h^2_{κ}; S^d) = \bigoplus_{n=0}^{∞} \mathcal{H}^d_n(h^2_{κ}) : \quad f = \bigoplus_{n=0}^{∞} \operatorname{proj}_{n}(h^2_{κ}; f),$$
The operator \( f \) convolution \( C_a \) written as an integral operator
\[
\text{proj}_n(h^2; f, x) = a_\kappa \int_{S^d} f(y) P_n(h^2; x, y) h^2_n(y) d\omega(y), \quad x \in S^d,
\]
where \( d\omega(y) \) denotes the usual Lebesgue measure on \( S^d \), \( a_\kappa \) is the normalization constant, and \( a_\kappa^{-1} = \int_{S^d} h^2_2(y) \omega(y) \) and \( P_n(h^2; \cdot, \cdot) \) is the reproducing kernel of \( \mathcal{H}_n^d(h^2) \). The kernel satisfies an explicit formula
\[
\text{(2.1) } P_n(h^2; x, y) = \frac{n + \lambda_n}{\lambda_n} V_\kappa \left[ C^\lambda_n((\cdot, y)) \right](x), \quad \lambda_n = \frac{d - 1}{2} + |\kappa|,
\]
where \( C^\lambda_n \) is the Gegenbauer polynomial of degree \( n \) and \( V_\kappa \) is the so-called intertwining operator defined by
\[
\text{(2.2) } V_\kappa f(x) = c_\kappa \int_{[-1,1]^{d+1}} f(x_1 t_1, \ldots, x_d t_d) \prod_{i=1}^{d+1} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt,
\]
in which \( c_\kappa \) is a constant such that \( V_\kappa 1 = 1 \). If some \( \kappa_i = 0 \), then the formula holds under the limit relation
\[
\lim_{\lambda \to 0} c_\lambda \int_{-1}^1 g(t)(1 + t)(1 - t)^{\lambda - 1} dt = g(1).
\]
Clearly (2.1) is the same as (1.5). The operator \( V_\kappa \) is called an intertwining operator since it satisfies \( D_1 V_\kappa = V_\kappa D_j \), \( 1 \leq j \leq d + 1 \).

Let \( w_\lambda(t) := (1 - t^2)^{\lambda - 1/2} \) on \([-1,1]\). The Gegenbauer polynomials are orthogonal with respect to \( w_\lambda \). The intertwining operator can be used to define a convolution \( f \ast_g g \) for \( f \in L^1(h^2; S^d) \) and \( g \in L^1(w_\lambda; [-1,1]) \) \([20]\):
\[
\text{(2.3) } f \ast_g g(x) := a_\kappa \int_{S^d} f(y) V_\kappa[g((x, \cdot))](y) h^2_n(y) d\omega(y).
\]
In particular, the projection operator \( \text{proj}_n(h^2; f) \) can be written as
\[
\text{(2.4) } \text{proj}_n(h^2; f) = f \ast_Z Z_n, \quad \text{where } Z_n(t) := \frac{n + \lambda_n}{\lambda_n} C^\lambda_n(t).
\]
This convolution satisfies the usual Young’s inequality (see \([20], p.6, \text{Proposition 2.2}\) for \( \kappa = 0 \), \( V_\kappa = id \), it becomes the classical convolution on the sphere \([5]\). For \( f \in L^1(h^2; S^d) \), we also have \([20]\)
\[
\text{(2.5) } \text{proj}_n(h^2; f \ast_g g) = b_{\lambda_n} \int_0^{2\pi} \frac{C^\lambda_n(\cos \theta)}{C^\lambda_n(1)} g(\cos \theta)(\sin \theta)^{2\lambda_n} d\theta \text{ proj}_n(h^2; f),
\]
where \( b_{\lambda_n} \) is the normalization constant of \( w_\lambda(t) \) on \([-1,1]\).

For \( \delta > -1 \), the Cesàro \((C, \delta)\) means of the \( h \)-harmonic expansion is defined by
\[
S^\delta_n(h^2; f, x) := (A_n^{-\delta})^{-1} \sum_{j=0}^n A_n^{-\delta_j} \text{proj}_j(h^2; f, x), \quad A_n^{-\delta} = \binom{n - j + \delta}{n - j}.
\]
The operator \( S^\delta_n(h^2) \) can be written as a convolution,
\[
S^\delta_n(h^2; f) = f \ast K^\delta_n(w_\lambda), \quad K^\delta_n(w_\lambda; t) := (A_n^{-\delta})^{-1} \sum_{j=0}^n A_n^{-\delta_j} Z_j^\kappa(t).
\]
Let \( K_n^\delta(h_n^2; x, y) \) denote the kernel of \( S_n^\delta(h_n^2) \); then

\[ K_n^\delta(h_n^2; x, y) = V_n [K_n(w_{\lambda_0}; \langle x, \cdot \rangle)](y). \]

### 2.2. Orthogonal expansions on the unit ball.

We denote the usual Euclidean norm of \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) by \( \|x\| = (x_1^2 + \cdots + x_d^2)^{1/2} \). The weight functions we consider on the unit ball \( B^d = \{x : \|x\| \leq 1\} \subset \mathbb{R}^d \) are defined by

\[ W_n^B(x) := \prod_{i=1}^d |x_i|^{\kappa_i (1 - \|x\|^2)^{\kappa_{d+1}-1/2}}, \quad \kappa_i \geq 0, \quad x \in B^d, \]

which is related to the \( h_n \) in (1.2) by \( h_n^2(x, \sqrt{1-\|x\|^2}) = W_n^B(x)/\sqrt{1-\|x\|^2}, \) in which \( 1/\sqrt{1-\|x\|^2} \) comes from the Jacobian of changing variables

\[ \phi : x \in B^d \mapsto (x, \sqrt{1-\|x\|^2}) \in S^d := \{y \in S^d : y_{d+1} \geq 0\}. \]

Furthermore, under the above changing variables, we have

\[ \int_{B^d} g(y) d\omega(y) = \int_{S^d} \left[ g(x, \sqrt{1 - \|x\|^2}) + g(x, -\sqrt{1 - \|x\|^2}) \right] \frac{dx}{\sqrt{1-\|x\|^2}}. \]

The orthogonal structure is preserved under the mapping (2.7) and the study of orthogonal expansions for \( W_n^B \) can be essentially reduced to that of \( h_n^2 \). In fact, let \( \mathcal{V}_n(W_n^B) \) denote the space of orthogonal polynomials of degree \( n \) with respect to \( W_n^B \) on \( B^d \). The orthogonal projection, \( \text{proj}_n(W_n^B; f) \), of \( f \in L^2(W_n^B; B^d) \) onto \( \mathcal{V}_n(W_n^B) \) can be expressed in terms of the orthogonal projection of \( F(x, x_{d+1}) := f(x) \) onto \( \mathcal{H}^d_{n+1}(h_n^2) \):

\[ \text{proj}_n(W_n^B; f, x) = \text{proj}_n(h_n^2; F, X), \quad \text{with } X := (x, \sqrt{1-\|x\|^2}). \]

This relation allows us to deduce results on the convergence of orthogonal expansions with respect to \( W_n^B \) from that of \( h \)-harmonic expansions.

For \( d = 1 \) the weight \( W_n^B \) in (2.6) becomes the weight function

\[ w_{\kappa_2, \kappa_1}(t) = |t|^{2\kappa_1 (1 - t^2)^{\kappa_2-1/2}}, \quad \kappa_i \geq 0, \quad t \in [-1, 1], \]

whose corresponding orthogonal polynomials, \( C_n^{(\kappa_1, \kappa_2)} \), are called generalized Gegenbauer polynomials, and they can be expressed in terms of Jacobi polynomials,

\[ C_{2n+1}^{(\lambda, \mu)}(t) = \frac{(\lambda + \mu)_n}{(\mu + \frac{1}{2})_n} P_n^{(\lambda-1/2, \mu-1/2)}(2t^2 - 1), \]

where \((a)_n = a(a+1) \cdots (a+n-1)\).

### 2.3. Orthogonal expansions on the simplex.

The weight functions we consider on the simplex \( T^d = \{x : x_1 \geq 0, \ldots, x_d \geq 0, 1 - |x| \geq 0\} \) are defined by

\[ W_n^T(x) := \prod_{i=1}^d x_i^{\kappa_i - 1/2} (1 - |x|)^{\kappa_{d+1}-1/2}, \quad \kappa_i \geq 0, \]

where \(|x| := x_1 + \cdots + x_d\). They are related to \( W_n^B \), hence to \( h_n^2 \). In fact, \( W_n^T \) is exactly the product of the weight function \( W_n^B \) and the mapping

\[ \psi : (x_1, \ldots, x_d) \in T^d \mapsto (x_1^2, \ldots, x_d^2) \in B^d \]
and the Jacobian of this change of variables. Furthermore, the change of variables shows
\begin{equation}
\int_{B^d} g(x_1^2, \ldots, x_d^2) dx = \int_{T^d} g(x_1, \ldots, x_d) \frac{dx}{\sqrt{x_1 \cdots x_d}}.
\end{equation}

The orthogonal structure is preserved under the mapping (2.13). Let \( \gamma_n(\kappa)(w^T) \) denote the space of orthogonal polynomials of degree \( n \) with respect to \( W_\kappa^T \) on \( T^d \). Then \( R \in \gamma_n(\kappa)(w^T) \) if and only if \( R \circ \psi \in \gamma_{2n}(w_B) \). The orthogonal projection, \( \text{proj}_n(\kappa)(\kappa; f) \), of \( f \in L^2(\kappa; T^d) \) onto \( \gamma_n(\kappa)(w^T) \) can be expressed in terms of the orthogonal projection of \( f \circ \psi \) onto \( \gamma_{2n}(w_B) \):
\begin{equation}
\text{proj}_n(\kappa)(\kappa; f) \circ \psi \right) = \frac{1}{2^n} \sum_{\varepsilon \in \mathbb{Z}^d} \text{proj}_{2n}(w_B; f \circ \psi, x\varepsilon).
\end{equation}

The fact that \( \text{proj}_n(\kappa)(\kappa) \) of degree \( n \) is related to \( \text{proj}_{2n}(w_B) \) of degree \( 2n \) suggests that some properties of the orthogonal expansions on \( B^d \) cannot be transformed directly to those on \( T^d \). We will also need the explicit formula for the kernel, \( P_n(\kappa; x, y) \), of \( \text{proj}_n(\kappa)(\kappa; f) \), which can be derived from (2.1) and the quadratic transform between Gegenbauer and Jacobi polynomials,
\begin{equation}
P_n(\kappa; x, y) = \frac{(2n + \lambda_\kappa)\Gamma(\frac{1}{2})\Gamma(n + \kappa)}{\Gamma(\lambda_\kappa + 1)\Gamma(n + \frac{\kappa}{2})} \times c_n \int_{[-1, 1]^{d+1}} P_n^{(\lambda_\kappa - \frac{1}{2}, -\frac{1}{2})} \left( \frac{2z(x, y, t)^2 - 1}{(1 - t^2)^{\kappa}} \right) dt,
\end{equation}
where \( z(x, y, t) = \sqrt{x_1 y_1 t_1 + \ldots + x_d y_d t_d + \sqrt{1 - |x|^2 \sqrt{1 - |y|^2} t_{d+1}}} \).

2.4. Some estimates. Throughout this paper we denote by \( c \) a generic constant that may depend on fixed parameters such as \( \kappa \), \( d \) and \( \rho \), whose value may change from line to line. Furthermore we write \( A \sim B \) if \( A \geq cB \) and \( B \geq cA \).

Let \( d(x, y) := \arccos(x, y) \) denote the geodesic distance of \( x, y \in S^d \). For \( 0 \leq \theta \leq \pi \), the set
\[ c(x, \theta) : = \{ y \in S^d : d(x, y) \leq \theta \} = \{ y \in S^d : \langle x, y \rangle \geq \cos \theta \} \]
is called the spherical cap centered at \( x \) with radius \( \theta \). It is shown in [7] that \( h_\kappa \) is a doubling weight and, furthermore, the following estimate holds:

**Lemma 2.1.** For \( 0 \leq \theta \leq \pi \) and \( x = (x_1, \ldots, x_{d+1}) \in S^d \),
\begin{equation}
\int_{c(x, \theta)} h_\kappa^2(y) d\omega(y) \sim \theta^\kappa \prod_{j=1}^{d+1} \langle |x_j| + \theta \rangle^{2\kappa_j},
\end{equation}
where the constant of equivalence depends only on \( d \) and \( \kappa \).

We refer to the remarkable paper [12] of Mastroianni and Totik for various polynomial inequalities with doubling weights.

The Jacobi polynomials \( P_n^{(\alpha, \beta)} \) are orthogonal with respect to the weight
\[ w^{(\alpha, \beta)}(t) := (1 - t)^\alpha (1 + t)^\beta, \quad t \in [-1, 1]. \]
We will need the following estimate from [17, p. 169]:

**Lemma 2.2.** For \( \alpha \geq \beta \) and \( t \in [0, 1] \),

\[
|P_n^{(\alpha, \beta)}(t)| \leq cn^{-1/2}(1 - t + n^{-2})^{-(\alpha+1/2)/2}.
\]

The estimate on \([-1, 0]\) follows from the fact that \( P_n^{(\alpha, \beta)}(t) = (-1)^n P_n^{(\beta, \alpha)}(-t) \).

We will also need the estimate of the \( L^p \) norm for the Jacobi polynomials ([17], p. 391): for \( \alpha, \beta, \mu > -1 \) and \( p > 0 \),

\[
\int_0^1 |P_n^{(\alpha, \beta)}(t)|^p (1 - t)^{\mu} dt \sim \left\{ \begin{array}{ll}
1 + \frac{np - 2\mu - 2}{2n} & (p > p_{\alpha, \mu}), \\
\frac{n^{-\frac{p}{2}} \log n}{2n^{-\frac{p}{2}} + 2} & (p = p_{\alpha, \mu}), \\
\frac{n^{-\frac{p}{2}}}{2n^{-\frac{p}{2}} - 2} & (p < p_{\alpha, \mu}).
\end{array} \right.
\]

Recall that \( |\kappa| = \kappa_1 + \cdots + \kappa_{d+1} \) and \( V_\kappa \) is defined in (2.2). The following lemma was proved in [8, Theorem 3.1].

**Lemma 2.3.** Assume \( \alpha \geq \max\{\beta, |\kappa| - \frac{1}{2}\} \). Then for \( x, y \in S^d \),

\[
\int_{[-1,1]^{d+1}} P_n^{(\alpha, \beta)}(x_1y_1t_1 + \cdots + x_{d+1}y_{d+1}t_{d+1}) \prod_{j=1}^{d+1} (1 + t_j)(1 - t_j)^{\kappa_j - 1} dt \leq cn^{-2|\kappa|} \prod_{j=1}^{d+1} |x_jy_j| + n^{-1}\|\bar{x} - \bar{y}\| + n^{-2})^{\kappa_j - 1} (1 + nd(\bar{x}, \bar{y}))^{|\alpha, \mu| - |\kappa|},
\]

where and throughout, \( \bar{z} = (|z_1|, \ldots, |z_{d+1}|) \) for \( z = (z_1, \ldots, z_{d+1}) \in S^d \).

This lemma plays an essential role in the proof of a sharp pointwise estimate for the kernel \( K_\kappa^{(\alpha)}(h_\kappa^2; f) \) in [8]. For the present paper we will only need the pointwise estimate for \( P_n(h_\kappa^2; x, y) \):

**Lemma 2.4.** Let \( x, y \in S^d \). Then

\[
|P_n(h_\kappa^2; x, y)| \leq c \prod_{j=1}^{d+1} (|x_jy_j| + n^{-1}\|\bar{x} - \bar{y}\| + n^{-2})^{\kappa_j - 1} n^{-d(d-1)/2(||\bar{x} - \bar{y}\| + n^{-1})^{d(d-1)/2}}.
\]

The kernel \( P_n(W_\kappa^R) \) can be derived from (2.21) and will not be needed. We will need, however, the estimate for the kernel \( P_n(W_\kappa^T) \), which is also proved in [8].

**Lemma 2.5.** For \( x = (x_1, \ldots, x_d) \in T^d \) and \( y = (y_1, \ldots, y_d) \in T^d \),

\[
|P_n(W_\kappa^T; x, y)| \leq c \prod_{j=1}^{d+1} (\sqrt{x_1y_1} + n^{-1}\|ar{x} - \bar{y}\| + n^{-2})^{\kappa_j - 1} n^{-d(d-1)/2(||\bar{x} - \bar{y}\| + n^{-1})^{d(d-1)/2}},
\]

where \( \xi := (\sqrt{x_1}, \ldots, \sqrt{x_d}, \sqrt{x_{d+1}}), \zeta := (\sqrt{y_1}, \ldots, \sqrt{y_d}, \sqrt{y_{d+1}}) \) with \( x_{d+1} := 1 - |x| \) and \( y_{d+1} := 1 - |y| \).

### 3. Main Results

#### 3.1. \( h \)-harmonic expansions.

For \( h_\kappa \) defined in (1.2), we denote the \( L^p \) norm of \( L^p(h_\kappa^2; S^d) \) by \( \| \cdot \|_{\kappa, p} \),

\[
\|f\|_{\kappa, p} := \left( a_\kappa \int_{S^d} |f(y)|^p h_\kappa^2(y) d\omega(y) \right)^{1/p}.
\]
for \(1 \leq p < \infty\) and with the usual understanding that it is the uniform norm on \(S^d\) when \(p = \infty\). Recall that

\[
\sigma_k := \frac{d+1}{2} + |\kappa| - \kappa_{\text{min}} \quad \text{with} \quad \kappa_{\text{min}} := \min_{1 \leq j \leq \theta+1} \kappa_j.
\]

Our main results on the Cesàro summation of \(h\)-harmonic expansions are the following two theorems:

**Theorem 3.1.** Suppose that \(f \in L^p(h^2_\kappa; S^d), 1 \leq p \leq \infty, \frac{1}{p} - \frac{1}{2} \geq \frac{1}{2\sigma_k + 2}\) and

\[
\delta > \delta_k(p) := \max\{(2\sigma_k + 1)(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}, 0\}.
\]

Then \(S^d_\kappa(h^2_\kappa; f)\) converges to \(f\) in \(L^p(h^2_\kappa; S^d)\) and

\[
\sup_{n \in \mathbb{N}} \|S^d_\kappa(h^2_\kappa; f)\|_{\kappa, p} \leq c\|f\|_{\kappa, p}.
\]

**Theorem 3.2.** Assume \(1 \leq p \leq \infty\) and \(0 < \delta \leq \delta_k(p)\). Then there exists a function \(f \in L^p(h^2_\kappa; S^d)\) such that \(S^d_\kappa(h^2_\kappa; f)\) diverges in \(L^p(h^2_\kappa; S^d)\).

For \(\kappa = 0\), \(h_\kappa(x) \equiv 1\) and the spherical \(h\)-harmonic becomes the ordinary spherical harmonics. Hence Theorem 3.1 is the complete analogue of the Sogge theorem, while Theorem 3.2 is the analogue of \cite{3} Theorem 5.2 for spherical harmonics.

For the projection operator \(\text{proj}_n(h^2_\kappa; f)\) we have the following theorem, which is a complete analogue of a theorem due to Sogge \cite{14} for spherical harmonics.

**Theorem 3.3.** Let \(d \geq 2\) and \(n \in \mathbb{N}\). Then

(i) for \(1 \leq p \leq \frac{2(\sigma_k + 1)}{\sigma_k + 2}\),

\[
\|\text{proj}_n(h^2_\kappa; f)\|_{\kappa, 2} \leq cn^{\delta_k(p)}\|f\|_{\kappa, p},
\]

with \(\delta_k(p)\) given in (3.1);

(ii) for \(\frac{2(\sigma_k + 1)}{\sigma_k + 2} \leq p \leq 2\),

\[
\|\text{proj}_n(h^2_\kappa; f)\|_{\kappa, 2} \leq cn^{\sigma_k(\frac{1}{2} - \frac{1}{p})}\|f\|_{\kappa, p}.
\]

Furthermore, the estimate (i) is sharp in terms of the order of \(n\).

The estimate in (ii) is sharp if \(\kappa = 0\) as shown in \cite{14}. We expect that it is also sharp for \(\kappa \neq 0\) but could not prove it at this moment. For further discussion on this point, see Remark 5.1 in Section 5.

For the spherical harmonics, the above theorem is enough for the proof of the boundedness of the Cesàro means. (See \cite{3} and \cite{15}.) For \(h\)-harmonics, however, a stronger result is needed since \(\delta_k(p) > \delta(p) := \max\{d\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\}\).

**Theorem 3.4.** Suppose that \(1 \leq p \leq \frac{2\sigma_k + 2}{\sigma_k + 2}\) and that \(f\) is supported in a spherical cap \(c(\varpi, \theta)\) with \(\theta \in (n^{-1}, 1]\) and \(\varpi \in S^d\). Then

\[
\|\text{proj}_n(h^2_\kappa; f)\|_{\kappa, 2} \leq cn^{\delta_k(p)\theta^\delta_k(p) + \frac{1}{2}}\left[\int_{c(\varpi, \theta)} h^2_\kappa(x) d\varpi(x)\right]^2 \|f\|_{\kappa, p}.
\]

The above theorems on the projection operators will be proved in Section 5 and the theorems on Cesàro means will be proved in Section 6.
3.2. Orthogonal expansions on the ball and on the simplex. Let $\Omega^d$ stand for either $B^d$ or $T^d$ and $W^\Omega_k$ stand for either $W^B_k$ or $W^T_k$, respectively. We denote the $L^p$ norm of $L_p(W^\Omega_k; \Omega^d)$ by $\| \cdot \|_{W^\Omega_k, p}$,

$$\| f \|_{W^\Omega_k, p} := \left( a_k^\Omega \int_{\Omega^d} |f(y)|^p W^\Omega_k(y) dy \right)^{1/p}$$

for $1 \leq p < \infty$ and with the usual understanding that it becomes the uniform norm on $\Omega^d$ when $p = \infty$.

Our main results on the Cesàro summation of orthogonal expansions on $B^d$ and $T^d$ are the following two theorems:

**Theorem 3.5.** Suppose that $f \in L_p(W^\Omega_k; \Omega^d)$, $1 \leq p \leq \infty$, $|\frac{1}{p} - \frac{1}{2}| \geq \frac{1}{2k+2}$ and $\delta > \delta_k(p) := \max\{ (2k + 1)(|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0) \}$.

Then $S_n^\delta(W^\Omega_k; f)$ converges to $f$ in $L_p(W^\Omega_k; \Omega^d)$ and

$$\sup_{n \in \mathbb{N}} \| S_n^\delta(W^\Omega_k; f) \|_{W^\Omega_k, p} \leq c \| f \|_{W^\Omega_k, p}.$$ 

**Theorem 3.6.** Assume $1 \leq p \leq \infty$ and $0 < \delta \leq \delta_k(p)$. Then there exists a function $f \in L_p(W^\Omega_k; \Omega^d)$ such that $S_n^\delta(W^\Omega_k; f)$ diverges in $L_p(W^\Omega_k; \Omega^d)$.

For $d = 1$ and $\Omega = T^1 = [0, 1]$, these theorems become results for the Jacobi polynomial expansions ([6]). For $d = 1$ and $\Omega = B^1 = [-1, 1]$, these theorems become results for the generalized Gegenbauer polynomial expansions with respect to $w_{\kappa_1, \kappa_2}$ in (2.10), which appear to be new if $\kappa_1 \neq 0$ while the case $\kappa = 0$ corresponds to the Gegenbauer polynomial expansions [2] [3]. We state the result as follows:

**Corollary 3.7.** Suppose that $1 \leq p \leq \infty$, $|\frac{1}{p} - \frac{1}{2}| \geq \frac{1}{2\max(\lambda, \mu) + 2}$ and $\delta > \delta(p) := \max\{ (2\max(\lambda, \mu) + 1)(|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0) \}$.

Then $S_n^\delta(w_{\lambda, \mu}; f)$ converges to $f$ in $L_p(w_{\lambda, \mu}; [-1, 1])$ and

$$\sup_{n \in \mathbb{N}} \| S_n^\delta(w_{\lambda, \mu}; f) \|_{w_{\lambda, \mu}, p} \leq c \| f \|_{w_{\lambda, \mu}, p}.$$ 

Furthermore, the condition $\delta > \delta(p)$ is sharp.

The result analogous to Theorem 3.3 also holds for the projection operator.

**Theorem 3.8.** Let $d \geq 2$ and $n \in \mathbb{N}$. Then

(i) for $1 \leq p \leq \frac{2(\sigma_k + 1)}{\sigma_k + 2}$,

$$\| \text{proj}_n(W^\Omega_k; f) \|_{W^\Omega_k, 2} \leq cn^\delta_k(p) \| f \|_{W^\Omega_k, p};$$

(ii) for $\frac{2(\sigma_k + 1)}{\sigma_k + 2} \leq p \leq 2$,

$$\| \text{proj}_n(W^\Omega_k; f) \|_{W^\Omega_k, 2} \leq cn^{\sigma_k(\frac{1}{p} - \frac{1}{2})} \| f \|_{W^\Omega_k, p}.$$ 

Furthermore, the estimate in (i) is sharp.

An analogue of Theorem 3.3 also holds. We state only the one for $W^T_k$ for which we define a distance on $T^d$,

$$d_T(x, y) := \arccos(\sqrt{x_1 y_1 + \ldots + \sqrt{x_d y_d} + \sqrt{1 - |x| \sqrt{1 - |y|}}},$$

where $|z| = |z_1| + \ldots + |z_d|$ for $z \in \mathbb{R}^d$. The analogue of the spherical cap on $T^d$ is defined as $c_T(x, \theta) := \{ y \in T^d : d_T(x, y) \leq \theta \}$. 

Suppose \(1 \leq p \leq \frac{2n+2}{\sigma_\kappa+2}\) and \(f\) is supported in the set \(c_T(x, \theta)\) with \(\theta \in (n^{-1}, \pi]\) and \(x \in T^d\). Then

\[
\|\text{proj}_n(W^T_\kappa : f)\|_{W^2_{\nu,2}} \leq c n^{\delta_\kappa(p)\delta_\kappa(p)+\frac{1}{2} + 1} \left[ \int_{c_T(x, \theta)} W^T_\kappa(y) \, dy \right]^{\frac{1}{2} - \frac{1}{p} + \frac{1}{2}} \|f\|_{W^2_{\nu,p}}.
\]

The analogous result for \(B^d\) holds with \(c_T(x, \theta)\) replaced by \(c_B(x, \theta)\) defined in terms of \(d_B(x, \theta) := \arccos(\langle x, y \rangle + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2})\).

These results will be proved in Section 7.

4. Local estimate of projection operator

The main effort in the proof of Theorem 3.4, given in the next section, lies in proving the following local estimate of the projection operator.

**Theorem 4.1.** Let \(\nu := \frac{2n+2}{\sigma_\kappa+2}\) and \(\nu' := \frac{\nu}{\nu-1}\). Let \(f\) be a function supported in a spherical cap \(c(\varpi, \theta)\) with \(\theta \in (n^{-1}, 1/(8d)]\) and \(\varpi \in S^d\). Then

\[
\|\text{proj}_n(h^2_\kappa : f)\chi_{c(\varpi, \theta)}\|_{\nu',\nu} \leq c n^{\frac{\nu'}{\nu-1}\theta^{\sigma_\kappa+1}} \left[ \int \h^2_\kappa(x) \, d\omega(x) \right]^{1-\frac{\nu}{\nu'}} \|f\|_{\nu',\nu}.
\]

Here \(\chi_E\) denotes the characteristic function of the set \(E\). Note the norm of the left-hand side is taken over \(c(\varpi, \theta)\), so that the above estimate is a local one.

Throughout this section, we shall fix the spherical cap \(c(\varpi, \theta)\). Without loss of generality, we may assume \(\varpi = (\varpi_1, \cdots, \varpi_0, 1)\) satisfying \(|\varpi_k| \geq 4\theta\) for \(1 \leq k \leq v\) and \(|\varpi_v| < 4\theta\) for \(v < k \leq d + 1\). Accordingly, we define

\[
\gamma = \gamma_{\varpi} := \begin{cases} 0, & \text{if } v = d + 1; \\ \sum_{i=v+1}^{d+1} \kappa_i, & \text{if } v < d + 1. \end{cases}
\]

Since \(\theta \in (0, 1/(8d)]\) and \(\varpi \in S^d\), it follows that

\[
0 \leq \gamma \leq |\kappa| - \min_{1 \leq i \leq d+1} \kappa_i = \sigma_\kappa - \frac{d-1}{2}.
\]

The proof of Theorem 4.1 consists of two cases, one for \(\gamma < \sigma_\kappa - \frac{d-1}{2}\) and the other for \(\gamma = \sigma_\kappa - \frac{d-1}{2}\), using different methods.

4.1. **Proof of Theorem 4.1 case I:** \(\gamma < \sigma_\kappa - \frac{d-1}{2}\). The proof is long and will be divided into several subsections.

4.1.1. *Decomposition of the projection operator.* Recall \(\lambda_\kappa = \frac{d-1}{2} + |\kappa|\). Let \(\xi_0 \in C_\infty[0, \infty)\) be such that \(\chi_{[0,1/2]}(t) \leq \xi_0(t) \leq \chi_{[0,1]}(t)\), and define \(\xi_1(t) := \xi_0(t/4) - \xi_0(t)\). Evidently \(\xi_1 \subset (1/2, 4)\) and \(\xi_0(t) + \sum_{j=1}^\infty \xi_1(4^{-j+1}t) = 1\) whenever \(t \in [0, \infty)\). Define, for \(u \in [-1, 1]\),

\[
C_{n,0}(u) := \frac{n + \lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(u) \xi_0 \left( n^2(1 - u^2) \right),
\]

\[
C_{n,j}(u) := \frac{n + \lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(u) \xi_1 \left( \frac{n^2(1 - u^2)}{4^{j-1}} \right), \quad j = 1, 2, \ldots, L_n,
\]

respectively.
where \( L_n := \lfloor \log_2 n \rfloor + 2 \). By (2.14), \( \text{proj}_n(h_n^2;f) \) can be decomposed as

\[
(4.3) \quad \text{proj}_n(h_n^2;f) = \sum_{j=0}^{L_n} Y_{n,j} f, \quad \text{where} \quad Y_{n,j} f := f * \kappa_{n,j}.
\]

By the definition of the convolution, the kernel of \( Y_{n,j} \) is \( V_n[C_{n,j}(\langle x, \cdot \rangle)](y) \).

4.1.2. Estimates of the kernels \( V_n[C_{n,j}(\langle x, \cdot \rangle)](y) \) and \( L^\infty \) estimate.

**Definition 4.2.** Given \( n, v \in \mathbb{N}_0 \), and \( \mu \in \mathbb{R} \), we say a continuous function \( F : [-1, 1] \to \mathbb{R} \) belongs to the class \( S_n^v(\mu) \) if there exist functions \( F_j, j = 0, 1, \ldots, v \) on \([-1, 1]\) such that \( F_j(t) = F(t), \ t \in [-1, 1], \ 0 \leq j \leq v, \) and

\[
|F_j(t)| \leq n^{-2j+\mu} \left( 1 + n \sqrt{1 - |t|} \right)^{-\mu - \frac{1}{2} + j}, \quad t \in [-1, 1], \ j = 0, 1, \ldots, v.
\]

By (2.18) and the following well-known formula [7] (4.21.7),

\[
(4.5) \quad \frac{d}{dt} P_n^{(\alpha, \beta)}(t) = \frac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(t),
\]

it follows that \( c_{v, \alpha} P_n^{(\alpha, \beta)} \in S_n^v(\alpha) \) for all \( v \in \mathbb{N}_0 \) whenever \( \alpha \geq \beta \).

**Lemma 4.3.** Assume that \( \delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m \) satisfying \( \min_{1 \leq j \leq m} \delta_j > 0 \) and \( \mu \in \mathbb{R} \). Let \( F \in S_n^v(\mu) \) with \( v \geq 2m + m \sum_{j=1}^m \delta_j + |\mu| \). Let \( \xi \) be a \( C^\infty \) function, supported in \([-8, 8]\) and equal to a constant in a neighborhood of 0. For \( \rho \in (n^{-1}, 4) \), define

\[
G(u) := F(u) \xi \left( \frac{1 - u^2}{\rho^2} \right), \quad u \in [-1, 1].
\]

Then for \( s \in [-1, 1] \) and \( a = (a_1, \ldots, a_m) \in [-1, 1]^m \) satisfying \( \sum_{j=1}^m |a_j| + |s| \leq 1, \)

\[
(4.6) \quad \left| \int_{[-1,1]^m} G(\sum_{j=1}^m a_j t_j + s) \prod_{j=1}^m (1 - t_j^2)^{\delta_j - 1} (1 + t_j) \ dt_j \right| \leq cn^{-\frac{1}{2} - |\delta|} \rho^{-|\mu| - \frac{1}{2} m} \prod_{j=1}^m (|a_j| + n^{-1} \rho)^{-\delta_j},
\]

where \( |\delta| = \sum_{j=1}^m \delta_j \).

**Proof.** Without loss of generality, we may assume that \( |a_j| \geq n^{-1} \rho \) for \( 1 \leq j \leq m, \) since otherwise we can modify the proof by replacing \( s \) with

\[
s + \sum_{j: |a_j| < n^{-1} \rho} a_j t_j.
\]

Let \( \eta_0 \in C^\infty(\mathbb{R}) \) be such that \( \eta_0(t) = 1 \) for \( |t| \leq \frac{1}{4} \) and \( \eta_0(t) = 0 \) for \( |t| \geq 1, \) and let \( \eta_1(t) = 1 - \eta_0(t) \). Set \( B_j := \frac{\eta_1(t)}{n|a_j|}, j = 1, \ldots, m. \) Given \( \varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m, \) we define \( \psi_\varepsilon : [-1, 1]^m \to \mathbb{R} \) by

\[
\psi_\varepsilon(t) := \xi \left( \frac{1 - \left( \sum_{j=1}^m a_j t_j + s \right)^2}{\rho^2} \right) \prod_{j=1}^m \eta_{\varepsilon_j} \left( \frac{1 - t_j^2}{B_j} \right) (1 + t_j) (1 - t_j^2)^{\delta_j - 1},
\]
where \( t = (t_1, \cdots, t_m) \). We then split the integral in (4.6) into a finite sum:
\[
\sum_{\varepsilon \in \{0,1\}^m} \int_{[-1,1]^m} F\left( \sum_{j=1}^m a_j t_j + s \right) \psi_\varepsilon(t) \, dt =: \sum_{\varepsilon \in \{0,1\}^m} J_\varepsilon.
\]
Thus, it is sufficient to prove that each term \( J_\varepsilon \) in the above sum satisfies the desired inequality. By symmetry and Fubini’s theorem, we need only to consider the case when \( \varepsilon_1 = \cdots = \varepsilon_{m_1} = 0 \) and \( \varepsilon_{m_1+1} = \cdots = \varepsilon_m = 1 \) for some \( 0 \leq m_1 \leq m \).

Let \( m_1 \) and \( \varepsilon \) be fixed as in the last line. Fix \( (t_1, \cdots, t_{m_1}) \in [-1,1]^{m_1} \) momentarily, and write \( s_1 = \sum_{j=1}^{m_1} a_j t_j + s \). Define
\[
\phi(t) := \xi \left( 1 - \frac{\sum_{j=1}^{m_1} a_j t_j + s}{\rho^2} \right) \prod_{j=m_1+1}^m \eta\left( \frac{1-t_j^2}{B_j} \right)(1+t_j)(1-t_j^2)^{\delta_j-1}.
\]
Since the support set of each \( \eta\left( \frac{1-t_j^2}{B_j} \right) \) is a subset of \( \{t_j : |t_j| \leq 1 - \frac{1}{2}B_j \} \), we can use integration by parts \( |t| = \sum_{j=m_1+1}^m \ell_j \) times on the function \( F = F_{|t|} \) as in Definition 4.2 where \( |t| = (\ell_{m_1+1}, \cdots, \ell_m) \in \mathbb{N}^{m-m_1} \) satisfies \( \ell_j > \delta_j \) and \( |t| \geq \mu + \frac{1}{2} \), which gives
\[
\left| \int_{[-1,1]^{m-m_1}} F\left( \sum_{j=m_1+1}^m a_j t_j + s_1 \right) \phi(t) \, dt \right|
= \prod_{j=m_1+1}^m |a_j|^{\ell_j} \left| \int_{[-1,1]^{m-m_1}} F_{|t|}\left( \sum_{j=m_1+1}^m a_j t_j + s_1 \right) \frac{\partial |t| \phi(t)}{\partial^{\ell_{m_1+1} t_{m_1+1} \cdots \partial^{\ell_m} t_m}} \, dt \right|
\leq \prod_{j=m_1+1}^m |a_j|^{\ell_j} \left| \int_{[-1,1]^{m-m_1}} F_{|t|}\left( \sum_{j=m_1+1}^m a_j t_j + s_1 \right) \left| \frac{\partial |t| \phi(t)}{\partial^{\ell_{m_1+1} t_{m_1+1} \cdots \partial^{\ell_m} t_m}} \right| \, dt \right|.
\]

Since \( \xi \) is supported in \((-8,8)\), the integrand of the last integral is zero unless
\[
(4.7) \quad 8\rho^2 \geq 1 - \left| \sum_{k=m_1+1}^m a_k t_k + s_1 \right|
\geq 1 - \left| \sum_{k=m_1+1}^m a_k \right| - |s_1| + (1 - |t_j|)|a_j| \geq |a_j|(1 - |t_j|),
\]
for all \( m_1 + 1 \leq j \leq m \); that is, \( |a_j| \leq 8(1 - |t_j|)^{-1} \) for \( j = m_1+1, \cdots, m \). Also, recall that \( \xi \) is constant near 0. Hence, taking the \( k \)-th partial derivative with respect to \( t_j \), the \( \xi \) part of \( \phi \) is bounded by \( c(1 - t_j)^{-k} \). Also bounded is the same derivative of the \( \eta \) part of \( \phi \) since \( B_j^{-1} \leq (1 - t_j^2)^{-1} \) in the support of \( \eta \). Consequently, by the Leibnitz rule, we conclude
\[
\left| \frac{\partial |t| \phi(t)}{\partial^{\ell_{m_1+1} t_{m_1+1} \cdots \partial^{\ell_m} t_m}} \right| \leq c \prod_{j=m_1+1}^m (1 - |t_j|)^{\delta_j-\ell_j-1}
\]
in the support of the integrand. Next, since \( \rho \geq n^{-1} \) and \( |t| \geq \mu + \frac{1}{2} \), (4.7) together with (4.4) implies
\[
\left| F_{|t|}\left( \sum_{k=m_1+1}^m a_k t_k + s_1 \right) \right| \leq cn^{-\frac{1}{2} - |t|} \rho^{-\mu - \frac{1}{2} + |t|}.
\]
It follows that
\[
\int_{[-1,1]^{m-1}} |F| \left( \sum_{j=m_1+1}^{m} a_j t_j + s_1 \right) \left| \frac{\partial^{m_1} \phi(t)}{\partial t_{m_1+1} \cdots \partial t_m} \right| dt \\
\leq cn^{-\frac{1}{2} - |l|} \rho^{-\frac{\mu}{2} - \frac{1}{2} + |l|} \prod_{j=m_1+1}^{m} \int_{0}^{1} (1 - t_j) \delta_j \, dt_j \\
\leq cn^{-\frac{1}{2} - |l|} \rho^{-\frac{\mu}{2} - \frac{1}{2} + |l|} m \quad B_j^{\delta_j - \ell_j} \\
\leq cn^{-\frac{1}{2} - \alpha} \rho^{\alpha - \frac{\mu}{2}} \prod_{j=m_1+1}^{m} |a_j|^{\ell_j - \delta_j},
\]
where \( \alpha = \sum_{j=m_1+1}^{m} \delta_j \). Thus, since
\[
\psi_\varepsilon(t) = \phi(t) \prod_{j=1}^{m_1} \eta_0 \left( \frac{1 - t_j^2}{B_j} \right) (1 + t_j)(1 - t_j^{2\delta_j - 1}),
\]
and \( \eta_0 \left( \frac{1 - t_j^2}{B_j} \right) \) is supported in \( \{ t_j : 1 - B_j \leq |t_j| \leq 1 \} \), integrating with respect to \( t_1, \ldots, t_{m_1} \) over \([-1, 1]^{m_1}\) yields
\[
J_\varepsilon \leq \int_{[-1,1]^{m_1}} \left| \int_{[-1,1]^{m-1}} F \left( \sum_{j=1}^{m} a_j t_j + s_1 \right) \phi(t) \, dt \right|_{m_1+1} dt_m \\
\times \prod_{j=1}^{m_1} \eta_0 \left( \frac{1 - t_j^2}{B_j} \right) (1 + t_j)(1 - t_j^{2\delta_j - 1}) dt_j \\
\leq cn^{-\frac{1}{2} - \alpha} \rho^{\alpha - \frac{\mu}{2}} \prod_{j=m_1+1}^{m} |a_j|^{-\delta_j} \prod_{j=1}^{m_1} \int_{1-B_j \leq |t_j| \leq 1} (1 - |t_j|)^{\delta_j - 1} dt_j \\
\leq cn^{-\frac{1}{2} - |\delta|} \rho^{\delta - \mu + \frac{1}{2}} \prod_{j=1}^{m} |a_j|^{-\delta_j},
\]
where we have used \( |a_j|^\ell_j \leq 1 \) in the second step. This completes the proof. \( \square \)

Using the relation between the Gegenbauer and the Jacobi polynomials, we have
\[
C_{n,j}(u) = a_n P_{n}^{(\lambda_n, -\frac{1}{2}, \lambda_n, -\frac{1}{2})}(u) \xi \left( \frac{1 - u^2}{(2j-1/n)^2} \right),
\]
where \( \xi = \xi_1 \) or \( \xi_0 \), and \( |a_n| \leq cn^{\lambda_n + \frac{1}{2}} \). Hence, using the fact that \( c_{\nu, \kappa} P_{n}^{(\lambda_n, -\frac{1}{2}, \lambda_n, -\frac{1}{2})} \in S_0^{n}(\lambda_n, -\frac{1}{2}) \) for all \( \nu \in \mathbb{N} \), Lemma 4.3 has the following corollary.

**Corollary 4.4.** For \( x, y \in S^d \) and \( j = 1, 2, \ldots, L_n \),
\[
\left| V_{\kappa} [C_{n,j}(\langle x, \cdot \rangle)](y) \right| \leq cn^{d-1} 2^{-j(d-1)/2} \prod_{i=1}^{d+1} (|x_i y_i| + 2^j n^2)^{-\kappa_i}.
\]
Lemma 4.5. If \( f \) is supported in \( c(\varpi, \theta) \), then

\[
\sup_{x \in c(\varpi, \theta)} |Y_{n,j}(f)(x)| \leq \left( \frac{d+1}{2} + \gamma \right) g \gamma + d \left[ \int_{c(\varpi, \theta)} h^2_k(x) \, d\varpi(x) \right]^{-1} \|f\|_{\kappa, 1}.
\]

Proof. Note that if \( x \in c(\varpi, \theta) \), then \( |x_i - \varpi_i| \leq ||x - \varpi|| \leq d(x, \varpi) \leq \theta \) so that \( \frac{\theta}{2} |\varpi_i| \leq \frac{\theta}{2} |x_i| \) for \( 1 \leq i \leq v \), and \( |x_i| \leq 5\theta \) for \( v+1 \leq i \leq d+1 \). It follows from Corollary \([4.1]\) that, for any \( x, y \in c(\varpi, \theta) \),

\[
|V_n C_{n,j}(\langle x, \cdot \rangle))(y)| \leq c n^{d-1} 2^{-j(d-1)/2} \prod_{i=1}^v |\varpi_i|^{-2\kappa_i} \prod_{i=v+1}^{d+1} n^{2\kappa_i} 2^{-j\kappa_i}.
\]

\[
\leq c n^{d-1} 2^{-j\left( \frac{d+1}{2} + \gamma \right) + (n\theta)^{2\gamma} \prod_{i=1}^{d+1} (|\varpi_i| + \theta)^{-2\kappa_i}.
\]

\[
\leq c n^{d-1} 2^{-j\left( \frac{d+1}{2} + \gamma \right) \theta^{2\gamma} g \gamma + d} \left[ \int_{c(\varpi, \theta)} h^2_k(z) \, d\omega(z) \right]^{-1},
\]

where the last step follows from the relation \((2.17)\). This implies that

\[
\sup_{x \in c(\varpi, \theta)} |Y_{n,j}(f)(x)| \leq \sup_{x \in c(\varpi, \theta)} \int_{c(\varpi, \theta)} |f(y)| |V_n C_{n,j}(\langle x, \cdot \rangle))(y)| h^2_k(y) \, d\omega(y)
\]

\[
\leq c n^{d-1} 2^j \gamma + d \left[ \int_{c(\varpi, \theta)} h^2_k(x) \, d\omega(x) \right]^{-1} \|f\|_{\kappa, 1},
\]

which is the desired inequality. \( \square \)

4.1.3. \( L^2 \) estimates. We prove the following estimate:

Lemma 4.6. For any \( f \in L^2(h^2_k; S^d) \),

\[
\|Y_{n,j}(f)\|_{\kappa, 2} \leq c n^{-1} 2^j \|f\|_{\kappa, 2}.
\]

Proof. For simplicity, we shall write \( \xi_j = \xi_1 \) for \( j \geq 1 \). Also let \( \lambda = \lambda_k \) in this proof. From \((2.5)\) and the definition of \( Y_{n,j} \) in \((4.3)\), it follows that each \( Y_{n,j} \) is a multiplier operator,

\[
Y_{n,j}(f) = \sum_{k=0}^{\infty} m_{n,j}(k) \text{proj}_k(h^2_k; f),
\]

where the equality is understood in a distributional sense, and

\[
m_{n,j}(k) := c_{n,k} \int_0^\pi C_n^\lambda(\cos t) C_k^\lambda(\cos t) \xi_j \left( \frac{n^2 \sin^2 t}{4^j + 1} \right) \sin 2\lambda t \, dt
\]

with \( |c_{n,k}| \leq c n^{-2\lambda + 1} \). Hence, it is enough to prove

\[
(4.8) \quad \sup_k |m_{n,j}(k)| \leq c n^{-1} 2^j.
\]
If \( k \geq \frac{n}{4} \), then using the fact that \(|(\sin \theta)^{\lambda} C_{n}^{\lambda}(\cos \theta)| \leq cn^{\lambda-1}\), a straightforward computation gives

\[
|m_{n,j}(k)| \leq |c_{n,k}| \int_{0}^{\pi} \left| C_{n}^{\lambda}(\cos t) C_{k}^{\lambda}(\cos t) \xi_{j} \left( \frac{n^2 \sin^2 t}{4j-1} \right) \right| \sin^{2\lambda} t \, dt
\]

\[
\leq c \int_{0}^{\pi} \left| \xi_{j} \left( \frac{n^2 \sin^2 t}{4j-1} \right) \right| \, dt \leq \frac{c2^j}{n},
\]

where the last step follows easily using the support of \( \xi_{j} \).

For \( k \leq \frac{n}{4} \), we shall use the following formula (cf. [1, p. 319, Theorem 6.8.2]):

\[
(4.9) \quad C_{n}^{\lambda}(t)C_{k}^{\lambda}(t) = \sum_{i=0}^{\min\{k,n\}} a(i, k, n) C_{k+n-2i}(t),
\]

where

\[
a(i, k, n) := \frac{\Gamma(k + n + \lambda - 2i)(\lambda)^{k}(\lambda)_{n-i}(2\lambda)_{k+n-i}(k + n - 2i)!}{\Gamma(k + n + \lambda - i)!(n - i)!^{2}(\lambda)_{k+n-i}(2\lambda)_{k+n-2i}.}
\]

For \( k \leq n/4 \), it is easy to see that

\[
|a(i, k, n)| \sim \left( \frac{(i + 1)(\min\{k, n\} - i + 1)(k + n - i + 1)}{k + n - 2i + 1} \right)^{\lambda-1}
\]

\[
(4.10) \quad \sim (i + 1)^{\lambda-1}(k - i + 1)^{\lambda-1}.
\]

Consequently, it follows that for \( k \leq n/4 \),

\[
|m_{n,j}(k)| \leq cnk^{-2\lambda+1} \sum_{i=0}^{k} (i + 1)^{\lambda-1}
\]

\[
\times (k - i + 1)^{\lambda-1} \left| \int_{0}^{\pi} C_{k+n-2i}^{\lambda}(\cos t) \xi_{j} \left( \frac{n^2 \sin^2 t}{4j-1} \right) \sin^{2\lambda} t \, dt \right|
\]

\[
\leq cn^{\lambda+\frac{1}{2}} \max_{3n/4 \leq m \leq 5n/4} \left| \int_{-1}^{1} \mathcal{P}_{m}^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(s) \xi_{j} \left( \frac{n^2 (1-s^2)}{4j-1} \right) (1-s^2)^{\lambda-\frac{1}{2}} \, ds \right|
\]

Then using the estimate (2.18) we obtain

\[
m_{n,0}(k) \leq cn^{\lambda-\frac{1}{2}} \int_{1-|s| \leq cn^{-2}} (1-|s|)^{\lambda-\frac{1}{2}} \, ds \leq cn^{-1}.
\]

If \( j \geq 1 \), then for all \( \ell \in \mathbb{N} \), it follows that

\[
\left| \frac{d^{\ell}}{ds^{\ell}} \left( \xi_{j} \left( \frac{n^2 (1-s^2)}{4j-1} \right) (1-s^2)^{\lambda-\frac{1}{2}} \right) \right| \leq c \left( \frac{2j}{n} \right)^{2\lambda-1-2\ell},
\]

since \( 1-s^2 \sim \left( \frac{2j}{n} \right)^2 \) in the support of \( \xi_{j}^{\prime} \); consequently, we obtain by integration by parts, (4.5) and (2.18) that

\[
m_{n,j}(k) \leq cn^{\lambda+\frac{1}{2}-\ell} \max_{3n/4 \leq m \leq 5n/4} \left| \int_{-1}^{1} \mathcal{P}_{m+\ell}^{(\lambda-\frac{1}{2}-\ell,\lambda-\frac{1}{2}-\ell)}(s) \frac{d^{\ell}}{ds^{\ell}} \left( \xi_{j} \left( \frac{n^2 (1-s^2)}{4j-1} \right) (1-s^2)^{\lambda-\frac{1}{2}} \right) \, ds \right|
\]

\[
\leq c2^{(\lambda-\ell)}2^{j}n^{-1} \leq c2^{j}n^{-1}
\]

upon choosing \( \ell \geq \lambda \). Thus, in both cases, we get the desired estimate. \( \Box \)
4.1.4. **Proof of Theorem 4.11** case I: \( \gamma < \sigma_\kappa - \frac{d-1}{2} \). Recall \( \nu = \frac{2+2\sigma_\kappa}{2+\sigma_\kappa} \). We set, in this subsection,

\[
A := \int_{c(\varpi, \theta)} h^2_n(y) \, d\omega(y).
\]

Recall the decomposition [13]. For a generic \( f \), we set

\[
T_{n,j} f := Y_{n,j}(f X_{c(\varpi, \theta)} X_{c(\varpi, \theta)}), \quad 0 \leq j \leq L_n.
\]

Clearly, if \( f \) is supported in \( c(\varpi, \theta) \) and \( x \in c(\varpi, \theta) \), then \( T_{n,j} f(x) = Y_{n,j} f(x) \).

Using Lemmas 4.5 and 4.6, we have

\[
\|T_{n,j} f\|_{\kappa,1} \leq c n^{\gamma+d-1} 2^{j} (\frac{d+1}{2} + \gamma) \theta^{2\gamma + d} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,1},
\]

\[
\|T_{n,j} f\|_{\kappa,2} \leq c n^{\gamma+d-1} \|f\|_{c(\varpi, \theta)} \|f\|_{\kappa,2}.
\]

Hence, by the Riesz-Thorin convexity theorem, we obtain (recall \( \nu = \frac{2+2\sigma_\kappa}{2+\sigma_\kappa} \))

\[
\|T_{n,j} f\|_{\kappa,2} \leq c n^{\gamma + d} 2^{-j} (\frac{d+1}{2} + \gamma) \theta^{2\gamma + d} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,2}.
\]

On the other hand, using (4.11), Hölder’s inequality and (2.17), we obtain

\[
\|T_{n,j} f\|_{\kappa,\nu'} \leq c n^{\gamma+d-1} 2^{-j} (\frac{d+1}{2} + \gamma) \theta^{2\gamma + d} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'}.
\]

(4.13)

Now assume that \( f \) is supported in \( c(\varpi, \theta) \) and \( \frac{2j_0}{n} \leq \theta \leq \frac{2j_0}{n} \) for some \( 1 \leq j_0 \leq L_n \). Using (4.13) and Minkowski’s inequality, we have

\[
\|\text{proj}_n(h^2_n f) X_{c(\varpi, \theta)}\|_{\kappa,\nu'} = \sum_{j=0}^{2j_0} \|T_{n,j} f\|_{\kappa,\nu'} + \sum_{j=2j_0+1}^{L_n} \|T_{n,j} f\|_{\kappa,\nu'} =: \Sigma_1 + \Sigma_2.
\]

For the first sum \( \Sigma_1 \), we use (4.12) to obtain

\[
\Sigma_1 \leq c n^{-1} (n\theta)^{\frac{2\gamma+d}{\sigma_\kappa+1}} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'} \sum_{j=0}^{2j_0} 2^{j(1-(\frac{d+1}{2}+\gamma) \frac{1}{\sigma_\kappa+1})}
\]

\[
\leq c n^{\gamma+d} \theta^{\frac{2\gamma+d}{\sigma_\kappa+1}} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'},
\]

since \( \gamma < \sigma_\kappa - \frac{d-1}{2} \) readily implies that \( 1 - (\frac{d+1}{2} + \gamma) \frac{1}{\sigma_\kappa+1} > 0 \). For the second sum \( \Sigma_2 \), we use (4.13) to obtain

\[
\Sigma_2 \leq c n^{-1} (n\theta)^{2\gamma+d} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'} \sum_{j=2j_0+1}^{\infty} 2^{-j(\frac{d+1}{2} + \gamma)}
\]

\[
\leq c n^{-1} (n\theta)^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'} \leq c n^{-1} (n\theta)^{\frac{2\gamma+d}{\sigma_\kappa+1}} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'}
\]

\[
= c n^{\gamma+d} \theta^{\frac{2\gamma+d}{\sigma_\kappa+1}} A^{1-\frac{2}{\nu}} \|f\|_{\kappa,\nu'}
\]

where in the third inequality we have used the fact that \( n\theta \geq 1 \).

Putting the above together proves Theorem 4.11 in the case \( \gamma < \sigma_\kappa - \frac{d-1}{2} \). \( \square \)
4.2. Proof of Theorem 4.1, case 2: \( \gamma = \sigma_\kappa - \frac{d+1}{2} \). Recall that \( |\varpi_j| \geq 4\theta \) for \( 1 \leq j \leq v, |\varpi_j| < 4\theta \) for \( v + 1 \leq j \leq d + 1 \), and \( \gamma = \gamma_{\varpi} = \sum_{j=v+1}^{d+1} \kappa_j \). In this case, either \( v = 1 \) and \( |\varpi_1| = \max_{1 \leq j \leq d+1} |\varpi_j| \geq \frac{2\kappa_1}{\sqrt{\kappa_1}} + 1 \); or \( v \geq 2 \) and \( \kappa_1 = \cdots = \kappa_v = 0 \). Therefore, by (2.14), we have

\[
\int_{c(\varpi, \theta)} h^2_k(x) \, d\omega(x) \sim \theta^d \left( \prod_{j=1}^{v} |\varpi_j|^{2\kappa_j} \right) \theta^{2\gamma} \sim \theta^{2\sigma_\kappa+1}.
\]

Hence, Theorem 4.1 in this case is equivalent to the following proposition:

**Proposition 4.7.** Let \( f \) be supported in \( c(\varpi, \theta) \) with \( \theta \in (n^{-1}, 1/(8d)] \) and let \( \nu := \frac{2\sigma_\kappa+2}{\sigma_\kappa+2} \) and \( \nu' := \frac{\nu}{\nu-1} \). Then

\[
\left\| \text{proj}_n(h^2_k; f) \right\|_{\kappa, \nu} \leq c \theta^{\frac{\sigma_\kappa}{\nu-\sigma_\kappa}} \| f \|_{\kappa, \nu}.
\]

To prove Proposition 4.7, we use the method of analytic interpolation [10]. For \( z \in \mathbb{C} \), define

\[
P^*_n f(x) := (f \ast G^*_n)(x) = a_n \int_{S^d} f(y) V_k \left[ G^*_n((x, \cdot)) \right](y) h^2_k(y) \, d\omega(y)
\]

for \( x \in S^d \), where

\[
G^*_n(t) = (\sigma_\kappa + 1)(1-z) \frac{n + \lambda_\kappa}{\lambda_\kappa} C^\lambda_\kappa(t)(1-t^2 + n^{-2})^{\frac{\sigma_\kappa - (\sigma_\kappa + 1)z}{2}}.
\]

From (2.14), it readily follows that

\[
P^*_n f = \text{proj}_n(h^2_k; f).
\]

For the rest of this subsection, we shall use \( c\_r \) to denote a general constant satisfying \( |c\_r| \leq c(1 + |\tau|)^\ell \) for some inessential positive number \( \ell \).

4.2.1. Estimate for \( z = 1 + i\tau \).

**Lemma 4.8.** For \( \tau \in \mathbb{R} \),

\[
\left\| P_n^{1+i\tau} f \right\|_{\kappa, 2} \leq c_r \| f \|_{\kappa, 2}.
\]

**Proof.** From (4.14), (4.16) and (26), it follows that

\[
\text{proj}_k(h^2_k; P_n^{1+i\tau} f) = J_n(k) \text{proj}_k(h^2_k; f), \quad k = 0, 1, \cdots,
\]

where

\[
J_n(k) := O(1) n^{k-2\lambda_\kappa+1}\tau' \int_0^\pi C^\lambda_\kappa(t) C^\lambda_\kappa(\cos t) \sin^2 t + n^{-2} \lambda_\kappa \sin^2 t \left( 1 - \lambda_\kappa \right) \sin(2\lambda_\kappa + 1) \sin t \, dt
\]

and \( \tau' = -\frac{2\lambda_\kappa+1}{\lambda_\kappa} \). Therefore, it is sufficient to prove

\[
|J_n(k)| \leq c_r, \quad \forall k, n \in \mathbb{N}.
\]

For \( k < \frac{n}{2} \), (4.16) can be shown as in the proof of Lemma 4.2. In fact, using (1.13) and (1.10), we obtain

\[
|J_n(k)| \leq c|\tau| n^{\lambda_\kappa+\frac{1}{2}}
\]

\[
\times \max_{3n/4 \leq s \leq 5n/4} \left| \int_{-1}^{1} P_m^{(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2})} (s)(1 - s^2 + n^{-2})^{-\frac{1}{2}+i\tau'} (1 - s^2)^{\lambda_\kappa - \frac{1}{2}} \, ds \right|,
\]

\[
\leq \frac{n^{\lambda_\kappa+\frac{1}{2}}}{4} \int_{-\frac{n}{2}}^{\frac{n}{2}} \left( \int_{-1}^{1} (1 - s^2 + n^{-2})^{-\frac{1}{2}+i\tau'} (1 - s^2)^{\lambda_\kappa - \frac{1}{2}} \, ds \right) \, ds
\]

\[
= \frac{n^{\lambda_\kappa+\frac{1}{2}}}{4} \int_{-\frac{n}{2}}^{\frac{n}{2}} \left( \int_{-1}^{1} (1 - s^2 + n^{-2})^{-\frac{1}{2}+i\tau'} (1 - s^2)^{\lambda_\kappa - \frac{1}{2}} \, ds \right) \, ds.
\]
which is controlled by
\[
c_r + c_r n^{\lambda_\alpha + \frac{1}{2} - \ell} \max_{3n/4 \leq m \leq 5n/4} \int_{-1}^{1} \left| P_m^{(\lambda_\alpha, -\frac{1}{2}, \lambda_\beta - \frac{1}{2})}(s) \right| ds \times \frac{d^\ell}{ds^\ell} \left( (1 - s^2 + n^{-2})^{-\frac{1}{2} + ir'} (1 - s^2) \lambda_\alpha - \frac{1}{2} \right) ds \leq c_r'
\]
using integration by parts \( \ell > \lambda_\kappa \) times. This proves (4.16) for \( k < \frac{n}{4} \).

For \( k \geq \frac{n}{4} \), (4.16) can be established exactly as in [14, pp. 54–55] (see also [13] pp. 76–81). For completeness, we sketch the proof as follows. Since \( C_{17}(t) = O(1) j^{\lambda - \frac{1}{2}} P_j^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(t) \) and \( P_j^{(\alpha, \beta)}(-t) = (-1)^j P_j^{(\alpha, \beta)}(t) \), we can write
\[
J_n(k) = O(1) k^{\lambda_\alpha + \frac{1}{2} n^{\lambda_\beta + \frac{1}{2} r'}} \left[ \int_0^{4n^{-1}} + \int_{4n^{-1}}^{\frac{\pi}{2}} \right] P_k^{(\lambda_\alpha, -\frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\cos t) (\cos t) \sin^2 t + n^{-2} \cos t \sin t 2^{2\lambda_\kappa} dt =: J_{n,1}(k) + J_{n,2}(k)
\]
Since \( |P_j^{(\alpha, \beta)}(t)| \leq cj^\alpha \), a straightforward calculation shows that \( |J_{n,1}(k)| \leq c_r \). To estimate \( J_{n,2}(k) \), we need the asymptotics of the Jacobi polynomials as given in [17, p. 198],
\[
P_j^{(\alpha, \beta)}(\cos t) = \pi^{-\frac{1}{4}} j^{-\frac{1}{2}} (\sin \frac{1}{2})^{-\frac{1}{2} - \frac{1}{2}} \left[ \cos(N_j t + \tau_\alpha) + O(1)(j \sin t)^{-1} \right] \]
for \( j^{-1} \leq t \leq j^{-1} \), where \( N_j = j + \frac{\alpha + \beta + 1}{2} \) and \( \tau_\alpha = -\frac{\pi}{2}(\alpha + \frac{1}{2}) \). Applying this asymptotic formula with \( \alpha = \beta = \lambda_\kappa - \frac{1}{2} \), we obtain, for \( k \geq \frac{n}{4} \) and \( 4n^{-1} \leq t \leq \frac{\pi}{2} \),
\[
k^{-\lambda_\alpha + \frac{1}{2} n^{\lambda_\beta + \frac{1}{2} r'}} P_k^{(\lambda_\alpha, -\frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\cos t) P_n^{(\lambda_\alpha, -\frac{1}{2}, \lambda_\kappa - \frac{1}{2})}(\cos t) \sin t 2^{2\lambda_\kappa} \]
\[
= O(1) \left[ \cos((k - n)t) + \cos((k + n - 2\lambda_\kappa)t - \lambda_\kappa \pi) \right] + O \left( \frac{1}{nt} \right)
\]
using the cosine addition formula. Also, note that
\[
\left( \frac{1}{\sin^2 t + n^{-2}} \right)^{\frac{1}{2} - ir'} = t^{-1 + 2ir'} + O(t) + O(n^{-2} t^{-3}) \quad 4n^{-1} \leq t \leq \frac{\pi}{2}
\]
It follows that
\[
|J_{n,2}(k)| \leq c_r + c_r \sup_{\ell \in \mathbb{R}} 2r' \int_{4n^{-1}}^{\frac{\pi}{2}} t^{-1 + 2ir'} e^{it\ell} dt \leq c_r' + c_r \sup_a \left| \int_a^b e^{it\ell} dt \right| \leq c_r \]
This proves the desired inequality (4.16) for \( k \geq \frac{n}{4} \). \( \square \)

4.2.2. Estimate for \( z = 1r \).

**Lemma 4.9.** If \( \tau \in \mathbb{R} \) and \( f \) is supported in \((\varpi, \theta)\), then
\[
\sup_{x \in (\varpi, \theta)} |P^{(\tau)}_n f(x)| \leq c_r n^{\rho_x} ||f||_{1, \kappa}.
\]
Proof. Since \( f \) is supported in \( c(\varphi, \theta) \), we have

\[
\sup_{x \in c(\varphi, \theta)} |\mathbb{P}^\tau_n(f)(x)| \leq \sup_{x \in c(\varphi, \theta)} \int_{c(\varphi, \theta)} |f(y)| V_n \left[ G^\tau_n \left( \langle x, \cdot \rangle \right) \right](y) \, \hat{h}_n^2(y) \, d\omega(y)
\]

\[
\leq \|f\|_{1, \kappa} \sup_{x, y \in c(\varphi, \theta)} |V_n \left[ G^\tau_n \left( \langle x, \cdot \rangle \right) \right](y)|.
\]

Thus, it is sufficient to prove

\[
\text{(4.17)} \quad |V_n \left[ G^\tau_n \left( \langle x, \cdot \rangle \right) \right](y)| \leq c_\tau n^{\kappa_\tau} \quad \text{for all } x, y \in c(\varphi, \theta).
\]

We note that (4.17) is trivial when \( \kappa_{\min} = 0 \) since in this case \( \|G^\tau_n\|_{\infty} \leq c_\tau n^{\lambda_\kappa} = c_\tau n^{\kappa_\tau} \). So we shall assume \( \kappa_{\min} > 0 \) for the rest of the proof.

To prove (4.17), we claim that it is enough to prove that

\[
\text{(4.18)} \quad \left| \int_{-1}^{1} G^\tau_n(at + s)(1 - t^2)^{\delta - 1} (1 + t) \, dt \right| \leq c_\tau n^{\kappa_\tau},
\]

whenever \( |a| \geq \varepsilon_d > 0, |a| + |s| \leq 1, \delta \geq \kappa_{\min} \), where \( c_\tau \) is independent of \( s \).

To see this, let \( x, y \in c(\varphi, \theta) \) and without loss of generality, assume \( \varphi_1 = \max_{1 \leq j \leq d + 1}|x_j| \). Then \( \varphi_1 \geq \sqrt{1 + d} \), which implies that \( |x_1|, |y_1| \geq \sqrt{d + 1} - \theta \geq \sqrt{d + 1} - 1/(8d) > 0 \), so that \( |x_1 y_1| \geq \varepsilon_d > 0 \). Thus, invoking (4.18) with \( a = x_1 y_1, \delta = \kappa_1 \) and \( s = \sum_{j=2}^{d+1} t_j x_j y_j \) gives

\[
\left| \int_{-1}^{1} G^\tau_n \left( \sum_{j=1}^{d+1} x_j y_j t_j \right) (1 - t^2)^{\kappa_1 - 1} (1 + t_1) \, dt_1 \right| \leq c_\tau n^{\kappa_\tau}.
\]

The desired inequality (4.17) then follows by the Fubini theorem and the integral representation of \( V_n \) in (2.2). This proves the claim.

For the proof of (4.18), by symmetry, it is sufficient to prove

\[
\text{(4.19)} \quad \left| \int_{-1}^{1} G^\tau_n(at + s)(1 - t)^{\delta - 1} \xi(t) \, dt \right| \leq c_\tau n^{\kappa_\tau},
\]

where \( \xi \) is a \( C^\infty \) function supported in \([-\frac{1}{2}, 1]\), whenever \( |a| \geq \varepsilon_d > 0, |a| + |s| \leq 1 \) and \( \delta \geq \kappa_{\min} \).

Let \( \eta_0 \in C^\infty(\mathbb{R}) \) be such that \( \chi_{[-\frac{1}{2}, \frac{1}{2}]} \leq \eta_0 \leq \chi_{[-1,1]}, \) and let \( \eta_1(t) := 1 - \eta_0(t) \).

Set, in this subsection,

\[
B := n^{-1} + \sqrt{1 - |a + s|}.
\]

We then split the integral in (4.19) into a sum \( I_0(a, s) + I_1(a, s) \) with

\[
I_j(a, s) := \int_{-1}^{1} G^\tau_n(at + s) \eta_j \left( \frac{1 - t}{B} \right) (1 - t)^{\delta - 1} \xi(t) \, dt, \quad j = 0, 1.
\]

It is easy to verify that \( 1 + n \sqrt{1 - |a + s|} \sim 1 + n \sqrt{1 - |a + s|} \) whenever \( t \in [1 - B, 1] \cap [-1, 1] \). Therefore, for \( 1 - B \leq t \leq 1 \), using (2.18),

\[
|G^\tau_n(at + s)| \leq c n^{\lambda_\kappa} (n^{-1} + \sqrt{1 - |a + s|})^{-\lambda_\kappa + \kappa_\tau} \leq c n^{\kappa_\tau} B^{-\kappa_{\min}},
\]

which implies that

\[
|I_0(a, s)| \leq c \int_{\max\{1 - B, -\frac{1}{2}\}}^{1} \left| G^\tau_n(at + s) \right|(1 - t)^{\delta - 1} \, dt \leq c n^{\kappa_\tau} B^{\delta - \kappa_{\min}} \leq c n^{\kappa_\tau}.
\]
To estimate $I_1(a, s)$, we write
\[
G_n \eta(1 - t) (1 - t)^{\delta-1} \xi(t) = c_n P_n(\lambda_n - \frac{1}{2}, \lambda_n - \frac{1}{2})(at + s) \varphi(t),
\]
where $|c_n| \leq c_n n^{\lambda_n + \frac{1}{2}}$ and
\[
\varphi(t) := (1 - (at + s)^2 + n^{-2})^{\frac{d}{2}} \xi(t)(1 - t)^{\delta-1}.
\]
Recall $|a| \geq \varepsilon_d > 0$. Using integration by parts $\ell$ times gives
\[
|I_1(a, s)| \leq c_n n^{\lambda_n + \frac{1}{2} - \ell} \int_{-1}^{1} \left| P_n(\lambda_n - \frac{1}{2} - \ell, \lambda_n - \frac{1}{2} - \ell)(at + s) \right| |\varphi(\ell)(t)| \, dt.
\]
If $-\frac{1}{2} \leq t \leq 1 - B/2$, then $1 - |at + s| \geq 1 - |a| - |s| + (1 - |t|)|a| \geq c(1 - t) \geq cB \geq cn^{-2}$, which implies, in particular, $(1 - (at + s)^2 + n^{-2})^{-1} \leq c(1 - t)^{-1}$. Since $\varphi$ is supported in $(-\frac{1}{2}, 1 - \frac{B}{2})$, which gives $B^{-1} \leq (1 - t)^{-1}$, it follows from Leibniz’ rule that
\[
|\varphi(\ell)(t)| \leq c_r (1 - |at + s|)^2 (1 - t)^{\delta-\ell}.
\]
Therefore, choosing $\ell > 2\delta$ and recalling that $\delta \geq \kappa_{\min}$, we have by (2.18) that
\[
|I_1(a, s)| \leq c_n n^{\lambda_n - \ell} \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - |a + s|)\frac{d}{2} \frac{1}{2} (1 - t)^{\delta-\ell} \, dt
\]
\[
\leq c_n n^{\lambda_n - \ell} \int_{\frac{1}{2}}^{\frac{1}{2}} (1 - |a + s| + u)\frac{d}{2} \frac{1}{2} u^{\delta-\ell} \, du.
\]
Using the fact that $(1 - |a + s| + u)^{\alpha} \leq c((1 - |a + s|)^{\alpha} + u^{\alpha})$ we break the last integral into a sum $J_1 + J_2$, where
\[
J_1 \leq c_n n^{\lambda_n - \ell} \int_{\frac{1}{2}}^{\frac{1}{2}} (1 - |a + s|)\frac{d}{2} \frac{1}{2} u^{\delta-1-\ell} \, du
\]
\[
\leq c_n n^{\lambda_n - \ell} (1 - |a + s|)\frac{d}{2} \frac{1}{2} B^{\delta-\ell} \leq c_n n^{\lambda_n - \ell} B^{\delta+\sigma_n - \lambda_n}
\]
\[
= c_n n^{\lambda_n - \delta} (B)^{\delta - \kappa_{\min}} \leq c_n n^{\sigma_n}
\]
and
\[
J_2 \leq c_n n^{\lambda_n - \ell} \int_{\frac{1}{2}}^{\infty} u^{\frac{d}{2} \frac{1}{2} - \frac{1}{2}} \frac{1}{2} u^{\delta-1-\ell} \, du \leq c_n n^{\lambda_n - \ell} B^{\frac{d}{2} \frac{1}{2} - \frac{1}{2} + \delta}
\]
\[
= c_n n^{\lambda_n - \delta} (n^2 B)^{\frac{\kappa_{\min} - \ell}{2}} (B)^{\delta - \kappa_{\min}} \leq c_n n^{\lambda_n - \delta} \leq c_n n^{\sigma_n}.
\]
Putting the above together, we obtain the desired estimate (4.19) and complete the proof of Lemma 4.9. \hfill \square

4.2.3. **Proof of Proposition 4.7** Define
\[
T^z f = n^{\sigma_n(z-1)} P_n(f x_{c(\varpi, \theta)}) x_{c(\varpi, \theta)}, \quad 0 \leq \Re z \leq 1.
\]
By Lemmas 4.9 and 4.8 we have
\[
\|T^{1+i} f\|_{1,2} \leq c_r \|f\|_{1,2} \quad \text{and} \quad \|T^{i} f\|_{\infty} \leq c_r \|f\|_{1,\kappa}.
\]
These allow us to apply Stein’s interpolation theorem [16, p. 205] to the analytic family of operators $T^z$, which yields
\[
\|T^{1+ir} f\|_{\kappa, \nu} \leq c \|f\|_{\kappa, \nu'}.
\]
Consequently, using the fact that
\[ T^{\frac{\alpha}{n}} f = n^{-\frac{\alpha}{n}} \mathbb{P}^{\frac{\alpha}{n}} (f \chi_{c(\varpi, \theta)}) \chi_{c(\varpi, \theta)} = n^{-\frac{\alpha}{n}} \text{proj}_{n}(h_{c}^{2}; f \chi_{c(\varpi, \theta)}) \chi_{c(\varpi, \theta)}, \]
we have proved Proposition 4.7. \hfill \Box

5. Boundedness of projection operator

The objective of this section is to prove Theorems 3.3 and 3.4.

5.1. Proof of Theorem 3.4 Assume that \( f \) is supported in a spherical cap \( c(\varpi, \theta) \). Without loss of generality, we may assume \( \theta < 1/(8d) \), since otherwise we can decompose \( f \) as a finite sum of functions supported on a family of spherical caps of radius \(< 1/(8d)\).

We start with the case \( p = 1 \). By the definition of the projection operator, it follows from the integral version of the Minkowski inequality and orthogonality that
\[
\| \text{proj}_{n}(h_{c}^{2}; f) \|_{\kappa, 2} \leq \sup_{y \in c(\varpi, \theta)} \left( \int_{S^{d}} \left| \frac{1}{n} P_{n}(h_{c}^{2}; x, y) \right|^2 h_{c}^{2}(x) d\omega(x) \right)^{1/2} \| f \|_{\kappa, 1}
\]
\[
= \left( \sup_{y \in c(\varpi, \theta)} P_{n}(h_{c}^{2}; y, y) \right)^{1/2} \| f \|_{\kappa, 1}.
\]

Using the pointwise estimate of the kernel in (2.21) and the fact that \( n\theta \geq 1 \), we then obtain
\[
\| \text{proj}_{n}(h_{c}^{2}; f) \|_{\kappa, 2} \leq cn^{\frac{d+1}{2}} \sup_{y \in c(\varpi, \theta)} \prod_{j=1}^{d+1} |y_{j}|^{\kappa - 1} \| f \|_{\kappa, 1}
\]
\[
\leq cn^{\frac{d+1}{2}} (n\theta)^{\varpi - \frac{d+1}{2}} \sup_{y \in c(\varpi, \theta)} \prod_{j=1}^{d+1} |y_{j}|^{- \kappa} \| f \|_{\kappa, 1}
\]
\[
\leq cn^{\varpi - \theta^{\varpi} + \frac{1}{2} \int_{c(\varpi, \theta)} h_{c}^{2}(y) d\omega(y)} \| f \|_{\kappa, 1},
\]
where the last step follows from (2.17). This proves Theorem 3.4 for \( p = 1 \).

Next, we use Hölder’s inequality and Theorem 4.1 to obtain
\[
\| \text{proj}_{n}(h_{c}^{2}; f) \|_{\kappa, 2}^2 = \int_{c(\varpi, \theta)} f(y) \text{proj}_{n}(h_{c}^{2}; f)(y) h_{c}^{2}(y) d\omega(y)
\]
\[
\leq \| f \|_{\kappa, \nu} \left( \int_{c(\varpi, \theta)} | \text{proj}_{n}(h_{c}^{2}; f, y)|^{\nu} h_{c}^{2}(y) d\omega(y) \right)^{\frac{1}{\nu}}
\]
\[
\leq cn^{\frac{\varpi}{\nu} + \frac{\alpha}{\nu} + \frac{1}{\nu} + 1} \left[ \int_{c(\varpi, \theta)} h_{c}^{2}(x) d\sigma(x) \right]^{-\frac{1}{\nu}} \| f \|_{\kappa, \nu}^{\frac{1}{\nu}},
\]
which proves Theorem 3.4 for \( p = \nu = \frac{2\pi + 2}{\varpi + 2} \).

Finally, Theorem 3.3 for \( 1 \leq p \leq \nu \) follows by applying the Riesz-Thorin convexity theorem to the linear operator \( g \mapsto \text{proj}_{n}(h_{c}^{2}; g \chi_{c(\varpi, \theta)}) \).

5.2. Proof of Theorem 3.3 Theorem 3.3(i) follows directly by invoking Theorem 4.1 with \( \theta = \pi \). Theorem 3.3(ii) follows from the Riesz-Thorin convexity theorem applied to the boundedness of \( f \mapsto \text{proj}_{n}(h_{c}^{2}; f) \) in \((2,2)\) and in \((\nu, 2)\). \hfill \Box
We now prove that the estimates are sharp. We start with a duality result whose proof is standard:

**Lemma 5.1.** Assume $1 \leq p \leq 2 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following are equivalent:

(i) $\| \text{proj}_n(h_{κ,q}^2; f) \|_{k,2} \leq A\| f \|_{k,p}$,

(ii) $\| \text{proj}_n(h_{κ,q}^2, f) \|_{k,q} \leq A\| f \|_{k,2}$.

To prove the sharpness of the estimates, we can assume without loss of generality that $κ_{\text{min}} = κ₁$. For the case in Theorem 3.3 (i) we define

$$f_n(x) := P_n(h_{κ,q}^2; x, e), \quad e = (1, 0, 0, \ldots, 0).$$

Since $f_n \in H_n^{κ+1}(h_{κ}^2)$, we have

$$\| \text{proj}_n(h_{κ,q}^2; f_n) \|_{κ,q} = \| f_n \|_{κ,q} = \left( \int |P_n(h_{κ,q}^2; x, e)|^q h_{κ}^2(x) dω(x) \right)^{1/q}.$$

Thus, it is sufficient to show that, for $n$ sufficiently large,

$$\| f_n \|_{κ,q} \sim n^{κ-\frac{2κ+1}{q}} \| f_n \|_{κ,2} \quad \text{for} q \geq \frac{2(κ_1 + 1)}{κ_1}. \tag{5.1}$$

Indeed, setting $p = q/(q - 1)$ and using Lemma 5.1 (5.1) shows that

$$\| \text{proj}_n(h_{κ,q}^2; f_n) \|_{κ,2} \sim n^{κ-\frac{2κ+1}{q}} \| f_n \|_{κ,p} = c_n (2κ_1 + 1) \left( \frac{1}{2} - \frac{κ_1 + 1}{κ_1} \right) \| f_n \|_{κ,p},$$

which proves the sharpness of (i).

Recall that $C_n^{λ,ρ}(t)$ denotes the generalized Gegenbauer polynomial. It is connected to $C_n^λ$ by an integral formula, which implies by (1.5) that

$$P_n(h_{κ}^2; x, e) = \frac{n + λ_n}{λ_k} C_n^{(σ_κ,κ_1)}(x).$$

Hence, using (2.11), in terms of Jacobi polynomials we have

$$P_n(h_{κ}^2; x, e) = O(n^{σ_κ+\frac{1}{2}} P_n^{(σ_κ-\frac{1}{2},κ_1-\frac{1}{2})}(2x_1^2 - 1). \tag{5.2}$$

Since this is a function that only depends on $x_1$, a standard change of variables leads to

$$\| f_n \|_{κ,q} \sim n^{σ_κ+\frac{1}{2}} \left( \int_0^1 |P_n^{(σ_κ-\frac{1}{2},κ_1-\frac{1}{2})}(2\cos^2 θ - 1)|^q \cos θ |\sin θ|^{2κ_1} dθ \right)^{1/q} \sim n^{σ_κ+\frac{1}{2}} \left( \int_{-1}^1 |P_n^{(σ_κ-\frac{1}{2},κ_1-\frac{1}{2})}(t)|^q |\sin(\frac{1}{2},κ_1-\frac{1}{2})| dt \right)^{1/q} \sim n^{σ_κ} \left( \int_{-1}^1 \right)^{1/q},$$

where in the last step we have used (2.19) and the condition $q \geq 2(κ_1 + 1)/κ_1 > (2κ_1 + 1)/κ_1$ to conclude that the integral over $[0, 1]$ has the stated estimate, whereas the integral over $[-1, 0]$, using $P_n^{(α,β)}(t) = P_n^{(β,α)}(-t)$, has an order dominated by the integral on $[0, 1]$. For $q = 2$, using (5.2), we get

$$\| f_n \|_{κ,2} = (P_n(h_{κ}^2; e, e))^\frac{1}{2} \sim n^{σ_κ}.$$

Together, these two relations establish (5.1) for even $n$. The proof for odd $n$ is similar. \qed
Remark 5.1. For the ordinary spherical harmonics, the sharpness of part (ii) in Theorem 3.3 was proved in [14] with the help of the function \((x_1 + ix_2)^n\). For \(h\)-harmonics, it is then natural to consider the function

\[ F_n(x) := V_n[(x_1 + ix_2)^n], \quad x \in \mathbb{R}^d, \]

where \(V_n\) is the intertwining operator associated with \(h^2\) and \(Z_d^2\). Since the Dunkl operator commutes with \(V_n\), so is the \(h\)-Laplacian, which leads to \(\Delta h V_n[(x_1 + ix_2)^n] = V_n[\Delta(x_1 + ix_2)^n] = 0\), proving that \(F_n(x)\) is an \(h\)-harmonic of degree \(n\). Furthermore, since \(V_n\) is a product form, it follows from [10] Prop. 5.6.10 that

\[ F_n(x) = a_n (x_1^2 + x_2^2)^{n/2} \times \left[ \frac{n + 2\kappa_2 + \delta_n}{2\kappa_2 + 2\kappa_1} C_n^{(\kappa_2, \kappa_1)} \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right) + ix_2 C_n^{(\kappa_2 + 1, \kappa_1)} \left( \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right) \right], \]

where \(a_n\) is a constant given explicitly in [10] and \(\delta_n = 2\kappa_2\) if \(n\) is even and \(\delta_n = 0\) if \(n\) is odd. This explicit formula allows us to compute the norm \(\|F_n\|_{\kappa, p}\) explicitly. For example, using [10] Lemma 3.8.9, we have

\[
\int_{\mathbb{R}^2} |F_n(x)|^q \tilde{h}_n^2(x) dx \omega(x) = \int_{B^2} |F_n(x)|^q |x_1|^{2\kappa_1} |x_2|^{2\kappa_2} (1 - \|x\|^2)^{\kappa - \kappa_1 - \kappa_2 + \frac{d-2}{2}} dx_1 dx_2,
\]

where, since \(F_n\) depends only on \((x_1, x_2)\), we have abused notation somewhat by using \(F_n(x)\) in the right-hand side as well. The above integral can then be evaluated, by (2.11) and in polar coordinates, by using (2.19). The result, however, does not yield the sharpness of (ii) in Theorem 3.3 when \(\kappa \neq 0\).

6. Boundedness of Cesàro means

In this section we prove Theorems 3.1 and 3.2. By a standard duality argument, it suffices to prove these theorems for \(1 \leq p \leq 2\). We shall assume \(1 \leq p \leq \nu := \frac{2\sigma_\nu + 2}{\sigma_\nu + 2}\) and \(\delta > \delta_\nu(p)\) for the rest of this section.

6.1. Proof of Theorem 3.1

We follow essentially the approach of [15], although there are still several difficulties that need to be overcome.

6.1.1. Decomposition. Let \(\varphi_0 \in C^\infty(0, \infty)\) be such that \(\chi_{[0,1]} \leq \varphi_0 \leq \chi_{[0,2]}\), and let \(\varphi(t) := \varphi_0(t) - \varphi_0(2t)\). Clearly, \(\varphi\) is a \(C^\infty\)-function supported in \((\frac{1}{2}, 2)\) and satisfying \(\sum_{n=0}^{\infty} \varphi(2^n t) = 1\) for all \(t > 0\). Setting

\[
\tilde{S}^\delta_{n,v}(j) := \varphi\left( 2^n \left( n - j \right) \right) \frac{A_j}{A_n},
\]

we define

\[ S_{n,v}^\delta := \sum_{j=0}^{n} \tilde{S}^\delta_{n,v}(j) \text{proj}_j(h^2; f), \quad v = 0, 1, \cdots, \lfloor \log_2 n \rfloor + 2. \]

Since \(\sum_{v=0}^{\lfloor \log_2 n \rfloor + 2} \varphi\left( 2^n (n-j) \right) = 1\) for \(0 \leq j \leq n - 1\), it follows that the Cesàro means are decomposed as

\[
S_n^\delta(h^2; f) = \sum_{v=0}^{\lfloor \log_2 n \rfloor + 2} S_{n,v}^\delta + \frac{1}{A_n} \text{proj}_n(h^2; f).
\]
Using Theorem 3.3 and the fact that $\delta > \delta_\kappa(p)$, we have
\[
\frac{1}{A_n^\delta} \| \text{proj}_n (h^2_n; f) \|_{\kappa, p} \leq c n^{-\delta} \| \text{proj}_n (h^2_n; f) \|_{\kappa, 2} \leq c n^{\delta_\kappa(p) - \delta} \| f \|_{\kappa, p} \leq c \| f \|_{\kappa, p}.
\]
On the other hand, using summation by parts $\ell \geq \lambda_\kappa$ follows from Theorem 3.3 of [4, pp. 413–414], we can write, for any integer $\ell \geq \lambda_\kappa$,
\[
S^\delta_{n,v,f} = \sum_{j=0}^n \Delta^{\ell} \left( \frac{\Delta j}{S_{n,v,j}}(j) \right) A_j^{\ell-1} S_j^{\ell-1}(h^2_n; f),
\]
where $\Delta$ denotes the forward difference and $\Delta^{\ell+1} := \Delta^{\ell} \Delta$. Since $S^\delta_{n,v}(j) = 0$ whenever $n - j > \frac{n}{2^{\ell+1}}$ or $n - j < \frac{n}{2^{\ell+1}}$, it is easy to verify by the Leibniz rule that
\[
|\Delta^{\ell}(S^\delta_{n,v,j}(j))| \leq c 2^{-\ell} \left( \frac{2^n}{n} \right)^{\ell}, \quad \forall \ell \in \mathbb{N}, 0 \leq j \leq n.
\]
Hence, choosing $\ell > \lambda_\kappa$ and using the fact that $S^\ell_{n}(h^2_n; f)$ is bounded in $L^p(h^2_n; S^\ell)$ for all $1 \leq p \leq \infty$ if $\ell > \lambda_\kappa$, we conclude that for $v = 0$ and 1,
\[
\| S^\delta_{n,v,f} \|_{\kappa, p} \leq c n^{-\ell} \sum_{j=0}^n j^{\ell-1} \| S_j^{\ell-1}(h^2_n; f) \|_{\kappa, p} \leq c \| f \|_{\kappa, p}.
\]
Therefore, by (6.2), it is sufficient to prove that
\[
\| S^\delta_{n,v,f} \|_{\kappa, p} \leq c 2^{-v\varepsilon_0} \| f \|_{\kappa, p}, \quad v = 2, \ldots, \lfloor \log_2 n \rfloor + 2,
\]
where $\varepsilon_0$ is a sufficiently small positive constant depending on $\delta$ and $p$, but independent of $n$ and $v$.

6.1.2. Estimate of the kernel of $S^\delta_{n,v,f}$. Let
\[
D^\delta_{n,v}(t) := \sum_{j=0}^n \frac{\bar{S}_{n,v,j}(j)}{\lambda_\kappa + j} C_j^{\lambda_\kappa}(t).
\]
The definition shows that $S^\delta_{n,v,f} = f \ast_n D^\delta_{n,v}$, so that the kernel of $S^\delta_{n,v,f}$ is defined by
\[
K^\delta_{n,v}(x,y) := V_\kappa \left[ D^\delta_{n,v}(\langle x, \cdot \rangle) \right](y).
\]

Lemma 6.1. Let $2 \leq v \leq \lfloor \log_2 n \rfloor + 2$. Then for any given positive integer $\ell$,
\[
|K^\delta_{n,v}(x,y)|h^2_n(y) \leq c n^{2\ell v(1-\delta)} (1 + nd(z, y))^{\ell - d + \lambda_\kappa + 1},
\]
where $\bar{z} = (|z_1|, \ldots, |z_{d+1}|)$ for $z = (z_1, \ldots, z_{d+1}) \in \mathbb{R}^{d+1}$.

Proof. We first define a sequence of functions $\{a_{n,v,\ell}(\cdot)\}_{\ell=0}^\infty$ by
\[
a_{n,v,0}(j) = 2(j + \lambda_\kappa) \bar{S}_{n,v,j}(j),
\]
\[
a_{n,v,\ell+1}(j) = \frac{a_{n,v,\ell}(j)}{2j + 2\lambda_\kappa + 2} - \frac{a_{n,v,\ell}(j + 1)}{2j + 2\lambda_\kappa + \ell + 2}, \quad \ell \geq 0.
\]
Following the proof of Lemma 3.3 of [3, pp. 413–414], we can write, for any integer $\ell \geq 0$,
\[
D^\delta_{n,v}(t) = c_n \sum_{j=0}^\infty a_{n,v,\ell}(j) \frac{\Gamma(j + 2\lambda_\kappa + \ell)}{\Gamma(j + 2\lambda_\kappa + \frac{\ell}{2})} P_j^{\lambda_\kappa + \frac{\ell}{2}}(\frac{x}{\lambda_\kappa - 1}(\lambda_\kappa - \frac{\ell}{2})(t),
\]
so that

\[ K_{n,v}^\delta(x,y) = c_n \sum_{j=0}^{\infty} a_{n,v}^\ell(j) \Gamma(j + 2\lambda_n + \ell) \Gamma(j + \lambda_n + \frac{1}{2}) V_n \left[ \prod_{j=1}^{j_n} (\lambda_n + \ell - \frac{1}{2}; \lambda_n - \frac{1}{2}) ((x,\cdot)) \right](y). \]

Note that \( a_{n,v}^\ell(j) = 0 \) if \( j + \ell \leq (1 - \frac{1}{n^{\alpha}})n \) or \( j \geq (1 - \frac{1}{n^{\alpha}})n \), so that the sum is over \( j \sim n \). Furthermore, it follows from the definition, \((6.4)\) and Leibniz rule that

\[ \left| \Delta^i a_{n,v}^\ell(j) \right| \leq c_2^{-v^2} n^{-t+1} \left( \frac{2n}{n} \right)^{i+\ell}, \quad i, \ell = 0, 1, \ldots. \]

Consequently, using the pointwise estimate of \((2.20)\), it follows by \((6.6)\) that

\[ |K_{n,v}^\delta(x,y)| \leq c_n 2^{n+2t-1-2|\kappa|} \sum_{n-j=n^{\alpha}} |a_{n,v}^\ell(j)| \prod_{j=1}^{d+1} \left( |x_j y_j| + n^{-1} d(\bar{x}, \bar{y}) + n^{-2} \right)^{-\kappa_j} 
\leq c_n 2^{n+2t-1-\delta} h_n^{-2} \left( y \right) |1 + nd(\bar{x}, \bar{y})|^{\lambda_n-\ell-1} \left| d(\bar{x}, \bar{y}) \right| |\kappa|, \]

where in the last inequality we have used the fact that

\[ \prod_{j=1}^{d+1} \left( |x_j y_j| + n^{-1} d(\bar{x}, \bar{y}) + n^{-2} \right)^{-\kappa_j} \leq c h_n^{-2} \left( y \right) d(\bar{x}, \bar{y})^{|\kappa|}, \]

which follows since if \( |y_j| \geq 2d(\bar{x}, \bar{y}) \), then \( |x_j - \bar{y}_j| \leq d(\bar{x}, \bar{y}) \leq |y_j|/2 \) so that \( |y_j|^2 \leq 2|x_j y_j| \), whereas if \( |y_j| < 2d(\bar{x}, \bar{y}) \), then \( |y_j|^2 \leq 2(n^{-1} d(\bar{x}, \bar{y})) \cdot nd(\bar{x}, \bar{y}) \). This completes the proof of Lemma \((6.4)\). \( \square \)

**Corollary 6.2.** For any \( \gamma > 0 \) there exists an \( \varepsilon_0 > 0 \) independent of \( n \) and \( v \) such that

\[ \sup_{x \in S^d} \int_{\{y : d(\bar{x}, \bar{y}) > 2^{(1+\gamma)v/n}\}} |K_{n,v}^\delta(x,y)| h_n^2(y) \, d\omega(y) \leq c 2^{-v \varepsilon_0}. \]

**Proof.** Invoking Lemma \((6.3)\) with \( \ell > \lambda_n + 1 + \frac{\lambda_n-\delta}{\delta} \), we see that the quantity to be estimated is bounded by

\[ c \sup_{x \in S^d} n^{d+2v(1-\delta)} \int_{\{y : d(\bar{x}, \bar{y}) > 2^{(1+\gamma)v/n}\}} \frac{1}{(1 + nd(\bar{x}, \bar{y}))^{\ell+d-\lambda_n-1}} d\omega(y) 
\leq c 2^{v(1-\delta)} \int_{(2^{(1+\gamma)v/n})} \int_{(2^{(1+\gamma)v/n})} \frac{n(n\theta)^{d-1}}{(1 + n\theta)^{\ell+d-\lambda_n-1}} \, d\theta 
\leq c 2^{v(1-\delta) - (1+\gamma)(\ell-\lambda_n-1)} = c 2^{-\varepsilon_0}, \]

which proves the corollary. \( \square \)

### 6.1.3. Proof of \((6.3)\). Now we are in a position to prove \((6.4)\). Recall that

\[ S_{n,v}^\delta f = \sum_{(1-2v^{-1})n \leq \lambda_n \leq (1-2v^{-1})n} \tilde{S}_{n,v}^\delta(j) \proj_j(h_n^2; f). \]

Assume \( \delta > \delta_n(p) \), and let \( \gamma > 0 \) be sufficiently small so that \( \delta > \delta_n(p) + \gamma (\delta_n(p) + \frac{\mu}{2}) \). Set \( v_1 = v(1+\gamma) \). Let \( \Lambda \) be a maximal \( \frac{d_\mu}{n} \)-separable subset of \( S^d \);
that is, \( \min_{x \neq \varpi \in \Lambda} d(\varpi, \varpi') \geq \frac{2^{v_1}}{n} \) and \( S^d \subseteq \bigcup_{\varpi \in \Lambda} c(\varpi, \frac{2^{v_1}}{n}) \). Define
\[
 f_{\varpi}(x) := f(x) \chi_{c(\varpi, \frac{2^{v_1}}{n})}(x)[A(x)]^{-1}, \quad A(x) := \sum_{\varpi \in \Lambda} \chi_{c(\varpi, \frac{2^{v_1}}{n})}(x).
\]
Then evidently \( 1 \leq A(x) \leq c, \ x \in S^d, \ |f_{\varpi}| \leq c|f|, \) and \( f(x) = \sum_{\varpi \in \Lambda} f_{\varpi}(x) \). Using the Minkowski inequality, we obtain
\[
\|S_n^d(f)\|_{\kappa, p} \leq \sum_{\varpi \in \Lambda} \|S_n^d(f_{\varpi})\|_{\kappa, p}.
\]
Thus, it is sufficient to show that for each \( \varpi \in \Lambda \), we have
\[
\|S_n^d(f_{\varpi})\|_{\kappa, p} \leq c2^{-v\varpi_0}\|f_{\varpi}\|_{\kappa, p}.
\]
To this end, we denote by \( c^\ast(\varpi, 2^{v_1+1}/n) \) the set
\[
c^\ast(\varpi, 2^{v_1+1}/n) = \{ x \in S^d : d(\bar{x}, \bar{\varpi}) \leq 2^{v_1+1}/n \}
\]
and further define \( J(v, n) : = \{ j : (1 - 2^{-v+1})n \leq j \leq (1 - 2^{-v-1})n \} \). Using (6.7) and orthogonality, we obtain
\[
\|S_n^d(f_{\varpi})\|_{\kappa, 2} = \left( \sum_{j \in J(v, n)} |\hat{S}_{n, v}^d(j)|^2 \| \text{proj}_j(h_{\kappa}^2; f_{\varpi}) \|^2_{\kappa, 2} \right)^{1/2}.
\]
Hence, by Hölder’s inequality, Theorem 3.4 and (2.17), and (6.3) with \( \ell = 0 \),
\[
\left( \int_{c^\ast(\varpi, 2^{v_1+1}/n)} |S_{n,v}^d(f_{\varpi})(x)|^p \ h_{\kappa}^2(x) \ d\omega(x) \right)^{1/p} \\
\leq c \left( \int_{c^\ast(\varpi, 2^{v_1+1}/n)} h_{\kappa}^2(x) \ d\omega(x) \right)^{1/p} \left( \sum_{j \in J(v, n)} |\hat{S}_{n, v}^d(j)|^2 \| \text{proj}_j(h_{\kappa}^2; f_{\varpi}) \|^2_{\kappa, 2} \right)^{1/2} \\
\leq c2^{v(\varpi_0)(p+\frac{1}{2})} n^{-\frac{1}{2}} \left( \sum_{j \in J(v, n)} |\hat{S}_{n, v}^d(j)|^2 \right)^{1/2} \|f_{\varpi}\|_{\kappa, p} \\
\leq c2^{-v(\varpi_0)(\varpi_0+\frac{1}{2})} \|f_{\varpi}\|_{\kappa, p} = c2^{-v\varpi_0} \|f_{\varpi}\|_{\kappa, p}.
\]
Finally, using Hölder’s inequality, we obtain, for \( x \notin c^\ast(\varpi, 2^{v_1+1}/n) \),
\[
|S_{n,v}^d(f_{\varpi})(x)|^p = \int_{\{y : d(y, \bar{x}) \leq 2^{v_1}/n \}} f_{\varpi}(y) K_{n,v}^d(x, y) h_{\kappa}^2(y) \ d\omega(y) \ \\
\leq \left( \int_{\{y : d(y, \bar{x}) \geq 2^{v_1}/n \}} |f_{\varpi}(y)|^p |K_{n,v}^d(x, y)| h_{\kappa}^2(y) \ d\omega(y) \right)^{1/p} \\
\times \left( \int_{\{y : d(y, \bar{x}) \geq 2^{v_1}/n \}} |K_{n,v}^d(x, y)| h_{\kappa}^2(y) \ d\omega(y) \right)^{p-1},
\]
which, together with Corollary 6.2, implies
\[
\left( \int_{S^d \setminus c^\ast(\varpi, 2^{v_1+1}/n)} |S_{n,v}^d(f_{\varpi})(x)|^p h_{\kappa}^2(x) \ d\omega(x) \right)^{1/p} \\
\leq c(2^{-v\varpi_0})^{1-\frac{1}{p}} \sup_{y \in S^d} \left( \int_{\{x : d(\bar{x}, \bar{y}) \geq 2^{v_1}/n \}} |K_{n,v}^d(x, y)| h_{\kappa}^2(x) \ d\omega(x) \right)^{1/p} \|f_{\varpi}\|_{\kappa, p} \\
\leq c2^{-v\varpi_0} \|f_{\varpi}\|_{\kappa, p}.
\]
Putting the above together, we deduce the desired estimate (6.8), hence (6.4), and complete the proof of Theorem 3.1.
6.2. Proof of Theorem [3.2]

6.2.1. Main body of the proof. For the proof of Theorem [3.1] we follow the approach in [2], which can be traced back to [13]. We start with the following lemma.

Lemma 6.3. If $Q$ is a polynomial of degree $n$ on $\mathbb{R}^{d+1}$, then for $1 \leq p < \infty$,
\[
\|Q\|_\infty := \max_{x \in S^d} |Q(x)| \leq cn^{(2\sigma_\kappa+1)/p}\|Q\|_{\kappa,p}.
\]

Proof. Let $S_n(h_\kappa^2; f)$ denote the partial sum operator of the $h$-harmonic expansion. Then $S_n(h_\kappa^2; Q) = Q$. The kernel of $S_n(h_\kappa^2; f)$ is
\[
K_n(h_\kappa^2; x, y) = \sum_{k=0}^{\infty} P_k(h_\kappa^2; x, y), \quad P_k(h_\kappa^2; x, y) = \sum_{j} Y_j^k(x)Y_j^k(y),
\]
where $\{Y_j^k\}_j$ forms an orthonormal basis of $\mathcal{H}_k^{d+1}(h_\kappa^2)$. If $y \in S^d$ and $|y_l| = \max_{1 \leq j \leq d+1} |y_j|$, then $|y_l| \geq 1/\sqrt{d+1}$. Hence, the pointwise estimate of (6.9) implies that $|P_k(h_\kappa^2; x, x)| \leq cn^{2\sigma_\kappa}$. By the Cauchy-Schwarz inequality, we obtain
\[
|K_n(h_\kappa^2; x, y)| \leq \sum_{k=0}^{\infty} P_k(h_\kappa^2; x, x)^2 P_k(h_\kappa^2; y, y)^2 \leq c \sum_{k=0}^{\infty} k^{2\sigma_\kappa} \leq cn^{2\sigma_\kappa+1}.
\]
Consequently, we conclude that
\[
\|Q\|_\infty = \|S_n(h_\kappa^2; Q)\|_\infty = \left\|a_\kappa \int_{S^d} \int_{S^d} Q(y)K_n(h_\kappa^2; x, y)h_\kappa^2(y)d\omega\right\|_\infty \leq cn^{2\sigma_\kappa+1}\|Q\|_{\kappa,1}.
\]
Furthermore, we clearly have $|Q|_\infty \leq \|Q\|_\infty$, and the case $1 < p < \infty$ follows immediately from interpolation of these two cases. \hfill \Box

Proof of Theorem [3.2]. Our main objective is to show that
\[
(6.9) \quad \sup_{n \in \mathbb{N}} \|S_n^{(\delta)}(h_\kappa^2; f)\|_{\kappa,p} \leq c\|f\|_{\kappa,p}
\]
does not hold if $1 \leq p \leq \frac{2\sigma_\kappa+1}{\sigma_\kappa+\delta}$ or $p \geq \frac{2\sigma_\kappa+1}{\sigma_\kappa+\delta-\delta}$. Let
\[
p_1 := \frac{2\sigma_\kappa+1}{\sigma_\kappa+\delta} \quad \text{and} \quad q_1 := \frac{p_1}{p_1-1} = \frac{2\sigma_\kappa+1}{\sigma_\kappa+1+\delta}.
\]
It is sufficient to prove that (6.9) does not hold for $p_1$, since it then follows from the Riesz-Thorin convexity theorem that (6.9) fails for $p_1 \leq p \leq \infty$ and the fact that (6.9) fails for $1 \leq p \leq q_1$ follows by duality.

Let $c \in S^d$ be fixed. Define a linear functional $T_\delta^c : L^p \to \mathbb{R}$ by
\[
T_\delta^c f := S_n^{(\delta)}(h_\kappa^2; f, c) = a_\kappa \int_{S^d} f(x)K_n^{(\delta)}(h_\kappa^2; x, c)h_\kappa^2(x)d\omega(x).
\]
Since this is an integral operator, a standard argument shows that
\[
\|T_\delta^c f\|_{\kappa,p} = \|K_n^{(\delta)}(h_\kappa^2; x, c)\|_{\kappa,q}, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]
where $\|T_\delta^c f\|_{\kappa,p} = \sup_{\|f\|_{\kappa,p}} |T_\delta^c f|$. On the other hand, by Lemma [6.3] if (6.9) holds, then we will have
\[
|T_\delta^c f| = |S_n^{(\delta)}(h_\kappa^2; f, c)| \leq \|S_n^{(\delta)}(h_\kappa^2; f)\|_{\kappa,\infty} \leq cn^{(2\sigma_\kappa+1)/p}\|S_n^{(\delta)}(h_\kappa^2; f)\|_{\kappa,p} \leq cn^{(2\sigma_\kappa+1)/p}\|f\|_{\kappa,p}.
\]
Consequently, the above two equations show that we will have
\begin{equation}
\|K_n^\delta(h_n^2; \cdot, e)\|_{\kappa,q} \leq cn^{(2\sigma_\kappa+1)/p}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\end{equation}

To complete the proof of Theorem 3.2, we show that (6.10) does not hold for \( p = p_1 \).

For this purpose we use the explicit formula for \( K_n^\delta(h_n^2; x, e) \) for \( e = e_j \) (see, for example, [11]), where \( e_1, \ldots, e_{d+1} \) denote the usual coordinate vectors in \( \mathbb{R}^{d+1} \) and
\[
K_n^\delta(h_n^2; x, e_j) := K_n^\delta(w_{\lambda_\kappa,\kappa_j}; 1, x_j),
\]
where \( K_n^\delta(w_{\lambda,\mu}; s, t) \) denotes the \((C, \delta)\) kernel of the generalized Gegenbauer polynomials with respect to the weight function \( (2.10) \). Hence, we have
\[
\int_{S^d} |K_n^\delta(h_n^2; x, e_j)|^q h_n^2(x) d\omega(x) = c \int_{-1}^1 |K_n^\delta(w_{\lambda_\kappa,\kappa_j}; 1, t)|^q w_{\lambda_\kappa,\kappa_j}(t) dt.
\]
Consequently, choosing \( j \) such that \( \kappa_j = \kappa_{\text{min}} \), we see that the proof of Theorem 3.2 follows by applying Proposition 6.4 below to \( \sigma = \sigma_\kappa \) and \( \mu = \kappa_{\text{min}} \).

\[\square\]

**Proposition 6.4.** Let \( w_{\sigma,\mu} \) be the weight function in (2.10) and \( \sigma \geq \mu \geq 0 \). Define
\[
\Phi_{n,q}(w_{\sigma,\mu}, s) := \int_{-1}^1 |K_n^\delta(w_{\sigma,\mu}; s, t)|^q w_{\sigma,\mu}(t) dt.
\]
Then for \( q_1 = \frac{2\sigma+1}{\sigma+1+\delta} \) and \( p_1 = \frac{2\sigma+1}{\sigma-\delta} \),
\[
\Phi_{n,q_1}(w_{\sigma,\mu}, 1) \geq \Phi_{n,q_1}(w_{\mu,\sigma}, 0) \geq cn^{(2\sigma+1)q_1/p_1} \log n.
\]

The proof of this proposition is given in the following subsection.

6.2.2. **Proof of Proposition 6.4** The case of \( q = 1 \) and \( \delta = \sigma \) has already been established in [8] [11]. We follow the approach in [8] and briefly sketch the proof.

Using the sufficiency part of Corollary 3.7, we can follow exactly the deduction in [11, p. 293] to obtain
\[
\Phi_{n,q_1}(w_{\sigma,\mu}, 1) \geq \Phi_{n,q_1}(w_{\mu,\sigma}, 0) = cn^{(\sigma+\mu-\delta+\frac{1}{2})q_1}
\times \int_0^1 \int_{-1}^1 P_{n}^{\sigma+\mu+\delta+\frac{1}{2}, \sigma+\mu-\frac{1}{2}}(st)(1-s^2)^{\mu-1} ds \left| 1-t^2 \right|^\frac{q_1}{2} dt + O(1).
\]
As a result, we see that Proposition 6.4 is a consequence of the lower bound of a double integral of the Jacobi polynomial given in the next proposition.

**Proposition 6.5.** Assume \( \sigma, \mu \geq 0 \) and \( 0 \leq \delta \leq \sigma + \mu \). Let \( a = \sigma + \mu + \delta, \) \( b = \sigma + \mu - 1 \) and \( q_1 = \frac{2\sigma+1}{\sigma+1+\delta} \). Then
\begin{equation}
\int_{-1}^1 \left| P_n^{\sigma+\mu+\delta+\frac{1}{2}, \sigma+\mu-\frac{1}{2}}(st)(1-s^2)^{\mu-1} ds \right| t^2(1-t^2)^{\sigma-1/2} dt \geq c n^{-(\mu+1/2)q_1} \log n.
\end{equation}

**Proof.** Denote the left-hand side of (6.11) by \( I_{n,q_1} \). First assume that \( 0 < \mu < 1 \). Following the proof of [8], we can conclude that
\[
I_{n,q_1} \geq cn^{-q_1/2} \int_{-1}^{\pi/4} |M_n(\phi)|^{q_1} (\sin \phi)^{2\sigma} d\phi - O(1) E_{n,q_1},
\]
where

\[
E_{n,q_1} := n^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \left[ \int_{\phi}^{\pi/4} \frac{(\cos^2 \phi - \cos^2 \theta)^{\mu-1}}{(\sin \frac{\theta}{2})^{\sigma+1}(\cos \frac{\theta}{2})^{\delta+1}} d\theta \right]^{q_1} (\sin \phi)^{2\sigma} d\phi
\]

and \( M_n(\phi) := K_n(\phi) + G_n(\phi) \) satisfies

\[
K_n(\phi) \geq cn^{-\mu} \phi^{-\sigma-\delta-1} (1 + \cos(2N\phi + 2\gamma)), \quad n^{-1} \leq \phi \leq \varepsilon,
\]

for a sufficiently small absolute constant \( \varepsilon > 0 \), where \( N = n + \frac{a+b}{2} + 1 \) and \( \gamma = -\frac{\pi}{2}(a + 1 - \mu) \), and

\[
|G_n(\phi)| \leq cn^{-\mu} \phi^{-\sigma-\delta-2}, \quad n^{-1} \leq \phi < \pi/4.
\]

From these estimates and the fact that \( q_1(\sigma + 1 + \delta) = 2\sigma + 1 \), it follows that

\[
\int_{n^{-1}}^{\varepsilon} |K_n(\phi)|^{q_1} (\sin \phi)^{2\sigma} d\phi \geq cn^{-\mu q_1} \int_{n^{-1}}^{\varepsilon} \phi^{2\sigma-q_1(\sigma+1+\delta)}(1 + \cos(2N\phi + 2\gamma))^{q_1} d\phi
\]

\[
= cn^{-\mu q_1} \int_{n^{-1}}^{\varepsilon} \phi^{-1}(1 + \cos(2N\phi + 2\gamma))^{q_1} d\phi \geq cn^{-\mu q_1} \log n,
\]

where in the last step we used \((1+A)^{q_1} \geq 1 + q_1 A\) for \( A \in [-1,1] \) and the fact that \( \int_{n^{-1}}^{\varepsilon} \phi^{-1} \cos(2N\phi + 2\gamma) d\phi \leq c \) upon using integration by parts once. Furthermore,

\[
\int_{n^{-1}}^{\varepsilon} |G_n(\phi)|^{q_1} (\sin \phi)^{2\sigma} d\phi \leq cn^{-\mu q_1} \int_{n^{-1}}^{\varepsilon} \phi^{2\sigma+q_1(\mu-\sigma-\delta-2)} d\phi
\]

\[
= cn^{-\mu q_1} \int_{n^{-1}}^{\varepsilon} \phi^{q_1(\mu-1)-1} d\phi \leq cn^{-q_1 \mu}.
\]

Together, these estimates yield that for \( 0 < \mu < 1 \),

\[
\int_{n^{-1}}^{\pi/4} |M_n(\phi)|^{q_1} (\sin \phi)^{2\sigma} d\phi \geq cn^{-q_1 \mu} \log n.
\]

Moreover, the remainder \( E_{n,q_1} \) term can be estimated as follows:

\[
E_{n,q_1} \leq cn^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \left[ \int_{\phi}^{\pi/4} \theta^{\mu-a-2}(\theta - \phi)^{\mu-1} d\theta \right]^{q_1} (\sin \phi)^{2\sigma} d\phi
\]

\[
\leq cn^{-\frac{3}{2}q_1} \int_{n^{-1}}^{\pi/4} \phi^{(\mu-\sigma-\delta-2)q_1+2\sigma} d\phi
\]

\[
\leq cn^{-\frac{3}{2}q_1-q_1(\mu-1)} = cn^{-(\mu+\frac{3}{2})q_1},
\]

where in the second step, we divided the inner integral into two parts, over \([\phi, 2\phi]\) and over \([2\phi, \pi/2]\), respectively, to derive the stated estimate.

Putting these two terms together, we conclude the proof for the case \( 0 < \mu < 1 \). The case \( \mu = 1 \) can be derived similarly upon taking an integration by parts for the inner integral in \([0,1]\). The case \( \mu > 1 \) reduces to the case \( 0 < \mu < 1 \) upon taking integration by parts \([\mu] \) times as in \( \mathbb{S} \). \( \square \)
7. Proof of theorems on the ball and on the simplex

7.1. Proof of results on the unit ball. Under the mapping \( \phi : x \in B^d \mapsto (x, \sqrt{1 - \|x\|^2}) \in S^d_+ \) in (2.7), orthogonal polynomials with respect to \( W^B_\kappa \) in (2.6) can be deduced from \( h \)-spherical harmonics that are even in \( x_{d+1} \). Moreover, the connection (2.9) shows that \( S^\delta_n(W^B_\kappa; f) \) is related to \( S^\delta_n(h^2_\kappa; F) \) by

\[
S^\delta_n(W^B_\kappa; f, x) = S^\delta_n(h^2_\kappa; F, X), \quad X := (x, \sqrt{1 - \|x\|^2}),
\]

where \( F(x, x_{d+1}) := f(x) \) for \( x \in B^d \) and \( (x, x_{d+1}) \in S^d \). Consequently, by (2.8), Theorem 3.5 with \( \Omega = B \) follows immediately from Theorem 3.3.

The proof of Theorem 3.6 follows almost exactly as that of Theorem 3.2. We have in this case ([11, pp. 287-288])

\[
\begin{align*}
K_n(W^B_\kappa; x, e_j) &= K_n(\mathbf{w}_{\lambda_n, e_j}; 1, x_j), \quad 1 \leq j \leq d, \\
K_n(W^B_\kappa; x, 0) &= K_n(\mathbf{w}_{0, x_{d+1}}; |x|, 0).
\end{align*}
\]

Hence, by (2.8), we can again reduce the proof of Theorem 3.6 to the lower bound of \( \Phi^\delta_{n,\tau}(\mathbf{w}_{\sigma,\mu}, 1) \) and \( \Phi^\delta_{n,q}(\mathbf{w}_{\mu,\sigma}, 0) \), which follows again from Proposition 6.4.

7.2. Proof of results on the simplex.

7.2.1. Projection operator. Under the mapping \( \psi : x \in T^d \mapsto (x_1^2, \ldots, x_d^2) \in B^d \), orthogonal polynomials with respect to \( W^T_\kappa \) on \( T^d \) and those with respect to \( W^B_\kappa \) on \( B^d \) are related. In particular, we have the connection between \( \text{proj}_n(W^T_\kappa; f) \) and \( \text{proj}_n(W^B_\kappa; f \circ \psi) \) given in (2.13), from which the result on the projection operators can be readily derived.

In fact, if \( \|\text{proj}_n(W^B_\kappa; f)\|_{W^B_\kappa,p} \leq A_n\|f\|_{W^B_\kappa,p} \), then by (2.15) and (2.14),

\[
\|\text{proj}_n(W^T_\kappa; f)\|_{W^T_\kappa,p} = \|\text{proj}_n(W^T_\kappa; f) \circ \psi\|_{W^T_\kappa,p} \leq \frac{1}{2d} \left\| \sum_{\varepsilon \in \mathbb{Z}_d^d} \text{proj}_2n(W^B_\kappa; f \circ \psi, \varepsilon) \right\|_{W^B_\kappa,p} \leq A \|f \circ \psi\|_{W^B_\kappa,p} = A_2n\|f\|_{W^B_\kappa,p},
\]

from which Theorem 3.8 for \( \Omega = T \) follows immediately from the case \( \Omega = B \). Furthermore, since the distance \( d_T(x, y) \) on \( T^d \) is related to the geodesic distance on \( S^d \) by

\[
d_T(x, y) = d(X, Y), \quad X = (x, \sqrt{1 - \|x\|^2}), \quad Y = (y, \sqrt{1 - \|y\|^2}),
\]

from (2.8) and (2.14) it follows readily that

\[
\int_{c_T(x, \theta)} W^T_\kappa(x)dx = \int_{c(X, \theta)} h^2_\kappa(y)d\omega(y),
\]

where \( X = \left( \sqrt{x_1}, \ldots, \sqrt{x_d}, \sqrt{1 - |x|} \right) \). Consequently, we conclude that Theorem 3.9 follows from Theorem 3.3.
7.2.2. Cesàro means. Since the connection (2.15) relates the projection operator of degree n for \( W^T_n \) to the projection operator of degree 2n for \( W^B_n \), we cannot deduce results for the Cesàro means \( S^δ_n(W^T_n; f) \) from those of \( S^δ_n(W^B_n; f) \) directly. We can, however, follow the proof of the theorems for the \( h \)-harmonics. Below we give a brief outline on how this will work.

Proof of Theorem 3.5 with \( \Omega = T \). We follow the decomposition in the subsection 6.1.1 to define

\[
S^δ_n(W^T_n; f) = \sum_{j=0}^{n} \hat{S}^δ_n(j) \text{proj}_j(W^T_n; f), \quad v = 1, 2, \ldots, \lfloor \log_2 n \rfloor + 2.
\]

The same argument shows that it suffices to prove the analogue of (6.4),

\[
(7.1) \quad \|S^δ_{n,v}(W^T_n; f)\|_{L^p_{x,v}} \leq c2^{-v\varepsilon}\|f\|_{L^p_{x,v}}, \quad v = 2, \ldots, \lfloor \log_2 n \rfloor + 2.
\]

Denote the kernel of \( S^δ_{n,v}(W^T_n; f) \) by \( K^δ_{n,v}(W^T_n; x, y) \). Then we have by (2.10) that

\[
K^δ_{n,v}(W^T_n; x, y) := c_k \int_{[-1,1]^{d+1}} D^δ_{n,v}(W^T_n; 2z(x, y, t)^2 - 1) \prod_{i=1}^{d+1} (1 - \bar{t}_i^{2\gamma})^{-1} dt,
\]

where

\[
D^δ_{n,v}(W^T_n; t) := \sum_{j=0}^{n} \hat{S}^δ_n(j) \frac{(2j + \lambda_k)\Gamma\left(\frac{1}{2}\right)\Gamma(j + \lambda_k)}{\Gamma(\lambda_k + 1)\Gamma(j + \frac{1}{2})} P^j_{\lambda_k}(-\frac{1}{2}, \frac{1}{2})(t).
\]

Consequently, defining analogues of \( a_{n,v,\ell} \) by

\[
a^T_{n,v,0}(j) = (2j + \lambda_k)\hat{S}^δ_n(j), \quad a^T_{n,v,\ell+1}(j) = \frac{a^T_{n,v,\ell}(j)}{2j + 2\lambda_k + \ell} - \frac{a^T_{n,v,\ell}(j + 1)}{2j + 2\lambda_k + \ell + 2},
\]

we can then write, again following the proof of Lemma 3.3 of [4], that

\[
D^δ_{n,v}(W^T_n; t) = c \sum_{j=0}^{n} a^T_{n,v,\ell}(j) \frac{\Gamma(j + 2\lambda_k + \ell)}{\Gamma(j + \lambda_k + \frac{1}{2})} P^j_{\lambda_k+\ell}(-\frac{1}{2}, \frac{1}{2})(t).
\]

The estimate (6.6) holds exactly for \( a_{n,v,\ell}(j) \). Thus, to follow the proof of Lemma 6.1 we need to estimate

\[
(7.2) \quad \int_{[-1,1]^{d+1}} P^j_{\lambda_k+\ell}(-\frac{1}{2}, \frac{1}{2})(2z(x, y, t)^2 - 1) \prod_{i=1}^{d+1} (1 - t_i^{2\gamma})^{-1} dt.
\]

Using the fact that \( P^j_0(2t^2-1) \) can be written in terms of \( P^j_0(\alpha, \alpha) \) (17 (4.1.5)), we can estimate (7.2) again by Lemma 2.20. The result is

\[
|K^δ_{n,v}(W^T_n; x, y)| \left| W^T_n(y) \right|^{-1} \leq c\alpha^d 2^{(\ell-1-\delta)} \left( 1 + nd(x, y) \right)^{-\lambda_k-\ell+d-1},
\]

which implies that the analogue of Corollary 6.2 holds; that is, for any \( \gamma > 0 \) there is an \( \varepsilon_0 > 0 \) such that

\[
\sup_{x \in T^{d}} \int_{\{y: d_T(x, y) \geq 2^{(1+\varepsilon)/n}\}} |K^δ_{n,v}(W^T_n; x, y)| W^T_n(y) dy \leq c2^{-v\varepsilon_0}.
\]
In order to prove (7.1), we then define \( \Lambda \) to be a maximal separate subset of \( T_d \) exactly as the one we defined in Subsection 6.1.3, except with \( d(\varpi, \varpi') \) replaced by \( d_T(x, x') \). Define

\[
f_y(x) := f(x) \chi_{c_T}(y, x')(x)[A(x)]^{-1}, \quad A(x) = \sum_{y \in \Lambda} \chi_{c_T}(y, x')(x).
\]

Then the same argument shows that it suffices to prove that

\[
\|S_{n,\nu}(W_T; f_y)\|_{\nu, p} \leq c2^{-v_\nu} \|f_y\|_{\nu, p}.
\]

This last inequality can be established exactly as in (6.8) and there is no need to introduce the additional set \( c^*(\varpi, \theta) \).

**Proof of Theorem 3.6.** We first note that the analogue of Lemma 6.3 holds; that is, for \( 1 \leq p < \infty \),

\[
\|Q\|_{\infty} := \max_{x \in T^d} |Q(x)| \leq c n(2\sigma_n + 1)/p \|Q\|_{\nu, p}
\]

for any polynomial of degree \( n \) on \( \mathbb{R}^d \). Furthermore, following the proof of Theorem 3.2, it is sufficient to prove that

\[
(7.3) \quad \|K_n^\delta(W_T; y)\|_{\kappa, q} \leq c n(2\sigma_n + 1)/p, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

where \( y \in T^d \) is fixed, does not hold for \( p = p_1 := \frac{2\gamma_n + 1}{\sigma_n - 1} \). To proceed, we then express \( K_n(W_T) \) in terms of the kernel for the Jacobi polynomial expansions ([11, p. 290])

\[
K_n^\delta(W_T; x, e_j) = K_n^\delta(w^{(\lambda_n - \epsilon_j^{(n)}, \frac{1}{2}, \sigma_j^{(n)} - \frac{1}{2})}; \kappa, q, x_j - 1), \quad 1 \leq j \leq d,
\]

\[
K_n^\delta(W_T; x, 0) = K_n^\delta(w^{(\lambda_n - \frac{1}{2}, \frac{1}{2})}; \kappa, q, 1),
\]

from which we can deduce by changing variables that

\[
\int_{T^d} |K_n^\delta(W_T; e_j, y)|^q W_T^\nu(y)dy = c \int_{-1}^1 |K_n^\delta(w^{(\lambda_n - \epsilon_j^{(n)}, \frac{1}{2}, \sigma_j^{(n)} - \frac{1}{2})}; 1, t)|^q w^{(\lambda_n - \frac{1}{2}, \frac{1}{2})}(t)dt
\]

for \( 1 \leq j \leq d \), and also for \( j = d + 1 \) if we agree that \( e_{d+1} = 0 \). As a result, we have reduced the problem to that of Jacobi polynomial expansions, so that the desired result follows from [6].

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