THE COX RING OF $\overline{M}_{0,6}$

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Abstract. We prove that the Cox ring of the moduli space $\overline{M}_{0,6}$, of stable rational curves with 6 marked points, is finitely generated by sections corresponding to the boundary divisors and divisors which are pull-backs of the hyperelliptic locus in $\overline{M}_3$ via morphisms $\rho: \overline{M}_{0,6} \to \overline{M}_3$ that send a 6-pointed rational curve to a curve with 3 nodes by identifying 3 pairs of points. In particular this gives a self-contained proof of Hassett and Tschinkel’s result about the effective cone of $\overline{M}_{0,6}$ being generated by the above mentioned divisors.

1. Introduction

A question of Fulton about the moduli space $\overline{M}_{0,n}$, of stable, $n$-pointed, rational curves, is whether the cone $\text{NE}^k(\overline{M}_{0,n})$ of effective cycles of codimension $k$ in $\overline{M}_{0,n}$ is generated by $k$-strata, i.e., loci in $\overline{M}_{0,n}$ corresponding to reducible curves with at least $k$ nodes. While the case when $k = n - 4$ (i.e., the cone of effective curves) is completely open (and an affirmative result would imply, by results of Gibney, Keel and Morrison [GKM], the similar statement for the moduli space $\overline{M}_{g,n}$, of stable, $n$-pointed, genus $g$ curves, thus determining the ample cone of $\overline{M}_{g,n}$), the case when $k = 1$ (i.e., the cone of effective divisors) was settled independently by Keel (unpublished; a reference to this may be found in [GKM], p.277) and Vermeire [V]: Fulton’s question has a negative answer when $n = 6$ (and therefore for any $n \geq 6$). Hassett and Tschinkel prove in [HT] that the Keel-Vermeire divisors (pull-backs of the locus of hyperelliptic curves in the moduli space $\overline{M}_3$, via morphisms $\overline{M}_{0,6} \to \overline{M}_3$ sending a 6-pointed rational curve to a curve with 3 nodes by identifying 3 pairs of points) together with the 2-strata (the boundary) generate the cone of effective divisors in $\overline{M}_{0,6}$. The proof in [HT] is based on a computer check. In this paper we give a proof of Hassett and Tschinkel’s result, by proving a stronger statement: we show that the sections corresponding to the above divisors generate the Cox ring of $\overline{M}_{0,6}$.

Recall that if $X$ is a smooth projective variety with Picard group freely generated by divisors $D_1, \ldots, D_r$, then the Cox ring (or total coordinate ring) of $X$ is the multi-graded ring:

$$\text{Cox}(X) = \bigoplus_{(m_1, \ldots, m_r) \in \mathbb{Z}^r} \text{H}^0(X, m_1 D_1 + \ldots + m_r D_r).$$

The Cox ring being finitely generated has strong implications for the birational geometry of $X$ ($X$ is a so-called Mori Dream Space): the effective cone and the nef
cone are both polyhedral and there are finitely many small modifications of $X$ (i.e., varieties $X'$ isomorphic in codimension one to $X$) such that any moving divisor on $X$ (i.e., a divisor whose base locus has codimension at least 2) is nef on one of the varieties $X'$ (see [HK] for the precise statements). It has been conjectured by Hu and Keel [HK] that any log-Fano variety has a finitely generated Cox ring. This has been recently proved in the groundbreaking paper [BCHM]. In [HK] Hu and Keel ask the following question:

**Question 1.1.** Is the Cox ring of $\mathcal{M}_{0,n}$ finitely generated?

As pointed out in [KM], the moduli space $\mathcal{M}_{0,n}$ is log-Fano only for $n \leq 6$.

We answer Question 1.1 for $n = 6$ by finding explicit generators. Our hope is that our method for finding generators, which proved to be useful in other circumstances (see [CT]), will eventually help answer Question 1.1 for larger $n$ as well.

Consider the Kapranov description of the moduli space $\mathcal{M} = \mathcal{M}_{0,6}$. If $p_1, \ldots, p_5$ are points in linearly general position in $\mathbb{P}^3$, then $\mathcal{M}$ is the iterated blow-up of $\mathbb{P}^3$ along $p_1, \ldots, p_5$ and along the proper transforms of the lines $l_{ij} = p_ip_j$ for all $i \neq j$. If $p$ is a general point in $\mathbb{P}^3$, there is a unique twisted cubic $C$ in $\mathbb{P}^3$ that contains the points $p_1, \ldots, p_5, p$. Then $(C, p_1, \ldots, p_5, p)$ is a 6-pointed rational curve, hence an element of $\mathcal{M}_{0,6}$. The point $p$ corresponds to the 6’th marking (the so-called moving point).

Denote by $H$ the hyperplane class on $\mathcal{M}$ and by $E_i$ and $E_{ij}$ the exceptional divisors in $\mathcal{M}$ corresponding to the points $p_i$ and the lines $l_{ij}$.

**Notation 1.2.** Let $\Lambda_{ijk}$ be the class of the proper transform of the plane $\overline{p_ip_jp_k}$:

$$\Lambda_{ijk} = H - E_i - E_j - E_k - E_{ij} - E_{ik} - E_{jk}.$$  

If $S \subset \{1, \ldots, 6\}$ and $|S| = 2$, or 3, let $\Delta_S$ be the boundary divisor in $\mathcal{M}$ with general element a curve with two irreducible components with the partition of the markings given by $S \cup S^c$. In the Kapranov description, the boundary divisors $\Delta_S$ have the following classes:

$$\Delta_{i6} = E_i, \quad \Delta_{ij6} = E_{ij}, \quad i, j = 1, \ldots, 5,$$

$$\Delta_{ij} = \Lambda_{abc}, \text{ if } \{i, j, a, b, c\} = \{1, \ldots, 5\}.$$  

**Notation 1.3.** Let $Q_{(ij)(kl)}$ be the class of the proper transform of the unique quadric that contains all the points $p_1, \ldots, p_5$ and the lines $l_k, l_{il}, l_{jk}, l_{jl}$:

$$Q_{(ij)(kl)} = 2H - \sum_i E_i - E_{ik} - E_{il} - E_{jk} - E_{jl}.$$  

The divisor classes $Q_{(ij)(kl)}$ are exactly the divisors considered by Keel and Vermeire: for example, if one considers the map $\mathcal{M}_{0,6} \to \mathcal{M}_3$ given by identifying the pairs of points $(12)(34)(56)$, then the class of the pull-back of the hyperelliptic locus in $\mathcal{M}_3$ is computed in [HT] to be the class of $Q_{(12)(34)}$. We call the divisors $Q_{(ij)(kl)}$ the Keel-Vermeire divisors. We prove the following:

**Theorem 1.4.** The Cox ring of $\mathcal{M}_{0,6}$ is generated by the sections (unique up to scaling) corresponding to the boundary divisors (i.e., $\Lambda_{ijk}$ and the exceptional divisors $E_i$ and $E_{ij}$) and the Keel-Vermeire divisors $Q_{(ij)(kl)}$.

The paper is divided as follows: Section 2 explains the strategy of proof; there are two main cases, the details of each are given in Section 3 respectively Section 4.
The remaining sections contain auxiliary results needed in the proof. Section 5 contains proofs for some basic inequalities, while Section 6 contains some general multiplicity estimates needed for Case II. Section 7 contains the proof of Lemma 2.21 (needed for Case II) that states that the Cox ring of the blow-up of \( \mathbb{P}^2 \) in seven (non-general) points is generated by sections corresponding to \(-1\) and \(-2\) curves. Section 8 gives necessary and sufficient conditions for a divisor on \( X \), the iterated blow-up of \( \mathbb{P}^3 \) in four general points and lines through them, to have sections. Finally, in Section 9 we compute the restrictions of an arbitrary divisor \( D \) to all the boundary divisors and Keel-Vermeire divisors on \( \mathcal{M} \). Moreover, we derive some necessary conditions for these restrictions to be effective (an assumption in our main proof).

2. Plan of proof

Consider an arbitrary divisor class on \( \mathcal{M} \):

\[
D = dH - \sum_i m_i E_i - \sum_{i,j} m_{ij} E_{ij}.
\]

In all that follows we assume \( H^0(\mathcal{M}, D) \neq 0 \).

Notation 2.1. Let \( l \) be the class of the proper transform in \( \mathcal{M} \) of a general line in \( \mathbb{P}^3 \). Let \( e_i \) be the class of a general line in \( E_i \). Let \( C \) be the class of the proper transform of a general cubic that passes through \( p_1, \ldots, p_5 \):

\[
C = 3l - e_1 - \ldots - e_5.
\]

The curves with class \( C \) cover a dense set of \( \mathcal{M} \); hence, \( D.C \geq 0 \) for any effective divisor \( D \).

Definition 2.2. Let \( x_i, x_{ij}, x_{ijk}, x_{(ij)(kl)} \) be the sections (unique up to scalar) corresponding to the divisors:

\[
E_i, E_{ij}, \Lambda_{ijk}, Q_{(ij)(kl)}.
\]

Definition 2.3. We call a section \( s \in H^0(\mathcal{M}, D) \) a distinguished section if

\[
s = x_i^{n_i} x_{ij}^{n_{ij}} x_{ijk}^{n_{ijk}} x_{(ij)(kl)}^{n_{(ij)(kl)}},
\]

where \( n_i, n_{ij}, n_{ijk}, n_{(ij)(kl)} \) are non-negative integers.

To show that \( H^0(\mathcal{M}, D) \) is generated by distinguished sections, we do an induction on \( D.C \). Note that we may assume that \( D \) contains none of the divisors (2.1) in its base locus, i.e., equivalently, if for \( E \) any of the divisors in (2.1), one has \( H^0(E, D|_E) \neq 0 \). To see this, note that if \( E \) is an effective divisor, say \( E \) is the zero locus of a section \( x_E \in H^0(\mathcal{M}, E) \), then there is an exact sequence,

\[
0 \to H^0(\mathcal{M}, D - E) \to H^0(\mathcal{M}, D) \to H^0(E, D|_E).
\]

If \( H^0(E, D|_E) = 0 \), then any \( s \in H^0(\mathcal{M}, D) \) is of the form \( x_E t \), where \( t \in H^0(\mathcal{M}, D - E) \). If in addition \( E \) is a divisor in (2.1), then we may replace \( D \) with \( D - E \) and \( s \) with \( t \). (Clearly, if \( t \) is generated by distinguished sections, then \( s \) is too.) Therefore, we may assume:

Assumption 2.4. \( H^0(E, D|_E) \neq 0 \) for all divisors \( E \) in (2.1).

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Denote by $r_E$ the restriction to $E$:

$$r_E : H^0(\overline{M}, D) \to H^0(E, D|_E).$$

To prove Theorem 1.4 it is enough to prove the following:

**Main Claim.** Let $D$ be a divisor on $\overline{M}$:

$$D = dH - \sum_i m_i E_i - \sum_{i,j} m_{ij} E_{ij},$$

such that $H^0(\overline{M}, D) \neq 0$ and that satisfies Assumption 2.4. Up to a renumbering, we may assume that $m_5 \leq m_i$, for $i = 1, \ldots, 4$. If $m_i = m_5$ for all $i = 1, \ldots, 4$, then we may assume that the maximum of the $m_{ij}$’s for all $i, j \in \{1, \ldots, 5\}$ is attained for $m_{i5}$ for some $i = 1, \ldots, 4$. Let $E = E_5$. Then for any $s \in H^0(D\overline{M}, D)$, there is $s' \in H^0(\overline{M}, D)$, generated by distinguished sections, such that $r_E(s) = r_E(s')$.

To see how the Main Claim implies Theorem 1.4 note that the kernel of the restriction $r_E$ is $H^0(\overline{M}, D - E)$ and the map $H^0(\overline{M}, D - E) \to H^0(\overline{M}, D)$ is given by multiplication with $x_E$. If $r_E(s) = r_E(s')$, then $s - s' = x_E t$, where $t \in H^0(\overline{M}, D - E)$. If $s'$ is generated by distinguished sections, then to show that $s$ is generated by distinguished sections is enough to show that $H^0(\overline{M}, D - E)$ is generated by distinguished sections. We may replace $D$ with $D - E$ and continue the procedure. Since $E$ is always among the $E_i$’s, note that $(D - E).C < D.C$ and $H^0(\overline{M}, D - E)$ is generated by distinguished sections by induction. The process has to stop as $D.C \geq 0$ for any effective divisor $D$. (In particular, note that $D.C$ also decreases when we subtract from $D$ any of the divisors $E$ in (2.1) for which $H^0(E, D|E) = 0$.)

**Notation 2.5.** Given any divisor $D$ on $\overline{M}$ we denote by $\overline{D}$ the restriction $D|_{E_5}$ of $D$ to $E_5$. By (3.3) one has

$$\overline{D} = m_5 \overline{H} - \sum_{i=1}^4 m_{i5} \overline{E}_i.$$  

Let $\rho_5 : \mathbb{P}^3 \to \mathbb{P}^2$ be the projection from $p_5$. Let $q_i = \rho_5(p_i)$ ($i \in \{1, \ldots, 4\}$). The divisor $E_5$ is isomorphic to the blow-up of $\mathbb{P}^2$ along the points $q_1, \ldots, q_4$ (as $q_i$ determines the direction of the line $l_{ij}$). The divisors $\overline{H}$, respectively $\overline{E}_i$, are the hyperplane class, respectively the exceptional divisors on $E_5$ (see also Section 9.1). The map $\rho_5$ is resolved by the morphism $\pi_5 : \overline{M} \to \overline{M}_{0,5}$ that forgets the 5’th marking (which is also a retract for the inclusion $E_5 \subset \overline{M}$).

**Notation 2.6.** Let $\overline{l}_{ij}$ be the line $q_i q_j$ in $\mathbb{P}^2$. Denote:

$$x = \overline{l}_{13} \cap \overline{l}_{24}, \quad y = \overline{l}_{14} \cap \overline{l}_{23}, \quad z = \overline{l}_{12} \cap \overline{l}_{34}.$$

**Notation 2.7.** Let $L_x$ be the proper transform in $\overline{M}$ of the unique line in $\mathbb{P}^3$ that passes through $p_5$ and intersects the skew lines $l_{13}$ and $l_{24}$. Similarly, let $L_y$ (respectively $L_z$) be the unique line that passes through $p_5$ and intersects the skew lines $l_{14}$ and $l_{23}$ (respectively $l_{12}$ and $l_{34}$).

Remark that $x = \rho_5(L_x)$, $y = \rho_5(L_y)$, $z = \rho_5(L_z)$.

In order to prove the Main Claim, we distinguish two cases.
Case I: Assume that $D.L_x \geq 0, \quad D.L_y \geq 0, \quad D.L_z \geq 0$.

Notation 2.8. Denote by $s_{ij}$ the section on $E_5$ corresponding to the proper transform of the line $l_{ij}$ in $\mathbb{P}^2$. Let $s_i$ ($i = 1, \ldots, 4$) be the sections corresponding to the exceptional divisors $E_i$.

Definition 2.9. We call a section $s \in H^0(E_5, \overline{D})$ a distinguished section on $E_5$ if $s$ can be written as a monomial in the sections $s_{ij}$ and $s_i$.

Since $E_5 \cong M_{0,5}$ is the blow-up of $\mathbb{P}^2$ along $q_1, \ldots, q_4$, by Lemma 7.3, the Cox ring $\text{Cox}(E_5)$ of $E_5$ is generated by distinguished sections. The Main Claim follows from the following:

Proposition 2.10. Under the assumptions of the Main Claim and the assumptions in Case I, the restriction map

$$r_{E_5}: H^0(M, D) \to H^0(E_5, \overline{D})$$

is surjective and one may lift any distinguished section (hence, any section) in $H^0(E_5, \overline{D})$ to a section generated by distinguished sections in $H^0(M, D)$.

The following is the main observation needed to prove Proposition 2.10:

Main Observation – Case I. Distinguished sections on $E_5$ may be lifted to distinguished sections on $M$ using the following rules:

$$x_{ij5|E_5} = s_{ij}, \quad x_{i5|E_5} = s_i.$$  \hspace{1cm} (2.2)

This is because $\Lambda_{ij5|E_5} = l_{ij}, \quad E_{i5|E_5} = \overline{E}_i$ (see Section 9, formula (9.3)).

Sketch of Proof of Proposition 2.10. We lift a distinguished section $s \in H^0(E_5, \overline{D})$ using the rules (2.2). Hence, there is a section $t'$ belonging to some $H^0(M, D')$, where $\overline{D}' = \overline{D}$ and $r_{E_5}(t') = s$.

Notation 2.11. Let $\Delta = D - D'$.

Notation 2.12. Denote by $X$ the iterated blow-up of $\mathbb{P}^3$ in $p_1, \ldots, p_4$ and proper transforms of lines $l_{ij}$ ($i, j \in \{1, \ldots, 4\}$).

Since $D$ and $D'$ have the same restriction to $E_5$, it follows from (9.3) that the divisor $\Delta$ is a divisor on $X$. Note, $X$ is a toric variety. The following is a standard result:

Lemma 2.13. The Cox ring of $X$ is generated by sections $x_i, x_{ij}, x_{ijk}$ corresponding to the exceptional divisors $E_i, E_{ij}$ and proper transforms of hyperplanes $\Lambda_{ijk}$ ($i, j, k \in \{1, \ldots, 4\}$).

Proposition 2.10 is now immediate if $H^0(\Delta) \neq 0$: Since the points $p_1, \ldots, p_5$ are general, the restriction to $E_5$ of any distinguished section in $\text{Cox}(X)$ is non-zero. In particular, if $t''$ is any non-zero section in $H^0(\Delta)$, then $t''_{E_5} \in H^0(E_5, \mathcal{O})$ is non-zero. Therefore, the section $s = t' t''$ is a section in $H^0(M, D)$ that restricts to (a non-zero multiple of) $s$ in $H^0(E_5, \overline{D})$. Since $t''$ is a distinguished section, it follows that $t$ is generated by distinguished sections.

Definition 2.14. We call a distinguished section $\overline{s}$ on $E_5$ a section with straightforward lifting to $D$ if after lifting using the rules (2.2) we end up with a divisor $D'$ for which $\Delta = D - D'$ has $H^0(\Delta) \neq 0$. 

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The following claim (proof in Section 3) finishes the proof of Proposition 2.10.

Claim 2.15. Under the assumptions of Proposition 2.10 any distinguished section \( \pi \in H^0(E_5, \mathcal{D}) \) is a linear combination of distinguished sections with straightforward lifting to \( D \).

Case II: Assume one of \( D.L_x, D.L_y, D.L_z \) is negative.

Definition 2.16. Let:

\[
m_x = \max \{0, -D.L_x\}, \quad m_y = \max \{0, -D.L_y\}, \quad m_z = \max \{0, -D.L_z\}.
\]

Notation 2.17. Denote by \( Y \) the blow-up of \( \mathbb{P}^2 \) along \( q_1, q_2, q_3, q_4, x, y, z \). Let \( E_i, E_x, E_y, E_z \) be the corresponding exceptional divisors. For a given divisor \( D \) on \( \mathcal{M} \) we consider the following divisor \( D^Y \) on \( Y \):

\[
D^Y = D - m_x E_x - m_y E_y - m_z E_z.
\]

Clearly, the linear system \( H^0(Y, D^Y) \) is a subspace of the linear system \( H^0(E_5, \mathcal{D}) \).

Claim 2.18. The restriction map \( r_{E_5} \) factors through \( H^0(Y, D^Y) \).

Proof. Clearly, Claim 2.18 is non-trivial only when one of \( m_x, m_y, m_z \) is positive. Take for example the case when \( m_x > 0 \) (the other cases are identical). By Proposition 6.1 the line \( L_x \) is contained in \( D \) with multiplicity \( m \geq m_x > 0 \). It follows that for any \( s \in H^0(\mathcal{M}, D) \) the section \( r_{E_5}(s) \) vanishes at \( x \) with multiplicity \( \geq m \); hence, \( r_{E_5}(s) \) lies in the subspace \( H^0(Y, D^Y) \). \( \square \)

In Case II we follow the exact same steps as in Case I, with the only difference being that we work on \( Y \) instead of \( E_5 \).

Notation 2.19. Denote by \( s'_{ij} \) the section corresponding to the proper transform in \( Y \) of the line \( \tilde{l}_{ij} \). Similarly, let \( s_{xy}, s_{xz}, s_{yz} \) be the sections corresponding to the proper transforms of the lines \( \tilde{w}, \tilde{w}_x, \tilde{w}_z \). Let \( s_i, s_x, s_y, s_z \) be the sections corresponding to the exceptional divisors \( E_i, E_x, E_y, E_z \).

Note:

\[
x \in \tilde{l}_{13}, \tilde{l}_{24}, \quad y \in \tilde{l}_{14}, \tilde{l}_{23}, \quad z \in \tilde{l}_{12}, \tilde{l}_{34}.
\]

Hence, for example, \( s_{13} \) is a section of the divisor \( \mathcal{H} - E_1 - E_3 - E_x \), and the section \( s_{13} \) (Notation 2.18) is given by \( s_{13} = s'_{13}s_x \). Moreover, if we let

\[
r_Y : H^0(\mathcal{M}, D) \rightarrow H^0(Y, D^Y)
\]

be the morphism of Claim 2.18 then \( r_{E_5}(s) = r_Y(s)s_x^{m_x}s_y^{m_y}s_z^{m_z} \).

Definition 2.20. We call a section \( s \in H^0(Y, D^Y) \) a distinguished section on \( Y \) if \( s \) can be written as a monomial in the sections \( s'_{ij}, s_{xy}, s_{xz}, s_{yz}, s_x, s_y, s_z \).

In Section 7 we prove the following:

Lemma 2.21. The Cox ring \( \text{Cox}(Y) \) of \( Y \) is generated by distinguished sections.
Note, by Lemma 2.21 the generators of \( \text{Cox}(Y) \) are given by the sections (unique up to scalar multiplication) corresponding to the \((-1)\) and \((-2)\) curves on \( Y \). The Main Claim follows from:

**Proposition 2.22.** Under the assumptions of the Main Claim, the restriction map

\[
r_Y : H^0(M, D) \rightarrow H^0(Y, D')
\]

is surjective and one may lift any distinguished section (hence, any section) in \( H^0(Y, D') \) to a section generated by distinguished sections in \( H^0(M, D) \).

The following is the main observation needed to prove Proposition 2.22:

**Main Observation – Case II.** Distinguished sections on \( Y \) may be lifted to distinguished sections on \( M \) using the following rules:

\[
(2.3) \quad r_Y(x_{ij5}) = s'_{ij}, \quad r_Y(x_{i5}) = s_i,
\]

\[
(2.4) \quad r_Y(x_{(13)(24)}) = s_{yz}, \quad r_Y(x_{(14)(23)}) = s_{xz}, \quad r_Y(x_{(12)(34)}) = s_{xy}.
\]

This is because when \( D = \Lambda_{ij5} \), one has

\[
D' = \bar{H} - \bar{E}_i - \bar{E}_j - \bar{E}_\alpha,
\]

where \( \alpha = x \) if \( ij \in \{13, 24\} \), \( \alpha = y \) if \( ij \in \{14, 23\} \), and \( \alpha = z \) if \( ij \in \{12, 34\} \).

Similarly,

\[
Q_{(13)(24)}' = \bar{H} - \bar{E}_y - \bar{E}_z, \quad Q_{(14)(23)}' = \bar{H} - \bar{E}_x - \bar{E}_z, \quad Q_{(12)(34)}' = \bar{H} - \bar{E}_x - \bar{E}_y.
\]

**Sketch of Proof of Proposition 2.22.** We lift a distinguished section \( \overline{\pi} \in H^0(E_5, \overline{D}) \) using the rules (2.3) and (2.4). Hence, there is a section \( t' \) in some \( H^0(M', D') \) and \( r_Y(t') = \overline{\pi} \). As in Case I, we let \( \Delta = D - D' \). The divisor \( \Delta \) is a divisor on \( X \) (Notation 2.12). As in Case I, Proposition 2.22 follows from Lemma 2.13 if \( H^0(\Delta) \neq 0 \).

**Definition 2.23.** We call a distinguished section \( \pi \) on \( Y \) a section with straightforward lifting to \( D \) if lifting using the rules (2.3) and (2.4) results in a divisor \( D' \) for which \( \Delta = D - D' \) has \( H^0(\Delta) \neq 0 \).

The following claim (proof in Section 3) finishes the proof of Proposition 2.22:

**Claim 2.24.** Under the assumptions of the Main Claim, any distinguished section \( \pi \in H^0(Y, D') \) is a linear combination of distinguished sections with straightforward lifting to \( D \).

3. **Proof of Claim 2.24**

The idea is that any distinguished section on \( E_5 \) can be rewritten, using the relations in \( \text{Cox}(E_5) \), as a linear combination of distinguished sections with straightforward lifting. To check that \( H^0(\Delta) \neq 0 \) we use Lemma 8.2. Assumption 2.24 is equivalent to inequalities (9.3), (9.6), (9.7), (9.8) (for all permutations of indices).

We use the notation from Section 3. Recall that \( e_{ij} \) is the class of a fiber of the \( \mathbb{P}^1 \)-bundle \( E_{ij} \rightarrow \overline{I}_{ij} \). One has \( D.l = d, D.e_i = m_i, D.e_{ij} = m_{ij} \) (see for example (9.1), (9.6)). The inequalities defining Case I are equivalent to

\[
(3.1) \quad D.(l - e_5 - e_{ij} - e_{kl}) \geq 0, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}.
\]
Lemma 3.1. Let $\pi$ be a distinguished section on $E_5$:
\begin{equation}
\pi = \prod_{i,j} s_{ij}^{a_{ij}} \prod_i s_i^{l_i},
\end{equation}
where $a_{ij}, l_i \geq 0$. If $\pi$ is a section $H^0(E_5, D)$, then $\pi$ has straightforward lifting to $D$ if and only if for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has
\begin{equation}
a_{ij} \leq D.(C_{k:l} - e_5),
\end{equation}
where $C_{k:l} = 2l - e_{ki} - e_{kj} - e_l$.

Remark 3.2. By (9.8) one has $D.(C_{k:l} - e_5) \geq 0$ for all $k, l \in \{1, 2, 3, 4\}$.

Remark 3.3. The condition that $\pi$ is in $H^0(D)$ is equivalent to
\begin{equation}
\sum a_{ij} = D.e_5, \quad a_{ij} + a_{ik} + a_{il} - l_i = D.e_{i5}
\end{equation}
(the coefficients of $H$ and $E_i$ in $D$). It follows from (3.4) that
\begin{equation}
a_{kl} - a_{ij} - l_k - l_i = D.(e_{k5} + e_{i5} - e_5),
\end{equation}
\begin{equation}
\sum_{i=1}^4 l_i = D.(2e_5 - \sum_{i \neq 5} e_{i5}),
\end{equation}
\begin{equation}
a_{jk} + a_{jl} + a_{kl} + l_i = D.(e_5 - e_{i5}).
\end{equation}

Proof of Lemma 3.1. If $D = 0$, $\pi = 1$ (i.e., $a_{ij} = 0, l_i = 0$), then the lift $D'$ is 0. Hence, $\Delta = D - D' = D$. Since $H^0(D) \neq 0$, there is nothing to prove in this case.

Assume now $D \neq 0$. Recall that $E_5 \subset M$ has a retract $\pi : M \to E_5 \cong M_{0,5}$ given by the morphism that forgets the 5'th marking. One has
\begin{equation}
\pi^*\mathcal{L}_{ij} = \Lambda_{ij5} + E_{ij}, \quad \pi^*E_i = E_{i5} + E_i.
\end{equation}
(This is a general fact about the forgetful morphisms $\pi_i : M_{0,n} \to M_{0,n-1}$ that forget a marking $i$. If $\Delta_S$ is a boundary divisor in $M_{0,n-1}$, corresponding to the partition $S \cup S^c$, then $\pi^*\Delta_S = \Delta_S + \Delta_{S \cup \{i\}}$.)

Since we lift $D$ to $D'$ by lifting $\mathcal{L}_{ij}$ to $\Lambda_{ij5}$ and $E_i$ to $E_{i5}$, it follows that
\begin{equation}
D' = \pi^*D - \Delta_0,
\end{equation}
where $\Delta_0$ is the effective divisor on $X$ given by
\begin{equation}
\Delta_0 = \sum_{i,j \in \{1,\ldots,4\}} a_{ij} E_{ij} + \sum_{i \in \{1,\ldots,4\}} l_i E_i.
\end{equation}

Then $\Delta = D - D' = D - \pi^*D + \Delta_0$.

Observation 3.4. If $(D - \pi^*D).C \geq 0$ for some nef curve $C$ on $X$, then $\Delta.C \geq 0$.

Below we show that $(D - \pi^*D).C \geq 0$ for all the nef curves $C$ in Lemma 3.2 giving inequalities (1)–(4). Hence, by Observation 3.2, $\Delta.C \geq 0$. For the remaining nef curves $C$ in general it will not be true that $(D - \pi^*D).C \geq 0$, but we show that we still have $\Delta.C \geq 0$ for the nef curves $C$ giving inequalities (5),(7),(8),(9) and that for $C = C_{k:l}$ (inequality (6)) $\Delta.C \geq 0$ is equivalent to (3.8). Note that
\begin{equation}
(\pi^*D).l = D.e_5, \quad (\pi^*D).e_i = D.e_{i5}, \quad (\pi^*D).e_{ij} = 0 \quad (i, j \neq 5).
\end{equation}
(It is enough to check this when $D = H, E_i, E_{ij}$. For this, use the formulas (3.8).)
We check one by one the inequalities (1) – (9) in Lemma 8.2.

(1) \( (D - \pi^* D).l = D.(l - e_5) \geq 0 \),
as \( l - e_5 \) is a nef curve on \( \mathcal{M} \). Similarly:

(2) \( (D - \pi^* D).(l - e_i) = D.(l - e_i - e_5 + e_{i5}) \geq 0 \) by (9.6),
(3) \( (D - \pi^* D).(l - e_{ij}) = D.(l - e_5 - e_{ij}) \geq 0 \) by (9.7),
(4) \( (D - \pi^* D).(l - e_{ij} - e_{kl}) = D.(l - e_5 - e_{ij} - e_{kl}) \geq 0 \) by (3.1).

For inequality (5) (recall \( C_{ij} = 2l - e_{ij} - e_k - e_l \)),
\[
(D - \pi^* D).C_{ij} = D.(C_{ij} - 2e_5 + e_{i5} + e_{j5}),
\]
\[
\Delta_0.C_{ij} = a_{ij} + l_k + l_t,
\]
\[
\Delta.C_{ij} = D.(C_{ij} - e_5) + D.(e_k + e_5 - e_5) + a_{ij} + l_k + l_t.
\]

By (3.6) \( \Delta.C_{ij} = D.(C_{ij} - e_5) + a_{kl} \) from (9.7) (and \( a_{kl} \geq 0 \)), \( \Delta.C_{ij} \geq 0 \). For inequality (7) (recall \( C_i = 2l - e_{ij} - e_{ik} - e_{il} \)),
\[
(D - \pi^* D).C_i = D.(C_i - 2e_5)
\]
and \( \Delta_0.C_i = a_{ij} + a_{ik} + a_{il} \) using (3.1), \( \Delta_0.C_i = D.e_{i5} + l_i \). Therefore,
\[
\Delta.C_i = D.(C_i - 2e_{i5} + e_{i5}) + l_i = 2D.(l - e_i - e_5 + e_{i5}) + D.(2e_i - \sum_{u \neq i} e_{iu}) + l_i.
\]

It follows from (9.4) and (9.6) that \( \Delta.C_i \geq 0 \). For inequality (8) (recall \( B = 3l - \sum_{i=1}^{4} e_i \)),
\[
(D - \pi^* D).B = D.(B - 3e_5 + \sum_{i=1}^{4} e_{i5})
\]
and \( \Delta_0.B = \sum_{i=1}^{4} l_i \). It follows from (3.6) that \( \Delta.B = D.(3l - \sum_{i=1}^{5} e_i) \geq 0 \).

For inequality (9) (recall \( B_i = 3l - 2e_i - e_{jk} - e_{jl} - e_{kl} \)),
\[
(D - \pi^* D).B_i = D.(B_i - 3e_5 + 2e_{i5})
\]
and \( \Delta_0.B_i = a_{jk} + a_{jl} + a_{kl} + \sum_{i} l_i \). From (3.7) one has \( \Delta_0.B_i = D.(e_5 - e_{i5}) + l_i \),
\[
\Delta.B_i = D.(B_i - 2e_5 + e_{i5}) = 2D.(l - e_i - e_5 + e_{i5}) + D.(l - e_{jk} - e_{jl} - e_{kl} - e_{ik}).
\]
It follows by (9.4) and (9.7) that \( D.B_i \geq 0 \).

There is at least one strict inequality in (4): assume \( \Delta.(l - e_{ij} - e_{kl}) = 0 \), for all \( \{i, j, k, l\} = \{1, 2, 3, 4\} \). From the computation above for case (4), we have
\[
(D - \pi^* D).(l - e_{ij} - e_{kl}) = D.(l - e_5 - e_{ij} - e_{kl}) \geq 0.
\]
As \( \Delta_0.(l - e_{ij} - e_{kl}) = 0 \) (\( l - e_{ij} - e_{kl} \) is a nef curve) it follows that
\[
(D - \pi^* D).(l - e_{ij} - e_{kl}) = \Delta_0.(l - e_{ij} - e_{kl}) = 0.
\]
Since \( \Delta_0.(l - e_{ij} - e_{kl}) = a_{ij} + a_{kl} \) it follows that \( a_{ij} = 0 \) for all \( i, j \). By (3.4),
\( D.e_5 = 0 \) and \( D.e_{i5} = 0, l_i = 0 \) for all \( i \neq 5 \). Hence, \( \overline{D} = 0, \overline{\pi} = 1 \), which contradicts our assumption.

We now show that inequality (6) is equivalent to (3.3). One has
\[
(D - \pi^* D).C_{ij} = D.(C_{i5} - 2e_5 + e_{j5})
\]
and \( \Delta_0.C_{ij} = a_{ik} + a_{il} + l_j \). From (3.7), \( \Delta_0.C_{ij} = D.(e_5 - e_{j5}) - a_{kl} \). Therefore,
\[
\Delta.C_{ij} = D.(C_{i5} - e_5) - a_{kl}.
\]
Hence, inequality (6) is equivalent to (3.3).

\[\Box\]
3.1. Proof of Claim 3.5. Let \(\pi\) be a distinguished section in \(\mathbb{H}^3(E_5, D)\) as in (3.2). If \(a_{ij} \leq D(C_{k,l} - e_5)\) for all \(\{i,j,k,l\} = \{1,2,3,4\}\), then by Lemma 3.1 \(\pi\) has straightforward lifting to \(D\). Assume now that \(a_{ij} > D(C_{k,l} - e_5)\) for some choice of \(i,j,k,l\). Without loss of generality, we may assume \(a_{12} > D(C_{3,4} - e_5)\). Note that by Remark 3.2 it follows that \(a_{12} > 0\).

Claim 3.5. If \(a_{12} > D(C_{3,4} - e_5)\), then either \(a_{34} > 0\) or \(l_1 + l_2 > 0\).

Proof. By (3.10) one has

\[
a_{12} - a_{34} - l_1 - l_2 = D(e_{15} + e_{25} - e_5).
\]

Assume \(a_{34} = l_1 = l_2 = 0\). It follows from (3.10) and \(a_{12} > D(C_{3,4} - e_5)\) that

\[
a_{12} > D(e_{15} + e_{25} - e_5) > D(C_{3,4} - e_5).
\]

This is a contradiction, as by (9.8) one has

\[
D(C_{3,4} - e_5) - D(e_{15} + e_{25} - e_5) = D(2l - e_4 - e_{13} - e_{25} - e_{15} - e_{25}) \geq 0.
\]

\[\square\]

3.2. Algorithm for replacing \(\pi\). We now give an algorithm for replacing \(\pi\) with another distinguished section \(\pi'\) for which \(a_{12} - D(C_{3,4} - e_5)\) is strictly smaller than for \(\pi\) and moreover, for all \(i,j\) for which \(a_{ij} - D(C_{k,l} - e_5)\) increases by this change, the section \(\pi'\) (still) satisfies \(a_{ij} - D(C_{k,l} - e_5) \leq 0\). We repeat the following two steps until \(a_{12} \leq D(C_{3,4} - e_5)\) (as by Claim 3.5 one of the two situations must happen if \(a_{12} > D(C_{3,4} - e_5)\)). The same argument works for any \(a_{ij}\).

Step 1: If \(l_1 + l_2 > 0\): We may assume without loss of generality that \(l_1 > 0\). Consider the following sections in the linear system \([\mathcal{P} - \mathcal{E}_2]\):

\[
s_{12} a_1, s_{23} s_3, s_{24} s_4.
\]

The linear system \([\mathcal{P} - \mathcal{E}_2]\) is 1-dimensional and any two of the above sections are linearly independent. Since \(a_{12} > 0, l_1 > 0\), we may replace \(s_{12} a_1\) in \(s\) with a linear combination of \(s_{23} s_3\) and \(s_{24} s_4\). The effect on the coefficients \(a_{ij}\) and \(l_i\) (of the corresponding two distinguished sections) is as follows: \(a_{12}\) and \(l_1\) both decrease by 1, while either \(a_{23}, l_3\) increase by 1, or \(a_{24}, l_4\) increase by 1 (everything else stays the same). But by Lemma 3.6 one has

\[
a_{2j} < D(C_{k,l} - e_5), \quad \text{for all } j \in \{3,4\}, \{j,k,l\} = \{1,3,4\}.
\]

Therefore, after increasing \(a_{23}\) or \(a_{24}\) by 1 one still has \(a_{2j} \leq D(C_{k,l} - e_5)\).

Step 2: If \(a_{34} > 0\): Consider the following sections in the linear system \([2\mathcal{P} - \mathcal{E}_1 - \ldots - \mathcal{E}_4]\):

\[
s_{12} s_{34}, s_{13} s_{24}, s_{14} s_{23}.
\]

The linear system \([2\mathcal{P} - \mathcal{E}_1 - \ldots - \mathcal{E}_4]\) is 1-dimensional, and any two of the above sections are linearly independent. Since \(a_{12} > 0, a_{34} > 0\), we may replace \(s_{12} s_{34}\) in \(s\) with a linear combination of \(s_{13} s_{24}\) and \(s_{14} s_{23}\). The effect on the coefficients \(a_{ij}\) is: \(a_{12}\) and \(a_{34}\) both decrease by 1, while either \(a_{13}, a_{24}\) increase by 1, or \(a_{14}, a_{23}\) increase by 1. By Lemma 3.6 one has

\[
a_{ij} < D(C_{k,l} - e_5), \quad \text{for all } i \in \{1,2\}, j \in \{3,4\}, \{i,j,k,l\} = \{1,2,3,4\}.
\]

Therefore after increasing \(a_{13}, a_{14}, a_{23}, a_{24}\) by 1, each of them still satisfies its corresponding inequalities.
Lemma 3.6. If \( a_{ij} > D.(C_{k,l} - e_5) \), then \( a_{iu} < D.(C_{v,w} - e_5) \) for all \( \{u, v, w\} = \{j, k, l\} \) such that \( u \in \{k, l\} \).

Proof. Assume the contrary. Then \( a_{ij} + a_{iu} > D.(C_{k,l} - e_5) + D.(C_{v,w} - e_5) \). But by (3.7) \( a_{ij} + a_{iu} \leq D.(e_5 - e_{u'5}) \), where \( \{u', u\} = \{k, l\} \). This is a contradiction with Claim 3.7. \( \square \)

Claim 3.7. \( D.(C_{k,l} + C_{v,w} - 2e_5) \geq D.(e_5 - e_{u'5}) \) for all \( v, w, u' \) such that \( \{u', u\} = \{k, l\} \) and \( \{v, w, u\} = \{j, k, l\} \) for some \( u \in \{k, l\} \).

Proof. There are four cases:

Case (i): \( v = j, w = l \) (\( u = k, u' = l \)). Using (9.6) and (9.7) one has

\[
D.(C_{k,l} + C_{j,l} - 2e_5) - D.(e_5 - e_{l5}) = 2D.(l - e_5 - e_l + e_{l5}) + D.(l - e_{i,j} - e_{i,k} - e_{j,k} - e_{l5}) + D.(l - e_5 - e_{j,k}) \geq 0.
\]

Case (ii): \( v = l, w = j \) (\( u = k, u' = l \)). Using (9.6) and (9.7) one has

\[
D.(C_{k,l} + C_{l,j} - 2e_5) - D.(e_5 - e_{l5}) = D.(l - e_5 - e_l + e_{l5}) + D.(l - e_{i,j} - e_{l,l} - e_j - e_{j5}) + D.(l - e_5 - e_{j,k}) \geq 0.
\]

Case (iii): \( v = j, w = k \) (\( u = l, u' = k \)). This is symmetric to Case (ii).

Case (iv): \( v = k, w = j \) (\( u = l, u' = k \)). Using (9.4), (9.6) and (9.7) one has

\[
D.(C_{k,l} + C_{k,j} - 2e_5) - D.(e_5 - e_{l5}) = 2D.(l - e_5 - e_k + e_{k5}) + D.(2l - e_5 - e_j - e_{l} - e_{i,k}) + D.(2e_k - \sum_{\alpha \neq k} e_{k\alpha}) \geq 0.
\]

\( \square \)

4. Proof of Claim 2.24

As in Section 3 we show that any distinguished section on \( Y \) can be rewritten, using the relations in \( \text{Cox}(Y) \), as a linear combination of distinguished sections with straightforward lifting. Assumption 2.4 is equivalent to the inequalities (9.4), (9.6), (9.7), (9.8) (for all permutations of indices). We use the notation from Section 8.

Notation 4.1. Let \( \chi : \{12, 13, 14, 23, 24, 34\} \rightarrow \{x, y, z\} \) be the function

\[
\chi(13) = \chi(24) = x, \quad \chi(14) = \chi(23) = y, \quad \chi(12) = \chi(34) = z.
\]

Note that one has

\[
L_\chi(ij) = l - e_5 - e_{ij} - e_{kl}, \quad \text{for all } \{i, j, k, l\} = \{1, 2, 3, 4\}.
\]

Remark 4.2. By Definition 2.16 one has \( m_\alpha + D.L_\alpha \geq 0 \) for all \( \alpha \in \{x, y, z\} \), with equality if and only if \( D.L_\alpha \leq 0 \).

Lemma 4.3. Let \( \overline{s} \) be a distinguished section on \( Y \):

\[
\overline{s} = \prod_{i,j} \iota_{ij}^{\alpha_{ij}} \prod_{l} \iota_{l}^{e_{y_l} \mu_{x_l} \nu_{z_l}} \iota_{y_l} \iota_{x_l} \iota_{z_l}, \tag{4.1}
\]

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where \( a_{ij}, l_i, c_x, c_y, c_z, l_x, l_y, l_z \geq 0 \). If \( \pi \) is a section \( H^0(Y, \mathcal{D}_Y^*) \), then \( \pi \) has straight-forward lifting to \( D \) if and only if for all \( \{i, j, k, l\} = \{1, 2, 3, 4\} \) and \( \alpha \in \{x, y, z\} \),

(i) \[ c_{\chi(ij)} - a_{ij} \leq D.(l - e_5 - e_{ij}), \]

(ii) \[ c_{\alpha} - l_{\alpha} \leq m_{\alpha} + D.L_{\alpha}, \]

(iii) \[ c_{\chi(ij)} - a_{ij} \leq D.(C_{kl} - e_5), \]

(iv) \[ a_{ij} + \sum_{\alpha \neq \chi(ij)} c_{\alpha} \leq D.(C_{k;l} - e_5), \]

(v) \[ c_{x} + c_{y} + c_{z} \leq D.C, \]

where \( C_{kl} = 2l - e_i - e_j - e_{kl}, C_{k;l} = 2l - e_{kl} - e_{k} - e_l, C = 3l - \sum_{i=1}^{5} e_i \).

Remark 4.4. Note that the right sides of the inequalities in Lemma 4.3 are non-negative due to (9.7) (for (i)), (9.8) (for (iv)), Remark 4.2 (for (ii)) and (9.9) (for (v)), Remark 4.1 (for (iii)) and because \( C \) is a nef curve on \( M \) (for (vi)).

Remark 4.5. The condition that \( \pi \) is in \( H^0(Y, \mathcal{D}_Y^*) \) is equivalent to

\[
\sum_{i} a_{ij} + (c_x + c_y + c_z) = D.e_5,
\]

(4.2)

\[
a_{ij} + a_{ik} + a_{il} - l_i = D.e_{i5}
\]

(4.3)

(the coefficients of \( \mathcal{H} \) and \( \mathcal{E}_i \) in \( \mathcal{D}_Y^* \)). From (4.2), (4.3) and (4.4) one has

\[
\sum_{i} l_i + 2(c_x + c_y + c_z) = D.(2e_5 - \sum_{i \neq 5} e_{i5}),
\]

(4.5)

\[
a_{jk} + a_{jl} + a_{kl} + (c_x + c_y + c_z) + l_i = D.(e_5 - e_{i5}),
\]

(4.6)

\[
(c_x + c_y + c_z) - (l_x + l_y + l_z) = (m_x + m_y + m_z) - D.e_5.
\]

(4.7)

Proof of Lemma 4.3. We lift \( \pi \) using the rules (2.3) and (2.4) (see also Remark 4.6) to a section of the divisor:

\[
D' = \sum_{i,j} a_{ij}A_{ij} + \sum_{i} l_iE_{i5} + c_xQ_{(13)(24)} + c_yQ_{(14)(23)} + c_zQ_{(12)(34)}
\]

\[
= \left( \sum_{i,j} a_{ij} + 2 \sum_{i} c_{\alpha} \right)H - \sum_{i \neq 5} (a_{ij} + a_{ik} + a_{il} + \sum_{\alpha} c_{\alpha})E_i - (\sum_{i,j} a_{ij} + \sum_{\alpha} c_{\alpha})E_5
\]

\[
\quad - \sum_{i,j \neq 5} (a_{ij} + \sum_{\alpha \neq \chi(ij)} c_{\alpha})E_{ij} - \sum_{i,j \neq 5} (a_{ij} + a_{ik} + a_{il} - l_i)E_{i5}.
\]

Using (4.2) and (4.3) one has

\[
D' = (D.e_5 + \sum_{i} c_{\alpha})H - \sum_{i \neq 5} (D.e_{i5} + l_i + \sum_{\alpha} c_{\alpha})E_i - (D.e_5)E_5
\]

\[
\quad - \sum_{i,j \neq 5} (a_{ij} + \sum_{\alpha \neq \chi(ij)} c_{\alpha})E_{ij} - \sum_{i,j \neq 5} (D.e_{i5})E_{i5}.
\]
Then $\Delta = D - D'$ is given by the following formula:

$$\Delta = (D.(l - e_5) - \sum c_\alpha)H - \sum_{i=1}^4 (D.(e_i - e_{i5}) - l_i - \sum c_\alpha)E_i$$

$$+ \sum_{i,j \neq 5} (D.e_{ij} - a_{ij} - \sum c_\alpha)E_{ij}.$$

We show that $\Delta.C \geq 0$ for the nef curves $C$ giving the inequalities (1), (2), (7), (9) in Lemma 8.2 and that for the nef curves $C$ giving the remaining inequalities, $\Delta.C \geq 0$ is equivalent to (1), (3), (5), (6), (7).

For inequality (1),

$$\Delta.l = D.(l - e_5) - \sum c_\alpha.$$  

By (4.6), one has $\sum c_\alpha \leq D.(e_5 - e_{i5})$. By the assumption in the Main Claim $D.e_5 \leq D.e_i$. Then $\Delta.l \geq D.(l - e_5 - e_i + e_{i5})$. It follows from (9.6) that $\Delta.l \geq 0$.

For inequality (2),

$$\Delta.(l - e_i) = D.(l - e_5 - e_i + e_{i5}) + l_i.$$  

It follows from (9.6) that $\Delta.(l - e_i) \geq 0$.

For inequality (3) is equivalent to (iv) as one has

$$\Delta.(l - e_{ij}) = e_5 - a_{12} \leq D.(l - e_5 - e_{ij}) + a_{ij} - c_{\chi(ij)}.$$  

For inequality (4),

$$\Delta.(l - e_{ij} - e_{kl}) = D.(l - e_5 - e_{ij} - e_{kl}) + a_{ij} + a_{kl} + \sum_{\alpha \neq \chi(ij)} -c_{\chi(ij)}.$$  

By using (4.4) to substitute $a_{ij} + a_{kl} + \sum_{\alpha \neq \chi(ij)}$ one has that $\Delta.(l - e_{ij} - e_{kl}) \geq 0$ is equivalent to (v). Note that in Lemma 8.2 we require that at least one of the inequalities is strict. As Lemma 4.7 shows, this is automatically satisfied in this case.

For inequality (5),

$$\Delta.C_{kl} = D.(C_{kl} - 2e_5 + e_{i5} + e_{j5}) + a_{kl} + l_i + l_j - c_{\chi(kl)}.$$  

Using (4.3) (to substitute $l_i, l_j$) and (4.2) $\Delta.C_{kl} \geq 0$ is equivalent to (iii).

For inequality (6),

$$\Delta.C_{kl} = D.(C_{kl} - 2e_5 + e_{i5}) + a_{ik} + a_{jk} + l_i + c_{\chi(kl)}.$$  

By using (4.6) to substitute $a_{ik} + a_{jk} + l_i + c_{\chi(kl)}$, $\Delta.C_{kl} \geq 0$ is equivalent to (iv).

For inequality (7) (recall that $C_i = 2l - e_{ij} - e_{ik} - e_{il}$),

$$\Delta.C_i = D.(C_i - 2e_5) + a_{ij} + a_{ik} + a_{il}.$$  

By using (4.3) to substitute $a_{ij} + a_{ik} + a_{il}$, $\Delta.C_i = D.(C_i - 2e_5 + E_{i5}) + l_i$. But,

$$D.(C_i - 2e_5 + E_{i5}) = 2D.(l - e_i - e_5 + m_{i5}) + D.(2e_i - \sum_{j \neq i} e_{ij}).$$  

From (9.6) and (9.4) it follows that $\Delta.C_i \geq 0$.

For inequality (8) (recall that $B = 3l - \sum_{i=1}^4 e_i$),

$$\Delta.B = D.(B - 3e_5 + \sum_{i \neq 5} e_{i5}) + \sum_{i} l_i + \sum c_\alpha.$$  

By using (4.5) to substitute $\sum l_i + 2\sum c_\alpha$, $\Delta.B \geq 0$ is equivalent to (v).
For inequality (9) (recall that $B = 3l - 2e_i - e_{jk} - e_{jl} - e_{kl}$),
\[
\Delta B_i = D_i (B_i - 3e_i + 2e_{i5}) + a_{jk} + a_{jl} + a_{kl} - 2l_i + \sum c_{\alpha}.
\]
By using (4.1) to substitute $a_{jk} + a_{jl} + a_{kl} + l_i + \sum c_{\alpha}$, $\Delta B_i = D_i (B_i - 2e_i + e_{i5})$.

But one has
\[
D_i (B_i - 2e_i + e_{i5}) = 2D_i (l - e_i - e_5 + e_{i5}) + D_i (l - e_{jk} - e_{jl} - e_{kl} - e_{i5}).
\]
It follows from (9.6) and (9.7) that $\Delta B_i \geq 0$. \hfill \Box

**Remark 4.6.** In order to lift $\sigma \in H^0(Y, D')$ we need to group $s_{ij}'$ with $s_{\chi(ij)}$, such that we may lift $s_{ij} = s_{ij}'s_{\chi(ij)}$ to $x_{ij5}$, etc. (so in fact we lift $s_{x,x}m^x s_{y,y} s_{z,z}^m$). For this, we need to have enough sections $s_x, s_y, s_z$. Take the case of $s_x$: one needs exactly $a_{13} + a_{24} + c_y + c_z$ of them (to be distributed to $s_{13}, s_{24}, s_y, s_z$). Since the image of the restriction map $r_Y$ in $H^0(E_5, D)$ is
\[
H^0(Y, D')s_{x,x}m^x s_{y,y} s_{z,z}^m,
\]
the number of $s_x$ appearing in $s_{x,x}m^x s_{y,y} s_{z,z}^m$ is $m_x + l_x$, and by (4.4) one has
\[
m_x + l_x = a_{13} + a_{24} + c_y + c_z.
\]

**Lemma 4.7.** It is not possible to have $c_{\alpha} - l_{\alpha} \geq m_{\alpha} + D.L_{\alpha}$ for all $\alpha \in \{x, y, z\}$.

**Proof.** Assume the contrary and add up the three inequalities. Then one has
\[
\sum c_{\alpha} - \sum l_{\alpha} \geq \sum m_{\alpha} + \sum D.L_{\alpha}.
\]
By (4.7), this is equivalent to $\sum D.L_{\alpha} \leq -D.e_5$, which contradicts Lemma 5.3. \hfill \Box

4.1. **Proof of Claim 2.24**

Let $\sigma$ be a distinguished section in $H^0(Y, D')$ as in (1.1). If inequalities (11-15) in Lemma 4.3 are satisfied, then by Lemma 4.3 $\sigma$ has straightforward lifting to $D$. Assume now that one of the inequalities (11-15) fails. We first show that we can keep replacing the section $\sigma$ with a linear combination of distinguished sections until we are in one of the following cases:

(A) $c_x = c_y = c_z = 0$,
(B) $l_x = l_y = l_z = 0, c_x + c_y + c_z > 0$,
(C) $c_x = c_y = l_x = l_y = 0, c_z > 0, l_z > 0$ (up to a permutation of $x, y, z$).

This follows from:

**Claim 4.8.** If $l_{\alpha} > 0$ and $c_{\beta} > 0$ for $\alpha, \beta \in \{x, y, z\}, \beta \neq \alpha$, then we may replace $\sigma$ with a sum of distinguished sections $\sigma'$ for which both $c_x + c_y + c_z$ and $l_x + l_y + l_z$ decreased.

**Proof.** We may assume without loss of generality that $l_x > 0, c_z > 0$. Consider the following sections in the linear system $[H - E_y]$:
\[
s_{xy}s_x, s_{14}s_1s_4, s_{23}s_2s_3.
\]

The linear system $[H - E_y]$ is 1-dimensional, and any two of the above sections are linearly independent. Hence, we may replace $s_{xy}s_x$ with a linear combination of the sections $s_{14}s_1s_4, s_{23}s_2s_3$. The effect is: $c_x, l_x$ decrease by 1, and either $a_{14}, l_1, l_4$ or $a_{23}, l_2, l_3$ increase by 1. Note that $c_x, c_y, l_y, l_z$ stay the same. Hence, both $c_x + c_y + c_z$ and $l_x + l_y + l_z$ decreased by 1. \hfill \Box
4.2. **Case (A):** \( c_x = c_y = c_z = 0 \). This case is very similar to Case I.

**Lemma 4.9.** If \( c_x = c_y = c_z = 0 \), then \( \Phi \) has straightforward lifting to \( D \) if and only if for all \( \{i, j, k, l\} = \{1, 2, 3, 4\} \) one has
\[
a_{ij} \leq D.(C_{k,l} - e_5).
\]

*Proof.* One may immediately see (use for example Remark 4.4) that the inequalities (i), (ii), (iii), (iv) in Lemma 4.3 are satisfied. The inequality (v) is satisfied (see Remark 4.2). Condition (vi) in Lemma 4.3 becomes \( a_{ij} \leq D.(C_{k,l} - e_5) \) in Case (A). \( \square \)

**Algorithm for replacing \( s \) – Case (A).** If for all \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), one has \( a_{ij} \leq D.(C_{k,l} - e_5) \), and by Lemma 4.9, \( \Phi \) has straightforward lifting to \( D \). If for some \( i, j, k, l \) one has \( a_{ij} > D.(C_{k,l} - e_5) \), we will replace \( \Phi \) with a sum of distinguished sections such that all the inequalities improve, while leaving \( c_x = c_y = c_z = 0 \). We do this in exactly the same way as we did in Case I, as Lemma 3.6 Claim 3.5, as well as the Algorithm 3.2 all apply word by word.

4.3. **Case (B):** \( l_x = l_y = l_z = 0, \ c_x + c_y + c_z > 0 \). This is impossible because of (4.7) and Lemma 5.3.

4.4. **Case (C):** \( c_x = c_y = l_x = l_y = 0, \ c_z > 0, l_z > 0 \).

**Remark 4.10.** Under the assumptions of Case (C) the relations in Remark 4.5 become
\[
\sum a_{ij} + c_z = D.e_5,
\]
\[
a_{13} + a_{24} + c_z = m_x, \quad a_{14} + a_{23} + c_z = m_y,
\]
\[
a_{12} + a_{34} - l_z = m_z,
\]
\[
c_z - l_z = m_x + m_y + m_z - D.e_5.
\]

From (4.9) one has
\[
0 < c_z \leq \min \{m_x, m_y\}.
\]

From the definitions of \( m_x, m_y \) it follows that \( m_x = -D.L_x, m_y = -D.L_y \). From (4.10) and (4.11) one has
\[
a_{12} + a_{34} - c_z = m_5 - m_x - m_y = D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}).
\]

**Lemma 4.11.** Under the assumptions of Case (C) \( \Phi \) has straightforward lifting to \( D \) if and only if:
\[
(iii') \quad a_{ij} \leq D.(C_{ij} - e_5) + D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}),
\]
\[
(iv') \quad a_{ij} \leq D.(C_{k,l} - e_5),
\]

whenever either \( ij = 12, kl = 34 \) or \( ij = 34, kl = 12 \).

**Remark 4.12.** By (9.7) and (9.8) the right hand sides of (iii'), (iv') are \( \geq 0 \).
Proof of Lemma 4.11 We claim that in Lemma 4.3 the inequalities (i), (ii) and (v) are satisfied and that (iii), respectively (iv), are equivalent to (iii′) and (iv′).

Inequality (i): by Remark 4.4 the inequalities involving $c_x, c_y$ are automatic. We claim that $c_z \leq D.(l - e_5 - e_{ij})$ whenever $ij = 12$ or $34$: by (4.12) one has $c_z \leq m_x, m_y$, hence $c_z \leq (m_x + m_y)/2$, and the claim follows from Lemma 5.2.

Inequality (ii): this is clearly satisfied for $l_z - c_z = 0$, $l_y - c_y = 0$. From (4.11) and Lemma 5.3 it follows that $c_z - l_z \leq 0$, and we are done by Remark 4.12.

Inequality (iii): the inequalities involving $c_z$ and $c_y$ are automatically satisfied. The inequalities (iii′) involving $c_z$ are of the form (here $ij = 12$ or $34$)

$$c_z - a_{ij} \leq D.(C_{kl} - e_5).$$

Using (4.12) to substitute $c_z - a_{ij}$ in (4.14), one obtains (iii′):

$$a_{kl} \leq D.(C_{kl} - e_5) + D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}).$$

Inequality (iv): We claim that the inequalities involving $a_{13}, a_{14}, a_{23}, a_{24}$ are satisfied: this is because by (4.12) $a_{ij} + c_z \leq m_x$ whenever $ij \not= 12, 34$. By Lemma 5.1, $m_x \geq D.(C_{kl} - e_5)$, and we are done. The inequalities (iv′) involving $a_{12}, a_{34}$ are exactly the inequalities (iv′).

Inequality (v): this follows from (4.12) and Lemma 5.2.

4.5. Algorithm for replacing $s$ in Case (C). If the inequalities in Lemma 4.11 are satisfied, then $s$ has straightforward lifting to $D$. Assume one of (iii′) or (iv′) is not satisfied, say for $a_{12}$ (the same argument applies for $a_{34}$). Then by Remark 4.12 one has $a_{12} > 0$. Then we make replacements to decrease $a_{12}$ as follows. Consider the following sections in the linear system $|2H - E_1 - E_2 - E_x - E_y|:

$$s_{12}'s_{xy}s_z, \quad s_{13}'s_{23}s_3^2, \quad s_{14}'s_{24}s_4^2.$$  

The linear system is 1-dimensional, and any two of the above sections are linearly independent. Since $a_{12}, c_z, l_z > 0$, we may replace $s_{12}'s_{xy}s_z$ in $s$ with a linear combination of $s_{13}'s_{23}s_3^2, s_{14}'s_{24}s_4^2$. The effect is: $a_{12}, c_z, l_z$ decrease by 1, while either $a_{13}, a_{23}$ increase by 1 or $a_{14}, a_{24}$ increase by 1. Note that besides the above changes and the changes affecting the $l_z$’s (which we ignore, since they do not appear in (iii′), (iv′)) no other changes occur. In particular, we still have $c_x = c_y = l_x = l_y = 0$.

The inequalities involving $a_{12}$ were improved (while the ones involving $a_{34}$ remained the same). If after the replacement $c_z = 0$ or $l_z = 0$ we are in Case (A) or Case (B), we apply the procedure described for those cases. If after the replacement we still have $c_z > 0$ and $l_z > 0$, then we are in Case (C), and therefore all inequalities are satisfied, except perhaps (iii′), (iv′) for $a_{12}$ or $a_{34}$.

5. Inequalities involving $m_x, m_y, m_z$

The assumptions in this section are the same as in the Main Claim. Recall:

$$L_x(ij) = L_x(kl) = l - e_5 - e_{ij} - e_{kl}.$$

Lemma 5.1. For any $\{i, j, k, l\} = \{1, 2, 3, 4\}$ one has

$$-D.L_x(kl) \leq D.(C_{kl} - e_5),$$

where $C_{k; l} = 2l - e_l - e_{ik} - e_{jk}$.
Proof. One has

\[ D.(C_{kl} - e_5) + D.L_{x(kt)} = D.(l - e_ik - e_{jk} - e_{ij} - e_{l5}) + D.(l - e_5 - e_i + e_{l5}) + D.(l - e_5 - e_{kl}) \geq 0. \]

It follows from (9.6) and (9.7) that \( D.(C_{kl} - e_5) + D.L_{x(kt)} \geq 0. \)

\[ \Box \]

**Lemma 5.2.** For any \( i, j \in \{1, 2, 3, 4\} \) one has

\[-\frac{1}{2} \sum_{\alpha \neq \chi(ij)} L_{\alpha} \leq \min \{d - m_5 - m_{ij}, 3d - \sum_{i=1}^{5} m_i\}.\]

**Proof.** Without loss of generality, we may assume \( ij = 12 \). One has

\[ 2D.(l - e_5 - e_{12}) + D.(L_x + L_y) \]

\[ = 2D(l - e_5 - e_{12}) + D.(2l - 2e_5 - e_{13} - e_{14} - e_{23} - e_{24}) \]

\[ = D.(l - e_{12} - e_{13} - e_{23} - e_{45}) + (l - e_{12} - e_{14} - e_{24} - e_{35}) \]

\[ + D.(l - e_5 - e_3 + e_{35}) + D.(l - e_5 - e_4 + e_{45}) + D.(e_3 + e_4 - 2e_5). \]

The first inequality follows from (9.7) and the assumption \( D.e_5 \leq D.e_i \). Moreover,

\[ 2D(3l - \sum_{i=1}^{5} e_i) + D.(L_x + L_y) \]

\[ = 2(3l - \sum_{i=1}^{5} e_i) + D.(2l - 2e_5 - e_{13} - e_{14} - e_{23} - e_{24}) \]

\[ = D.(2l - e_1 - e_3 - e_5 - e_{24}) + D.(2l - e_1 - e_4 - e_5 - e_23) \]

\[ + D.(2l - e_2 - e_3 - e_5 - e_{14}) + D.(2l - e_2 - e_4 - e_5 - e_{13}). \]

The second inequality now follows from (9.7).

\[ \Box \]

**Lemma 5.3.** One has \( (m_x + m_y + m_z) \leq D.e_5, -D.(L_x + L_y + L_z) < D.e_5 \).

**Proof.** Note by definition of \( m_x \) that if \( m_x > 0 \), then \( m_x = -D.L_x \) (similarly for \( y, z \)), if \( m_x = m_y = m_z = 0 \). The claim follows from (9.4).

Case 1) Assume just one of \( m_x, m_y, m_z \) is \( > 0 \), say \( m_x > 0, m_y = m_z = 0 \):

\[ D.e_5 - (m_x + m_y + m_z) = D.(l - e_{13} - e_{24}). \]

However, \( D.(l - e_{13} - e_{24}) \geq 0 \) (see Lemma 5.2). The other cases are similar.

Case 2) Assume two of \( m_x, m_y, m_z \) is \( > 0 \), say \( m_x, m_y > 0, m_z = 0 \):

\[ D.e_5 - (m_x + m_y + m_z) = D.(2l - e_5 - e_{13} - e_{14} - e_{23} - e_{24}). \]

By (9.8) \( D.e_5 - (m_x + m_y + m_z) \geq 0 \). The other cases are similar.

Case 3) Assume \( m_x, m_y, m_z > 0 \):

\[ D.e_5 - (m_x + m_y + m_z) = D.(L_x + L_y + L_z + e_5) = D.(3l - 2e_5 - \sum_{i,j=1,...,4} e_{ij}) \]

\[ = D.(2e_i - \sum_{j \neq i} e_{ij}) + 2D(l - e_5 - e_i + e_{l5}) + D.(l - e_{jk} - e_{kl} - e_{jl} - e_{l5}), \]

for any \( \{i, j, k, l\} = \{1, 2, 3, 4\} \). By (9.4), (9.6), (9.7) \( D.e_5 - (m_x + m_y + m_z) \geq 0 \).
If $-D.(L_x + L_y + L_z) = D.e_5$, by the above computation one has (here for simplicity, we let $d = D.l, m_i = D.e_i, m_{ij} = D.e_{ij}$)

$$2m_i - \sum_{j \neq i} m_{ij} = 0, \quad d - m_5 - m_i + m_{i5} = 0, \quad m_{jk} + m_{kl} + m_{jl} + m_{l5} = d.$$ 

It follows that

(5.1) $m_{ij} + m_{ik} + m_{il} = d - m_5 + m_i,$

(5.2) $m_{jk} + m_{kl} + m_{jl} = 2d - m_5 - m_i.$

Adding up all relations (5.1) and (5.2), one has:

$$2 \sum_{i,j=1,...,4} m_{ij} = 4d - 4m_5 + \sum_{i=1}^4 m_i, \quad 2 \sum_{i,j=1,...,4} m_{ij} = 8d - 4m_5 - \sum_{i=1}^4 m_i.$$ 

It follows that $\sum_{i=1}^4 m_i = 2d$. However, by assumption $m_i \geq m_5$ for all $i$, hence $m_5 \leq d/2$. As $0 \leq m_5 = m_i + m_5 - d$ it follows that $m_i \geq d - m_5 \geq d/2$. Since $\sum_{i=1}^4 m_i = 2d$ it follows that $m_i = d/2, m_{i5} = 0$. Moreover, $m_{ij} + m_{ik} + m_{il} = d$. As $d > 0$ it follows that $m_{ij} > 0$ for some $i, j \in \{1, \ldots, 4\}$. This contradicts the assumption in the Main Claim. $\square$

6. Multiplicity estimates

Let $l$ be the unique line in $\mathbb{P}^3$ that passes through $p_5$ and intersects lines $l_{13}$ and $l_{24}$ (the other cases are similar). Let $L$ be the proper transform of $l$ in $\overline{\mathcal{M}}$.

**Proposition 6.1.** Let $D = dH - \sum m_i E_i - m_{ij} E_{ij}$ be an effective divisor on $\overline{\mathcal{M}}$. Let $m$ be the multiplicity of $D$ along $L$. Then

$$m \geq m_5 + m_{13} + m_{24} - d.$$ 

**Proof.** Let $\rho : X \to \overline{\mathcal{M}}$ be the blow-up of $\overline{\mathcal{M}}$ along $L$ and let $E$ be the exceptional divisor. Let $\tilde{D}$ be the proper transform of $D$. Then $\rho^* D = \tilde{D} + mE$. Restricting to $E$, one has

$$\rho^* D|_E = \tilde{D}|_E + mE|_E.$$ 

Let $N$ be the normal bundle of $L$ in $\overline{\mathcal{M}}$ and let $N_{l|\mathbb{P}^3}$ be the normal bundle of $l$ in $\mathbb{P}^3$. If $l'$ is the proper transform of $l$ in the blow-up $X$ of $\mathbb{P}^3$ along $p_1, \ldots, p_5$, let $N'$ be the normal bundle of $l'$ in $\overline{\mathcal{M}}$. One has

(6.2) $N_{l|\mathbb{P}^3} = \mathcal{O}(-1) \oplus \mathcal{O}(-1), \quad N' = \pi^* N_{l|\mathbb{P}^3}(-E_5) = \mathcal{O} \oplus \mathcal{O}.$

It is easy to see that $\deg(N) = \deg(N') - 2 = -2$. In fact we have the following:

**Claim 6.2.** $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$

**Proof.** Note that one could obtain $\overline{\mathcal{M}}$ by blowing up $\mathbb{P}^3$ first along the points $p_1, \ldots, p_4$, then the proper transforms of the lines $l_{13}$ and $l_{24}$, and then the point $p_5$ and the proper transforms of the lines $l_{ij}$, for all $ij \neq 13, 24$. Let $\Lambda$ be the plane in $\mathbb{P}^2$ spanned by the line $l$ and $l_{13}$. Then the proper transform $\tilde{\Lambda}$ of $\Lambda$ in $\overline{\mathcal{M}}$ is the blow-up of $\Lambda \cong \mathbb{P}^2$ along $p_1, p_3, p_5, q$, where $q = l_{24} \cap \Lambda$. If $N_{l|\tilde{\Lambda}}$ is the normal bundle of $L$ in $\tilde{\Lambda}$ and $N_{\tilde{\Lambda}|\overline{\mathcal{M}}}$ is the normal bundle of $\tilde{\Lambda}$ in $\overline{\mathcal{M}}$, one has an exact sequence:

(6.3) $0 \to N_{l|\tilde{\Lambda}} \to N \to (N_{\tilde{\Lambda}|\overline{\mathcal{M}}})|_L \to 0.$
It is easy to see that $N_{L|\bar{A}} = \mathcal{O}(-1)$. Since $\text{deg}(N) = -2$ and $\mathcal{O}(-1)$ is a subbundle of $N$ (the quotient is a line bundle), it follows that $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Then $E = \mathbb{P}(N) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $p : E \to \bar{I} = \mathbb{P}^1$ be the restriction of $\rho$ to $E$. Let $q : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the other projection. Then

$$E|_{E} = \mathcal{O}_E(-1) = q^*\mathcal{O}(-1) \otimes p^*\mathcal{O}(-1).$$

Note that $(\rho^*D)|_E = p^*(D|_L)$ and $D|_L = \mathcal{O}(a)$, where we let $a = D.L$. One has $H.L = E_5.L = E_{13}.L = E_{24}.L = 1, E_i.L = E_{ij}, L = 0$, for all other indices $i, j$. It follows that $a = d - m_5 - m_{13} - m_{24}$. From (6.1) one has

$$\tilde{D}|_E = p^*\mathcal{O}(a + m) \otimes q^*\mathcal{O}(m).$$

Since $\tilde{D}|_E$ is effective, it follows that $m \geq -a = m_5 + m_{13} + m_{24} - d$.

7. Proof of Lemma [2.21]

Recall that $Y$ is the blow-up of $\mathbb{P}^2$ along $q_1, \ldots, q_4, x, y, z$.

![Figure 1. The configuration of the points $q_1, q_2, q_3, q_4, x, y, z$.](image)

Let

$$D = d\bar{H} - \sum_{i=1}^{4} m_i \bar{E}_i - m_x \bar{E}_x - m_y \bar{E}_y - m_z \bar{E}_z$$

be a divisor on $Y$. Assume $D$ is effective and let $s$ be a section in $\mathbb{H}^0(Y, D)$.

We show that $s$ is generated by distinguished sections on $Y$ by induction on $d$. Let $\bar{t}_{ij}$ (respectively $\bar{t}_{xy}, \bar{t}_{yz}, \bar{t}_{xz}$) be the proper transforms in $Y$ of the lines $\bar{t}_{ij}$ (respectively $\bar{t}_{xy}, \bar{t}_{yz}, \bar{t}_{xz}$). We may assume $D.C \geq 0$ for $C$ among the classes:

$$\bar{t}_{ij}, \bar{t}_{xy}, \bar{t}_{xz}, \bar{t}_{yz}, \bar{E}_i, \bar{E}_x, \bar{E}_y, \bar{E}_z.$$

This is because if $D.C < 0$, then $s = x_Cs'$, where $s' \in \mathbb{H}^0(Y, D - C)$ and $x_C$ is a generator of $\mathbb{H}^0(Y, C)$, and $s'$ is generated by distinguished sections by induction. Hence, we assume

$$(*) \quad d \geq m_i + m_j + m_{\chi(ij)}, \quad d \geq m_x + m_y, \quad d \geq m_x + m_z, \quad d \geq m_y + m_z, \quad d \geq m_i \geq 0, \quad d \geq m_x \geq 0, \quad d \geq m_y \geq 0, \quad d \geq m_z \geq 0.$$
If $d = 0$, then it follows by \( D = 0 \). Assume $d > 0$. We may assume without loss of generality that \( m_4 \leq m_1 \leq m_2 \leq m_3 \).

Consider the restriction map:
\[
r : H^0(Y, D) \to H^0(E_4, D_{E_4}) = H^0(\mathbb{P}^1, \mathcal{O}(m_4)).
\]

It is easy to show that we may lift any $t \in H^0(\mathbb{P}^1, \mathcal{O}(m_4))$ to a section in $H^0(Y, D)$ generated by distinguished sections on $Y$. This is because by the same argument as in Section 2, if $s, s'$ are sections in $H^0(Y, D)$ are such that $r(s) = r(s')$, then $s - s'$ is in $H^0(Y, D - E_4)$ and we are done by induction.

Let $t_i$ be the restriction in $H^0(\mathbb{P}^1, \mathcal{O}(1))$ of the section $s_{i4}$ corresponding to $\mathcal{I}_{i4}$. Any two of $t_1, t_2, t_3$ generate $H^0(\mathbb{P}^1, \mathcal{O}(1))$. In particular, it is enough to lift $t = t_1 k t_3 m_4 - k$ (for any $0 \leq k \leq m_4$) to a combination of distinguished sections. We lift $t_i$ to $s_{i4}$, hence $t$ to $s_{14} s_{34} m_4 - k$ (a section of $D' = k \mathcal{I}_{14} + (m_4 - k) \mathcal{I}_{34}$). Let:
\[
\Delta = D - D' = (d - m_4) \mathcal{H} - (m_1 - k) \mathcal{E}_1 - m_2 \mathcal{E}_2 - (m_3 - m_4 + k) \mathcal{E}_3
\]
\[\quad - m_x \mathcal{E}_x - (m_y - k) \mathcal{E}_y - (m_z - m_4 + k) \mathcal{E}_z.
\]

**Claim 7.1.** There is a section $u \in H^0(\Delta)$, generated by distinguished sections and such that $u|_{E_4} \in H^0(E_4, \mathcal{O})$ is non-zero.

Assuming Claim 7.1 we lift $t$ to $u s_{14} s_{34} m_4 - k$ and we are done.

**Proof of Claim 7.1.** Let
\[
\Delta' = \Delta - (m_1 - k) \mathcal{I}_{12} = (d - m_4 - m_1 + k) \mathcal{H} - (m_2 - m_1 + k) \mathcal{E}_2
\]
\[\quad - (m_3 - m_4 + k) \mathcal{E}_3 - m_x \mathcal{E}_x - (m_y - k) \mathcal{E}_y - (m_z - m_1 - m_4 + 2k) \mathcal{E}_z.
\]

Note that since $k \leq m_4 \leq m_1$ and since a section corresponding to $\mathcal{I}_{12}$ has non-zero restriction to $E_4$, it is enough to show that there is a section $u' \in H^0(\Delta')$, generated by distinguished sections and such that $u'|_{E_4} \in H^0(E_4, \mathcal{O})$ is non-zero.

**Case when $m_y - k < 0$.** Let
\[
\Delta'' = \Delta + (m_y - k) \mathcal{E}_y = (d - m_4 - m_1 + k) \mathcal{H} - (m_2 - m_1 + k) \mathcal{E}_2
\]
\[\quad - (m_3 - m_4 + k) \mathcal{E}_3 - m_x \mathcal{E}_x - (m_z - m_1 - m_4 + 2k) \mathcal{E}_z.
\]

It is enough to show that there is a section $u'' \in H^0(\Delta'')$, generated by distinguished sections and such that $u''|_{E_4} \in H^0(E_4, \mathcal{O})$ is non-zero. Since $\Delta''$ is a divisor on the blow-up of $\mathbb{P}^2$ along the points $q_2, q_3, x, z$, it follows from Lemma 7.3 (a direct check shows that all inequalities (7.1) hold; use $k \leq m_4 \leq m_1$ and \((*)\)) and Lemma 7.4 applied to the lines $q_2 x$ and $q_3 z$, that there is a section $u'' \in H^0(\Delta'')$, generated by distinguished sections and not containing $q_4$ in its zero-locus.

**Case when $m_y - k \geq 0$.** Denote
\[
N_1 = m_1 + m_4 + m_x + m_y - d - 2k,
\]
\[
N_2 = 2d - m_2 - m_3 - m_x - m_z - 2k.
\]

**Claim 7.2.** $N_1 \leq N_2$, $0 \leq N_2$, $N_1 \leq m_y - k$. 


Proof Claim [7.2] We have
\[ N_2 - N_1 = (d - m_1 - m_2 - m_3) + (d - m_3 - m_4 - m_5) + (d - m_5 - m_6) \geq 0, \]
using (*) and \( m_4 \leq m_1 \). Similarly, as \( 0 \leq k \leq m_4 \), we have \( N_2 \geq 0 \) and \( N_1 \leq m_6 - k \) (using (*) and \( m_4 \leq m_1 \)). □

By Claim [7.2] we may choose \( \alpha, \beta \geq 0 \) be integers such that \( \alpha + \beta = m_6 - k \) and \( N_1 \leq \alpha \leq N_2 \). Let
\[
\Delta'' = \Delta - \alpha t_{xy} - \beta t_{23} = (d - m_1 - m_4 - m_5 + 2k)H - (m_2 - m_1 + k - \beta)E_2 \\
-(m_3 - m_4 + k - \beta)E_3 - (m_5 - \alpha)E_5 - (m_6 - m_1 - m_4 + 2k)E_6.
\]

Since \( t_{xy} \) and \( t_{23} \) have non-zero restriction to \( E_4 \), it is enough to find \( u \in H^0(\Delta'') \) such that \( u|_{E_4} \neq 0 \). As before, since \( \Delta'' \) is a divisor on the blow-up of \( \mathbb{P}^2 \) along the points \( q_2, q_3, x, z \), it follows from Lemma [7.3] and Lemma [7.4] applied to the lines \( \overline{q_2, x} \) and \( \overline{q_3, z} \), that there is a section \( u'' \in H^0(\Delta'') \), generated by distinguished sections and not containing \( q_4 \) in its zero-locus. All inequalities follow in a straightforward way from (*) and \( m_4 \leq m_1 \), except for:

- \( \Delta'' \). \( (H - E_2) \geq 0 \), (equivalent to \( \alpha \geq N_1 \))
- \( \Delta'' \). \( (H - E_3) \geq 0 \), (use that \( m_1 \leq m_2 \))
- \( \Delta'' \). \( (2H - E_2 - E_3) \geq 0 \), (equivalent to \( \alpha \leq N_2 \)). □

Lemma 7.3. Let \( Z \) be the blow-up of \( \mathbb{P}^2 \) along points \( q_1, \ldots, q_4 \) (no three collinear). One has \( H^0(D) \neq 0 \) for a divisor \( D = dH - \sum_{i=1}^{4} m_i E_i \) if and only if
\[
(7.1) \quad d \geq 0, \quad d \geq m_4, \quad 2d \geq \sum_{i=1}^{4} m_i.
\]

The Cox ring \( \text{Cox}(Z) \) is generated by sections corresponding to the lines \( \overline{1, j} \) and the exceptional divisors \( E_i \).

Proof. It is a well known result that the Cox ring \( \text{Cox}(Z) \) is generated by sections corresponding to the lines \( \overline{1, j} \) and the exceptional divisors \( E_i \); see for example [BP].

If \( D \) is an effective divisor, then clearly, the inequalities (7.1) hold. Conversely, assume (7.1) hold. We write \( D \) as an effective combination of the classes of the lines \( \overline{1, j} = H - E_i - E_j \) and the exceptional divisors \( E_i \). Consider the table with 2 rows and \( d \) columns filled with \( E_i \)'s in the following way. Start in the upper left corner and write \( m_1 E_1 \)'s in the first row. Then write \( m_2 E_2 \)'s passing to the second row if necessary, and so on. Fill the remaining entries with zeros. For example, if \( D = 5H - 3E_1 - 3E_2 - 2E_3 - E_4 \),
\[
\begin{array}{cccccc}
E_1 & E_1 & E_3 & E_3 & E_4 & E_4 \\
E_2 & E_3 & E_4 & 0.
\end{array}
\]

Our conditions guarantee that all entries of a given column are different. Therefore \( D \) is the sum of classes \( H - (E_i + E_j) \), one for each column, where \( E_i, E_j \) are the entries of the column. In the example above,\n\[
D = (H - E_1 - E_2) + (H - E_1 - E_3) + (H - E_1 - E_3) + (H - E_2 - E_4) + (H - E_2).
\]
\[ \square \]
Lemma 7.4. In the notations of Lemma 7.3, let $D$ be a divisor such that $H^0(D) \neq 0$. Let $q$ be the intersection of the lines $\overline{ij}$ and $\overline{kl}$. The linear system $|D|$ does not contain $q$ as a base point if and only if

$$D.(H - E_1 - E_2) \geq 0, \quad D.(H - E_3 - E_4) \geq 0.$$  

Proof. The conditions are clearly necessary. It is enough to show that $D$ can be written as an effective combination of lines $l_{ij}$ ($l_{ij} \neq l_{12}, l_{34}$) and the exceptional divisors $E_i$. Let

$$D = \sum k_{ij}l_{ij} + \sum k_iE_i, \quad k_{ij}, k_i \geq 0.$$  

Assume $k_{12} > 0$. Note that the only generators $E$ of Cox($Z$) with the property that $E.l_{12} > 0$ are $l_{34}, E_1, E_2$. Since $D.l_{12} \geq 0$, it follows that one of $k_{34}, k_1, k_2 > 0$. If $k_1 > 0$ we may replace $l_{12} + E_1$ with a divisor in the pencil $|H - E_2|$ that does not contain $l_{12}$ (for example $l_{23} + E_3$). The case $k_2 > 0$ is similar. If $k_{34} > 0$, we replace $l_{12} + l_{34}$ with, for example, $l_{13} + l_{24}$. The case when $k_{34} > 0$ is similar. At the end of this process, we have $k_{12} = k_{34} = 0$. \hfill $\square$

8. Inequalities for the effective cone of $X$

Let $X$ be the iterated blow-up of $\mathbb{P}^3$ in points $p_1, \ldots, p_4$ (in a linearly general position) and proper transforms of lines $l_{ij} (i, j = 1, \ldots, 4, i \neq j)$. Let $E_i, E_{ij}$ be the exceptional divisors. Let $l$ be the class on $X$ of the proper transform of a general line in $\mathbb{P}^3$. Let $e_i$ be the class of (the proper transform of) a general line in $E_i$. Let $e_{ij}$ be the class of a fiber of the $\mathbb{P}^1$-bundle $E_{ij} \to l_{ij}$.

Notation 8.1. For $\{i, j, k, l\} = \{1, 2, 3, 4\}$ let

$$C_{ij} = 2l - e_{ij} - e_k - e_l,$$

$$C_{ij} = 2l - e_{ik} - e_{il} - e_j,$$

$$C_i = 2l - e_{ij} - e_{ik} - e_{il},$$

$$B = 3l - \sum_{i=1}^4 e_i,$$

$$B_i = 3l - 2e_i - e_{jk} - e_{jl} - e_{kl}.$$  

Lemma 8.2. Let $D$ be a divisor on $X$. Then $D$ is an effective sum (with integer, non-negative coefficients) of boundary divisors $\Lambda_{ijk}, E_{ij}, E_i$ (in particular, $H^0(D) \neq 0$) if $D.C \geq 0$ for all $C$ in the list below (for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$): (1) $l$; (2) $l - e_i$; (3) $l - e_{ij}$; (4) $l - e_{ij} - e_k$; (5) $C_{ij}$; (6) $C_{ij}$; (7) $C_i$; (8) $B$; (9) $B_i$, and moreover, if one has $D.(l - e_{ij} - e_{kl}) > 0$ for some $i, j, k, l$.

It is easy to see that each of the classes $C$ in (1)-(9) in Lemma 8.2 cover a dense set of $X$; hence, for any effective divisor $D$ one has $D.C \geq 0$, i.e., $C$ is a nef curve.

Remark 8.3. It is a standard fact that the divisor $D$ is in the convex hull of the effective divisors $\Lambda_{ijk}, E_{ij}, E_i$ (where $\Lambda_{ijk}$ is the proper transform of the plane $\overline{ijl} \overline{jk}$) if and only if inequalities (1)-(9) hold. However, the extra condition of having at least one strict inequality in (4) is necessary for $H^0(D) \neq 0$, as the
Observation 8.4. If

\[ D = dh - \sum m_i E_i - \sum m_{ij} E_{ij} \]

is such that \( d \geq 0, d \geq m_i, m_{ij} \) (for all \( i, j \)) and there is an \( i \) such that \( m_i \leq 0 \) and \( m_{ij} \leq 0 \) for all \( j \neq i \), then \( D \) is an effective sum of boundary, as one has

\[ D = d \Lambda_{jkl} + \sum_{j \neq i} (d - m_j) E_j + \sum_{u,v \neq i} (d - m_{uv}) E_{uv} + (-m_i) E_i + \sum_{j \neq i} (-m_{ij}) E_{ij}. \]

Proof of Lemma 8.2. Let \( D = dh - \sum m_i E_i - \sum m_{ij} E_{ij} \). One has

\[ d = D.l, \quad m_i = D.e_i, \quad m_{ij} = D.e_{ij}. \]

We do an induction on \( d \). If \( d = 0 \), then from (2) and (3) \( m_i, m_{ij} \leq 0 \) and we are done by Observation 8.4. Assume \( d > 0 \). We show that there are \( i, j, k \) such that \( D' = D - \Lambda_{ijk} \) also satisfies (1)-(9), and hence, we are done by induction.

Note that \( D' = D - \Lambda_{ijk} \) (for any \( i, j, k \)) always satisfies (1), (4), (5), (8), (9). Moreover, one has at least one strict inequality in (4). Inequality (2) fails for \( D' \) if and only if \( m_i = d \), where \( l \neq i, j, k \), and (3) fails for \( D' \) if and only if one of \( m_{ij}, m_{jl}, m_{kl} \) equal \( d \). Inequality (6) fails for \( D' \) if and only if one has

\[ 2d = m_{il} + m_{ij} + m_k \]

(or the similar inequalities for a permutation of indices \( i, j, k \)). Inequality (7) fails for \( D' \) if and only if \( m_{il} + m_{jl} + m_{kl} \in \{2d - 1, 2d\} \).

Case 1: \( m_{ij} = d \) for some \( i, j \). We may assume \( d = m_{12} \). From (4), \( m_{34} \leq 0 \).

Case 1: \( m_i = d \) for \( i \in \{3, 4\} \). We may assume \( m_4 = d \). Then by (5) one has that \( m_3 \leq 0 \) and by (6) one has \( m_{13}, m_{23} \leq 0 \), and we are done by Observation 8.4.

Case 2: Assume \( m_3 < d, m_4 < d \).

We may assume that \( m_{13} \) is the largest among \( m_{13}, m_{14}, m_{23}, m_{24} \).

Claim 8.5. One has \( m_{14}, m_{24} < d \).

Proof. Assume \( m_{14} = d \) for \( i \in \{1, 2\} \). Since by assumption \( m_{14} \leq m_{13} \) and \( m_{13} \leq d \), one has \( m_{13} = d \). If \( m_{14} = d \), since \( m_{12} = d \), one has a contradiction with (7). If \( m_{24} = d \) one contradicts (4). □

Claim 8.6. The divisor \( D' = D - \Lambda_{123} \) satisfies (1)-(9).

Proof. Inequality (2) holds, as \( m_4 < d \). Since by Claim 8.5 \( m_{14}, m_{24} < d \) and since \( m_{34} \leq 0 \), inequality (3) holds. If (7) is not satisfied, i.e., \( m_{14} + m_{24} + m_{34} = 2d - 1 \) or \( 2d \), one has a contradiction, as \( m_{34} \leq 0 \) and (by Claim 8.5) \( m_{14}, m_{24} < d \). If (6) is not satisfied for \( D' \), then one has

\[ 2d = m_{i4} + m_{j4} + m_k, \]

for some \( \{i, j, k\} = \{1, 2, 3\} \). If \( k = 3 \), then by (6) one has \( 2d \geq m_{12} + m_{24} + m_4 \). Since \( m_{12} = d \) one has \( d \geq m_{24} + m_3 \) and hence, by (8.1), \( m_{14} = d \). This contradicts Claim 8.5. If \( k \in \{1, 2\} \) (say \( i = 3 \)), one has \( 2d = m_{34} + m_{j4} + m_k \). Since \( m_{34} \leq 0 \), \( m_{j4} < d \) (Claim 8.5) and \( m_k \leq d \); this is a contradiction. □
Case II: $m_i = d$ for some $i$, $m_{ij} < d$ for all $i, j$. We may assume that $d = m_4$. We may also assume that $d \geq m_3 \geq m_2 \geq m_1$. By (9) and (5) one has
\begin{align}
  (8.2) & \quad m_{12} + m_{13} + m_{23} \leq d, \\
  (8.3) & \quad m_{ij} + m_k \leq d,
\end{align}
where $\{i, j, k\} = \{1, 2, 3\}$.

Case 1: $m_{23} > 0$. By (8.2) one has
\begin{equation}
  (8.4) \quad m_{12} + m_{13} < d.
\end{equation}

Claim 8.7. The divisor $D' = D - \Lambda_{234}$ satisfies (1)-(9).

Proof. Inequality (3) is satisfied by the assumption $m_{ij} < d$. Inequality (2) is not satisfied if and only if one has $m_1 = d$. If $m_1 = d$, it follows from the assumptions that $m_2 = m_3 = d$. As $m_4 = d$, one has a contradiction with (8). If (7) is not satisfied, i.e., $m_{12} + m_{13} + m_{14} = \{2d - 1, 2d\}$, one has a contradiction with $m_{14} < d$ and $m_{12} + m_{13} < d$ (8.4). If (6) is not satisfied, then one has
\begin{equation}
  (8.5) \quad 2d = m_{11} + m_{1j} + m_k,
\end{equation}
for some $\{i, j, k\} = \{2, 3, 4\}$. If $k = 4$, since $m_4 = d$ one has from (8.5) $d = m_{12} + m_{13}$, which contradicts (8.2). If $k \in \{2, 3\}$ (say $i = 4$) one has $2d = m_{14} + m_{1j} + m_k$ for $\{k, j\} = \{2, 3\}$. But $m_{14} < d$ and $m_{1j} + m_k \leq d$ (8.3). This is a contradiction.

Case 2: $m_{23} \leq 0$. If $m_2 = d$, then it follows from the assumptions that $m_3 = d$. As $m_4 = d$, one has from (8) that $m_1 \leq 0$. It follows from (5) that $m_{1i} \leq 0$ for all $i = 2, 3, 4$. Then we are done by Observation 8.4. Hence, we may assume $m_2 < d$.

Claim 8.8. The divisor $D' = D - \Lambda_{134}$ satisfies (1)-(9).

Proof. Inequality (2) is satisfied as $m_2 < d$. Inequality (3) is satisfied by assumption. If (7) is not satisfied, i.e., $m_{12} + m_{23} + m_{24} \in \{2d - 1, 2d\}$, one has a contradiction with $m_{24} < d$ and $m_{12} + m_{23} \leq m_{12} < d$. If (6) is not satisfied, then
\begin{equation}
  (8.6) \quad 2d = m_{2i} + m_{2j} + m_k,
\end{equation}
for some $\{i, j, k\} = \{1, 3, 4\}$. If $k = 4$, since $m_4 = d$ one has from (8.6) $d = m_{12} + m_{23}$. But $m_{12} < d$ and $m_{23} \leq 0$. This is a contradiction. If $k \in \{1, 3\}$ (say $i = 4$) one has $2d = m_{24} + m_{j2} + m_k$ for $\{k, j\} = \{1, 3\}$. But $m_{24} < d$ and $m_{j2} + m_k \leq d$ (8.3). This is a contradiction. □

Case III: $m_i < d$, $m_{ij} < d$ for all $i, j$.

By Claim 8.10 we may assume $D.C_{i;j} > 0$ for $i = 1, 2, 3$, and all $j \neq i$.

Claim 8.9. One of $D_1 = D - \Lambda_{234}$, $D_2 = D - \Lambda_{134}$, $D_3 = D - \Lambda_{124}$ satisfies all the inequalities (1)-(9).

Proof. Inequalities (2), (3) follow from the assumptions. If (6) is not satisfied for $D_i$, then $D.C_{i;j}$, for some $j \neq i$, which we assume does not happen. Hence, (6) is satisfied for all $D_i$. If (7) fails for all $D_i$, then one has for all $i \in \{1, 2, 3\}$
\begin{equation}
  (8.7) \quad m_{ij} + m_{ik} + m_{il} \geq 2d - 1.
\end{equation}
Adding up (8.7) for $i = 1$ and $i = 2$ one has
\begin{equation}
  (8.8) \quad 2m_{12} + (m_{13} + m_{24}) + (m_{14} + m_{23}) \geq 4d - 2.
\end{equation}
From (4) \( d \geq m_{ij} + m_{kl} \). As \( m_{12} < d \), it follows from (8.8) that \( m_{13} + m_{24} = m_{14} + m_{23} = d \). Similarly, by adding (8.8) for \( i = 1 \) and \( i = 3 \) one has \( m_{12} + m_{34} = d \). This contradicts our assumption that one of the inequalities in (4) is strict. \( \square \)

**Claim 8.10.** There are at least three indices \( i \in \{1, 2, 3, 4\} \) such that \( D.C_{i;j} > 0 \) for all \( j \neq i \).

**Proof.** Assume \( D.C_{i;j} = 0 \), for some \( i, j \). We may assume without loss of generality that \( D.C_{1;2} = 0 \):

\[
2d = m_{13} + m_{14} + m_2.
\]

We claim that for all \( i \in \{2, 3, 4\} \) one has \( D.C_{i;j} > 0 \) for all \( j \neq i \). This follows from \( D.(C_{1;2} + C_{i;j}) > 0 \) for all \( i \in \{2, 3, 4\}, j \neq i \). This is because

\[
D.(C_{1;2} + C_{3;j}) = D.C_{3k} + D.(l - e_{23} - e_{14}) + D.e_{13} \quad (\{j, k\} = \{1, 4\}).
\]

It follows from (5), (4) and \( m_{13} < d \) that \( D.(C_{1;2} + C_{3;j}) > 0 \). Similarly,

\[
D.(C_{1;2} + C_{3;2}) = D.B_2 + D.e_{13}.
\]

By (9) and \( m_{13} < d \), \( D.(C_{1;2} + C_{3;2}) > 0 \). By symmetry, \( D.C_{4;j} > 0 \), for all \( j \neq 4 \).

If \( \{j, k\} = \{3, 4\} \), one has

\[
D.(C_{1;2} + C_{2;j}) = D.C_{1k} + D.(l - e_{1j} - e_{2k}) + D.e_{12}.
\]

From (5), (4) and \( m_{12} < d \) one has \( D.(C_{1;2} + C_{2;j}) > 0 \). Similarly,

\[
D.(C_{1;2} + C_{2;1}) = D.(2l - e_1 - e_2) + D.(l - e_{13} - e_{24}) + D.(l - e_{14} - e_{23}).
\]

From (4) and \( m_1, m_2 < d \) one has \( D.(C_{1;2} + C_{2;1}) > 0 \). \( \square \)

### 9. Restrictions to Generators

Let \( \pi' : \overline{\mathcal{M}} \to \mathbb{P}^3 \) be the blow-up along \( p_1, \ldots, p_5 \) and let \( E'_i \) be the corresponding exceptional divisors. Let \( \pi : \overline{\mathcal{M}} \to \overline{\mathcal{M}}' \) be the blow-up of the proper transforms of the lines \( l_{ij} \). In what follows, we compute the classes of the restrictions of an arbitrary divisor \( D \) on \( \overline{\mathcal{M}} \) to the divisors \( E_i, E_{ij}, \lambda_{ijk}, Q_{(ij)(kl)} \).

#### 9.1. Restrictions to \( E_i \)

The divisor \( E_i \) is the inverse image \( \pi^{-1}(E'_i) \). By Fact 9.1, the divisor \( E_j \) is the blow-up of \( E'_j \cong \mathbb{P}^2 \) along the 4 points corresponding to the directions of the lines \( l_{ij} \), for \( j \neq i \). Denote by \( \mathcal{E}_j \) the corresponding exceptional divisors. Denote by \( \mathcal{H} \) the hyperplane class on \( E_i \). One may easily see the following:

\[
(H|_{E_i} = 0, \quad E_i|_{E_i} = -\mathcal{H}, \quad E_{ij}|_{E_i} = 0, \quad E_{ij}|_{E_j} = \mathcal{E}_j, \quad E_{jk}|_{E_i} = 0).
\]

where \( j, k \neq i, j \neq k \). This is clear from Fact 9.1

**Fact 9.1 ([DH] Prop. IV.21, p. 167).** Let \( Y \) and \( Z \) be closed subschemes in a scheme \( X \) and let \( X \) be the blow-up of \( X \) along \( Z \). Let \( E \) be the exceptional divisor. The proper transform \( Y' \) of \( Y \) is the blow-up of \( Y \) along the scheme theoretic intersection \( Y \cap Z \), and the exceptional divisor is \( Y' \cap E \). In particular, if \( Z \) is contained in \( Y \), the scheme \( Y' \) is the blow-up of \( Y \) along \( Z \).

Consider an arbitrary divisor \( D \) on \( \overline{\mathcal{M}} \):

\[
D = dH - \sum_i m_i E_i - \sum_{ij} m_{ij} E_{ij}, \quad \text{where } d, m_i, m_{ij} \in \mathbb{Z}.
\]
It follows from (9.1) that the restriction of \( D \) to \( E_i \) is given by
\[
D|_{E_i} = m_i \overline{H} - \sum_{j \neq i} m_{ij} \overline{E}_j.
\]

**Lemma 9.2.** The divisor \( D|_{E_i} \) is an effective divisor if and only if
\[
m_i \geq 0, \quad m_i \geq m_{ij} \quad (j \neq i), \quad 2m_i \geq \sum_{j \neq i} m_{ij}.
\]

**Proof.** This is Lemma 7.3 applied to (9.3). \( \square \)

9.2. **Restrictions to \( E_{ij} \).** The normal bundle \( N \) of the proper transform of the line \( l_{ij} \) in \( \overline{M} \) is given by
\[
N = (\pi^* N_{{l_{ij}}|\mathbb{P}^3})(-E_i - E_j) = \mathcal{O}(-1) \oplus \mathcal{O}(-1).
\]

The divisor \( E_{ij} = \mathbb{P}(N) \) is isomorphic to \( \mathbb{P}(\mathcal{O} \oplus \mathcal{O}) = \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the projection map given by the blow-up map \( E_{ij} \to l_{ij} = \mathbb{P}^1 \) and let \( p_2 \) be the other projection. Since \( \mathcal{O}(E_{ij})|_{E_{ij}} = \mathcal{O}(\mathbb{P}(N)|_{\mathbb{P}^1}(-1) \) and
\[
\mathcal{O}(\mathbb{P}(N)|_{\mathbb{P}^1}(-1) \cong \mathcal{O}(\mathcal{O} \oplus \mathcal{O})|_{\mathbb{P}^1}(-1) \otimes p_1^* \mathcal{O}(-1),
\]
it follows that
\[
E_{ij}|_{E_{ij}} = p_1^* \mathcal{O}(-1) \otimes p_2^* \mathcal{O}(-1).
\]
Moreover, one may easily see, for all distinct \( i, j, k, l, \)
\[
H|_{E_{ij}} = E_i|_{E_{ij}} = p_1^* \mathcal{O}(1), \quad E_k|_{E_{ij}} = 0, \quad E_{k|E_{ij}} = E_{k|E_{ij}} = 0.
\]
It follows from (9.5) that the restriction of \( D \) in (9.2) to \( E_{ij} \) is given by
\[
D|_{E_{ij}} = p_1^* \mathcal{O}(d - m_i - m_j + m_{ij}) \otimes p_2^* \mathcal{O}(m_{ij}).
\]
Clearly, the divisor \( D|_{E_{ij}} \) is an effective divisor if and only if
\[
m_{ij} \geq 0, \quad d - m_i - m_j + m_{ij} \geq 0.
\]

9.3. **Restrictions to \( \Lambda_{ijk} \).** Take the case of \( \Lambda_{123} \) (the other cases are similar). Let \( \Lambda \) be the plane \( \mathbb{P}_{123} \). Then \( \Lambda_{123} \) is the proper transform \( \tilde{\Lambda} \) of \( \Lambda \) in \( \overline{M} \). Denote by \( \Lambda' \) the proper transform of \( \Lambda \) in \( \overline{M} \). Let \( q \) be the point \( l_{145} \cap \Lambda \). Note that by Fact 9.1 \( \Lambda' \) is the blow-up \( \Lambda' = \mathbb{P}^2 \) along \( p_1, p_2, p_3 \) and \( \Lambda' \) is isomorphic to the blow-up of \( \Lambda' \) in \( q \), i.e., \( \Lambda \) is isomorphic to the blow-up of \( \mathbb{P}^2 \) along \( p_1, p_2, p_3, q \). Let \( \overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{E}_4 \) be the exceptional divisors and \( \overline{H} \) the hyperplane class. One may easily see that
\[
H|_{\tilde{\Lambda}} = \overline{H}, \quad E_i|_{\tilde{\Lambda}} = 0 \quad (i = 4, 5), \quad E_{ij}|_{\tilde{\Lambda}} = 0 \quad (ij \neq 12, 13, 23, 45).
\]
Using Fact 9.1, one has that
\[
E_{ij}|_{\tilde{\Lambda}} = \overline{E}_i \quad (i = 1, 2, 3), \quad E_{145}|_{\tilde{\Lambda}} = \overline{E}_q, \quad E_{ij}|_{\tilde{\Lambda}} = \overline{H} - \overline{E}_i - \overline{E}_j \quad (ij \in \{12, 13, 23\}).
\]
It follows that the restriction of \( D \) in (9.2) to \( \Lambda_{123} \) is given by
\[
D|_{\Lambda_{123}} = (d - m_{12} - m_{13} - m_{23}) \overline{H} - \sum_{\{i, j, k\} = \{1, 2, 3\}} (m_i - m_{ij} - m_{ik}) \overline{E}_i - m_{45} \overline{E}_q.
\]
By permuting indices and applying Lemma 7.3 one has the following:

**Lemma 9.3.** If the divisor \( D|_{\Lambda_{ijk}} \) is effective and \( \{i, j, k, u, v\} = \{1, 2, 3, 4, 5\} \), then
\[
d \geq m_i + m_{jk}, \quad d \geq m_{ij} + m_{ik} + m_{jk} + m_{uv}, \quad 2d \geq m_i + m_j + m_{ij} + m_{uv}.
\]
9.4. Restrictions to the Keel-Vermeire divisors $Q_{(ij)(kl)}$. Take the case of $Q_{(12)(34)}$. There is a unique (smooth) quadric $Q$ in $\mathbb{P}^3$ that contains the points $p_1, \ldots, p_5$ and the lines $l_{13}, l_{14}, l_{23}, l_{24}$. Since $Q_{(12)(34)}$ has class,

$$Q_{(12)(34)} = 2H - \sum_i E_i - E_{13} - E_{14} - E_{23} - E_{24},$$

it follows that $Q_{(12)(34)}$ is the proper transform $\tilde{Q}$ of $Q$ in $\overline{M}$. Denote by $Q'$ the proper transform of $Q$ in $\overline{M}$. By Fact 9.1 it follows that $Q'$ is the blow-up of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ along the points $p_1, \ldots, p_5$. Moreover $\tilde{Q} \cong Q'$.

Let $F_1$, respectively $F_2$, be the class of the lines in the ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ that contains $l_{13}$ and $l_{24}$, respectively $l_{14}$ and $l_{23}$. Let $E_1, \ldots, E_5$ be the exceptional divisors on $\tilde{Q}$, considered as a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ along $p_1, \ldots, p_5$. By Fact 9.1

$$H_{i|\tilde{Q}} = F_1 + F_2, \quad E_{ij|\tilde{Q}} = E_i,$$

$$E_{ij|\tilde{Q}} = F_1 - E_i - E_j \quad (ij = 13, 24), \quad E_{ij|\tilde{Q}} = F_2 - E_i - E_j \quad (ij = 14, 23),$$

$$E_{ij|\tilde{Q}} = 0 \quad \text{for all other cases.}$$

It follows that restriction $D_{i|\tilde{Q}}$ of the divisor $D$ in (9.2) to $\tilde{Q}$ is given by

$$D_{i|\tilde{Q}} = (d - m_{13} - m_{24})F_1 + (d - m_{14} - m_{23})F_2 - (m_1 - m_{13} - m_{14})E_1 - (m_2 - m_{23} - m_{24})E_2 - (m_3 - m_{13} - m_{23})E_3 - (m_4 - m_{14} - m_{24})E_4 - m_5E_5.$$  

Alternative description of $\tilde{Q}$. Let $\rho: \mathbb{P}^3 \setminus \{p_5\} \rightarrow \mathbb{P}^2$ be the projection from $p_5$ and let $q_i = \rho(p_i)$ ($i = 1, \ldots, 4$). Let $l_1$ (respectively $l_2$) be the unique line through $p_5$ in the ruling of $F_1$ (respectively $F_2$).

Let $y$ (respectively $x$) be the image $l_1$ (respectively $l_2$). The blow-up of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ in $p_5$ is isomorphic to the blow-up of $\mathbb{P}^2$ in $x, y$. Hence, $\tilde{Q}$ is isomorphic to the blow-up of $\mathbb{P}^2$ along $p_1, \ldots, p_4, x, y$. Denote by $E_1, \ldots, E_4, E_x, E_y$ the exceptional divisors corresponding to the points $p_1, \ldots, p_4, x, y$ and let $\overline{H}$ be the hyperplane class. One may immediately see

$$\overline{H} = \rho^*O(1) = F_1 + F_2 - E_5.$$  

Note that lines in the ruling $F_2$ (respectively $F_1$) intersect $l_1$ (respectively $l_2$); therefore their images in $\mathbb{P}^2$ all pass through $y$ (respectively $x$). In particular, the lines $\overline{q_1q_3}$ and $\overline{q_2q_4}$ intersect in $x$, while the lines $\overline{q_1q_4}$ and $\overline{q_2q_3}$ intersect in $y$. Moreover, one has

$$F_1 = \overline{H} - E_x, \quad F_2 = \overline{H} - E_y.$$  

It follows that

$$E_5 = \overline{H} - E_x - E_y, \quad E_i = E_i \quad (i = 1, \ldots, 4).$$  

Hence, the restriction $D_{i|\tilde{Q}}$ of the divisor $D$ in (9.2) to $\tilde{Q}$ is given by

$$D_{i|\tilde{Q}} = (2d - m_5 - m_{13} - m_{14} - m_{23} - m_{24})\overline{H} - (m_1 - m_{13} - m_{14})E_1 - (m_2 - m_{23} - m_{24})E_2 - (m_3 - m_{13} - m_{23})E_3 - (m_4 - m_{14} - m_{24})E_4 - (d - m_5 - m_{13} - m_{14} - m_{23} - m_{24})E_y.$$  

Lemma 9.4. If the divisor $D_{i|\tilde{Q}}$ is effective, then

$$(9.8) \quad 2d \geq m_5 + m_{13} + m_{14} + m_{23} + m_{24}, \quad 2d \geq m_1 + m_5 + m_{23} + m_{24}.$$
Proof. This follows from Lemma 7.3. □

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