

PRESCRIBING CURVATURES ON THREE DIMENSIONAL RIEMANNIAN MANIFOLDS WITH BOUNDARIES

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ABSTRACT. Let (M, g) be a complete three dimensional Riemannian manifold with boundary ∂M . Given smooth functions $K(x) > 0$ and $c(x)$ defined on M and ∂M , respectively, it is natural to ask whether there exist metrics conformal to g so that under these new metrics, K is the scalar curvature and c is the boundary mean curvature. All such metrics can be described by a prescribing curvature equation with a boundary condition. With suitable assumptions on K, c and (M, g) we show that all the solutions of the equation can only blow up at finite points over each compact subset of \bar{M} ; some of them may appear on ∂M . We describe the asymptotic behavior of the blow-up solutions around each blow-up point and derive an energy estimate as a consequence.

1. INTRODUCTION

In geometric analysis two well-known problems are closely related: the Nirenberg problem (or the Kazdan-Warner problem) and the Yamabe problem. The Nirenberg problem asks what function $K(x)$ on \mathbb{S}^n is the scalar curvature of a metric g on \mathbb{S}^n conformal to the standard metric g_0 . From another point of view, on any given compact Riemannian manifold (M, g) without boundary, the Yamabe problem, which was solved through the works of Yamabe [36], Trudinger [35], Aubin [3] and Schoen [32], concerns whether it is possible to deform g conformally to get a new metric with constant scalar curvature. Similar questions can also be asked on general Riemannian manifolds with boundaries: Let (M, g) be a Riemannian manifold with boundary, and let $K \in C^1(M)$ and $c \in C^1(\partial M)$ be C^1 functions defined on M and ∂M , respectively. Then, is it possible to deform g conformally to another metric g_1 so that K is the scalar curvature and c is the boundary mean curvature under g_1 ? By writing g_1 as $g_1 = u^{\frac{4}{n-2}}g$, K and c are related to the scalar curvature R_g and the mean curvature h_g under the metric g by

$$(1.1) \quad \begin{cases} K = -\frac{4(n-1)}{n-2}u^{-\frac{n+2}{n-2}}(\Delta_g u - \frac{n-2}{4(n-1)}R_g u), \\ c = \frac{2}{n-2}u^{-\frac{n}{n-2}}(\partial_{\nu_g} u + \frac{n-2}{2}h_g u), \end{cases}$$

where ν_g is the unit normal vector pointing to the outside of ∂M . Δ_g is the Laplace-Beltrami operator, which can be written as $\Delta_g = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)$ in local coordinates.

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If K and c are constants and (M, g) is compact, the existence of a metric with constant scalar curvature and boundary mean curvature is always referred to as the boundary Yamabe problem (BYP). Many cases of the BYP have been solved by Escobar [17, 18, 19], Han and Li [22] and Marques [28]. But unlike the completely solved Yamabe problem, some cases of the BYP are still open. In general, the boundary terms in (1.1) make the nature of equation (1.1) very different from its counterpart without the boundary condition. Some difficulties created by the boundary terms are still not completely understood. In this article, we focus on three dimensional Riemannian manifolds with boundaries and consider the corresponding prescribing curvature equations defined on these manifolds.

Let (M, g) be a smooth three dimensional complete Riemannian manifold with boundary ∂M . Suppose $K(x) > 0$ and $c(x)$ are C^1 functions defined on M and ∂M , respectively. If $g_1 = u^4 g$ ($u > 0$ smooth) is a metric conformal to g that takes K as the scalar curvature and c as the mean curvature on ∂M , one can write the equation as

$$(1.2) \quad \begin{cases} \Delta_g u - \frac{1}{8} R_g u + \frac{1}{8} K(x) u^5 = 0, & \text{on } M, \\ \partial_{\nu_g} u + \frac{1}{2} h_g u = \frac{1}{2} c(x) u^3, & \text{on } \partial M. \end{cases}$$

The purpose of this article is to understand the bubbling phenomena of (1.2) under natural assumptions on K and c . Without the boundary condition, the bubbling phenomenon of equation (1.2) and its high dimensional variants have been studied extensively under various assumptions. The reader is referred to the following incomplete list and the references therein [4, 5, 6, 7, 11, 12, 13, 15, 16, 20, 21, 23, 24, 25, 26, 27, 28, 31, 32, 37]. However, much less work can be found to address the case with boundary conditions. In this article we describe the blow-up phenomenon for (1.2) under weak assumptions on K and c .

To state our main result, we first remark that M may not be compact, so our result concerns the blow-up phenomenon for every compact subset of M that shares with M a part of its boundary. Namely, let V be a subset of M such that \bar{V} is compact and let $\Gamma = \partial V \cap \partial M$ be the part of the boundary that V shares with M . Then our main result can be stated as follows:

Theorem 1.1. *Let (M, g) , R_g , h_g , V , Γ , K , c be described as above. Let $u > 0$ be a classical solution of (1.2). Suppose Γ is umbilic and there exists $\epsilon_0 > 0$ such that for every point $p \in \Gamma$, g is conformal to the Euclidean metric δ in $B(p, \epsilon_0) \cap M$. Assume in addition that for some $\Lambda > 0$,*

$$\frac{1}{\Lambda} \leq K(x) \leq \Lambda, \\ \max \left(\|\nabla K(x)\|_{L^\infty(M)}, \|c\|_{C^1(\partial M)} \right) \leq \Lambda.$$

Then for some $C(M, g, \Lambda, V, \epsilon_0) > 0$ and integer $m(M, g, \Lambda, V, \epsilon_0) > 0$, there exist local maximum points of u , denoted as $S := \{P_1, \dots, P_m\}$, such that

$$(1.3) \quad \begin{aligned} \text{dist}_g(P_i, P_j) &\geq \frac{1}{C}, \quad \frac{1}{C} u(P_i) \leq u(P_j) \leq C u(P_i), \quad \forall i \neq j, \\ \frac{1}{C} \sum_{l=1}^m \xi_{P_l, u(P_l)^2} &\leq u \leq C \sum_{l=1}^m \xi_{P_l, u(P_l)^2}, \quad \text{on } \bar{V}, \end{aligned}$$

where

$$\xi_{Q,\mu}(P) := \left(\frac{\mu}{1 + \mu^2 \text{dist}(P, Q)^2}\right)^{\frac{1}{2}}, \quad P \in \bar{V}.$$

As a consequence, we have the following energy estimate:

Corollary 1.1. *Under the assumptions of Theorem 1.1, we have*

$$(1.4) \quad \int_V (|\nabla_g u|^2 + u^6) \leq C(M, g, \Lambda, V, \epsilon_0).$$

If (M, g) is compact, Theorem 1.1 concludes that if the metric is locally conformally flat near ∂M , which is umbilic, then there are only a finite number of blow-up points in \bar{M} and the solutions of (1.2) can be estimated by (1.3).

The results of Theorem 1.1 and Corollary 1.1 are closely related to a priori estimates of the solutions. When (M, g) is compact with a boundary and K and c are constants, various compactness results and related discussions can be found in [2], [20], [22], [30] and the reference therein. The C^1 assumptions on K and c in Theorem 1.1 should be sharp. The reason is that even for the interior equations the C^1 assumption on K is necessary to have a description of the blow-up phenomenon by (1.3). So Theorem 1.1 implies that for this three dimensional situation, the C^1 assumption on c does not lead to more restrictions on K . For dimensions 4 and higher, we expect the situation to be much more subtle: first the assumption on K will be more delicate and should be made in neighborhoods of its critical points; then the sign of c at the blow-up point will make a more significant difference. Also, the flatness assumptions in the neighborhood of the critical points of K and c might be related. As the dimension grows higher, the relations will become more subtle and new phenomena will come out.

The major step in the proof of Theorem 1.1 is to establish a Harnack type inequality near Γ . This is done by the well-known method of moving planes. Unlike most of the previous works on this type of Harnack inequality established only for the interior equations, the equation in Theorem 1.1 has new features that were not dealt with before. Generally speaking, the method of moving planes requires a delicate construction of some test functions. Now this becomes more difficult for two reasons: first, the symmetry of the domain is destroyed; second, when blowup happens on the boundary or close to the boundary, the function c creates more error terms for the test functions to control. In this article we found a way to handle all the difficulties for dimension three; our approach is motivated by Caffarelli’s magnificent ideas in [9], [10] on free boundary problems.

In the next few sections, we mainly focus on the proof of the Harnack type inequality on upper half Euclidean balls and we shall mention how to obtain the interior estimate away from the boundary Γ .

2. HARNACK TYPE INEQUALITIES ON THE BOUNDARY

The proof of Theorem 1.1 can be divided into two parts. First we consider the region close to Γ : $N(\Gamma, \epsilon_0)$, i.e. the ϵ_0 neighborhood of Γ . In this region we shall establish a Harnack type inequality. Then outside this region, we use the techniques developed in [25] to get the same type of Harnack inequality. Note that for this interior part we don’t need to assume the metric to be locally conformally flat.

Since (M, g) is locally conformally flat near Γ , at each point $p \in \Gamma$ we can find a positive smooth function ϕ such that $\delta = \phi^4 g$, where δ is the Euclidean metric. The

umbilicity of Γ implies that ∂M near p is either a piece of a sphere or a hyperplane in the new metric. Since both neighborhoods are conformally equivalent through the inversion map $f(x) = x|x|^{-2}$, we just assume that near p the boundary is a hyperplane. Then the equation in the neighborhood of p can be written as

$$\begin{cases} \Delta(\frac{u}{\phi}) + K(\frac{u}{\phi})^5 = 0 & B_{\epsilon_1}^+ \subset \mathbb{R}^3, \\ \partial_{x_3}(\frac{u}{\phi}) = -c(\frac{u}{\phi})^3 & \text{on } \partial' B_{\epsilon_1}^+, \end{cases}$$

where $\epsilon_1 > 0$ is a positive constant that only depends on $\epsilon_0, M, g,$ and Γ . $B_{\epsilon_1}^+$ is the upper half ball of radius $\epsilon_1, \partial' B_{\epsilon_1}^+ = \partial B_{\epsilon_1}^+ \cap \partial \mathbb{R}_+^3$. The computation above is based on the following conformal covariant properties of the two operators:

$$\begin{aligned} \Delta(\frac{u}{\phi}) &= \phi^{-5}(\Delta_g u - c(n)R_g u), \\ -\partial_{x_3}(\frac{u}{\phi}) &= \phi^{-3}(\partial_{\nu_g} u + \frac{1}{2}h_g u). \end{aligned}$$

So the whole equation is reduced to the Euclidean case. From now on in this section we just consider the following case (by abusing the notation we still use K to denote a positive C^1 function on an upper half disk B_{3R}^+ and c to denote a C^1 function defined on $\partial' B_{3R}^+$, the lower part of ∂B_{3R}^+):

$$(2.1) \quad \begin{cases} \Delta u + K(x)u^5 = 0 & B_{3R}^+ \subset \mathbb{R}^3, \\ \frac{\partial u}{\partial x_3} = c(x')u^3 & \text{on } \partial' B_{3R}^+, \end{cases}$$

where $\partial' B_{3R}^+ = \partial B_{3R}^+ \cap \partial \mathbb{R}_+^3$, and x' is the projection of x to $\partial' B_{3R}^+$. The main result of this section is

Proposition 2.1. *Let $u \in C^1(\overline{B_{3R}^+}) \cap C^2(B_{3R}^+)$ be a positive function that solves (2.1). Assume for some $\Lambda > 0$ that K and c satisfy*

$$(2.2) \quad \frac{1}{\Lambda} \leq K(x) \leq \Lambda, \quad |\nabla K(x)| \leq \Lambda R^{-1}, \quad \forall x \in \overline{B_{3R}^+}$$

and

$$(2.3) \quad |c(x')| \leq \Lambda, \quad |\nabla c(x')| \leq \Lambda R^{-1}, \quad \forall x' \in \partial' B_{3R}^+$$

respectively. Then for some $C(\Lambda) > 0$,

$$(2.4) \quad \max_{\overline{B_R^+}} u \cdot \min_{\overline{B_{2R}^+}} u \leq C(\Lambda)R^{-1}.$$

This Harnack type inequality reveals important information about the interaction of bubbles (large local maximum points) of u , which can be seen in the following.

Corollary 2.1. *Under the assumptions in Theorem 2.1, there exist a positive constant $C(\Lambda)$, a positive integer $m(\Lambda)$, and a set of m local maximum points of u in $\overline{B_R^+}$, denoted as $S := \{P_1, \dots, P_m\}$, such that*

$$(2.5) \quad \begin{aligned} \text{dist}(P_i, P_j) &\geq R/C, \quad \frac{1}{C}u(P_j) \leq u(P_i) \leq Cu(P_j), \quad \forall i \neq j, \\ \frac{1}{C} \sum_{l=1}^m \xi_{P_l, u(P_l)^2} &\leq u \leq C \sum_{l=1}^m \xi_{P_l, u(P_l)^2}, \quad \text{on } \overline{B_R^+}, \end{aligned}$$

where

$$\xi_{Q,\mu}(P) := \left(\frac{\mu}{1 + \mu^2 \text{dist}(P, Q)^2} \right)^{\frac{1}{2}}, \quad P \in \bar{B}_R^+$$

Consequently u satisfies

$$(2.6) \quad \int_{B_R^+} (|\nabla u|^2 + u^6) \leq C(\Lambda).$$

In the proof of Proposition 2.1 we shall omit a selection process of Schoen since it is well known to experts. We only include it in the appendix. At the end of this section we indicate the outline of the proof of Corollary 2.1 based on Proposition 2.1. The proof can be found in [24] with obvious changes, but we indicate the key points for the convenience of the readers.

2.1. The proof of Proposition 2.1. We only need to consider the case $R = 1$. The general case can be reduced to the case $R = 1$ by considering the function $R^{\frac{1}{2}}u(R\cdot)$. The proof is by a contradiction. Suppose (2.4) does not hold. Then there exists a sequence $u_i(x_i)$ such that

$$(2.7) \quad u_i(x_i) \min_{\bar{B}_2^+} u_i \geq i,$$

where $u_i(x_i) = \max_{\bar{B}_1^+} u_i$ and $x_i \in \bar{B}_1$. Clearly the above inequality implies $u_i(x_i) \rightarrow \infty$. By a standard selection process of Schoen [31] and the classification theorems of Caffarelli-Gidas-Spruck [8] and Li and Zhu [27], one can consider x_i as a local maximum of u_i . Moreover the following sequence of functions

$$(2.8) \quad \bar{v}_i(y) := u_i(x_i)^{-1} u_i(u_i(x_i)^{-2}y + x_i)$$

converges in C^2 over any finite domain in the following two cases:

- (1) If $\lim_{i \rightarrow \infty} u_i^2(x_i)x_{i3} \rightarrow \infty$, the $\bar{v}_i(y)$ converge in $C_{loc}^2(\mathbb{R}^3)$ to $U(y)$ which satisfies

$$(2.9) \quad \begin{cases} \Delta U(y) + \lim_{i \rightarrow \infty} K_i(x_i)U(y)^5 = 0, & \text{on } \mathbb{R}^3, \\ U(0) = 1 = \max_{\mathbb{R}^3} U. \end{cases}$$

- (2) If $\lim_{i \rightarrow \infty} u_i^2(x_i)x_{i3}$ is bounded, by passing to a subsequence we assume

$$\lim_{i \rightarrow \infty} u_i^2(x_i)x_{i3} = t_0.$$

In this case the \bar{v}_i converge uniformly to U which satisfies

$$(2.10) \quad \begin{cases} \Delta U(y) + \lim_{i \rightarrow \infty} K_i(x_i)U(y)^5 = 0, & y_3 > t_0, \\ \partial_3 U(y) = \lim_{i \rightarrow \infty} c_i(x'_i)U^3, & y_3 = -t_0, \\ U(0) = 1 = \max U. \end{cases}$$

Since the selection process and the application of the classification theorems are standard, we put the details in the Appendix. Similar techniques can also be found in [11], [24], [37], etc.

The proof of Proposition 2.1 that follows can be divided into three steps. First we rule out case one; i.e., we shall show that the blow-up points cannot be far away from the boundary. In the second step we prove the case of $\lim_{i \rightarrow \infty} c(x'_i) \leq 0$. Then in the final step we prove the case of $\lim_{i \rightarrow \infty} c(x'_i) > 0$.

2.1.1. *Step one.* In this subsection we derive a contradiction to

$$\lim_{i \rightarrow \infty} T_i := \lim_{i \rightarrow \infty} u_i^2(x_i)x_{i3} \rightarrow \infty.$$

With no loss of generality we assume $\lim_{i \rightarrow \infty} K_i(x_i) = 3$. In this case, the U in (2.9) is of the form

$$U(y) = (1 + |y|^2)^{-\frac{1}{2}}.$$

Let $M_i = u_i(x_i)$. In this subsection we use v_i instead of \bar{v}_i , i.e.

$$v_i(y) = M_i^{-1}u_i(M_i^{-2}y + x_i).$$

The selection process implies $B(x_i, \frac{1}{2}) \cap \mathbb{R}_+^3 \subset B_2^+$, so the rescaled domain for v_i will have a part of the boundary (the upper part) whose distance to 0 is comparable to M_i^2 . By (2.7) we know on this part of the boundary that $v_i(y)|y| > i$. By choosing $\epsilon_i \rightarrow 0$ slowly we can still obtain $v_i(y)|y| > \sqrt{i}$ for $|y| = \epsilon_i M_i^2$. So we define

$$\Omega_i = \{y : M_i^{-2}y + x_i \in B(x_i, \epsilon_i) \cap \mathbb{R}_+^3\}$$

and

$$\partial' \Omega_i = \partial \Omega_i \cap \{y_3 = -T_i\}, \quad \partial'' \Omega_i = \partial \Omega_i \setminus \partial' \Omega_i.$$

Then v_i satisfies

$$(2.11) \quad \begin{cases} \Delta v_i(y) + K_i(M_i^{-2}y + x_i)v_i(y)^5 = 0, & y \in \Omega_i, \\ \partial_3 v_i(y) \leq \Lambda v_i^3 & y \in \partial \Omega_i \cap \{y_3 = -T_i\}, \quad T_i := M_i^2 x_{i3}. \end{cases}$$

Note that in this case we don't need the specific equation for v_i on $\partial' \Omega_i$.

By the discussion on the behavior of v_i on $\partial'' \Omega_i$ we have

$$(2.12) \quad v_i(y)|y| \rightarrow \infty \quad \text{for } y \in \partial'' \Omega_i.$$

Let $\lambda \in [\frac{1}{2}, 2]$ and

$$v_i^\lambda(y) := \left(\frac{\lambda}{|y|}\right)v_i\left(\frac{\lambda^2 y}{|y|^2}\right).$$

Then a direct computation gives

$$(2.13) \quad \Delta v_i^\lambda(y) + K_i(M_i^{-2}y^\lambda + x_i)v_i^\lambda(y)^5 = 0 \quad \text{for } |y| > \lambda/2.$$

Here we use $y^\lambda = \lambda^2 y / |y|^2$.

In this step we assume $\lambda \in [\frac{1}{2}, 2]$. Let $\Sigma_\lambda = \Omega_i \setminus \bar{B}_\lambda$. Note that for simplicity we shall omit i in some notation. Let $w_\lambda = v_i - v_i^\lambda$ and consider the equation for w_λ :

$$(2.14) \quad \begin{cases} \Delta w_\lambda + b_\lambda w_\lambda = Q_\lambda, & \text{on } \Sigma_\lambda, \\ \partial_3 w_\lambda \leq \Lambda \xi_i w_\lambda + \Lambda(v_i^\lambda)^3 - \partial_3(v_i^\lambda), & \text{on } \partial' \Omega_i \setminus \bar{B}_\lambda, \end{cases}$$

where b_λ and ξ_i are obtained from the mean value theorem:

$$\begin{aligned} b_\lambda(y) &= 5K_i(M_i^{-2}y + x_i) \int_0^1 (tv_i(y) + (1-t)v_i^\lambda(y))^4 dt, \\ \xi_i(y) &= 3 \int_0^1 (tv_i + (1-t)v_i^\lambda)^2 dt, \end{aligned}$$

and

$$Q_\lambda(y) = (K_i(M_i^{-2}y^\lambda + x_i) - K_i(M_i^{-2}y + x_i))(v_i^\lambda(y))^5$$

is estimated as follows:

$$(2.15) \quad |Q_\lambda(y)| \leq C_0(\Lambda)M_i^{-2}|y|^{-4}, \quad \text{on } \Sigma_\lambda.$$

Since v_i converges in the C^2 norm over U over any fixed finite domain, v_i^λ is close to U^λ , the Kelvin transformation of U . By direct computation $U > U^\lambda$ for $|y| > \lambda$ and $\lambda \in (0, 1)$. On the other hand, $U < U^\lambda$ for $\lambda > 1$ and $|y| > \lambda$. So the strategy of the proof is to find a test function h_λ (i is omitted in this notation) so that the moving sphere method works for $w_\lambda + h_\lambda$, and the h_λ is just a perturbation of w_λ , which means $h_\lambda(y) = o(1)|y|^{-1}$ in Σ_λ . Then it is possible to move the spheres from a position less than 1 to a position greater than 1, keeping $w_\lambda + h_\lambda > 0$ in Σ_λ . But this is a contradiction since $w_\lambda + h_\lambda$ converges to $U - U^\lambda$ in finite domains and $U < U^\lambda$ for $\lambda > 1$ and $|y| > \lambda$.

The test function in this section is

$$(2.16) \quad h_\lambda(y) = C_0 M_i^{-2} \left(-\frac{1}{2}|y|^{-2} + \lambda^{-1}|y|^{-1} - \frac{1}{2}\lambda^{-2} \right),$$

h_λ is a radial function, and the function $h_\lambda(r)$ satisfies

$$\begin{cases} h_\lambda''(r) + \frac{2}{r}h_\lambda'(r) = -C_0 M_i^{-2} r^{-4}, & r > \lambda, \\ h_\lambda(\lambda) = h_\lambda'(\lambda) = 0. \end{cases}$$

By the maximum principle, $h_\lambda(r) < 0$ for $r > \lambda$. Note that the C_0 in (2.16) is the same as the one in (2.15).

Now we consider the equation for $w_\lambda + h_\lambda$. From (2.14) we have

$$(2.17) \quad (\Delta + b_\lambda)(w_\lambda + h_\lambda) = Q_\lambda + \Delta h_\lambda + b_\lambda h_\lambda \leq 0 \quad \text{on } \Sigma_\lambda.$$

The reason is that $\Delta h_\lambda + Q_\lambda \leq 0$ and $h_\lambda \leq 0$ in Σ_λ .

On $\partial'\Omega_i \setminus \bar{B}_\lambda$ we have

$$(2.18) \quad \partial_3(w_\lambda + h_\lambda) \leq \Lambda \xi_i(w_\lambda + h_\lambda) + \Lambda(v_i^\lambda)^3 - \partial_3(v_i^\lambda) + \partial_3 h_\lambda - \Lambda \xi_i h_\lambda.$$

By the definition of h_λ one can verify that $h_\lambda = o(1)|y|^{-1}$ over Σ_λ . Let

$$O_\lambda = \{y \in \Sigma_\lambda \mid v_i(y) \leq 2v_i^\lambda(y)\}.$$

Then we observe that the maximum principle in the moving sphere process only needs to be applied over O_λ because outside O_λ , $w_\lambda + h_\lambda > 0$. Now we claim that

$$(2.19) \quad \Lambda(v_i^\lambda)^3 - \partial_3(v_i^\lambda) + \partial_3 h_\lambda - \Lambda \xi_i h_\lambda < 0, \quad \text{on } \partial'\Omega_i \cap \bar{O}_\lambda.$$

Once we have (2.19), (2.18) becomes

$$(2.20) \quad \partial_3(w_\lambda + h_\lambda) \leq \Lambda \xi_i(w_\lambda + h_\lambda) \quad \text{on } \partial'\Omega_i \cap \bar{O}_\lambda,$$

which is the form for the application of the moving sphere method.

To see (2.19), we first observe that

$$\begin{aligned} \partial_3(v_i^\lambda) &= \lambda|y|^{-3} T_i v_i(y^\lambda) + 2\lambda^3|y|^{-5} \sum_{j=1}^3 \partial_j v_i(y^\lambda) y_j T_i \\ &\quad + \lambda^3|y|^{-3} \partial_3 v_i(y^\lambda) \quad \text{on } \partial'\Omega_i \setminus \bar{B}_\lambda. \end{aligned}$$

The second term and the third term on the right are negligible compared to the first term on the right. The reason is that $v_i \rightarrow U$ in $C_{loc}^2(\mathbb{R}^3)$, which implies

$$\nabla v_i(y^\lambda) \rightarrow \nabla U(0) = 0.$$

On the other hand, by the assumption $T_i \rightarrow \infty$, we see that the first term dominates the other two terms.

Next we see that the assumption $T_i \rightarrow \infty$ makes $|\partial_3(v_i^\lambda)|$ dominate $\Lambda(v_i^\lambda)^3$ as the latter is of the order $O(|y|^{-3})$. By the definition of h_λ and the definition of the

domain Ω_i , we see easily that $|\partial_3(v_i^\lambda)|$ also dominates $\partial_3 h_\lambda$. Finally in \bar{O}_λ , ξ_i is comparable to $|y|^{-2}$, so $\xi_i h_\lambda$ is comparable to $O(M_i^{-2}|y|^{-2})$, which is much smaller than $|\partial_3(v_i^\lambda)|$. (2.19) is proved.

The process of making the moving sphere process start is standard, even though the boundary condition makes it different from the interior case.

Lemma 2.1. *For any fixed $\lambda_0 \in (\frac{1}{2}, 1)$ and all large i ,*

$$w_{\lambda_0} + h_{\lambda_0} > 0 \quad \text{in } \Sigma_{\lambda_0}.$$

Proof of Lemma 2.1. Since $v_i \rightarrow U$ in $C_{loc}^2(\mathbb{R}^3)$ and $U > U^\lambda$ for $|y| > \lambda$ if $\lambda \in (\frac{1}{2}, 1)$, by the convergence of v_i to U and the fact $h_\lambda = o(1)$ over finite domains we can check easily that for any fixed $R_1 \gg 1$,

$$w_{\lambda_0} + h_{\lambda_0} > 0 \quad \text{in } \Sigma_{\lambda_0} \cap B_{R_1}.$$

Since R_1 is chosen sufficiently large, one can find $\sigma(\lambda_0) > 0$ such that

$$v_i^{\lambda_0}(y) \leq (1 - 3\sigma)|y|^{-1} \quad |y| \geq R_1$$

and

$$v_i(y) > (1 - \sigma)|y|^{-1} \quad \text{on } |y| = R_1.$$

The definitions of h_{λ_0} and Ω_i imply $|h_{\lambda_0}(y)| \leq \frac{\sigma}{5}|y|^{-1}$ for $|y| > R_1$. So to finish the proof of this lemma it is enough to show

$$(2.21) \quad v_i(y) > (1 - 2\sigma)|y|^{-1} \quad \text{in } \Omega_i \setminus \overline{B_{R_1}}.$$

On $|y| = R_1$ and $\partial''\Omega_i$ we certainly have $v_i > (1 - 2\sigma)|y|^{-1}$. To apply the maximum principle to the superharmonic function $v_i - (1 - 2\sigma)|y|^{-1}$, we need

$$(2.22) \quad \partial_3(v_i - (1 - 2\sigma)|y|^{-1}) \leq \Lambda(v_i^3 - (1 - 2\sigma)^3|y|^{-3}) \quad \text{on } \partial'\Omega_i.$$

Using $T_i \rightarrow \infty$ we obtain by direct computation that

$$\partial_3((1 - 2\sigma)|y|^{-1}) > \Lambda((1 - 2\sigma)|y|^{-1})^3 \quad \text{on } \partial'\Omega_i.$$

(2.22) follows immediately. (2.22) means that the non-positive minimum cannot be attained on $\partial'\Omega_i$. Lemma 2.1 is established. \square

Lemma 2.1 means that the moving sphere process can start at $\lambda = \lambda_0$. The purpose of the moving spheres is to show that

$$(2.23) \quad w_\lambda + h_\lambda > 0 \quad \text{in } \Sigma_\lambda \quad \text{for all } \lambda \in [\lambda_0, \lambda_1].$$

Once this is proved, we have $U > U^{\lambda_1}$ for some $|y| > \lambda_1$, which is a contradiction to the fact that $\lambda_1 > 1$. To see why (2.23) holds, first, (2.17) and (2.20) mean that the maximum principle holds for $w_\lambda + h_\lambda$, and second, (2.12) means that there is no touch on $\partial''\Omega_i$. Step one is established.

2.1.2. *Step two.* In this step, we deal with the case $\lim_{i \rightarrow \infty} c_i(x'_i) \leq 0$. Let

$$-c_0 = \lim_{i \rightarrow \infty} c_i(x'_i) \leq 0.$$

In this case the limit of \bar{v}_i is U , which satisfies

$$\begin{cases} \Delta U + 3U^5 = 0, & y \in \mathbb{R}^3, y_3 > -\lim_{i \rightarrow \infty} T_i, \\ \partial_3 U = -c_0 U^3, & y_3 = -\lim_{i \rightarrow \infty} T_i, \\ U(0) = 1 = \max U. \end{cases}$$

By Li and Zhu's classification theorem [27], $\lim_{i \rightarrow \infty} T_i = 0$. So we define v_i as

$$v_i(y) = M_i^{-1} u_i(M_i^{-2}y + x'_i),$$

where x'_i is the projection of x_i on $\partial\mathbb{R}_+^3$. Then $v_i \rightarrow U$ in $C_{loc}^2(\bar{\mathbb{R}}_+^3)$. Since $U(0) = 1$, we know, from Li and Zhu's theorem, that

$$(2.24) \quad U(y) = (1 + c_0^2)^{\frac{1}{2}}(1 + (1 + c_0^2)|y'|^2 + (1 + c_0^2)(y_3 + \frac{c_0}{1 + c_0^2})^2)^{-\frac{1}{2}}.$$

Let $v_i^\lambda(y) = (\frac{\lambda}{|y|})v_i(y^\lambda)$. In this step we let

$$\lambda_0 = \frac{1}{2}(1 + c_0^2)^{-\frac{1}{2}}, \quad \lambda_1 = 2(1 + c_0^2)^{-\frac{1}{2}}$$

and we require $\lambda \in [\lambda_0, \lambda_1]$ and the moving sphere method will be applied on $\Sigma_\lambda = B^+(0, \epsilon_i M_i^2) \setminus \bar{B}_\lambda$. A direct computation gives

$$(2.25) \quad \begin{cases} \Delta v_i^\lambda(y) + K_i(M_i^{-2}y^\lambda + x'_i)(v_i^\lambda(y))^5 = 0 & \text{on } \Sigma_\lambda, \\ \partial_3 v_i^\lambda(y) = c_i(M_i^{-2}y^\lambda + x'_i)(v_i^\lambda(y))^3 & \text{on } \partial\Sigma_\lambda \cap \partial\mathbb{R}_+^3. \end{cases}$$

Also by a direct computation one can verify that

$$\begin{cases} U(y) > U^\lambda(y), & |y| > \lambda, \quad 0 < \lambda < (1 + c_0^2)^{-\frac{1}{2}}, \\ U(y) < U^\lambda(y), & |y| > \lambda, \quad \lambda > (1 + c_0^2)^{-\frac{1}{2}}. \end{cases}$$

For this reason we require $\lambda \in [\lambda_0, \lambda_1]$ in this step. The equation for $w_\lambda = v_i - v_i^\lambda$ becomes

$$(2.26) \quad \begin{cases} \Delta w_\lambda + K_i(M_i^{-2} \cdot + x'_i)\xi_1 w_\lambda = Q_1 & \text{on } \Sigma_\lambda, \\ \partial_3 w_\lambda = c_i(M_i^{-2} \cdot + x'_i)\xi_2 w_\lambda + Q_2 & \text{on } \partial\Sigma_\lambda \cap \partial\mathbb{R}_+^3, \end{cases}$$

where ξ_1 and ξ_2 are obtained by the mean value theorem:

$$\xi_1 = 5 \int_0^1 (tv_i + (1-t)v_i^\lambda)^4 dt, \quad \xi_2 = 3 \int_0^1 (tv_i + (1-t)v_i^\lambda)^2 dt,$$

and Q_1 and Q_2 are error terms to be controlled by test functions:

$$\begin{aligned} Q_1(y) &= (K_i(M_i^{-2}y^\lambda + x'_i) - K_i(M_i^{-2}y + x'_i))v_i^\lambda(y)^5 & \text{on } \Sigma_\lambda, \\ Q_2(y) &= (c_i(M_i^{-2}y + x'_i) - c_i(M_i^{-2}y^\lambda + x'_i))v_i^\lambda(y)^3 & \text{on } \partial\Sigma_\lambda \cap \partial\mathbb{R}_+^3. \end{aligned}$$

For Q_1 and Q_2 we use

$$(2.27) \quad \begin{aligned} |Q_1(y)| &\leq C_2(\Lambda)M_i^{-2}|y|^{-4} & \text{on } \Sigma_\lambda, \\ |Q_2(y)| &\leq C_2(\Lambda)M_i^{-2}|y|^{-2}(1 - \frac{\lambda}{|y|}) & \text{on } \partial\Sigma_\lambda \cap \partial\mathbb{R}_+^3. \end{aligned}$$

Note that in Q_2 it is important to have the $1 - \frac{\lambda}{|y|}$ term, even though we don't need this term to appear in the estimate of Q_1 .

The construction of the test function here consists of two parts, h_1 and h_2 . We first define

$$(2.28) \quad h_1(r) = C_3(\Lambda)(r^{-\frac{1}{2}} - \frac{1}{2}\lambda^{\frac{1}{2}}r^{-1} - \frac{1}{2}\lambda^{-\frac{1}{2}})M_i^{-2},$$

where $C_3(\Lambda) \gg C_2$ is to be determined. $h_1(r)$ is the solution of

$$\begin{cases} h_1''(r) + \frac{2}{r}h_1'(r) = -\frac{C_3}{4}M_i^{-2}r^{-\frac{5}{2}}, & r > \lambda, \\ h_1(\lambda) = h_1'(\lambda) = 0. \end{cases}$$

By the maximum principle, $h_1(r) < 0$ for $r > \lambda$. $h_1(|y|)$ is the first part of the test function h_λ . To define the second part of the test function, we let $\phi : [1, \infty) \rightarrow [0, \infty)$ be a smooth non-negative function that satisfies

$$\begin{cases} \phi(1) = 0, & \phi(r) = r^{-2} \text{ for } r \geq 2, \\ \phi'(1) > 0, & \phi(r) > 0 \text{ for } r > 1. \end{cases}$$

Let $\phi_\lambda(\cdot) = \phi(\lambda \cdot)$ and define

$$(2.29) \quad h_2(y) = -C_4(\Lambda)M_i^{-2}y_3\phi_\lambda(|y|),$$

where $C_4(\Lambda)$ is chosen so large that

$$(2.30) \quad C_4\phi_\lambda(|y|) > 2C_2|y|^{-2}\left(1 - \frac{\lambda}{|y|}\right), \quad |y| > \lambda.$$

Note that on $\partial'\Sigma_\lambda := \partial\Sigma_\lambda \cap \partial\mathbb{R}_+^3$,

$$(2.31) \quad \partial_3 h_2(y) = -C_4M_i^{-2}\phi_\lambda(|y|), \quad \partial_3 h_1(y) = 0 \quad \text{on } \partial'\Sigma_\lambda.$$

Letting $h_\lambda = h_1 + h_2$, the equation for $w_\lambda + h_\lambda$ is

$$\begin{aligned} (\Delta + K_i(M_i^{-2} \cdot + x'_i)\xi_1)(w_\lambda + h_\lambda) &= Q_1 + \Delta h_\lambda + K_i(M_i^{-2} \cdot + x'_i)\xi_1 h_\lambda \quad \text{on } \Sigma_\lambda, \\ (\partial_3 - c_i(M_i^{-2} \cdot + x'_i)\xi_2)(w_\lambda + h_\lambda) &= -c_i(M_i^{-2} \cdot + x'_i)\xi_2 h_\lambda + \partial_3 h_2 + Q_2 \quad \text{on } \partial'\Sigma_\lambda. \end{aligned}$$

For the application of the moving sphere method we show that the right-hand sides of the above are non-positive. Namely we shall show that

$$(2.32) \quad Q_1 + \Delta h_1 + \Delta h_2 + K_i(M_i^{-2} \cdot + x'_i)\xi_1 h_\lambda \leq 0 \quad \text{on } \Sigma_\lambda$$

and

$$(2.33) \quad -c_i(M_i^{-2} \cdot + x'_i)\xi_2 h_\lambda + \partial_3 h_2 + Q_2 \leq 0 \quad \text{on } \partial'\Sigma_\lambda \cap \overline{O_\lambda},$$

where $O_\lambda = \{y \in \Sigma_\lambda \mid v_i(y) \leq 2v_i^\lambda(y)\}$ is the only place where the maximum principle needs to hold, because by the definition of h_λ , $h_\lambda(y) = o(1)|y|^{-1}$ in Σ_λ , which means $w_\lambda + h_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{O_\lambda}$.

To see (2.32), first by (2.27) and (2.28), one sees that if C_3 is large enough, then

$$\Delta h_1 + Q_1 \leq -\frac{1}{8}C_3M_i^{-2}|y|^{-2.5} \quad \text{on } \Sigma_\lambda.$$

For h_2 we have

$$\Delta h_2 = -C_4M_i^{-2}(\phi''_\lambda(|y|) + \frac{4}{|y|}\phi'_\lambda(|y|))y_3 \quad \text{on } \Sigma_\lambda.$$

So by choosing C_3 larger if necessary we have

$$\Delta(h_1 + h_2) + Q_1 \leq -\frac{1}{16}C_3M_i^{-2}|y|^{-2.5} \quad \text{on } \Sigma_\lambda.$$

The term $K_i(M_i^{-2} \cdot + x_i)\xi_1 h_\lambda$ is non-positive. So (2.32) is proved.

To see (2.33), first by the definition of h_2 we have

$$\partial_3 h_2 + Q_2 < -\frac{C_4}{2}M_i^{-2}\phi_\lambda(|y|) \quad \text{on } \partial'\Sigma_\lambda.$$

Since $\lim_{i \rightarrow \infty} c_i(x'_i) = -c_0 \leq 0$, one uses the fact that $|y| \leq \epsilon_i M_i^2$ and the uniform bound of the C_1 norm of c_i to get

$$\begin{aligned}
 & -c_i(M_i^{-2} \cdot + x'_i)\xi_2 h_\lambda \\
 & \leq \circ(1)\xi_2|h_1 + h_2| \leq \circ(1)M_i^{-2}|y|^{-2}\left(1 - \frac{\lambda}{|y|}\right) \quad \text{on } \partial'\Sigma_\lambda \cap \bar{O}_\lambda.
 \end{aligned}$$

The second inequality above is because in \bar{O}_λ , $\xi_2 \sim |y|^{-2}$. We also use $h_\lambda(\lambda) = 0$ in the above. Then (2.33) follows immediately.

Next we show that the moving sphere process can get started. Namely we have

Lemma 2.2. $w_{\lambda_0} + h_{\lambda_0} > 0$ in Σ_{λ_0} for all large i .

Proof of Lemma 2.2. The proof is similar to the one in step one. For $|y| \leq R_1$ ($R_1 \gg 1$), $w_{\lambda_0} + h_{\lambda_0} > 0$ is guaranteed by the expressions of U , U^{λ_0} and the convergence of v_i to U over finite domains. Also $h_{\lambda_0} = \circ(1)|y|^{-1}$ means h_{λ_0} is just a perturbation over finite domains. Moreover on $|y| = R_1$, there is $\epsilon_0 > 0$ such that

$$U(y) - U^{\lambda_0}(y) > 5\epsilon_0|y - e_3|^{-1}, \quad |y| = R_1, \quad e_3 = (0, 0, 1).$$

Also we have

$$v_i^{\lambda_0}(y) \leq (1 - 4\epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1}, \quad |y| > R_1, y \in \Sigma_{\lambda_0}$$

and

$$|U^{\lambda_0}(y) - v_i^{\lambda_0}(y)| \leq \epsilon_0|y - e_3|^{-1}, \quad |y| > R_1$$

for all i large. So for $y \in \Sigma_\lambda \setminus B_{R_1}$ we only need to show that

$$v_i(y) > (1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1}, \quad y \in \Sigma_\lambda \setminus B_{R_1}.$$

Since $v_i(y) - (1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1}$ is a superharmonic function in $\Sigma_\lambda \setminus B_{R_1}$ and is positive on $|y| = R_1$ and $|y| = \epsilon_i M_i^2$, the only place to consider is $\partial'\Sigma_{\lambda_0} \setminus B_{R_1}$, on which we have

$$\begin{aligned}
 & \partial_3(v_i(y) - (1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1}) \\
 (2.34) \quad & = c_i(M_i^{-2}y + x'_i)v_i^3(y) - (1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-3}.
 \end{aligned}$$

Since $\lim_{i \rightarrow \infty} c_i(x'_i) \leq 0$, (2.34) means that

$$\partial_3(v_i(y) - (1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1}) < 0$$

at places where v_i is close to $(1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1}$ on $\partial'\Sigma_{\lambda_0} \setminus B_{R_1}$. This means that

$$v_i > (1 - \epsilon_0)(1 + c_0^2)^{-\frac{1}{2}}|y - e_3|^{-1} \quad \text{on } \partial'\Sigma_{\lambda_0} \setminus B_{R_1}.$$

Lemma 2.2 is established. □

Once we have Lemma 2.2, the moving sphere process can start at $\lambda = \lambda_0$. Note that the equation for $w_\lambda + h_\lambda$ becomes

$$\begin{cases} (\Delta + K_i(M_i^{-2} \cdot + x'_i)\xi)(w_\lambda + h_\lambda) \leq 0 & \text{on } \Sigma_\lambda, \\ (\partial_3 - c_i(M_i^{-2} \cdot + x'_i)\xi)(w_\lambda + h_\lambda) \leq 0 & \text{on } \partial'\Sigma_\lambda \cap \bar{O}_\lambda. \end{cases}$$

This means that the maximum principle always holds for $w_\lambda + h_\lambda$ as long as it is positive on the boundary $\partial''\Omega_i$, which is certainly the case because $v_i \gg v_i^\lambda$ on $\partial'\Omega_i$ and h_λ is a perturbation. So the spheres can be moved to $\lambda_1 = 2(1 + c_0^2)^{-\frac{1}{2}}$ to get a contradiction.

2.1.3. *Step three.* In this subsection we deal with the case $\lim_{i \rightarrow \infty} c_i(x'_i) = c_0 > 0$. v_i converges uniformly to U in all finite subsets of $\{y \in \mathbb{R}^3; y_3 \geq -\lim_{i \rightarrow \infty} M_i^2 x_{i3}\}$. U satisfies (still assuming $\lim_{i \rightarrow \infty} K_i(x_i) = 3$)

$$\begin{cases} \Delta U + 3U^5 = 0, & y_3 > -\lim_{i \rightarrow \infty} M_i^2 x_{i3}, \\ \partial_3 U = c_0 U^3, & y_3 = -\lim_{i \rightarrow \infty} M_i^2 x_{i3}, \\ U(0) = 1. \end{cases}$$

By Li and Zhu's classification theorem, $\lim_{i \rightarrow \infty} M_i^2 x_{i3} = c_0$ and

$$U(y) = (1 + |y|^2)^{-\frac{1}{2}}.$$

Let

$$v_i(y) = M_i^{-1} u_i(M_i^{-2} y + x'_i).$$

Then

$$v_i \rightarrow U_1(y) = (1 + |y - c_0 e_3|^2)^{-\frac{1}{2}}.$$

Let v_i^λ be the Kelvin transformation of v_i . In this case we let

$$\lambda_0 = \frac{1}{2}(1 + c_0^2)^{\frac{1}{2}}, \quad \lambda_1 = 2(1 + c_0^2)^{\frac{1}{2}}$$

because

$$\begin{aligned} U_1 > U_1^\lambda & \text{ for } \lambda \in (0, (1 + c_0^2)^{\frac{1}{2}}), \quad |y| > \lambda, \\ U_1 < U_1^\lambda & \text{ for } \lambda > (1 + c_0^2)^{\frac{1}{2}}, \quad |y| > \lambda. \end{aligned}$$

Let $w_\lambda = v_i - v_i^\lambda$. Then the equation for w_λ is still described by (2.26) with ξ_1, ξ_2, Q_1, Q_2 defined as in step two and the estimates for Q_1 and Q_2 are still (2.27). Since $\lim_{i \rightarrow \infty} c_i(x'_i) = c_0 > 0$, the construction of the test function h_λ in this case is much more delicate.

The construction of h_λ consists of two parts. First we shall construct h_1 to control the region close to ∂B_λ ; then this function is connected smoothly to 0. Clearly h_1 creates new difficulties at the regions where it is connected to 0. Next we use h_2 to control the region far enough to ∂B_λ . h_2 is 0 in regions close to ∂B_λ and becomes negative as y is far away from ∂B_λ . A parameter of h_2 is chosen to be so large that h_2 not only controls the difficulties from Q_1 and Q_2 , but also those from h_1 . One delicate thing is that h_2 does not create new difficulties. By choosing all the parameters carefully we shall obtain the following properties for h_λ :

$$\begin{aligned} \Delta h_\lambda + K_i(M_i^{-2} y + x'_i) \xi_1 h_\lambda + Q_1 &\leq 0 \quad \text{on } \Sigma_\lambda, \\ \partial_3 h_\lambda - c_i(M_i^{-2} y + x'_i) \xi_2 h_\lambda + Q_2 &\leq 0 \quad \text{on } \partial \Sigma_\lambda \cap \partial \mathbb{R}_+^3 \cap \overline{O_\lambda}, \\ h_\lambda &= 0 \quad \text{on } \partial B_\lambda \cap \mathbb{R}_+^3, \\ (2.35) \quad h_\lambda &= o(1)|y|^{-1} \quad \text{on } \Sigma_\lambda. \end{aligned}$$

We first construct h_2 . Let

$$\phi(y) = 1 - A^{\frac{1}{4}} |y + r e_3|^{-\frac{1}{4}}, \quad r = |y|,$$

where $A > 5\lambda_1$ is to be determined. Let $\Omega_1 \subset \mathbb{R}_+^3$ be the region where ϕ is positive. Let Γ_1 be the 0 level surface of ϕ . A direct computation shows that

$$\mathbb{R}_+^3 \setminus B(0, 2A) \subset \Omega_1 \subset \mathbb{R}_+^3 \setminus B(0, A/3).$$

Let

$$h_2(y) = \begin{cases} 0 & \text{on } \mathbb{R}_+^3 \setminus (\Omega_1 \cup B_\lambda), \\ -C_6 M_i^{-2} \phi^2 & \text{on } \Sigma_\lambda \cap \Omega_1, \end{cases}$$

where C_6 is a large constant to be determined. From here we see that h_2 is C^1 across Γ_1 . By a direct computation,

$$\begin{aligned}
 \Delta h_2 &= -C_6 M_i^{-2} (2|\nabla\phi|^2 + 2\phi\Delta\phi) \\
 &= -C_6 M_i^{-2} \left(\frac{1}{8} A^{\frac{1}{2}} (2r^2 + 2ry_3)^{-\frac{9}{4}} (5r^2 + 8ry_3 + 3y_3^2) \right. \\
 &\quad \left. + \frac{1}{2} \phi A^{\frac{1}{4}} (2r^2 + 2ry_3)^{-\frac{17}{8}} \left(\frac{3}{4} r^2 + 2ry_3 + \frac{5}{4} y_3^2 \right) \right) \\
 (2.36) \quad &\leq -\frac{C_6}{50} M_i^{-2} A^{\frac{1}{2}} r^{-\frac{5}{2}} \quad \text{on } \Omega_1 \cap \Sigma_\lambda.
 \end{aligned}$$

From here we observe that h_2 is weakly superharmonic in Σ_λ since $\nabla h_2 = 0$ on Γ_1 . On $\partial\Omega_1 \cap \partial\mathbb{R}_+^3 \cap \overline{O_\lambda}$ we need to estimate

$$\partial_3 h_2 - c_i(M_i^{-2} \cdot + x'_i)\xi_2 h_2 + Q_2.$$

Since

$$c_i(M_i^{-2}y + x'_i)\xi_2 \leq \frac{M}{1+r^2} \quad \text{on } \overline{O_\lambda} \cap \partial\mathbb{R}_+^3$$

for some $M > 0$ depending on c_0 , it is enough to show that

$$(2.37) \quad \partial_3 h_2 - \frac{M}{1+r^2} h_2 + Q_2 \leq 0 \quad \text{on } \partial\Omega_1 \cap \partial\mathbb{R}_+^3 \cap \overline{O_\lambda}.$$

By the definition of h_2 , if $A(c_0)$ is large, we have

$$\begin{aligned}
 \partial_3 h_2 - \frac{M}{1+r^2} h_2 &= -2C_6 M_i^{-2} \left(\partial_3 \phi - \frac{M}{2(1+r^2)} \phi \right) \phi \\
 &\leq -2C_6 M_i^{-2} \left(\partial_3 \phi - \frac{M}{2(1+r^2)} \right) \phi \\
 &= -2C_6 M_i^{-2} \left(\frac{1}{4} A^{\frac{1}{4}} (2r^2)^{-\frac{9}{8}} r - \frac{M}{2(1+r^2)} \right) \phi \\
 (2.38) \quad &\leq -\frac{C_6}{8} M_i^{-2} A^{\frac{1}{4}} r^{-\frac{5}{4}} \phi \quad \text{on } \partial\Omega_1 \cap \partial\mathbb{R}_+^3.
 \end{aligned}$$

Here A is fixed. We observe that if $|y| > 16A$, $\phi(y) > \frac{1}{2}$. So by choosing C_6 large enough we have

$$(2.39) \quad (\partial_3 - c_i(M_i^{-2} \cdot + x'_i)\xi_2) h_2 + Q_2 \leq -\frac{C_6}{16} M_i^{-2} A^{\frac{1}{4}} r^{-\frac{5}{4}} \quad \text{on } \partial(\Sigma_\lambda \setminus B_{16A}) \cap \partial\mathbb{R}_+^3 \cap \overline{O_\lambda}.$$

By (2.38) we also know

$$(2.40) \quad (\partial_3 - c_i(M_i^{-2} \cdot + x'_i)\xi_2) h_2 \leq 0 \quad \text{on } \partial(\Sigma_\lambda \cap B_{16A}) \cap \partial\mathbb{R}_+^3 \cap \overline{O_\lambda}.$$

So the most important feature of h_2 is that it does not create new difficulties, even though it does not control all the error terms in the whole Σ_λ .

Now we define h_1 to be

$$(2.41) \quad h_1(y) = \begin{cases} -C_5 M_i^{-2} (r - \lambda) e^{Ny_3} & \text{in } \Sigma_\lambda \cap B_{16A}, \\ 0 & \text{in } \Sigma_\lambda \setminus B_{17A}, \\ \text{smooth connection such that } h_1 \leq 0 & \text{in } \Sigma_\lambda \cap \overline{B_{17A}} \setminus B_{16A}, \end{cases}$$

where C_5 and N are large positive constants to be determined. A is the one in the definition of h_2 , which has been determined. Also the ‘‘smooth connection’’ in the

definition means that $|\nabla h_1(y)|, |\nabla^2 h_1(y)| \leq C_7(C_5, A)M_i^{-2}$ for $16A < |y| < 17A$, $y \in \Sigma_\lambda$.

By the definition of h_1 ,

$$\begin{aligned} \Delta h_1(y) &= -C_5 M_i^{-2} e^{Ny_3} \left(\frac{2}{r} + \frac{2N}{r} y_3 + N^2(r - \lambda) \right) \\ (2.42) \quad &\leq -2C_5 M_i^{-2} r^{-1}, \quad y \in B_{16A} \cap \Sigma_\lambda. \end{aligned}$$

Here we require C_5 to satisfy

$$(2.43) \quad \Delta h_1 + Q_1 \leq 0, \quad y \in B_{16A} \cap \Sigma_\lambda.$$

On the boundary we have

$$\begin{aligned} \partial_3 h_1 - c_i(M_i^{-2} y' + x'_i) \xi_2 h_1 \\ = -C_5 M_i^{-2} (r - \lambda) (N - c_i(M_i^{-2} y' + x'_i) \xi_2) \quad \text{on } \partial(\Sigma_\lambda \cap B_{16A}) \cap \partial \mathbb{R}_+^3 \cap \overline{O_\lambda}. \end{aligned}$$

So by choosing $N(c_0)$ large and C_5 larger if necessary we have

$$(2.44) \quad \partial_3 h_1 - c_i(M_i^{-2} y' + x'_i) \xi_2 h_1 + Q_2 \leq 0 \quad \text{on } \partial(\Sigma_\lambda \cap B_{16A}) \cap \partial \mathbb{R}_+^3 \cap \overline{O_\lambda}.$$

So far C_5 , N , and A are all determined. We finally determine C_6 . One last requirement for C_6 is to control the bad part of h_1 in $(B_{17A} \setminus B_{16A}) \cap \Sigma_\lambda$. By choosing C_6 larger if necessary we see from (2.38) and (2.39) that the errors caused by bending h_1 to 0 can be controlled by h_2 . Therefore the first two equations of (2.35) are established. Here we also use the fact that $h_\lambda \leq 0$ in Σ_λ . By the definitions of h_1 and h_2 , the other two equations of (2.35) are also satisfied. We are left with how to let the moving sphere process start. This part is similar to step two. We leave the details to the interested readers. Proposition 2.1 is established. \square

Proposition 2.1 is the major step in deriving the energy estimate in Corollary 2.1. Based on the previous works of Y. Y. Li [23], Han and Li [22] and Li and Zhang [24], it is standard to derive Corollary 2.1. Therefore we only mention the major steps and the main idea in this argument. First one uses Schoen's selection process to find all large maximums of u in, say, B_{2R}^+ . Around each of these local maximums there is a small neighborhood in which u looks like a standard bubble which has most of its energy in it. The distance between these local maximums is the crucial information to find and this is the place where the Harnack inequality in Proposition 2.1 is used. The essential difference between this locally defined equation and those globally defined equations (such as the ones in [23]) is that one cannot find two bubbles closest to each other. For each bubble, there certainly exists a bubble closest to it, but one certainly cannot assume the first bubble is the closest one to the second bubble. This difficulty, which comes from the local nature of the equation, requires a different approach than those in [23]. The way to overcome this difficulty is to rescale the equation so that after the scaling, the equation is centered at the first bubble and the distance between the first bubble and the second bubble is one. Then the Harnack inequality in Proposition 2.1 applied to the local region implies that the first bubble and the second bubble must have comparable magnitudes. Then it is possible to show that two bubbles cannot tend to the same blow-up point because otherwise a harmonic function with a positive second order term can be found. This second term will lead to a contradiction in the Pohozaev Identity. Note that the important second order term can only be proved to be positive if the adjacent bubbles have comparable magnitudes, which is the key information revealed by the Harnack inequality. Once we have known that

all bubbles are far apart, it is possible to use standard elliptic estimates to show that the behavior of u near each large local maximum is like a harmonic function with fast decay. So (2.5) as well as (2.6) can be obtained. \square

2.2. Harnack inequality in the interior of M . Now we consider $V \setminus N(\Gamma, \epsilon_0)$. Let d be the distance between $V \setminus N(\Gamma, \epsilon_0)$ and ∂M . Over this interior region we don't need to assume M to be locally conformally flat. We shall establish the following inequality:

Proposition 2.2. *Let $p \in V \setminus N(\Gamma, \epsilon_0)$. There exist $\epsilon_2(M, g, \Lambda, d) > 0$ and $C(M, g, \Lambda, d) > 0$ such that*

$$(2.45) \quad \sup_{B(p, \epsilon_2/3)} u \cdot \inf_{B(p, 2\epsilon_2/3)} u \leq C\epsilon_2^{-1}.$$

The proof of Proposition 2.2 follows from the argument in [25]. Only a small modification is needed to adjust to the current situation. The outline is as follows. Suppose (2.45) does not hold. Then we can select x_i as a sequence of local maximums of u_i . Then the equation can be written in conformal normal coordinates centered at x_i . After rescaling, u_i becomes v_i , a sequence that converges in $C_{loc}^2(\mathbb{R}^3)$ to a standard bubble whose maximum is 1. Then consider the Kelvin transformation of $v_i(y)$: $v_i^\lambda(y)$. By comparing the equation for v_i and v_i^λ we see that the only new term in this context is the term:

$$(K_i(M_i^{-2}y^\lambda + x_i) - K_i(M_i^{-2}y + x_i))(v_i^\lambda)^5 \quad \text{in } \Sigma_\lambda,$$

where Σ_λ becomes a symmetric domain defined appropriately. The above term is of the harmless order $O(M_i^{-2}|y|^{-4})$. By using the same test function we used in [25], a contradiction can be obtained correspondingly. We leave the details to the interested readers. Proposition 2.2 is proved. \square

Proposition 2.1 and Proposition 2.2 lead to Theorem 1.1 by the arguments in [24] and [26].

3. APPENDIX

In this section we provide some details of the Schoen selection process for the convenience of the readers. We only provide details for case one and case two. The details for case three are similar.

3.1. A calculus lemma. First we use a calculus lemma to simplify some computations. This calculus lemma was first used in [24].

Lemma 3.1. *Let $u \in C^0(\overline{B_1} \cap \{t > -T\})$, $T \geq 0$. Then for every $a > 0$, there exists $x \in B_1 \cap \{t \geq -T\}$ such that*

$$u(x) \geq \frac{1}{2^a} \max_{\overline{B_\sigma^T}(x)} u \quad \text{and} \quad \sigma^a u(x) \geq \frac{1}{2^a} u(0),$$

where $B_\sigma^T(x) = B(x, \sigma) \cap \{t \geq -T\}$, $\sigma = (1 - |x|)/2$.

Proof of Lemma 3.1. First we remark that if $T \geq 1$, the selection is over the whole B_1 . To prove the lemma, consider

$$v(y) = (1 - |y|)^a u(y).$$

Let $x \in \overline{B_1^T}$ be a maximum point of v and let $\sigma = (1 - |x|)/2$. By comparing $v(x)$ with $v(0)$ and $v(z)$ for all z in $\overline{B_\sigma^T}(x)$, we see that x and σ have the desired properties. \square

3.2. The selection process for case one in the proof of Proposition 2.1.

In this subsection we explain why x_i can be considered as a local maximum of u_i , assuming $u_i^2(x_i)x_{i3} \rightarrow \infty$. We shall apply Lemma 3.1 for $T > 1$; i.e., the selection is over the whole B_1 . Let $r_i = |x_{i3}|/2$, so $u_i(x_i)r_i^{\frac{1}{2}} \rightarrow \infty$. Apply Lemma 3.1 to the function $u_i(x_i + r_i \cdot)$ over B_1 . Then the conclusion can be translated as follows. There is $a_i \in B(x_i, r_i)$ such that

$$u_i(a_i) \geq \max_{\bar{B}(a_i, \sigma_i)} 2^{-\frac{1}{2}} u_i(x),$$

where $\sigma_i := \frac{1}{2}(r_i - |a_i - x_i|)$ and

$$\sigma_i^{\frac{1}{2}} u_i(a_i) \geq 2^{-\frac{1}{2}} u_i(x_i) r_i^{\frac{1}{2}} \rightarrow \infty.$$

Let

$$\tilde{v}_i(y) = u_i(a_i)^{-1} u_i(u_i(a_i)^{-2} y + a_i), \quad |y| \leq u_i(a_i)^2 \sigma_i.$$

Then

$$\tilde{v}_i(y) \leq 2^{\frac{1}{2}} \quad \text{for } |y| \leq u_i(a_i)^2 \sigma_i \rightarrow \infty.$$

Clearly \tilde{v}_i satisfies

$$\Delta \tilde{v}_i(y) + K_i(u_i(a_i)^{-2} y + a_i) \tilde{v}_i(y)^5 = 0, \quad |y| \leq u_i(a_i)^2 \sigma_i.$$

So by standard elliptic theory we know there is a subsequence of $\tilde{v}_i(y)$ (still denoted by $\tilde{v}_i(y)$) that converges uniformly to $\tilde{U}_0(y)$ on all compact subsets of \mathbb{R}^3 . $\tilde{U}_0(y)$ satisfies

$$(3.1) \quad \Delta \tilde{U}_0(y) + \tilde{K}_0 \tilde{U}_0(y)^5 = 0, \quad y \in \mathbb{R}^3,$$

where $\tilde{K}_0 = \lim_{i \rightarrow \infty} K_i(a_i)$. With no loss of generality we assume $\tilde{K}_0 = 3$. So by the well-known classification theorem of Caffarelli-Gidas-Spruck [8],

$$\tilde{v}_i(y) = \left(\frac{\mu}{1 + \mu^2 |y - z|^2} \right)^{\frac{1}{2}}$$

for some $z \in \mathbb{R}^3$ and $\mu \geq 1$. So $\tilde{U}_0(y)$ has an absolute maximum at z . Consequently $\tilde{v}_i(y)$ has a local maximum at z_i close to z when i is large.

Let $\bar{x}_i = u_i(a_i)^{-2} z_i + a_i$. Then the $\{\bar{x}_i\}$ are local maximum points of u_i . Also it is easy to verify that

$$u_i(\bar{x}_i) \geq \frac{\mu}{2} u_i(a_i) \geq \frac{\mu}{2} u(x_i)$$

and $\bar{x}_i \in B(a_i, \sigma_i) \subset B(x_i, r_i)$, which means $\bar{x}_{i3} \geq \frac{1}{2} x_{i3}$. Consequently,

$$u_i^2(\bar{x}_i) \bar{x}_{i3} \rightarrow \infty.$$

So we can consider x_i as \bar{x}_i at the beginning. The $\bar{v}_i(y)$ defined in (2.8) converge in $C_{loc}^2(\mathbb{R}^3)$ to

$$U(y) = (1 + |y|^2)^{-\frac{1}{2}}.$$

3.3. Selection for case two in the proof of Proposition 2.1. In this subsection we show that x_i can be considered as a local maximum of u_i under the assumption $u_i^2(x_i)x_{i3} = O(1)$.

By applying Lemma 3.1 to $u_i(x_i + \frac{1}{2}\cdot)$ with $a = 1/2$ and $T = -x_{i3}$ we can find $\bar{x}_i \in \overline{B(x_i, 1/2)} \cap \{x_3 > -x_{i3}\}$ such that

$$u_i(\bar{x}_i) \geq 2^{-\frac{1}{2}} \max u_i \quad \text{over} \quad \overline{B(x_i, \sigma_i) \cap \{x_3 > -x_{i3}\}}$$

and $\sigma_i^{\frac{1}{2}} u_i(\bar{x}_i) \geq 2^{-\frac{1}{2}} u_i(x_i)$, where $\sigma_i = \frac{1}{2}(\frac{1}{2} - |x_i - \bar{x}_i|)$. By these two inequalities we have

$$u_i(\bar{x}_i) \geq u_i(x_i) \rightarrow \infty, \quad u_i(\bar{x}_i)\sigma_i^{\frac{1}{2}} \rightarrow \infty,$$

and $u_i(\bar{x}_i) \inf_{\partial B_2^+} u_i \geq i$. Now we can assume that $\bar{T}_i := u_i^2(\bar{x}_i)\bar{x}_{i3}$ is bounded because otherwise by replacing x_i by \bar{x}_i we go back to case one, which has been discussed. Let

$$\hat{v}_i(y) = u_i(\bar{x}_i)^{-1} u_i(u_i(\bar{x}_i)^{-2}y + \bar{x}_i).$$

Then \hat{v}_i satisfies

$$\begin{cases} \Delta \hat{v}_i(y) + K_i(u_i(\bar{x}_i)^{-2}y + \bar{x}_i)\hat{v}_i(y)^5 = 0, & y \in \bar{\Omega}_i, \\ \partial_3 \hat{v}_i(y) = c_i(u_i(\bar{x}_i)^{-2}y' + \bar{x}'_i)\hat{v}_i(y)^3 & \text{on } \{y_3 = -\bar{T}_i\} \cap \partial \bar{\Omega}_i, \end{cases}$$

where $\bar{\Omega}_i = \{y \mid u_i(\bar{x}_i)^{-2}y + \bar{x}_i \in B_3^+\}$. It is clear that the lower part of $\partial \bar{\Omega}_i$ is a subset of $\{y_3 = -\bar{T}_i\}$. Moreover by the facts $u_i(\bar{x}_i)\bar{x}_{i3}^{\frac{1}{2}} \leq C$ and $u_i(\bar{x}_i)\sigma_i^{\frac{1}{2}} \rightarrow \infty$ we know that \hat{v}_i is uniformly bounded on all compact subsets of $\{y_3 \geq -\lim_{i \rightarrow \infty} \bar{T}_i\}$. With no loss of generality we assume that $K_i(\bar{x}_i) \rightarrow 3$. Then by elliptic estimates, $\hat{v}_i(y - \bar{T}_i e_3)$ converges in the C^2 norm on all compact subsets of \mathbb{R}_+^3 to \bar{U} , which solves

$$\begin{cases} \Delta \bar{U} + 3\bar{U}^5 = 0 & \text{on } \mathbb{R}_+^3, \\ \partial_3 \bar{U} = c\bar{U}^3 & \text{for } y_3 = 0, \end{cases}$$

where $c = \lim_{i \rightarrow \infty} c_i(\bar{x}'_i)$. By Li and Zhu's classification result [27], we know that \bar{U} has a unique maximum point z_0 in \mathbb{R}_+^3 . By the C^2 convergence of $\hat{v}_i(\cdot - \bar{T}_i e_3)$ to \bar{U} , we can find $\{y_i\}_{i=1,2,\dots}$ as local maximum points of $\hat{v}_i(\cdot - \bar{T}_i e_3)$ that approach z_0 as $i \rightarrow \infty$. Then by the definition of \hat{v}_i , we know that $u_i(\bar{x}_i)^{-2}(y_i - \bar{T}_i e_3) + \bar{x}_i$ are local maximum points of u_i . So we redefine x_i as $u_i(\bar{x}_i)^{-2}(y_i - \bar{T}_i e_3) + \bar{x}_i$. So the \bar{v}_i defined in (2.8) converge to the function U_1 in (2.10).

REFERENCES

- [1] A. Ambrosetti, Y. Y. Li, A. Malchiodi, On the Yamabe problem and the scalar curvature problems under boundary conditions. *Math. Ann.* 322 (2002), no. 4, 667–699. MR1905107 (2003e:53038)
- [2] H. Araújo, Existence and compactness of minimizers of the Yamabe problem on manifolds with boundary. *Comm. Anal. Geom.* 12 (2004), no. 3, 487–510. MR2128601 (2005m:53052)
- [3] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* 55 (1976), 269–296. MR0431287 (55:4288)
- [4] M. Berti, A. Malchiodi, Non-compactness and multiplicity results for the Yamabe problem on S^n . *J. Funct. Anal.* 180 (2001), no. 1, 210–241. MR1814428 (2002b:53049)
- [5] S. Brendle, A generalization of the Yamabe flow for manifolds with boundary. *Asian J. Math.* 6 (2002), no. 4, 625–644. MR1958085 (2003m:53052)
- [6] S. Brendle, Blow-up phenomena for the Yamabe equation, *J. Amer. Math. Soc.* 24 (2008), no 4, 951–979.
- [7] S. Brendle, F. C. Marques, Blow-up phenomena for the Yamabe PDE II, preprint 2007.

- [8] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear equations with critical Sobolev growth, *Comm. Pure Appl. Math.* 42 (1989), 271–297. MR982351 (90c:35075)
- [9] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$. *Rev. Mat. Iberoamericana* 3 (1987), no. 2, 139–162. MR990856 (90d:35306)
- [10] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz. *Comm. Pure Appl. Math.* 42 (1989), no. 1, 55–78. MR973745 (90b:35246)
- [11] C. C. Chen, C. S. Lin, Estimates of the conformal scalar curvature equation via the method of moving planes, *Comm. Pure Appl. Math.* 50 (1997), 971–1017. MR1466584 (98k:35051)
- [12] C. C. Chen, C. S. Lin, Estimate of the conformal scalar curvature equation via the method of moving planes. II. *J. Differential Geom.* 49 (1998), no. 1, 115–178. MR1642113 (2000h:35045)
- [13] C. C. Chen, C. S. Lin, Prescribing scalar curvature on S^N . I. A priori estimates. *J. Differential Geom.* 57 (2001), no. 1, 67–171. MR1871492 (2002j:53038)
- [14] Z. Djadli, A. Malchiodi, M. O. Ahmedou, Prescribing scalar and boundary mean curvature on the three dimensional sphere, *J. Geom. Anal.* 13 (2003), no. 2, 255–289. MR1967027 (2004d:53032)
- [15] O. Druet, Compactness for Yamabe metrics in low dimensions. *Int. Math. Res. Not.* 2004, no. 23, 1143–1191. MR2041549 (2005b:53056)
- [16] O. Druet, Compactness for Yamabe metrics in low dimensions. *Int. Math. Res. Not.* 2004, no. 23, 1143–1191. MR2041549 (2005b:53056)
- [17] J. Escobar, Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary. *Indiana Univ. Math. J.* 45 (1996), no. 4, 917–943. MR1444473 (98d:53051)
- [18] J. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)* 136 (1992), no. 1, 1–50. MR1173925 (93e:53046)
- [19] J. Escobar, The Yamabe problem on manifolds with boundary. *J. Differential Geom.* 35 (1992), no. 1, 21–84. MR1152225 (93b:53030)
- [20] V. Felli, M. Ould Ahmedou, Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries. *Math. Z.* 244 (2003), no. 1, 175–210. MR1981882 (2004d:53034)
- [21] M. Khuri, F.C. Marques, R. Schoen, A compactness theorem for the Yamabe problem, preprint, 2007.
- [22] Z. Han, Y. Y. Li, The Yamabe problem on manifolds with boundary: Existence and compactness results. *Duke Math. J.* 99 (1999), no. 3, 489–542. MR1712631 (2000j:53045)
- [23] Y. Y. Li, Prescribing scalar curvature on S^n and related problems, Part I, *J. Diff. Equations* 120 (1995), 319–410. MR1347349 (98b:53031)
- [24] Y. Y. Li, L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, *J. Anal. Math.* 90(2003), 27–87. MR2001065 (2004i:35118)
- [25] Y.Y. Li, L. Zhang, A Harnack type inequality for the Yamabe equation in low dimensions, *Calc. Var. Partial Differential Equations* 20 (2004), 133–151. MR2057491 (2005f:53053)
- [26] Y.Y. Li, L. Zhang, Compactness of solutions to the Yamabe problem II, *Calc. Var. Partial Differential Equations*, 24 (2005), no. 2, 185–237. MR2164927 (2006f:53049)
- [27] Y. Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, *Duke Math. J.* 80, No. 2 (1995), 383–417. MR1369398 (96k:35061)
- [28] F. C. Marques, Existence results for the Yamabe problem on manifolds with boundary. *Indiana Univ. Math. J.* 54 (2005), no. 6, 1599–1620. MR2189679 (2006j:53047)
- [29] F. C. Marques, A priori estimates for the Yamabe problem in the non-locally conformally flat case. *J. Differential Geom.* 71 (2005), no. 2, 315–346. MR2197144 (2006i:53046)
- [30] M. Ould Ahmedou, On the prescribed scalar and zero mean curvature on 3-dimensional manifolds with umbilic boundary. *Adv. Nonlinear Stud.* 6 (2006), no. 1, 13–46. MR2196889 (2007b:53074)
- [31] R. Schoen, Courses at Stanford University, 1988, and New York University, 1989.
- [32] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, in “Topics in Calculus of Variations”, Lecture Notes in Mathematics, Vol. 1365 (M. Giaquinta, Ed.), Springer-Verlag, Berlin/New York, 1989. MR994021 (90g:58023)

- [33] K. Taira, The Yamabe problem and nonlinear boundary value problems. *J. Differential Equations* 122 (1995), no. 2, 316–372. MR1355895 (96g:53050)
- [34] S. Taliaferro, L. Zhang, Asymptotic symmetry for conformal scalar curvature equation with singularities, *Calc. Var. Partial Differential Equations* 26 (2006), no. 4, 401–428. MR2235880 (2007d:35105)
- [35] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Scuola Norm. Sup. Cl. Sci. (3)* 22 (1968), 265–274. MR0240748 (39:2093)
- [36] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* 12 (1960), 21–37. MR0125546 (23:A2847)
- [37] L. Zhang, Refined asymptotic estimates for conformal scalar curvature equations via moving sphere method, *J. Funct. Anal.* 192 (2002), 491–516. MR1923411 (2003j:35113)

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