LOG CANONICAL MODELS FOR THE MODULI SPACE OF CURVES: THE FIRST DIVISORIAL CONTRACTION

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Abstract. In this paper, we initiate our investigation of log canonical models for \((\mathcal{M}_g, \alpha \delta)\) as we decrease \(\alpha\) from 1 to 0. We prove that for the first critical value \(\alpha = 9/11\), the log canonical model is isomorphic to the moduli space of pseudostable curves, which have nodes and cusps as singularities. We also show that \(\alpha = 7/10\) is the next critical value, i.e., the log canonical model stays the same in the interval \((7/10, 9/11]\). In the appendix, we develop a theory of log canonical models of stacks that explains how these can be expressed in terms of the coarse moduli space.

1. Introduction

This article is the first of a series of papers investigating the birational geometry of the moduli space \(\mathcal{M}_g\) of stable curves of genus \(g\). The guiding problem is to understand the canonical model of \(\mathcal{M}_g\),

\[
\text{Proj} \left( \bigoplus_{n \geq 0} \Gamma(\mathcal{M}_g, nK_{\mathcal{M}_g}) \right),
\]

when \(\mathcal{M}_g\) is of general type. The moduli space is known to be of general type for \(g \geq 24\) by work of Eisenbud, Harris, and Mumford \cite{HMS2}; recently, Farkas has announced results for \(g = 22\) \cite{Far06}.

It is a fundamental conjecture of birational geometry (recently proven by Birkar, Cascini, Hacon and McKernan \cite{BCHM06}) that canonical rings of varieties of general type are finitely generated. The projective variety associated with this graded ring is called the canonical model. Explicit canonical models of \(\mathcal{M}_g\) are not known for any genus \(g\). To get around this difficulty, we study the intermediate log canonical models

\[
\mathcal{M}_g(\alpha) := \text{Proj} \left( \bigoplus_{n \geq 0} \Gamma(n(K_{\mathcal{M}_g} + \alpha \delta)) \right)
\]

for various values of \(\alpha \in \mathbb{Q} \cap [0, 1]\). Here \(\mathcal{M}_g\) is the moduli stack, \(\delta_i, i = 0, \ldots, \lfloor g/2 \rfloor\), its boundary divisors, and \(K_{\mathcal{M}_g}\) its canonical divisor. Let \(\mathcal{M}_g \to \mathcal{M}_g\) denote the morphism to the coarse moduli space, which has boundary divisors \(\Delta_i\) and canonical class \(K_{\mathcal{M}_g}\).
A general discussion of log canonical models for stacks is offered in Appendix A, but we sketch the basic idea here. Assume that \( g \geq 4 \); the \( g = 2 \) case is discussed in [Has05]. Then the \( \mathbb{Q} \)-divisor on the coarse moduli space

\[
K_{\overline{M}_g} + \alpha(\Delta_0 + \Delta_2 + \ldots + \Delta_{[g/2]}) + \frac{1 + \alpha}{2}\Delta_1
\]

pulls back to \( K_{\overline{M}_g} + \alpha\delta \) on the moduli stack (see Remark 4.9 of [Has05]). We shall often abuse notation by conflating these divisors. The universal property of the coarse moduli space implies that sections of invertible sheaves on \( \overline{M}_g \) are all pullbacks of sections of the corresponding reflexive sheaves on \( M_g \). The log canonical model of \( \overline{M}_g \) with respect to \( K_{\overline{M}_g} + \alpha\delta \) can thus be identified with the log canonical model of \( M_g \) with respect to \( \langle \delta \rangle \) (see Proposition A.13).

For large values of \( \alpha \) these log canonical models are well-understood. Results of Mumford, Cornalba, and Harris imply that \( \overline{M}_g(\alpha) \simeq M_g \) if and only if \( \frac{9}{11} < \alpha \leq 1 \). Indeed, by the uniqueness of log canonical models, it suffices to show the pair \( (\overline{M}_g, \alpha\delta) \) has log canonical singularities and \( K_{\overline{M}_g} + \alpha\delta \) is ample on \( M_g \), for \( \alpha \) in this range. By Theorem 2 of [HMS2], \( M_g \setminus \Delta_1 \) actually has canonical singularities; the pair \( (M_g, \alpha\delta) \) is a smooth stack with a normal-crossings divisor and thus is log canonical. The main result of [CHSS] says that

\[
a\lambda - \delta = \frac{a}{13}(K_{\overline{M}_g} + (2 - \frac{13}{a})\delta)
\]

is ample for \( a > 11 \). Indeed, \( (\overline{M}_g, \alpha\delta) \) is a strict log canonical model in the sense that all divisors with negative discrepancies lie over \( \delta \) (cf. [Has05], §4.3).

In this paper, we describe what happens when \( \alpha = \frac{9}{11} \). The pair \( (\overline{M}_g, \frac{9}{11}\delta) \) remains log terminal, but \( K_{\overline{M}_g} + \frac{9}{11}\delta \) is not ample on \( M_g \). Mumford [Mum77] described an extremal ray \( R \) for this divisor, parametrizing the elliptic tails. Recall that the normalization of \( \delta_1 \) is isomorphic to \( \overline{M}_{1,1} \times \overline{M}_{g-1,1} \) and \( \overline{M}_{1,1} \simeq \mathbb{P}^1 \). Our ray is the class of the fiber under projection to the second factor. Explicitly, for each \((C_2, p)\) of \( \overline{M}_{g-1,1} \) consider the stable curves \( C = C_1 \cup_p C_2 \), where \((C_1, p)\) is a varying curve in \( \overline{M}_{1,1} \simeq \mathbb{P}^1 \), meeting \( C_2 \) at a single node \( p \). These are parametrized by a rational curve \( R(C_2, p) \); the class \( R = [R(C_2, p)] \) is independent of \( C_2 \) and \( p \) and

\[
(K_{\overline{M}_g} + \frac{9}{11}\delta), R = 0.
\]

This is the unique extremal ray for \( K_{\overline{M}_g} + \frac{9}{11}\delta \); it meets every other curve class in \( M_g \) positively (by the analysis of [CHSS] or §6 of [GKM02]).

Applying the basepoint-freeness theorem to \( K_{\overline{M}_g} + \frac{9}{11}\delta \) (see Corollary A.14), we obtain an extremal contraction of \( R \),

\[
Y : \overline{M}_g \to \overline{M}_g(9/11).
\]

Since \( R . \Delta_1 < 0 \), \( Y \) is birational with exceptional locus contained in \( \Delta_1 \). More precisely, its restriction to the generic point of \( \Delta_1 \) is a projection onto the \( \overline{M}_{g-1,1} \) factor. Thus \( Y \) is a divisorial contraction, and \( \overline{M}_g(9/11) \) remains \( \mathbb{Q} \)-factorial.

Is \( \overline{M}_g(9/11) \) the coarse moduli space of some moduli stack of curves with prescribed properties? Such a moduli functor is implicit in the work of D. Schubert [Sch91]. He constructed the moduli space \( \overline{M}_g^{ps} \) of pseudostable curves applying geometric invariant theory (GIT) to tricanonical curves. Recall that a cusp is a
curve singularity analytically-locally isomorphic to \( y^2 = x^3 \). A complete curve is pseudostable if

1. it is connected, reduced, and has only nodes and cusps as singularities;
2. every subcurve of genus one meets the rest of the curve in at least two points;
3. the canonical sheaf of the curve is ample.

The last condition means that each subcurve of genus zero meets the rest of the curve in at least three points. The corresponding moduli stack is denoted \( \overline{M}_g^{ps} \).

**Theorem 1.1.** There is a morphism of stacks

\[ T : \overline{M}_g \to \overline{M}_g^{ps} \]

which is an isomorphism in the complement of \( \delta_1 \). For a stable curve \( C \in \delta_1 \), \( T(C) \) is obtained by replacing each elliptic tail of \( C \) with a cusp. \( \overline{M}_g^{ps} \) and \( \overline{M}_g(9/11) \) are isomorphic compactifications of the moduli space of curves, and the induced morphism on coarse moduli spaces

\[ T : \overline{M}_g \to \overline{M}_g^{ps} \]

coincides with the extremal contraction \( \Upsilon \).

See [3] for a precise formulation of what it means to replace an elliptic tail with a cusp. We can characterize the \( \alpha \) for which \( \overline{M}_g(\alpha) \) is isomorphic to the moduli space of pseudostable curves:

**Theorem 1.2.** For \( 7/10 < \alpha \leq 9/11 \), \( \overline{M}_g(\alpha) \) exists as a projective variety and is isomorphic to \( \overline{M}_g^{ps} \).

We shall describe what happens at the next critical value \( \alpha = 7/10 \) in our forthcoming paper [HH08].

Finally, we survey recent results of a similar flavor. Shepherd-Barron has analyzed the canonical model of the moduli space of principally polarized abelian varieties of dimension \( \geq 12 \) [SB06, Theorem 0.2]. Hacking, Keel, and Tevelev have considered log canonical models for moduli of hyperplane arrangements [KT06, 1.6] (cf. [HKT06]). These can be understood explicitly in special cases, e.g., \( (\overline{M}_{0,n}, \delta) \) is a log canonical model [Has03, 7.1].

Throughout, we work over an algebraically closed field \( k \) of characteristic zero.

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**2. Moduli stack of pseudostable curves**

To lay groundwork for the construction of \( T : \overline{M}_g \to \overline{M}_g^{ps} \) in the subsequent section, we shall sketch Schubert’s construction of the moduli space of pseudostable curves [Sch91] and reformulate the main results using language of stacks. Throughout this section we assume \( g > 2 \). (The special features of the \( g = 2 \) case are addressed in [HL07].)

**Definition 2.1.** An \( n \)-canonical curve

\[ C \subset \mathbb{P}^N, \quad N := \begin{cases} g - 1 & \text{if } n = 1, \\ (2n - 1)(g - 1) - 1 & \text{if } n > 1 \end{cases} \]

is a Deligne-Mumford stable curve imbedded by the sections of the pluricanonical bundle \( \omega_C^{\otimes n} \). (\( \omega_C^{\otimes n} \) is very ample for \( n \geq 3 \) [DM69, Theorem 1.2].)
These form a locally closed subset \( \text{Chow}_n \) of the Chow variety parametrizing curves in \( \mathbb{P}^N \) of degree \( d := 2n(g - 1) \). Schubert investigated the stability of points in \( \text{Chow}_3 \). He proves that a Chow point is GIT stable if and only if the corresponding cycle is a pseudostable curve of genus \( g \). Furthermore, there are no strictly semistable points. Here is the idea of the argument: Using the methods of [Mum77], smooth tricanonical curves have GIT stable Chow points. Let \( C \subset \mathbb{P}^{5g-6} \) be a tricanonically-embedded pseudostable curve. A deformation-theoretic argument shows there exists a flat family of curves in \( \mathbb{P}^{5g-6} \)

\[ \pi : C \rightarrow \Delta = \text{Spec} \; k[[t]] \]

with closed fiber \( C \) and generic fiber smooth and tricanonically embedded. Consider the induced morphism to the GIT quotient

\[ \mu : \Delta \rightarrow \text{Chow}_3^{ss} / \text{SL}_{5g-5} ; \]

of course, \( \mu(0) \) must correspond to certain semistable points in \( \text{Chow}_3 \). After a base change \( \tilde{\Delta} \rightarrow \Delta \) (substitute \( t = t^M \) for some \( M \)), we obtain a lift

\[ \tilde{\mu} : \tilde{\Delta} \rightarrow \text{Chow}_3^{ss} \]

into the semistable locus of the Chow variety. The generic point maps into the stable locus. Let \( D \rightarrow \tilde{\Delta} \) be the family of pseudostable curves associated to \( \tilde{\mu} \). Note that \( D \) and the pull-back \( \tilde{C} : = \tilde{C} \times_{\tilde{\Delta}} \Delta \) agree up to the action of \( \text{SL}_{5g-1} \). To prove that \( C \) is semistable, we show that the closed fibers of \( \tilde{C} \) and \( D \) agree. We require the following uniqueness result, which may be regarded as the valuative criterion for separatedness for the moduli functor of pseudostable curves:

**Proposition 2.2** (cf. Lemma 4.2 of [Sch91]). Let \( D_1, D_2 \rightarrow \Delta \) be flat families of pseudostable curves with smooth generic fibers, isomorphic over \( k((t)) \). Then the closed fibers are isomorphic as well.

Finally, suppose there were strictly semistable points: Then some point of \( \text{Chow}_3^{ss} \) would have a positive-dimensional stabilizer, but pseudostable curves have finite automorphism groups [Sch91, pp. 312]. Implicit in Schubert’s work is the moduli stack \( \mathcal{M}_g^{ss} \) of pseudostable curves:

\[ \mathcal{M}_g^{ss}(S) = \left\{ f : C \rightarrow S \mid \begin{array}{l} (i) \; f \text{ is a proper and flat morphism,} \\
(ii) \; \text{geometric fibres of } f \text{ are pseudostable curves of genus } g. \end{array} \right\} \]

where \( S \) is a scheme of finite type over \( k \). This is a Deligne-Mumford stack, as pseudostable curves have finite automorphism groups when \( g > 2 \). Its coarse moduli space is denoted \( \overline{\mathcal{M}}_g \).

**Theorem 2.3** (cf. [Sch91 Theorem 5.4]). The quotient stack \([\text{Chow}_3^S / \text{PGL}_{5g-5}]\) is isomorphic to \( \mathcal{M}_g^{ss} \) and \( \text{Chow}_3 / \text{SL}_{5g-5} \simeq \mathcal{M}_g^{ss} \).

**Remark 2.4.** It is shown in [Mor08] that most of Schubert’s analysis goes through in the four-canonical case except that curves with elliptic tails as well as the corresponding cuspidal curves (Proposition 3.1) are strictly Chow semistable and are
identified in the quotient space \( \text{Chow}_4/\text{SL}_{g-7} \). Although this is isomorphic to the coarse moduli space of \( \overline{\mathcal{M}}_g^{ss} \), the quotient stack \( \text{Chow}_4^{ss}/\text{PGL}_{g-7} \) is not separated.

When the Hilbert scheme \( \text{Hilb}_4 \) (with asymptotic linearization [HH08]) is used in place of the Chow variety, a four-canonical curve is Hilbert stable if and only if it is pseudostable, and there are no strictly Hilbert semistable points. Hence the quotient stack \( [\text{Hilb}_4]/\text{PGL}_{g-7} \) is isomorphic to \( \overline{\mathcal{M}}_g^{ss} \) and \( \text{Hilb}_4/\text{SL}_{g-7} \) is isomorphic to the coarse moduli space \( \overline{\mathcal{M}}_g^{ss} \).

3. Constructing the natural transformation \( \mathcal{T} \)

Here we prove the first part of Theorem 1.1. Throughout, we assume \( g > 2 \).

3.1. Replacing elliptic tails with cusps.

Proposition 3.1. Let \( C \) be a stable curve of genus \( g > 2 \) with elliptic tails \( E_1, \ldots, E_r \), i.e., connected genus-one subcurves meeting the rest of \( C \) in a single node. Let \( D \) denote the union of the components of \( C \) other than elliptic tails and \( p_i \) the node where \( E_i \) meets \( D \), \( i = 1, \ldots, r \). Then there exists a unique curve \( T(C) \) characterized by the following properties:

1. there is a birational morphism \( \nu : D \to T(C) \), which is an isomorphism away from \( p_1, \ldots, p_r \);
2. \( \nu \) is bijective and maps each \( p_i \) to a cusp \( q_i \in T(C) \), formally isomorphic to \( y^2 = x^3 \).

There is a unique replacement morphism \( \xi_C : C \to T(C) \) with \( \xi_C|_D = \nu \) and \( \xi_C|_{E_i} \) constant. Note that \( T(C) \) has arithmetic genus \( g \).

Proof. To determine \( T(C) \), it suffices to specify the subrings

\[ O_{T(C), q_i} \subset O_{D, p_i} . \]

Let \( \mathfrak{m}_{D, p_i} \subset O_{D, p_i} \) be the maximal ideal and define \( O_{T(C), q_i} \) to be the algebra generated by the constants and \( \mathfrak{m}_{T(C), q_i}^2 \). A local computation shows that \( T(C) \) has a cusp at \( q_i \). Indeed, \( \mathfrak{m}_{T(C), q_i} \) is generated by two elements \( \hat{x} \) and \( \hat{y} \), vanishing at \( p_i \) to orders two and three respectively, whence \( \hat{y}^2 = c \hat{x}^3 \) for some unit \( c \in O_{D, p_i} \). Conversely, any germ of a cuspidal curve normalized by \( (D, p_i) \) is obtained in this way. The morphism \( \nu \) is the normalization of the cusps \( q_1, \ldots, q_r \). \( \square \)

A morphism of stacks

\[ T : \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g^{ss} \]

assigns to each stable curve \( f : C \to S \) (over a scheme \( S \) of finite type) a pseudostable curve \( T(f) : T(C) \to S \). This recipe must be compatible with base extension and isomorphisms. (This is just the definition of a 1-morphism between two categories fibered in groupoids over the category of schemes; cf. [LMB00] chapter 2.)

Our main tool is a replacement morphism \( \xi_C : C \to T(C) \) defined over \( S \), with the following properties:

(a) over the open subset \( S_0 \subset S \) mapping to the complement to \( \delta_1 \), \( \xi_C \) is an isomorphism;
(b) for \( s \in S \) mapped to \( \delta_1 \), \( \xi_C : C_s \to \mathcal{T}(C)_s \) replaces each elliptic tail in \( C_s \) with a cusp;
(c) \( \xi_C \) is compatible with base extension and isomorphisms as indicated above.

We shall specify \( \mathcal{T} \) and \( \xi \) using a suitable atlas of \( \overline{M}_g \) in the smooth or flat topology. Let \( \pi : Z \to U \) be a stable curve over a scheme so that the classifying map \( \rho_Z : U \to \overline{M}_g \) is faithfully flat. For example, we could take \( U \) to be \( \text{Chow}^*_1 \) (or the corresponding Hilbert scheme) and \( \pi : Z \to U \) the universal family. Let \( R \to U \) be the corresponding presentation of \( \overline{M}_g \), where \( R \) encodes the isomorphisms among the fibers of \( \pi \). To construct \( \mathcal{T} \) and \( \xi \) over \( \overline{M}_g \), it suffices to construct \( \mathcal{T}(\pi) : T(Z) \to U \) and \( \xi_Z \) compatibly with the isomorphism relation \( R \). Indeed, given an arbitrary stable curve \( f : C \to S \) with classifying map \( \rho_f : S \to \overline{M}_g \), we can pull back to the fiber product

\[
S' := S \times_{\overline{M}_g} U \xrightarrow{pr_U} U \xrightarrow{\rho_f} \overline{M}_g
\]

Consider the resulting stable curves

\[ pr^*_S f : pr^*_S C \to S', \quad pr^*_U \pi : pr^*_U Z \to S', \]

with classifying maps \( \rho_f \circ pr_S \) and \( \rho_\pi \circ pr_U \). The commutativity of the fiber product diagram implies that our two stable curves are isomorphic over \( S' \). By basechange, we have

\[ pr^*_U T(\pi) : pr^*_U T(Z) \to S', \quad pr^*_U \xi_Z : pr^*_U Z \to pr^*_U T(Z), \]

and thus the corresponding constructions for \( pr^*_S f : pr^*_S C \to S' \). Now \( pr_S \) is the basechange of \( \rho_\pi \) and thus is faithfully flat. Since \( \mathcal{T} \) is compatible with isomorphisms, the desired

\[ \mathcal{T}(f) : T(C) \to S, \quad \xi_C : C \to T(C) \]

exist by descent.

3.2. **Sketch construction of the morphism.** Assume \( \pi : Z \to U \) is a stable curve over a smooth base with a faithfully flat classifying map \( \rho_\pi : U \to \overline{M}_g \). Let \( W = \rho_\pi^* \delta_1 \). Let \( \mu_\pi : Z \to \overline{M}_{g,1} \) denote the classifying morphism to the universal family, \( \delta_{1,\{1\}} \subset \overline{M}_{g,1} \) the boundary divisor corresponding to the elliptic tails, and \( E = \mu_\pi^* \delta_{1,\{1\}} \), which is also a Cartier divisor on \( Z \).

**Example 3.2.** It may seem counterintuitive that the divisor of elliptic tails should always be Cartier. For example, consider a stable curve

\[ \pi : Z \to \Delta = \text{Spec} k[[t]] \]

with smooth generic fiber and special fiber

\[ \pi^{-1}(0) = C_1 \cup \cup C_2, \]

where \( C_1 \) is smooth of genus one, \( C_2 \) is smooth of genus \( g - 1 \), and \( \{ p \} = C_1 \cap C_2 \) is a single node. If \( Z \) has a local analytic equation \( xy = t^m \) at \( p \), then \( C_1 \) is *not* Cartier at \( p \) unless \( m = 1 \). However, note that \( E \equiv mC_1 \) is a Cartier divisor.
Fix
\[ L = \omega_\pi(E), \]
where \( \omega_\pi \) is the dualizing sheaf for the morphism \( \pi \). Away from \( W \) we have \( L = \omega_\pi \), and for each \( n > 0 \), \( \pi_\ast L^n \) is locally free of rank
\[ k_n := \begin{cases} g & \text{if } n = 1, \\ (2n - 1)(g - 1) & \text{if } n \geq 2, \end{cases} \]
over \( U \setminus W \). Furthermore, the higher-direct image
\[ R^1\pi_\ast L^n \simeq \begin{cases} O_U & \text{if } n = 1, \\ 0 & \text{if } n \geq 2, \end{cases} \]
over \( U \setminus W \). In particular, \( \pi_\ast L^n|_u \simeq \Gamma(Z_u, \omega_{Z_u}^n) \) for each \( u \in U \setminus W \).

The family \( \{ \pi_\ast L^n|_u \}_{u \in U} \) of linear systems over \( U \) will induce the \( U \)-morphism \( Z \to T(Z) \). For the remainder of this section, we focus our attention on the behavior of these linear systems near a point \( u_0 \in W \). In [3.3] we shall prove the local freeness of \( \pi_\ast L^n \). The proof depends on identifying the limits of linear series \( \pi_\ast L^n|_u \), which can be obtained by using [3.4]. We do this in [3.4 and also show that each linear system \( \pi_\ast L^n|_u \) replaces the elliptic tails of \( Z_u \) with cusps.

### 3.3. An explicit complex computing the cohomology of \( L \)
Suppose that \( Z_{u_0} := C \) has \( r \) elliptic tails \( E_1, \ldots, E_r \), and let \( D \) denote the union of the remaining components. In analyzing whether \( \pi_\ast L^n \) is locally free, we may liberally apply faithfully flat base-extensions \( U' \to U \). (Cohomology commutes with flat basechange, and being locally free is a local property in the faithfully flat topology.) For simplicity, we will not use new notation for these base extensions.

After such a base extension, we may assume there exist sections \( s_1, \ldots, s_r : U \to Z \) so that \( s_i(u_0) \in E_i \) as a smooth point of \( Z_{u_0} \). Over this neighborhood, we consider the exact sequence
\[ 0 \to L^n \to L^n(s_1 + \ldots + s_r) \to L^n(s_1 + \ldots + s_r)|_{s_1,\ldots,s_r} \to 0. \]
The last term is supported in \( s_1(u_0), \ldots, s_r(u_0) \), a finite set, and thus has vanishing higher cohomology. As for the second term,
\[ H^1(Z_{u_0}, L^n(s_1 + \ldots + s_r)|_{s_1,\ldots,s_r}) \]
\[ = H^1(D, L^n|_D) \oplus \left( \bigoplus_{i=1}^r H^1(E_i, O_{E_i}(s_i(u_0))) \right) = 0. \]
The resulting long exact sequence takes the form
\[ 0 \to \pi_\ast L^n \to F^0 \xrightarrow{\delta} F^1 \to R^1\pi_\ast L^n \to 0 \]
with \( F^0 := \pi_\ast(L^n(s_1 + \ldots + s_r)) \) and \( F^1 := \pi_\ast(L^n(s_1 + \ldots + s_r)|_{s_1,\ldots,s_r}) \) locally free of rank \( r_0 \) and \( r_1 \), respectively. The cokernel sheaf \( Q := \text{Coker}(\pi_\ast L^n \to F^0) \) is a subsheaf of a locally-free sheaf, hence is locally-free of rank \( r_0 - k_0 \) away from a subset \( Y \subset U \) of codimension \( \geq 2 \). Note that \( Y \) is contained in the locus where \( R^1\pi_\ast L^n \) fails to be locally free, and thus is a subset of \( W = \rho_\ast \delta_1 \).
3.4. Some results from limit linear series. Let $B$ be spectrum of a DVR over $k$, with special point $0$ and closed point $b$. Consider a morphism $\beta : (B, 0) \to (U, u_0)$ with $\beta(0) = u_0 \in W$ and $\beta(b) \notin W$. We have the fiber square

$$
\begin{array}{ccc}
Z_B & \xrightarrow{\beta'} & Z \\
\downarrow{\pi_B} & & \downarrow{\pi} \\
B & \xrightarrow{\beta} & U
\end{array}
$$

and we write $L_B = \beta'^* L$. The direct image $\pi_B^* L_B^n$ is locally free of rank $k_n$—it is a subsheaf of $\beta^* F_0$.

**Proposition 3.3** ([CF91, 4.3.iii]). For $n \geq 1$, $V_n := \pi_B^* L_B^n|_0$ is naturally identified with

$$
\Gamma(D, \omega_D^n((2n - 2) \sum_{j=1}^r p_j)) + \langle \sigma_1^n, \ldots, \sigma_r^n \rangle
$$

as a subspace of $H^0(C, L^n|_C)$, where $\sigma_j$ are sections of $V_1$ such that $\sigma_j(p_i) \neq 0$ and $\sigma_j$ vanishes to order two at $p_i$ for $i \neq j$.

**Corollary 3.4.** For $n \geq 2$, the linear series $V_n$ defines a morphism

$$
C \to \mathbb{P}^{k_n-1}
$$

with image $T(C)$. The induced $C \to T(C)$ is the morphism introduced in Proposition 3.1.

**Proof.** The linear series $V_n$ contains

$$
\Gamma(D, \omega_D^n((2n - 2) \sum_{j=1}^r p_j)).
$$

For any pointed stable curve $(D, p_1, \ldots, p_r)$, $(\omega_D(p_1 + \ldots + p_r))^2$ is very ample unless $g(D) = 2$ and $r = 0$, which cannot occur. Thus the sections of $V_n$ induce an imbedding of $D$ away from $p_1, \ldots, p_r$. Proposition 3.3 implies that the vanishing sequence of $V_n$ at each $p_j$ is $(0, 2, 3, \ldots)$, and $p_j$ is thus mapped to a cusp. Also, the images of $p_j$ are distinct, as they are separated by $\sigma_j^n$. Finally, sections in $V_n$ are constant along the $E_i$, but not identically zero, so each $E_i$ is collapsed to the corresponding cusp. \hfill \Box

3.5. Local freeness of $\pi_\ast L^n$.

**Proposition 3.5.** For each integer $n \geq 1$, $\pi_\ast L^n$ is locally free of rank $k_n$.

**Proof.** Given a locally-free sheaf $F$ on $U$ of rank $r$, let $\text{Grass}(m, F)$ denote the scheme representing the functor

$$
\text{Grass}(m, F) : \text{U-schemes} \to \text{Sets}
$$

that associates to a $U$-scheme $\beta : B \to U$ the set of rank-$m$ subbundles of $\beta^* F$, or equivalently, rank-$(r - m)$ locally-free quotients of $\beta^* F$. Our map $\varphi : F^0 \to F^1$ induces a morphism

$$
\tau : U \setminus Y \to \text{Grass}(r_0 - k_n, (F^0)^*) \times \text{Grass}(r_0 - k_n, F^1)
$$
and a rational map over $Y$. Via the Plücker and Segre embeddings

$$ Grass(r_0 - k_n, (F^0)^*) \times Grass(r_0 - k_n, F^1) \hookrightarrow \mathbb{P}(\bigwedge^{r_0-k_n} F^0) \times \mathbb{P}(\bigwedge^{r_0-k_n} F^1) $$

we consider $\tau$ as a rational map $U \dashrightarrow \mathbb{P}(\mathcal{H}om(\bigwedge^{r_0-k_n} F^0, \bigwedge^{r_0-k_n} F^1))$ given by the section $\bigwedge^{r_0-k_n} \varphi$. The indeterminacy locus of $\tau$ is precisely the zero locus of $\bigwedge^{r_0-k_n} \varphi$ which is defined by the ideal generated by $(r_0 - k_n) \times (r_0 - k_n)$ minors of $\varphi$. This is the $(r_1 - r_0 + k_n)\text{th}$ Fitting ideal $I$ of $R^1 \pi_* L^n$. Blowing up along $I$ to resolve indeterminacy, we get

$$ \tilde{U} := Bl_I(U) $$

Let $\tilde{\tau}$ denote the lift of $\tau$ to $\tilde{U}$. We claim that $\sigma$ is an isomorphism so that $\tilde{\tau}$ is defined on $U$ and extends $\tau$. This implies that there exists a subbundle $K$ of $F^0$ over $U$ such that $K|_{U \setminus Y} \simeq \pi_* L^n|_{U \setminus Y}$. For a general discussion of flattening and blowing up, we refer the reader to [Ray72, chapter 4].

Since $\sigma$ is proper birational and $\tilde{U}$ is normal, it suffices to show that $\sigma$ is quasi-finite. Suppose that it is not quasi-finite and there exist two distinct closed points $x_1, x_2 \in \tilde{U}$ such that $\tau$ is not regular at $u := \sigma(x_1) = \sigma(x_2)$. Choose curves $\tilde{\beta}_i : B = \text{Spec } R \to \tilde{U}$ with $\tilde{\beta}_i(0) = x_1$ and $\tilde{\beta}_i(r) \notin \delta_i, i = 1, 2$. Here $R = k[\{t\}]$, $0$ denotes the closed point and $r$, the generic point. Let $\beta_i = \sigma \circ \tilde{\beta}_i$. Due to the functoriality of $Grass(k_n, F^0)$, $\tilde{\tau} \circ \tilde{\beta}_i$ corresponds to a locally-free quotient $0 \to K_i \to \beta_i^* F^0 \to Q_i \to 0$ such that $\text{rank}(K_i) = k_n$. Consider the fibre square:

$$ \begin{array}{ccc} Z_B & \xrightarrow{\mu_i} & Z \\ \pi_i \downarrow & & \downarrow \pi \\ B & \xrightarrow{\beta_i} & U \end{array} $$

We have the exact sequence

$$ 0 \longrightarrow \pi_* \mu_*^0 L^n \longrightarrow \beta_i^* F^0 \xrightarrow{\beta_i^* \varphi} \beta_i^* F^1 \longrightarrow R^1(\pi_1)_* \mu_*^1 L^n \longrightarrow 0. $$

Note that the quotient sheaf $\beta_i^* F^0 / \pi_* \mu_*^0 L^n$ is locally free, and $\pi_* \mu_*^0 L^n$ and $K_i$ are subbundles of $\beta_i^* F^0$ that agree on $B \setminus \{0\}$. It follows that they are isomorphic on $B$. By Proposition [3.3] $[(\pi_1)_* \mu_*^1 L^n]|_0$ and $[(\pi_2)_* \mu_*^1 L^n]|_0$ are identified in $H^0(Z|_{\beta_1(0)}, L^n|_{Z|_{\beta_1(0)}})$. This implies that $\tilde{\tau} \circ \tilde{\beta}_1(0) = \tilde{\tau} \circ \tilde{\beta}_2(0)$ and that there exists a subbundle $K$ of $F^0$ over $U$ such that $K|_{U \setminus Y} \simeq \pi_* L^n|_{U \setminus Y}$. Let $j$ denote the natural inclusion $U \setminus Y \hookrightarrow U$. Since $\pi_* L^n$ is the kernel of a homomorphism of locally-free sheaves, it is reflexive, and $j_* (\pi_* L^n|_{U \setminus Y}) \simeq \pi_* L^n$ since $\text{codim}(Y) \geq 2$. Therefore, we have

$$ K \simeq j_* (K|_{U \setminus Y}) = j_* (\pi_* L^n|_{U \setminus Y}) \simeq \pi_* L^n, $$

and hence $\pi_* L^n$ is a vector subbundle of $F^0$. \qed
3.6. Analysis of the morphisms.

**Proposition 3.6.** For $n \geq 2$, the sections of $L^n$ relative to $\pi$ induce a morphism 
$Z \to \mathbb{P}(\pi_*L^n)$ over $U$, which factors 
$$Z \xrightarrow{\xi} T(Z) \to \mathbb{P}(\pi_*L^n),$$
where the second arrow is a closed embedding.

**Proof.** By Proposition 3.5, $\pi_*L^n$ is locally free. For $u \in U \setminus W$, $\pi_*L^n|_u = \Gamma(\omega^n_u)$ and $\omega^n_u$ is very ample on stable curves of genus $g > 2$. For $u \in W$, Corollary 3.3 implies $\pi_*L^n|_u = V_n$ induces the morphism replacing elliptic tails with cusps.

We therefore have a morphism $Z \to \mathbb{P}(\pi_*L^n)$ whose Stein factorization is $T(Z)$. Due to the functoriality of the relative dualizing sheaf and the line bundle $\delta$, this construction is compatible with isomorphism relation and commutes with base extension. \qed

4. $\overline{M}_g^{ps}$ as a log canonical model for $\overline{M}_g$

In this section, we identify $\overline{M}_g^{ps}$ and $\overline{M}_g(9/11)$, thus completing our proof of Theorem 1.1. We also prove Theorem 1.2: A log canonical model for the pair $(\overline{M}_g, K_{\overline{M}_g} + \alpha \delta)$ exists as a projective variety for $7/10 < \alpha \leq 9/11$.

We want to relate the contraction of the extremal ray $R$ parametrizing elliptic tails
$$\Upsilon: \overline{M}_g \to \overline{M}_g(9/11)$$
to the natural birational morphism
$$T: \overline{M}_g \to \overline{M}_g^{ps}.$$

Since $\overline{M}_g^{ps}$ is a GIT quotient, it is a normal projective variety. Our pointwise description of the natural transformation $T: \overline{M}_g \to \overline{M}_g^{ps}$ in Proposition 3.1 implies that

- $T$ is an isomorphism over $\overline{M}_g \setminus \Delta_1$;
- $T$ takes $R(C_2, p)$ to the cuspidal curve $T(C_1 \cup p, C_2)$, i.e., the curve normalized by $C_2$ with conductor in $p$;
- $T(C) = T(C')$ if and only if $C$ and $C'$ have the same number of elliptic tails and

$$(D, p_1, \ldots, p_r) \simeq (D', p'_1, \ldots, p'_r),$$

in the notation of Proposition 3.1.

In particular, the only curves contracted by $T$ are in the class $R$, i.e., $T$ is also an extremal contraction of $R$. Uniqueness of extremal contractions [KM98, 1.26] implies $T = \Upsilon$ and $\overline{M}_g^{ps} \simeq \overline{M}_g(9/11)$, and the proof of Theorem 1.1 is complete.

We turn to the proof of Theorem 1.2. Let $\delta^{ps}$ denote the boundary divisor of $\overline{M}_g^{ps}$; it is the image of $\delta$ under $T$. The proper transform of $\delta^{ps}$ in $\overline{M}_g$ is $\delta - \delta_1$.

**Lemma 4.1** (Log discrepancy formula).

$$K_{\overline{M}_g^{ps}} + \alpha \delta = T^*(K_{\overline{M}_g} + \alpha \delta^{ps}) + (9 - 11\alpha)\delta_1.$$
Proof. We determine the value of the discrepancy $c$ in

$$K_{\overline{M}_g} + \alpha \delta = T^*(K_{\overline{M}_g} + \alpha \delta^{ps}) + c \delta_1$$

by intersecting both sides with the contracted extremal ray $R$. We have

$$(K_{\overline{M}_g} + \alpha \delta).R = (13\lambda - 2\delta + \alpha \delta).R = 13\lambda R - (2 - \alpha)(\delta_0 R + \delta_1 R)$$

$$= 13/12 - (2 - \alpha)(1 - 1/12) = (11\alpha - 9)/12$$

and $c \delta_1 R = -c/12$; the term $T^*(K_{\overline{M}_g} + \alpha \delta^{ps})$ obviously yields zero. \qed

Thus $(\overline{M}_g^{sa}, \alpha \delta^{ps})$ has log canonical singularities for $\alpha \leq 9/11$ provided $(\overline{M}_g, \alpha \delta - (9 - 11\alpha) \delta_1)$ has log canonical singularities. The argument sketched in the Introduction still applies away from $\Delta_1$. However, reducing the coefficient of $\delta_1$ can only increase the discrepancies of divisors lying over $\delta_1$ \cite{GK92}, 2.17.3; this does not affect whether the singularities are log canonical.

To analyze whether $K_{\overline{M}_g} + \alpha \delta^{ps}$ is ample for $7/10 < \alpha < 9/11$, we describe the ample cone of $\overline{M}_g^{sa}$ in terms of the ample cone of $\overline{M}_g$. We have

$$T^* \text{NS}(\overline{M}_g^{sa}) = R^{1+} \subset \text{NS}(\overline{M}_g),$$

the hyperplane spanned by a facet of the nef cone of $\overline{M}_g$. The interior of this facet corresponds to the ample divisors of $\overline{M}_g^{sa}$. It suffices then to check:

**Proposition 4.2.** For $7/10 < \alpha \leq 9/11$, the divisor

$$K_{\overline{M}_g} + \alpha \delta - (9 - 11\alpha) \delta_1$$

lies in the interior of the facet $R^{1+}$ of the nef cone of $\overline{M}_g$.

The following conjectural description of the ample cone of $\overline{M}_{g,n}$ would end the discussion in one stroke:

**Conjecture 4.3** (Fulton’s conjecture \cite{GKM02}). Every one-dimensional facet of the closed cone of effective curves $NE_1(\overline{M}_{g,n})$ is generated by a one-dimensional boundary stratum. Equivalently, any effective curve in $\overline{M}_{g,n}$ is an effective combination of one-dimensional strata.

We recapitulate the description of the one-dimensional strata in \cite{Fab96} and \cite{GKM02}: Let $X_0$ be a 4-pointed stable curve of genus zero with one point moving and the other three fixed.

(a) a family of elliptic tails;
(b) attach a fixed 4-pointed curve of genus $g - 3$ to $X_0$;
(c) attach a fixed pointed curve of genus $g - 2 - i$ to $X_0$, with $g - 2 \geq i \geq 0$;
(d) attach two fixed 2-pointed curves of genus $i$ and $g - 2 - i$, respectively, to $X_0$, with $(g - 2)/2 \geq i \geq 0$;
(e) attach two 1-pointed curves of genus $i$ and $j$, respectively, and a 2-pointed curve of genus $g - 1 - i - j$ to $X_0$, with $i, j \geq 1$ and $i + j \leq g - 1$;
(f) attach four pointed curves of genus $i, j, k$, and $l$, respectively, to $X_0$, with $i + j + k + l = g$ and $i, j, k \geq 1$.

Where we attach a 2-pointed curve of genus zero, we stabilize the resulting ‘rational bridge’ to a node.
The intersection of a divisor $a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ with the one-dimensional strata is as follows:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$a/12 - b_0 + b_1/12$</td>
</tr>
<tr>
<td>(b)</td>
<td>$b_0$</td>
</tr>
<tr>
<td>(c)</td>
<td>$b_i$</td>
</tr>
<tr>
<td>(d)</td>
<td>$2b_0 - b_{i+1}$</td>
</tr>
<tr>
<td>(e)</td>
<td>$b_i + b_{j} - b_{i+j}$</td>
</tr>
<tr>
<td>(f)</td>
<td>$b_i + b_j + b_k + b_{i+j} - b_{i+k} - b_{i+i}$</td>
</tr>
</tbody>
</table>

Here we use the convention that $b_i = b_{g-1}$ for $i > g/2$.

Fortunately, we do not need the full strength of the conjecture. The following special case is sufficient:

**Proposition 4.4 ([GKM02, 6.1])**. Let $D = a\lambda - \sum_{0 \leq i \leq g/2} b_i \delta_i$ be a divisor on $\mathcal{M}_g$ such that either $b_i = 0$ or $b_i \geq b_0$, $1 \leq i \leq \lfloor g/2 \rfloor$. If $D$ has non-negative intersection with all the one-dimensional strata, then $D$ is nef.

In our case, we have $a = 13$, $b_i = \begin{cases} 2 - \alpha, & i \neq 1, \\ 11 - 12\alpha, & i = 1, \end{cases}$ and our divisor satisfies the hypothesis of Proposition 4.4 as long as $\alpha \leq 9/11$. We consider the following when each class of one-dimensional stratum intersects our divisor positively:

• For (b), (c) and (f), we have positivity provided $\alpha \leq 11/12$.
• For (d), we have positivity provided $2(2-\alpha) - (11 - 12\alpha) > 0$, i.e., $\alpha > 7/10$; the case $i = 0$ shows this is sharp.
• For (e), we have positivity provided $\alpha > 7/10$; the case $i + j = g - 1, i \neq 1, j \neq 1$ shows this is sharp.

Proposition 4.4 therefore follows from Proposition 4.4. This completes the proof of Theorem 1.2.

**Remark 4.5.** $T^*(K_{\mathcal{M}_g} + 7/10 \delta^{ps})$ contracts $\Delta_1$ and the following loci:

• the stratum (d) with $i = 0$ induces the contraction of $T_0 = \{C_1 \cup_p C_2 \mid g(C_1) = 1, g(C_2) = g - 2\}$, the locus of two-pointed elliptic tails or elliptic bridges;
• the stratum (e) with $i + j = g - 1$ and $i, j \neq 1$ induces the contraction of $T_i = \{C_1 \cup_p C_2 \cup_q C_3 \mid g(C_1) = i, g(C_2) = 1, g(C_3) = g - 1 - i\}$, $i = 2, \ldots, \lfloor (g-1)/2 \rfloor$. These have codimension two in the moduli space; at $\alpha = 7/10$ we obtain a small contraction.

**Remark 4.6.** For a special value of $\alpha \in (7/10, 9/11)$, we can prove the ampleness of $K_{\mathcal{M}_g} + \alpha \delta^{ps}$ using invariant theory. Recall from [22] that

$\mathcal{M}_g^{ps} \simeq \text{Chow}_3//\text{SL}_{3(g-1)}$,

where the Chow variety has its natural linearization. By Theorem 5.15 of [Mum77], the corresponding line bundle on $\mathcal{M}_g$ equals $3(g-1)(3(12\lambda - \delta) - 4\lambda)$, which is proportional to $32\lambda - 3\delta \sim K_{\mathcal{M}_g} + 25/32 \delta$. This is not ample on $\mathcal{M}_g$ since $\Delta_1$ is a
fixed component. However, the corresponding divisor $K_{\overline{\mathcal{M}}_g}^{ps} + 25/32\delta_{ps}$ on $\overline{\mathcal{M}}_g^{ps}$ is ample.

APPENDIX A. LOG CANONICAL MODELS OF STACKS

Here are conventions for this section: A ‘scheme’ is a separated scheme of finite type over $k$. A ‘stack’ is a Deligne-Mumford stack, separated and of finite type over $k$; a ‘morphism’ is a 1-morphism over the base field. For a general discussion of local properties of representable morphisms, we refer the reader to [LMB00, 3.10] and [Gro65, 2.6, 2.7]; we use these without specific attribution. Roughly, every property that behaves well under étale basechange makes sense for Deligne-Mumford stacks.

**Definition A.1.** A birational morphism of stacks is a morphism $f : X_1 \to X_2$ such that there exist dense open substacks $U_1 \subset X_1$ and $U_2 \subset X_2$ with $U_2 = f^{-1}(U_1)$ and $f : U_1 \to U_2$ an isomorphism. We say that $X_1$ and $X_2$ are properly birational if there exists a stack $Y$ and birational proper morphisms $g_1 : Y \to X_1$ and $g_2 : Y \to X_2$.

It is straightforward to check that this is an equivalence relation. There is a distinguished open substack $U \subset X_1$ which is the largest open set over which $f$ is an isomorphism; its complement is called the exceptional locus $\text{Ex}(f)$. For any closed substack $D \subset X_2$ with $f(U) \cap D$ dense in $D$, the birational transform $f^{-1}_*D$ is defined as the closure of $f^{-1}(D \cap f(U))$ in $X_1$.

**Proposition A.2.** For each reduced stack $\mathcal{X}$, there exists a smooth stack $\tilde{\mathcal{X}}$ and a birational proper morphism $f : \tilde{\mathcal{X}} \to \mathcal{X}$; this is called a resolution of $\mathcal{X}$. Furthermore, $\text{Ex}(f)$ can be taken to be a normal crossings divisor in $\tilde{\mathcal{X}}$. If $Z \to \mathcal{X}$ is a closed substack with ideal sheaf $I_Z \subset \mathcal{O}_X$, then $f$ can be chosen so that $f^*I_Z$ is an invertible sheaf $\mathcal{O}_{\tilde{\mathcal{X}}}(-\tilde{D})$ and $\text{Ex}(f) \cup \tilde{D}$ is simple normal crossings. This is called a log resolution of $Z$ in $X$.

**Proof.** This statement should be compared to Theorem 0.2 of [KM98]. It is an application of the functorial procedure for resolving singularities [BM97]: Such procedures commute with étale base extension [Hau03], pp. 329. In particular, given an étale presentation of $\mathcal{X}$

$$R \to U$$

canonical resolutions $\tilde{R} \to R$ and $\tilde{U} \to U$ form a presentation

$$\tilde{R} \to \tilde{U}$$

for a smooth stack $\tilde{X}$. The induced representable morphism $\tilde{\mathcal{X}} \to \mathcal{X}$ is birational and proper. If $E \subset U$ denotes the closed subscheme corresponding to $Z$, then the canonical embedded resolution of $E$ in $U$ naturally induces a resolution of $Z$ in $\mathcal{X}$. □

**Remark A.3.** Dan Abramovich has pointed out that the 1992 Harvard thesis of Andrew Joel Schwartz, *Functorial Smoothing Of Morphisms In Equal Characteristic 0*, should also suffice for our purposes.

Let $X$ be a normal connected scheme, separated and of finite type over a field, as usual. Let $D = \sum a_jD_j$ be a $\mathbb{Q}$-divisor on $X$, with the $D_j \subset X$ distinct codimension-one reduced closed subschemes; we assume each $a_j \in [0, 1]$. 

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Proposition A.4. The following properties of \((X, D)\) are local in the étale topology on \(X\):

1. \(X\) is normal;
2. \(D_j\) is a codimension-one reduced closed subscheme;
3. \(m(K_X + D)\) is Cartier for some \(m > 0\).

A pair \((X, D)\) satisfying the first three conditions is said to be admissible. Given a coherent \(\mathcal{O}_X\)-module \(F\), the following properties are local in étale topology:

1. \(F\) satisfies Serre’s condition \(S_k\) for some fixed \(k > 0\);
2. \(F\) is locally free.

Proof. These are standard properties of descent: See [Gro65], §6.1, 6.4, 6.5, and 6.8 for the behavior of dimension, property \(S_k\), property \(R_k\), and other standard singularity conditions under faithfully flat basechange. See [Gro65], 2.5.1 for the stability of local-freeness under faithfully flat base extension. □

Proposition A.5. Let \(X\) be a normal integral scheme with field of fractions \(K(X)\), and \(D\) be an integral Weil divisor on \(X\). Let \(\mathcal{O}_X(D)\) denote the \(\mathcal{O}_X\)-module associated to

\[ \{ f \in K(X) : \text{div}(f) + D \geq 0 \}. \]

This is a coherent \(\mathcal{O}_X\)-module of rank one, satisfying Serre’s condition \(S_2\). The formation of \(\mathcal{O}_X(D)\) commutes with étale base-extension.

Proof. Let \(j : U \subset X\) be the locus where \(X\) is smooth, which is compatible with étale basechange. Then \(\mathcal{O}_X(D) = j_* \mathcal{O}_U(D \cap U)\), which is coherent, \(S_2\), and compatible with base extension. □

Let \(\mathcal{X}\) denote a normal connected stack. Let \(D = \sum a_j D_j\) be a \(\mathbb{Q}\)-divisor on \(\mathcal{X}\), where the \(D_j \subset \mathcal{X}\) are distinct codimension-one reduced closed substacks; we assume each \(a_j \in [0, 1]\).

Definition A.6. Suppose \((\mathcal{X}, D)\) is proper and admissible in the sense of Proposition [A.4]. Its log canonical ring is the graded ring

\[ R(\mathcal{X}, D) := \bigoplus_{m \geq 0} \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}(mK_\mathcal{X} + \lfloor mD \rfloor)), \]

where

\[ \lfloor mD \rfloor = \sum_j |ma_j|D_j, \]

and the summands are defined via Proposition [A.5].

We refer the reader to §2.3 of [KM98] for definitions of the various singularities arising in birational geometry.

Proposition A.7 (Proposition 5.20 of [KM98]). Let \((X, D)\) be an admissible pair, as in Proposition [A.4], defined over a field of characteristic zero. The following singularity conditions are local in the étale topology: terminal, canonical, Kawamata log terminal, or log canonical.

Hence the following notions are well-defined:

Definition A.8. \((\mathcal{X}, D)\) is terminal (resp. canonical, Kawamata log terminal, or log canonical) if it admits an étale presentation with the same property. It is strictly log canonical if it is log canonical and \(\mathcal{X} \setminus \bigcup_i D_i\) is canonical.
The reader will recall that these notions are defined in terms of discrepancies [KM98, §2.3, measuring how the canonical divisor changes under birational morphisms:

**Proposition A.9.** The admissible pair \((\mathcal{X}, D)\) is terminal (resp. canonical, Kawamata log terminal, or log canonical) if and only if there exists a log resolution \(f : \tilde{\mathcal{X}} \to \mathcal{X}\) such that

1. \(\text{Ex}(f) = \bigcup E_i\) is a divisor, \(\text{Ex}(f) \cup f^{-1}(D)\) is normal crossings, and \(\sum J f^{-1}D_j\) is smooth;
2. we have a \(\mathbb{Q}\)-linear equivalence \((\ast)\)

\[ K_{\tilde{\mathcal{X}}} + \sum J a_j f^{-1}D_j \equiv f^*(K_{\mathcal{X}} + D) + \sum d(\mathcal{E}_i; \mathcal{X}, D)\mathcal{E}_i, \quad d_i := d(\mathcal{E}_i; \mathcal{X}, D), \]

with each \(a_j < 1\) and \(d_i > 0\) (resp. \(a_j \leq 1\) and \(d_i \geq 0\), \(a_j < 1\) and \(d_i > -1\), or \(a_j \leq 1\) and \(d_i \geq -1\)).

**Proof.** Proposition A.2 guarantees there exists a resolution with the prescribed properties. By definition, if \((\mathcal{X}, D)\) has the prescribed singularities, then the discrepancies \(d_i\) and coefficients \(a_j\) satisfy the listed properties. The converse direction is an application of Corollary 2.32 of [KM98].

**Definition A.10.** The discrepancy \(d(\mathcal{E}_i; \mathcal{X}, D)\) is defined via equation \((\ast)\) for any integral codimension-one substack in the exceptional locus of some birational proper morphism \(\mathcal{Y} \to \mathcal{X}\). By convention, we set

\[ d_j = d(D_j; \mathcal{X}, D) = -a_j \]

and take \(d(D_0; \mathcal{X}, D) = 0\) for any integral codimension-one substack \(D_0 \subset \mathcal{X}\) not in the support of \(D\).

**Definition A.11.** Two admissible pairs \((\mathcal{X}, D)\) and \((\mathcal{X}', D')\) are properly birational if there exists an admissible pair \((\tilde{\mathcal{X}}, \tilde{D})\) and proper birational morphisms \(f : \tilde{\mathcal{X}} \to \mathcal{X}, f' : \tilde{\mathcal{X}} \to \mathcal{X}'\), so that \(K_{\tilde{\mathcal{X}}} + \tilde{D} - f^*(K_{\mathcal{X}} + D)\) (resp. \(K_{\tilde{\mathcal{X}}} + \tilde{D} - f'^*(K_{\mathcal{X}'} + D')\)) is effective and \(f\)-exceptional (resp. \(f'\)-exceptional).

Our main interest is the case where \((\mathcal{X}, D)\) and \((\mathcal{X}', D')\) are log canonical and \((\tilde{\mathcal{X}}, \tilde{D})\) is a log resolution with suitably chosen boundary.

**Proposition A.12.** If the admissible pairs \((\mathcal{X}, D)\) and \((\mathcal{X}', D')\) are properly birational, then there is a natural isomorphism

\[ R(\mathcal{X}, D) \cong R(\mathcal{X}', D') \]

of graded rings.

**Proof.** It suffices to prove the isomorphism

\[ R(\mathcal{X}, D) \cong R(\tilde{\mathcal{X}}, \tilde{D}). \]

We have the discrepancy equation

\[ K_{\mathcal{X}} + \tilde{D} = q f^*(K_{\mathcal{X}} + D) + \sum d_i E_i \]
where the $d_i \geq 0$ and the $\mathcal{E}_i$ are $f$-exceptional. This was formulated on the level of $\mathbb{Q}$-Cartier divisors, but there is a refined interpretation: For each $m \geq 0$, consider the homomorphisms

$$f^* \mathcal{O}_X(mK_X + |mD|) \to \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} + |m(\tilde{D} - \sum_i d_iE_i)|) \subset \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} + |m\tilde{D}|).$$

These deserve some explanations: First, there is no harm passing to an étale neighborhood which is a scheme; for simplicity, we leave the notation unchanged. The divisors $|mf^{-1}D|$ and $|m(\tilde{X} - \sum_i d_iE_i)|$ agree on all the components which have codimension one in $\mathcal{X}$, so we can focus on the $f$-exceptional components. Every section $s \in \mathcal{O}_X(mK_X + |mD|)$ extends to a section with poles of some order on the exceptional locus of $f$. Replacing $s$ by a suitable power $s^N$, we may assume that $mN(K_X + D)$ is integral and Cartier, and our discrepancy equation gives the requisite bound on the pole order. The inclusions (**) induce injections

$$\Gamma(mK_X + |mD|) \hookrightarrow \Gamma(mK_{\tilde{X}} + |m\tilde{D}|).$$

This is surjective: Each element in the target of the map is a section of

$$\Gamma(\mathcal{X} - \bigcup_i f(E_i), mK_X + |mD|),$$

which extends to all of $\mathcal{X}$ because $\mathcal{X}$ is normal. \qed

**Proposition A.13.** Let $(\mathcal{X}, \mathcal{D})$ be a proper log canonical (resp. Kawamata log terminal) pair, and let $\pi : \mathcal{X} \to X$ denote the coarse moduli space of $X$. Then there exists an effective $\mathbb{Q}$-divisor $\tilde{D} = \sum_i \tilde{c}_iD_i, 0 \leq \tilde{c}_i \leq 1$ on $X$, with the $D_i$ distinct and irreducible with the following properties:

1. $(X, \tilde{D})$ is log canonical (resp. Kawamata log terminal);
2. for each positive integer $m$ such that $m(K_X + \tilde{D})$ is integral and Cartier, there is an equivalence of Cartier divisors
   $$m(K_X + \tilde{D}) = \pi^*m(K_X + \tilde{D});$$
3. for every $m \geq 0$ we have
   $$\Gamma(\mathcal{X}, mK_X + |m\tilde{D}|) \xrightarrow{\sim} \Gamma(X, mK_X + |mD|).$$

Together, these yield a natural isomorphism

$$\pi^* : R(\mathcal{X}, \tilde{D}) \xrightarrow{\sim} R(\mathcal{X}, D).$$

**Proof.** We first produce the divisor $\tilde{D}$; to extract its coefficients, we localize along each codimension-one integral closed substack $B \subset \mathcal{X}$. Let $a(B)$ denote the coefficient of $D$ at $B$, which might be zero if $B$ does not appear in $D$. Suppose $\pi$ ramifies to order $e(B)$ at the generic point $\eta(B)$; i.e., the order of the automorphism group at the generic point of $B$ is $e(B)$ times the order of the automorphism group at the generic point of $\mathcal{X}$. If $B \subset \mathcal{X}$ is the reduced image of $B$, then the ramification formula gives

$$\pi^*\omega_{\mathcal{X}, \eta(B)} = \omega_{\mathcal{X}, \eta(B)}(-(e(B) - 1)B), \quad \pi^*B = e(B)B,$$

which yields the equation of $\mathbb{Q}$-Cartier divisors

$$\pi^*(K_X + \sum_B \frac{e(B) - 1 + a(B)}{e(B)}B) \equiv K_X + \mathcal{D}.$$
We therefore set
\[ \hat{D} = \sum_B \frac{e(B) - 1 + a(B)}{e(B)} B; \]
the pair \((X, \hat{D})\) is log canonical (resp. Kawamata log terminal) by [KM98], 5.20.

It then remains to check
\[ \Gamma(X, mK_X + \sum_B [m e(B) - 1 + a(B)] B) \simeq \Gamma(X, mK_X + \sum_B [ma(B)] B). \]

By the ramification formulas, we have
\[ \Gamma(X, mK_X + \sum_B [m e(B) - 1 + a(B)] B) \rightarrow \Gamma(X, mK_X + \sum_B (e(B)[m e(B) - 1 + a(B)] B - m(e - 1) B). \]

This is surjective: For any \(O_X\)-module \(F\) we have \(\Gamma(X, F) = \Gamma(X, \pi^*F)\), as every section of \(\pi^*F\) is pulled back from the coarse moduli space. We must verify that the inclusion
\[ \Gamma(X, mK_X + \sum_B (e(B)[m e(B) - 1 + a(B)] B - m(e - 1) B)) \subset \Gamma(X, mK_X + \sum_B [ma(B)] B) \]

is an isomorphism. We analyze sections locally in an étale neighborhood of \(\eta(B)\), where the stack can be presented as a \(\mu_e\)-quotient, with \(e = e(B)\) (Lemma 2.2.3 of [AV02]; fix \(a = a(B)\). Let \(\omega\) be a local generator of the canonical class and \(F\) the local defining equation for \(B\), so that \(\mu_e\) acts on both \(\omega\) and \(F\) via multiplication by \(\zeta\), a primitive \(e\)th root of unity. The sheaf \(O_X(mK_X + \sum_B [ma(B)] B)\) has a local generator \(\omega^{\sum m/F[ma]}\), on which \(\mu_e\) acts by \(\zeta^{m-[ma]}\). Of course, \(\mu_e\) acts trivially on global sections, so global sections must be multiples of
\[ F[ma] - m - e([ma] - m) e^{-1} \omega^{\sum m/F[ma]} = \omega^{\sum m/F^{m+e([ma] - m)/e}}. \]

Thus global sections of \(O_X(mK_X + \sum_B [ma(B)] B)\) are forced to have zeros along \(B\); precisely, we find
\[ \Gamma\left( mK_X + \sum_B \left( m + e(B) \left[ \frac{ma(B) - m}{e(B)} \right] B \right) \right) = \Gamma(mK_X + \sum_B [ma(B)] B). \]

The final step is the combinatorial statement
\[ e[m \frac{e - 1 + a}{e}] - m(e - 1) = m + e([ma] - m)/e, \]
which is equivalent to
\[ \left[ \frac{ma - m}{e} \right] = \left[ ([ma] - m)/e \right]. \]

Here \(e\) and \(m\) are positive integers and \(a \in [0, 1] \cap \mathbb{Q}\). If this were false, then we could find an integer \(n\) with
\[ ([ma] - m)/e < n \leq (ma - m)/e. \]
and hence 
\[ \lfloor ma \rfloor < en + m \leq ma, \]
violating the definition of the round-down operation. \qed

The Kawamata basepoint-freeness theorem \([\text{KM98}]\) yields

**Corollary A.14 (Basepoint-freeness for stacks).** Let \((X, D)\) be a proper Kawamata log terminal pair. Assume \(K_X + D\) is nef and big, i.e., \(K_X + \tilde{D}\) is nef and big, where \((X, \tilde{D})\) is the pair constructed in Proposition A.13. Consider the quotient stack and coarse moduli space \(\mathcal{Y} := \left[ \{ \text{Spec } R(X, D) \} / \mathbb{G}_m \right] \quad Y := \text{Proj } R(X, \tilde{D}) \),

where the action of \(\mathbb{G}_m\) arises from the grading. Then there is a morphism of stacks 
\(\psi : \mathcal{X} \to \mathcal{Y}\)

inducing on coarse moduli spaces the contraction from \(X\) to the log canonical model of \((X, \tilde{D})\).

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**References**


LOG CANONICAL MODELS FOR $\overline{\mathcal{M}}_g$


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