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ON THE COHOMOLOGY GROUPS OF HOLOMORPHIC BANACH BUNDLES

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Abstract. We consider a compact complex manifold M and introduce the notion of two holomorphic Banach bundles E, F over M being compact perturbations of one another. Given two such bundles we show that if the cohomology groups $H^q(M, E)$ are finite dimensional, then so are the cohomology groups $H^q(M, F)$; as well as a more precise result in the same spirit.

Introduction

A basic result in complex geometry is the finiteness theorem of Cartan and Serre [CS], according to which the cohomology groups of a finite rank holomorphic vector bundle $E \to M$ over a compact base are finite dimensional. (More generally, the same holds for the cohomology groups of coherent sheaves over M.) It is no surprise that holomorphic *Banach* bundles $E \to M$ over a compact base can easily have infinite dimensional cohomology groups. For example, J. Kim observes that if E is the trivial bundle with fiber a Banach space A, then $H^q(M, E) \approx H^q(M, \mathcal{O}) \otimes A$ (tensor product over $\mathbb{C}$); the special case when $H^q(M, \mathcal{O}) = 0$ is due to Leiterer [K, Li2]. In particular, $\dim H^q(M, E) = \infty$ if $\dim A = \infty$ and $H^q(M, \mathcal{O}) \neq 0$.

However, already in [G], Gohberg was led to a class of holomorphic (first Hilbert, then) Banach bundles, for which finiteness theorems can be proved. The terminology is *bundles of compact type*, meaning their structure group can be reduced to the group of invertible operators of the form id + compact. Let $F \to \mathbb{P}_1$ be a holomorphic Banach bundle of compact type over the Riemann sphere. The Riemann–Hilbert type result of [G, GL], coupled with Bungart's work [B], implies that $F \approx T \oplus V$, where T is trivial and V has finite rank. Hence

$$H^1(\mathbb{P}_1, F) \approx H^1(\mathbb{P}_1, T) \oplus H^1(\mathbb{P}_1, V) \approx H^1(\mathbb{P}_1, V)$$

is finite dimensional by what has been said above. More recently Leiterer in [Li2] considers an arbitrary compact complex manifold M and holomorphic Banach bundles $F \to M$ of compact type, $V \to M$ of finite rank. Assuming $H^q(M, V) = 0$ for some $q \geq 0$, and putting a mild restriction on the fiber type of F, he proves $\dim H^q(M, F \otimes V) < \infty$.

Our goal in this paper is to generalize and thereby better understand the phenomenon discovered by Gohberg and Leiterer. Instead of bundles of compact type, we shall work with a bundle and its compact perturbation, by which we mean

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the following. Let N be a complex manifold and $E, F \to N$ holomorphic Banach bundles.

Definition 1.1. (a) A fredomorphism $E \to F$ is a homomorphism $\varphi\colon E \to F$ such that $\varphi|E_x : E_x \to F_x$ is a Fredholm operator for each $x \in N$.

(b) We say that F is a compact perturbation of E if there are an open cover $\mathfrak{U}$ of N and fredomorphisms $\varphi_U\colon E|U \to F|U$ such that $\varphi_U - \varphi_V|E_x\colon E_x \to F_x$ is compact whenever $U, V \in \mathfrak{U}$ and $x \in U \cap V$.

It is easy to see (cf. Proposition 4.2) that being a compact perturbation is a symmetric and transitive relation—for example, any two finite rank bundles $E, F \to N$ are compact perturbations of each other, the zero homomorphism $E \to F$ being a (global) fredomorphism. Further, if F is a holomorphic Banach bundle of compact type whose fibers are isomorphic to a Banach space A, then F is a compact perturbation of the trivial bundle $A \times N \to N$.

We recall that the sheaf cohomology groups $H^q(N, E)$ of holomorphic Banach bundles are topological vector spaces; see Section 2.

Theorem 1.2. *Let M be a compact complex manifold, E, F holomorphic Banach bundles over M that are compact perturbations of one another, and $q = 0, 1, \ldots$. If $H^{q+1}(M, E)$ is Hausdorff (in particular if $\dim H^{q+1}(M, E) < \infty$) and if $\dim H^q(M, E) < \infty$, then $H^q(M, F)$ is also finite dimensional (and Hausdorff).*

The assumption that $H^{q+1}(M, E)$ be Hausdorff is not automatically satisfied, and without it the conclusion would not neccessarily hold; see the discussion at the end of the paper.

There should be a sheaf version of this theorem, involving cohesive sheaves from [LP] or Leiterer's Banach coherent analytic Fréchet sheaves from [Li1], but so far we have only been able to prove an extension of Theorem 1.2 to certain very special subsheaves of E, F above.

Compact perturbations can be defined more generally for topological Banach bundles. If M is a compact topological space and $E, F \to M$ are Banach bundles, any system of fredomorphisms $\varphi_U : E|U \to F|U$, $U \in \mathfrak{U}$ as in Definition 1.1b induces a virtual vector bundle of finite rank over M, i.e., an element of the group $K(M)$. In particular, one can speak of characteristic classes in this situation. Perhaps even in the holomorphic category one should regard compact perturbations as generalizations of holomorphic vector bundles of finite rank; Theorem 1.2 seems to point in this direction.

The finiteness results of Gohberg and Leiterer are special cases of Theorem 1.2. Indeed, as said, a bundle F of compact type is a compact perturbation of a trivial bundle $E = A \times M \to M$, and $F \otimes V$ is a compact perturbation of $E \otimes V$, when rk $V < \infty$. Now in [K], J. Kim proves $H^p(M, E \otimes V) \approx H^p(M, V) \otimes A$ as topological vector spaces; in particular all $H^p(M, E \otimes V)$ are Hausdorff. (In general there are many natural ways to topologize the tensor product $Y \otimes A$ of Banach spaces; but there is no ambiguity when $\dim Y = n < \infty$ as above. In this case $Y \otimes A \approx A \oplus \ldots \oplus A$, n times.) The Cartan–Serre theorem, at least for bundles over manifolds, is also a special case.

In analysis and geometry finite dimensionality results typically depend on functional analysis: Fredholm theory or Schwartz's theorem on compact perturbations of Fréchet space operators. We shall also help ourselves to these tools, but the present proof will have a nonlinear ingredient as well, Michael's selection theorem

[M], whose application in complex geometry is pretty much the only novelty in this work.

2. Holomorphic Fréchet bundles and their cohomology groups. Generalities

By a Fréchet space we shall mean here a complete, metrizable locally convex topological vector space; a sequence of seminorms defining the topology need not be specified. Similarly, by a Banach space we mean a completely normable locally convex space. A continuous linear map between locally convex spaces will be called a homomorphism. A (locally trivial) holomorphic Fréchet bundle over a finite dimensional complex manifold M is a holomorphic map $\pi\colon E \to M$ of a complex Fréchet manifold E on M, with each fiber $E_x = \pi^{-1}(x)$, $x \in M$, endowed with the structure of a complex vector space. It is required that for every $x \in M$ there be a neighborhood $U \subset M$ of x, a complex Fréchet space A, and a biholomorphism (trivialization) $E|U = \pi^{-1}U \to A \times U$ mapping the fibers E_y linearly on $A \times \{y\} \cong A$, $y \in U$. If A above is a Banach space, we speak of a Banach bundle. For more about these notions, see [Le1, Sections 1,2]. We fix a holomorphic Fréchet bundle $E \to M$ and in this section we describe basic properties of its sheaf cohomology groups $H^q(M, E)$ as locally convex spaces. In addition to the notation and definitions introduced in this section, the results that we will need later on are Lemma 2.5 and Corollary 2.6.

Let $\varphi_U\colon E|U \to A \times U$ be a trivialization. For each compact $K \subset U$ and continuous seminorm p on A we define seminorms

$$\|e\|_{K,p} = \sup_{x\in K} p(\varphi_U e(x)), \qquad e \in \Gamma(U, E),$$

on the space $\Gamma(U, E)$ of holomorphic sections. They endow $\Gamma(U, E)$ with the structure of a complete locally convex space, in fact a Fréchet space, since a countable collection of K, p suffices to define the topology. Clearly, this Fréchet space structure is independent of the choice of the trivialization.

If $q = 0, 1, \ldots$, and $\mathfrak{U}$ is a countable open cover of M such that $E|U$ is trivial for each $U \in \mathfrak{U}$, then the space of (not necessarily alternating) q–cochains

$$C^q(\mathfrak{U}, E) = \prod_{U_0,\ldots,U_q\in\mathfrak{U}} \Gamma\Big(\bigcap_{i=0}^{q} U_i, E\Big)$$

is again a Fréchet space; we use the convention that the space of functions (and later, forms) on the empty set is the 0-dimensional space (0). The coboundary operator

$$\delta = \delta^q_{\mathfrak{U}} : C^q(\mathfrak{U}, E) \to C^{q+1}(\mathfrak{U}, E)$$

is continuous, $Z^q(\mathfrak{U}, E) = \mathrm{Ker}\delta^q_{\mathfrak{U}}$ is a closed subspace, and the cohomology groups

$$H^q(\mathfrak{U}, E) = Z^q(\mathfrak{U}, E)/\delta^{q-1}_{\mathfrak{U}} C^{q-1}(\mathfrak{U}, E)$$

are locally convex vector spaces, complete and semimetrizable, but possibly non-Hausdorff. A refinement map $\mathfrak{V} \to \mathfrak{U}$ between two covers induces homomorphisms $\rho^{\mathfrak{U}}_{\mathfrak{V}}\colon H^q(\mathfrak{U}, E) \to H^q(\mathfrak{V}, E)$. We endow the vector space direct limit $H^q(M, E)$ of the system $\{H^q(\mathfrak{U}, E), \rho^{\mathfrak{U}}_{\mathfrak{V}}\}$ with the finest locally convex topology for which the canonical maps $H^q(\mathfrak{U}, E) \to H^q(M, E)$ are continuous. (It can be seen from Theorem 2.2 below that this is simply the finest topology which makes the canonical maps continuous.)

A variant of this construction provides better control of the topology obtained. If $\varphi_U\colon E|U \to A \times U$ is a trivialization as above and $W \Subset U$ is open, then for each continuous seminorm p on A and $e \in \Gamma(W, E)$ we let $\|e\|_p = \sup_{x\in W} p(\varphi_U e(x)) \le \infty$ and define the space of bounded holomorphic sections by

$$\Gamma_b(W, E) = \{e \in \Gamma(W, E)\colon \|e\|_p < \infty \text{ for each } p\}.$$

This space, with the seminorms $\|\ \ \|_p$ is a Fréchet space, whose topology is again independent of the choice of $U \Supset W$ and the trivialization φ_U. Obviously, the embedding $\Gamma_b(W, E) \to \Gamma(W, E)$ is continuous. If $\mathfrak{W}$ is a countable open cover of M consisting of such W, then the space of bounded q–cochains,

$$C_b^q(\mathfrak{W}, E) = \prod_{W_0,\ldots,W_q\in\mathfrak{W}} \Gamma_b\big(\bigcap_{i=0}^{q} W_i, E\big),$$

is a Fréchet space, and the coboundary δ restricts to a homomorphism $C_b^q(\mathfrak{W}, E) \to C_b^{q+1}(\mathfrak{W}, E)$. Denoting the kernel of $\delta|C_b^q(\mathfrak{W}, E)$ by $Z_b^q(\mathfrak{W}, E) \subset C_b^q(\mathfrak{W}, E)$, the bounded cohomology groups

$$H_b^q(\mathfrak{W}, E) = Z_b^q(\mathfrak{W}, E)/\delta_{\mathfrak{W}}^{q-1} C^{q-1}(\mathfrak{W}, E)$$

are complete semimetrizable locally convex spaces. With a cover $\mathfrak{U}$ finer than $\mathfrak{W}$, restriction induces homomorphisms

$$H_b^q(\mathfrak{W}, E) \to H^q(\mathfrak{U}, E). \tag{2.1}$$

Composing these with the canonical homomorphisms $H^q(\mathfrak{U}, E) \to H^q(M, E)$ we obtain canonical homomorphisms

$$H_b^q(\mathfrak{W}, E) \to H^q(M, E). \tag{2.2}$$

Definition 2.1. We say that a countable open cover $\mathfrak{W}$ of M is special (with respect to E) if (a) for every $W \in \mathfrak{W}$ there is a biholomorphism of a neighborhood of $\overline{W}$ into some $\mathbb{C}^n$ that maps W on a bounded, strongly pseudoconvex domain with smooth boundary; also E is trivial on this neighborhood, and (b) the boundaries of $W \in \mathfrak{W}$ are in general position. This latter means that if $k \in \mathbb{N}$ and ρ_i are smooth defining functions of $W_i \in \mathfrak{W}$, $i = 1, \ldots, k$, then $d\rho_1, \ldots, d\rho_k$ are linearly independent at each point of the set $\{\rho_1 = \ldots = \rho_k = 0\}$.

Sard's lemma implies that M has arbitrarily fine special covers.

Theorem 2.2. *If $\mathfrak{W}$ is a special cover of M, then* (2.2) *is an isomorphism; if, in addition, $\mathfrak{U}$ consists of Stein open sets, then* (2.1) *is also an isomorphism.*

The proof uses routine sheaf theory plus certain deeper facts of several complex variables. We start by introducing a few more spaces generalizing $\Gamma(U, E)$, $\Gamma_b(W, E)$, $C^q(\mathfrak{U}, E)$, etc. Fix smooth vector fields $\xi_1, \ldots, \xi_m$ that span each tangent space T_xM. Let $\varphi_U\colon E|U \to A\times U$ be a trivialization over an open $U \subset M$. We endow the space $\Omega^{0,r}(U, E)$ of smooth E–valued $(0, r)$ forms with seminorms $\|\ \ \|_{K,p,j}$, one for each compact $K \subset U$, continuous seminorm p on A, and $j = 0, 1, 2, \ldots$:

$$\|f\|_{K,p,j} = \max \max_K p\left(\xi_{i_1}\xi_{i_2}\ldots\xi_{i_j}(\varphi_U f(\xi_{i_{j+1}}, \ldots, \xi_{i_{j+r}}))\right), \tag{2.3}$$

the outer max being taken over all tuples $1 \le i_1, \dots, i_{j+r} \le m$. If $\mathfrak{U}$ is a countable cover of M consisting of such open sets, we define a double complex

$$C^{qr}(\mathfrak{U}) = \prod_{U_0,\dots,U_q \in \mathfrak{U}} \Omega^{0,r}\big(\bigcap_{i=0}^{q} U_i, E\big), \quad q, r \ge 0,$$

of Fréchet spaces. We let

$$\delta = \delta^{qr} \colon C^{qr}(\mathfrak{U}) \to C^{q+1,r}(\mathfrak{U}), \qquad \overline{\partial} = \overline{\partial}^{qr} \colon C^{qr}(\mathfrak{U}) \to C^{q,r+1}(\mathfrak{U})$$

denote the Čech coboundary, resp. the homomorphism obtained by applying $\overline{\partial}$ componentwise. Set

$$Z^{qr}(\mathfrak{U}) = \mathrm{Ker}\overline{\partial}^{qr} \cap \mathrm{Ker}\delta^{qr}.$$

With U as above and $W \Subset U$ open, we also consider the space $\Omega_b^{0,r}(W, E) \subset \Omega^{0,r}(W, E)$ of bounded $(0, r)$ forms f, meaning that for each continuous seminorm p on A, $\|f\|_p = \max \sup_W p(\varphi_U f(\xi_{i_1}, \dots, \xi_{i_r})) < \infty$, where the max is taken over all $1 \le i_1, \dots, i_r \le m$. We endow $\Omega_b^{0,r}(W, E)$ with these seminorms $\| \ \|_p$ to obtain a metrizable locally convex space, no longer complete. Observe that the embedding $\Omega_b^{0,r}(W, E) \to \Omega^{0,r}(W, E)$ is not continuous when $r > 0$. Nevertheless, given a countable cover $\mathfrak{W}$ consisting of such W there are the spaces

$$C_b^{qr}(\mathfrak{W}) = \prod_{W_0,\dots,W_q \in \mathfrak{W}} \Omega_b^{0,r}\big(\bigcap_{i=0}^{q} W_i, E\big) \subset C^{qr}(\mathfrak{W})$$

of bounded cochains, again with a metrizable locally convex topology. Thus δ^{qr} restricts to a homomorphism $C_b^{qr}(\mathfrak{W}) \to C_b^{q+1,r}(\mathfrak{W})$ (but $\overline{\partial}^{qr}$ is not continuous from $C_b^{qr}(\mathfrak{W})$ to $C_b^{q,r+1}(\mathfrak{W})$). Finally, we let $Z_b^{qr}(\mathfrak{W}) = Z^{qr}(\mathfrak{W}) \cap C_b^{qr}(\mathfrak{W})$ and endow it with the topology inherited from $C_b^{qr}(\mathfrak{W})$.

Lemma 2.3. *If $\mathfrak{W}$ is a special cover of M, $r \ge 1$, and $W_0, \dots, W_q \in \mathfrak{W}$, then there is a homomorphism*

$$T = T_{W_0 \dots W_q} \colon \Omega_b^{0,r}\big(\bigcap_{i=0}^{q} W_i, E\big) \to \Omega_b^{0,r-1}\big(\bigcap_{i=0}^{q} W_i, E\big) \tag{2.4}$$

such that $\overline{\partial} T f = f$ if $\overline{\partial} f = 0$.

Proof. Because of our hypotheses, we can assume that $M = \mathbb{C}^n$ and E is the trivial bundle $A \times \mathbb{C}^n \to \mathbb{C}^n$. The key is a result of Range and Siu, who construct, when $A = \mathbb{C}$, homomorphisms T as in (2.4) with the property that $\overline{\partial} T + T \overline{\partial} = \mathrm{id}$; see [RS, (3.9)]. In fact, their operator T is a (locally uniform) limit of integral operators T^δ; see [RS, (2.8)–(2.9)]. Now the integrals defining T^δ make sense if instead of $\mathbb{C}$–valued forms we substitute in them A–valued forms, and by the Banach–Hahn theorem they satisfy the same estimates as the scalar–valued operators. It follows that the A–valued integral operators converge to a homomorphism T claimed by the lemma.

A simple consequence is

Theorem 2.4. *If M is Stein and $E = A \times M \to M$ is a trivial Fréchet bundle, then the Dolbeault cohomology groups $H_{\overline{\partial}}^{0,r}(M, E)$ vanish for $r \ge 1$.*

The only surprising thing about this theorem is that it cannot be generalized much. Vogt in [V] shows that $H^{0,1}_{\overline\partial}(M,E)$ need not vanish even for $M=\mathbb{C}$ but with a trivial bundle E with general complete locally convex fibers, and Leiterer explained to me that nontrivial Fréchet bundles over Stein manifolds may have nonvanishing $H^{0,1}_{\overline\partial}$. Such Fréchet bundles can be constructed from locally trivial fiber bundles with Stein base and fibers whose total space is not Stein ("counterexamples to Serre's conjecture"; see [Sk]).

Proof. First assume that M is an open subset of some $\mathbb{C}^k$ and exhaust it by holomorphically convex compact sets $K_1, K_2, \ldots$, with $K_i \subset \operatorname{int} K_{i+1}$. It follows from Lemma 2.3 that if f is a smooth $\overline\partial$–closed E–valued $(0,r)$ form in a neighborhood of K_i, then on a possibly smaller neighborhood, f is $\overline\partial$–exact, provided $r\ge 1$. We shall also need the following approximation result, true for any $r\ge 0$: the above f can be approximated by $\overline\partial$–closed forms in $\Omega^{0,r}(M,E)$, the approximation taking place in any seminorm $\|\ \ \|_{K_{i-1},p,j}$; see (2.3).

Indeed, if $r\ge 1$, this follows by writing $f=\overline\partial g$ and approximating g in $\|\ \ \|_{K_{i-1},p,j+1}$ by forms $g'\in\Omega^{0,r-1}(M,E)$, so that $f'=\overline\partial g'$ will be the required approximation of f. If $r=0$, we are talking about approximating holomorphic functions, and the result is a rather simple special case of [Le2, Théorème 1.1].

Now let $r\ge 1$ and $f\in\Omega^{0,r}(M,E)$ be $\overline\partial$–closed. Fix seminorms $p_1\le p_2\le\ldots$ on A defining its topology. In a neighborhood of K_i write $f=\overline\partial g_i$. We can arrange recursively that

$$\|g_{i+1}-g_i\|_{K_{i-1},p_i,i}<2^{-i},\qquad i=1,2,\ldots. \tag{2.5}$$

Indeed, we note that in general $\overline\partial(g_{i+1}-g_i)=0$ in a neighborhood of K_i; hence there is a closed $h\in\Omega^{0,r-1}(M,E)$ such that

$$\|g_{i+1}-g_i-h\|_{K_{i-1},p_i,i}<2^{-i}.$$

Therefore replacing g_{i+1} by $g_{i+1}-h$ we achieve (2.5), which in turn implies that $g=\lim g_i$ is in $\Omega^{0,r-1}(M,E)$ and satisfies $\overline\partial g=f$. Thus we are done if M is open in $\mathbb{C}^k$.

The general case then follows, since a connected Stein manifold M can be embedded in a Stein domain $M'\subset\mathbb{C}^k$ as a holomorphic retract, and the holomorphic retraction induces a monomorphism $H^{0,r}_{\overline\partial}(M,E)\to H^{0,r}_{\overline\partial}(M',E')$, where E' is the trivial bundle $A\times M'\to M'$.

Now we return to a general complex manifold and a Fréchet bundle $E\to M$. If $\mathfrak{U},\mathfrak{V}$ are open covers of M for which we defined the spaces $C^{qr}(\mathfrak{U}), C^{qr}(\mathfrak{V})$, and if $\mathfrak{U}\to\mathfrak{V}$ is a refinement map, then restriction induces homomorphisms $C^{qr}(\mathfrak{V})\to C^{qr}(\mathfrak{U})$. The image of $f\in C^{qr}(\mathfrak{V})$ will be denoted $f|\mathfrak{U}\in C^{qr}(\mathfrak{U})$.

Lemma 2.5. *If $\mathfrak{V}$ is a special cover of M, $\mathfrak{U}$ a countable refinement of $\mathfrak{V}$ consisting of Stein open sets, and $q,r\ge 0$, then there is a homomorphism*

$$\varepsilon=\varepsilon^{qr}\colon Z^{qr}(\mathfrak{U})\to Z^{qr}_b(\mathfrak{V})$$

with the following properties:

(a) *if $f\in Z^{0r}(\mathfrak{U})$, then $\varepsilon f|\mathfrak{U}=f$;*

(b) *if $q\ge 1$ and $f\in Z^{qr}(\mathfrak{U})$, then there is an $h\in C^{q-1,r}(\mathfrak{U})$ such that*

$$\overline\partial h=0\qquad\text{and}\qquad \varepsilon f|\mathfrak{U}-f=\delta h;$$

(c) *the restriction of* ε *to* $Z_b^{qr}(\mathfrak{U})$ *is continuous;*

(d) *if* $\mathfrak{U}$ *is also special and* $f \in Z_b^{qr}(\mathfrak{U})$*, then* h *in (b) can be chosen in* $C_b^{q-1,r}(\mathfrak{U})$*.*

Note that (c) is not automatic, as the topology of $Z_b^{qr}(\mathfrak{U})$ is inherited from $C_b^{qr}(\mathfrak{U})$ and is different from the one inherited from $Z^{qr}(\mathfrak{U}) \subset C^{qr}(\mathfrak{U})$ when $r > 0$.

Proof. Define homomorphisms

$$R = R^{qr} \colon Z^{qr}(\mathfrak{U}) \to C^{q-1,r}(\mathfrak{U}), \qquad S = S^{qr} \colon Z_b^{qr}(\mathfrak{V}) \to C_b^{q,r-1}(\mathfrak{V})$$

as follows. Let $\{\chi_U \colon U \in \mathfrak{U}\}$ be a smooth partition of unity subordinate to $\mathfrak{U}$; then for $f = (f_{U_0 \dots U_q}) \in Z^{qr}(\mathfrak{U})$, $q \geq 1$,

$$(Rf)_{U_1 \dots U_q} = \sum_{U \in \mathfrak{U}} \chi_U f_{U U_1 \dots U_q}. \tag{2.6}$$

If $g = (g_{V_0 \dots V_q}) \in Z_b^{qr}(\mathfrak{V})$, $r \geq 1$, apply the $\overline{\partial}$ solution operator of Lemma 2.3 componentwise to construct $Sg = (T_{V_0 \dots V_q} g_{V_0 \dots V_q})$. Thus $\delta R f = f$ and $\overline{\partial} S g = g$.

We prove the lemma by induction on q. The base case $q = 0$ holds since both $Z^{0r}(\mathfrak{U})$ and $Z_b^{0r}(\mathfrak{V})$ are naturally identified with the space of $\overline{\partial}$–closed elements of $\Omega^{0,r}(M, E)$, the topology coming from $Z^{0r}(\mathfrak{U})$ being finer than the topology coming from $Z_b^{0r}(\mathfrak{V})$. If $q \geq 1$, assume that $\varepsilon^{q-1,r}$ has been constructed for every r, and let

$$\varepsilon^{qr} = \delta S^{q-1,r+1} \varepsilon^{q-1,r+1} \overline{\partial} R^{qr}.$$

To see that this is indeed a homomorphism $Z^{qr}(\mathfrak{U}) \to Z_b^{qr}(\mathfrak{V})$, notice that $\delta \overline{\partial} R = \overline{\partial} \delta R = 0$ and $\overline{\partial} \delta S = \delta \overline{\partial} S = 0$, so that $\overline{\partial} R^{qr}$ maps $Z^{qr}(\mathfrak{U})$ in $Z^{q-1,r+1}(\mathfrak{U})$ and $\delta S^{q-1,r+1}$ maps $Z_b^{q-1,r+1}(\mathfrak{V})$ in $Z_b^{qr}(\mathfrak{V})$. If $f \in Z^{qr}(\mathfrak{U})$, let

$$h_1 = (S^{q-1,r+1} \varepsilon^{q-1,r+1} \overline{\partial} R^{qr} f)|\mathfrak{U} - R^{qr} f \in C^{q-1,r}(\mathfrak{U}). \tag{2.7}$$

Then $\delta h_1 = \varepsilon^{qr} f|\mathfrak{U} - f$, and by the inductive assumption

$$\overline{\partial} h_1 = (\varepsilon^{q-1,r+1} \overline{\partial} R^{qr} f)|\mathfrak{U} - \overline{\partial} R^{qr} f$$

either vanishes (if $q = 1$), and we take $h = h_1$ in (b); or else there is $h_2 \in C^{q-2,r+1}(\mathfrak{U})$ such that $\overline{\partial} h_2 = 0$ and $\overline{\partial} h_1 = \delta h_2$. By Theorem 2.4 there is an $h_3 \in C^{q-2,r}(\mathfrak{U})$ with $\overline{\partial} h_3 = h_2$. Then $h = h_1 - \delta h_3$ satisfies $\overline{\partial} h = \overline{\partial} h_1 - \delta \overline{\partial} h_3 = 0$ and $\delta h = \varepsilon^{qr} f|\mathfrak{U} - f$, which completes the proof of (b).

Next (c) follows since $\overline{\partial} R$ is continuous from $Z_b^{qr}(\mathfrak{U})$ to $Z^{q-1,r+1}(\mathfrak{U})$, as one checks by applying $\overline{\partial}$ to (2.6). Finally, if $\mathfrak{U}$ is also special and $f \in Z_b^{qr}(\mathfrak{U})$, $q \geq 1$, then by induction in (2.7) $h_1 \in C_b^{q-1,r}(\mathfrak{U})$ and h_2 above is in $C_b^{q-2,r+1}(\mathfrak{U})$. By Lemma 2.3 one can choose $h_3 \in C_b^{q-2,r}(\mathfrak{U})$, whence $h = h_1 - \delta h_3 \in C_b^{q-1,r}(\mathfrak{U})$, which proves (d).

Proof of Theorem 2.2. First we show that (2.2) and (2.1) are monomorphisms. Being a special cover is stable under small C^∞ perturbations of the boundaries of the covering sets. Therefore we can construct another special cover $\mathfrak{V}$ such that the closure of each $W \in \mathfrak{W}$ is contained in some $V \in \mathfrak{V}$. Suppose $f \in Z_b^q(\mathfrak{W}, E) = Z_b^{q0}(\mathfrak{W})$ represents a cohomology class $[f] \in H_b^q(\mathfrak{W}, E)$ that (2.2) sends to $0 \in H^q(M, E)$. We apply Lemma 2.5a,b with $\mathfrak{U} = \mathfrak{W}$ to conclude that the image of $[\varepsilon^{q0} f] \in H^q(\mathfrak{V}, E)$ in $H^q(M, E)$ factors through $[f] \in H_b^q(\mathfrak{W}, E)$; hence this image is zero. On the other hand, Theorem 2.4 and the Dolbeault isomorphism show that $\mathfrak{V}$ is a Leray cover for the bundle E, so that $H^q(\mathfrak{V}, E) \to H^q(M, E)$ is a bijection. Hence

$[\varepsilon^{q0}f] = 0$, i.e., $\varepsilon^{q0}f = \delta g$ with some $g \in C^{q-1,0}(\mathfrak{V})$. Since $g|\mathfrak{W} \in C_b^{q-1,0}(\mathfrak{W})$ and $\varepsilon^{q0}f|\mathfrak{W} = \delta(g|\mathfrak{W})$, Lemma 2.5a,d imply f is the coboundary of a bounded cochain: $[f] = 0$. Thus (2.2) is indeed a monomorphism, and with a Stein refinement $\mathfrak{U}$ of $\mathfrak{W}$, so is (2.1), since the canonical homomorphism $H^q(\mathfrak{U}, E) \to H^q(M, E)$ is a bijection, for the same reason as $H^q(\mathfrak{V}, E) \to H^q(M, E)$ was.

Next we take an arbitrary Stein refinement $\mathfrak{U}$ of $\mathfrak{W}$ and apply Lemma 2.5 with $\mathfrak{V} = \mathfrak{W}$. Suppose $f \in Z^q(\mathfrak{U}, E) = Z^{q0}(\mathfrak{U})$ is δ–exact. Again, Lemma 2.5a,b imply that the image of $[\varepsilon^{q0}f] \in H_b^q(\mathfrak{W}, E)$ in $H^q(M, E)$ is 0; hence by what has been proved, $[\varepsilon^{q0}f] = 0$. Thus ε^{q0} maps exact cocycles in $Z^{q0}(\mathfrak{U})$ to exact cocycles in $Z_b^{q0}(\mathfrak{W})$. It induces a homomorphism

$$H^q(\mathfrak{U}, E) \to H_b^q(\mathfrak{W}, E),$$

which, by Lemma 2.5a,b is a right inverse to the restriction homomorphism in (2.1). Passing to the direct limit we obtain a right inverse $H^q(M, E) \to H_b^q(\mathfrak{W}, E)$ of (2.2). Since (2.1), (2.2) are monomorphisms, the right inverses are left inverses as well.

Corollary 2.6. *If $\mathfrak{W}$ is a special cover of M and $\mathfrak{U}$ is a Stein refinement of $\mathfrak{W}$, then*

$$H^q(M, E) \approx H_b^q(\mathfrak{W}, E) \approx H^q(\mathfrak{U}, E).$$

Corollary 2.7. *If M is compact and $E \to M$ is a holomorphic Banach bundle, then the cohomology groups $H^q(M, E)$ are complete, seminormable topological vector spaces.*

Proof. Let $\mathfrak{W}$ be a finite special cover of M. Then $H_b^q(\mathfrak{W}, E)$ is a quotient of a Banach space, hence complete and seminormable; it is also isomorphic to $H^q(M, E)$.

3. σ–compact sets in Fréchet spaces

Recall that a topological space is σ–compact if it is the countable union of compact subspaces. A homomorphism of Fréchet spaces is compact if the image of some nonempty open set has compact closure.

Lemma 3.1. *The range of a compact homomorphism of Fréchet spaces is contained in a σ–compact set. Conversely, any σ–compact subset K of a Fréchet space Z is contained in the range of a compact homomorphism $Y \to Z$, where Y is a Banach space.*

Proof. The first part being obvious, we prove the converse only. Assume first that K is compact; by passing to its hull, we may as well assume K is convex and balanced. Such a set defines a Banach space $Y = Z_K = \bigcup\{\lambda K \colon \lambda \geq 0\}$, endowed with the norm $\|z\| = \min\{\lambda \geq 0 \colon z \in \lambda K\}$. Then the map $\varphi(z) = z$ from Y to Z is a compact homomorphism and $\varphi(Y) \supset K$. In general $K = \bigcup K_j$ with $K_j \subset Z$ compact, $j \in \mathbb{N}$. For each j there are Banach spaces $(Y_j, \|\ \ \|_j)$ and compact homomorphisms $\varphi_j \colon Y_j \to Z$ such that $\varphi_j(Y_j) \supset K_j$. Choose seminorms $p_1 \leq p_2 \leq \ldots$ on Z defining its topology; we can assume that $p_j(\varphi_j(y)) \leq \|y\|_j/2^j$ for $y \in Y_j$. The l^1–sum of the Y_j's,

$$Y = \{y = (y_j) \colon y_j \in Y_j,\ \|y\| = \sum_j \|y_j\|_j < \infty\},$$

with the norm $\| \ \|$ is a Banach space, $\varphi\colon Y \to Z$ given by

$$\varphi(y) = \sum_j \varphi_j(y_j), \qquad y = (y_j) \in Y,$$

is compact, and $\varphi(Y) \supset \bigcup K_j = K$.

Lemma 3.2. *Let C, Z be Fréchet spaces and $\delta\colon C \to Z$ a homomorphism. If there is a σ–compact $L \subset Z$ such that $Z = L + \delta C$, then δC is closed and* $\dim Z/\delta C < \infty$.

Proof. By Lemma 3.1 there exist a Banach space Y and a compact homomorphism $\varphi\colon Y \to Z$ with $\varphi Y \supset L$. Define $\psi, \theta\colon Y \oplus C \to Z$ by

$$\psi(y, c) = \varphi(y), \qquad \theta(y, c) = \delta(c),$$

so that $\psi + \theta\colon Y \oplus C \to Z$ is onto. Since ψ is compact, by Schwartz's theorem [Sc], $\theta(Y \oplus C) = \delta C$ is indeed closed and of finite codimension.

4. Compact perturbations

Let M be a complex manifold and E, F holomorphic Banach bundles over M. If $U \subset M$ is open, we shall say that a homomorphism $k\colon E|U \to F|U$ is fiberwise compact if $k|E_x \in \mathrm{Hom}(E_x, F_x)$ is compact for every $x \in U$.

Proposition 4.1. *Suppose $U \Subset V \subset M$ are open and $E|V$, $F|V$ are trivial. If $k \in Hom(E|V, F|V)$ is fiberwise compact, then the homomorphism*

$$\alpha\colon \Gamma(V, E) \ni f \mapsto kf|U \in \Gamma(U, F)$$

is compact.

This would not be true for Fréchet bundles, which is essentially the only reason why Theorem 1.2 is restricted to Banach bundles.

Proof. Let $(A, \| \ \|_A), (B, \| \ \|_B)$ be Banach spaces such that $E|V$, $F|V$ are (isomorphic to) $A \times U$, resp. $B \times U$. The set

$$\mathcal{F} = \{f \in \Gamma(V, E)\colon \sup_{x \in U} \|f(x)\|_A < 1\}$$

is open in $\Gamma(V, E)$. Its image under α is a locally equicontinuous family on U by Cauchy estimates; also $\{\alpha f(x)\colon f \in \mathcal{F}\}$ has compact closure in B for every $x \in U$. By the Arzelà–Ascoli theorem any sequence in $\alpha(\mathcal{F})$ has a subsequence convergent in the topology of $\Gamma(U, F)$, and so α is indeed compact.

Proposition 4.2. *If F is a compact perturbation of E, then E is also a compact perturbation of F. More precisely, if $\mathfrak{U}$ is an open cover of M and $\varphi_U\colon E|U \to F|U$ are fredomorphisms such that $\varphi_U - \varphi_V$ are fiberwise compact for $U, V \in \mathfrak{U}$, then there exist an open cover $\mathfrak{U}'$ of M and fredomorphisms $\psi_{U'}\colon F|U' \to E|U'$ for $U' \in \mathfrak{U}'$ such that $\psi_{U'} - \psi_{V'}$, $\psi_{U'}\varphi_U - id_{E|U' \cap U}$, and $\varphi_U \psi_{U'} - id_{F|U' \cap U}$ are fiberwise compact, $U \in \mathfrak{U}$, $U', V' \in \mathfrak{U}'$.*

Proof. Given $x \in U \in \mathfrak{U}$, there is a finite codimensional subspace $E'_x \subset E_x$ on which φ_U is injective. In fact, E'_x can be extended to a holomorphic subbundle E' of E, over a neighborhood $N(x) \subset U$ of x, and $\varphi_U|E'$ is still injective. It follows that $F' = \varphi_U(E') \subset F|N(x)$ is a holomorphic subbundle of finite corank, and φ_U restricts to an isomorphism $\varphi'\colon E' \to F'$. Upon shrinking $N(x)$ we can assume F' has a complementary bundle $F'' \subset F|N(x)$, and in fact, at the price of refining $\mathfrak{U}$

we can assume that the bundles E', F', F'' are defined over all of $U = N(x)$. If $p'\colon F|U = F' \oplus F'' \to F'$ denotes the projection, then

$$\psi_U = \varphi'^{-1} p' \in \mathrm{Hom}(F|U, E|U)$$

is a fredomorphism. Furthermore $\psi_U \varphi_U - \mathrm{id}_{E|U} = k_U$ and $\varphi_U \psi_U - \mathrm{id}_{F|U} = l_U$ are fiberwise compact (in fact, of finite rank). With ψ_U, $U \in \mathfrak{U} = \mathfrak{U}'$ thus constructed,

$$\psi_V - \psi_U = \psi_U(\varphi_U - \varphi_V)\psi_V - k_U \psi_V + \psi_U l_V$$

are also fiberwise compact, whence the claim follows.

From now on we assume M is compact. It will be convenient to use indexed covers $\mathfrak{U} = \{U_i \in I\}$, write $U_{ij\ldots} = U_i \cap U_j \cap \ldots$, and denote the components $f_{U_i U_j \ldots}$ of a cochain by $f_{ij\ldots}$.

Let $\mathfrak{U} = \{U_i\colon i \in I\}$ and $\mathfrak{V} = \{V_i\colon i \in I\}$ be finite open covers, $U_i \Subset V_i$. Assume that $E|V_i$ and $F|V_i$ are trivial, and that there are fredomorphisms $\varphi_i\colon E|V_i \to F|V_i$ such that $\varphi_i - \varphi_j$ are fiberwise compact. Define homomorphisms $\Phi^q\colon C^q(\mathfrak{V}, E) \to C^q(\mathfrak{U}, F)$ between spaces of holomorphic cochains by

$$(\Phi^q f)_{i_0 \ldots i_q} = \varphi_{i_0} f_{i_0 \ldots i_q} | U_{i_0 \ldots i_q}, \qquad f \in C^q(\mathfrak{V}, E). \tag{4.1}$$

Proposition 4.3. *$\Phi^q \delta - \delta \Phi^{q-1}\colon C^{q-1}(\mathfrak{V}, E) \to C^q(\mathfrak{U}, F)$ is compact for $q \geq 1$.*

Proof. If $f = (f_{i_1 \ldots i_q}) \in C^{q-1}(\mathfrak{V}, E)$, then

$$(\Phi^q \delta f)_{i_0 \ldots i_q} = \varphi_{i_0} \sum_{j=0}^{q} (-1)^j f_{i_0 \ldots \hat{i}_j \ldots i_q} | U_{i_0 \ldots i_q}$$

and

$$(\delta \Phi^{q-1} f)_{i_0 \ldots i_q} = \varphi_{i_1} f_{i_1 \ldots i_q} + \varphi_{i_0} \sum_{j=1}^{q} (-1)^j f_{i_0 \ldots \hat{i}_j \ldots i_q} | U_{i_0 \ldots i_q}.$$

Hence the claim follows from Proposition 4.1, as

$$(\Phi^q \delta f - \delta \Phi^{q-1} f)_{i_0 \ldots i_q} = (\varphi_{i_0} - \varphi_{i_1}) f_{i_1 \ldots i_q} | U_{i_0 \ldots i_q}.$$

Proof of Theorem 1.2. Construct three finite Stein covers $\mathfrak{U} = \{U_i\colon i \in I\}$, $\mathfrak{V} = \{V_i\colon i \in I\}$, and $\mathfrak{W} = \{W_i\colon i \in I\}$ of M, $U_i \Subset V_i \Subset W_i$. Make sure $\mathfrak{W}$ is special for both E and F (see Definition 2.1) and so fine that there are fredomorphisms $\varphi_i\colon E|W_i \to F|W_i$ and $\psi_i\colon F|W_i \to E|W_i$ such that

$$\varphi_i - \varphi_j, \quad \psi_i - \psi_j, \quad \varphi_i \psi_j - \mathrm{id}_{F|W_{ij}}, \text{ and } \psi_i \varphi_j - \mathrm{id}_{E|W_{ij}}$$

are all fiberwise compact; see Proposition 4.2. Define homomorphisms Φ^q as in (4.1) and $\Psi^q\colon C^q(\mathfrak{W}, F) \to C^q(\mathfrak{V}, E)$ by

$$(\Psi^q f)_{i_0 \ldots i_q} = \psi_{i_0} f_{i_0 \ldots i_q} | V_{i_0 \ldots i_q}, \qquad f \in C^q(\mathfrak{W}, F).$$

It follows from Proposition 4.1 that

$$\Phi^q \Psi^q f = f|\mathfrak{U} + \kappa f, \qquad \text{where} \quad \kappa\colon C^q(\mathfrak{W}, F) \to C^q(\mathfrak{U}, F) \tag{4.2}$$

is compact. By the case $r = 0$ of Lemma 2.5 there are homomorphisms

$$\varepsilon = \varepsilon^q\colon Z^q(\mathfrak{U}, F) \to Z^q(\mathfrak{W}, F)$$

such that $\varepsilon f|\mathfrak{U}$ is cohomologous to f for every $f \in Z^q(\mathfrak{U}, F)$. In view of (4.2), then

$$\Phi^q \Psi^q \varepsilon \equiv \mathrm{id}_{Z^q(\mathfrak{U}, F)} + \kappa \varepsilon \qquad \mathrm{mod}\ \delta C^{q-1}(\mathfrak{U}, F). \tag{4.3}$$

By Corollary 2.6, $H^{q+1}(\mathfrak{V},E)\approx H^{q+1}(M,E)$ is Hausdorff, that is, $\delta C^q(\mathfrak{V},E)\subset Z^{q+1}(\mathfrak{V},E)$ is closed. Michael's selection theorem [M, Theorem 3.2″] implies there is a continuous map

$$\mu\colon \delta C^q(\mathfrak{V},E)\to C^q(\mathfrak{V},E)$$

with $\delta\mu=\mathrm{id}_{\delta C^q(\mathfrak{V},E)}$. (Typically, Michael's theorem is formulated for Banach, rather than Fréchet spaces. However, it has always been understood that the proof applies for Fréchet spaces as well; see e.g. the last paragraph on page 364 in [M].) Consider the continuous map

$$\beta=\Psi^q\varepsilon-\mu\delta\Psi^q\varepsilon\colon Z^q(\mathfrak{U},F)\to C^q(\mathfrak{V},E). \tag{4.4}$$

In fact, β maps into $Z^q(\mathfrak{V},E)$, since $\delta\beta=\delta\Psi^q\varepsilon-\delta\mu\delta\Psi^q\varepsilon=0$.

Now we come to the main point: for every σ–compact $K\subset Z^q(\mathfrak{V},E)$ there is a σ–compact $L\subset Z^q(\mathfrak{U},F)$ such that

$$\beta^{-1}(K+\delta C^{q-1}(\mathfrak{V},E))\subset L+\delta C^{q-1}(\mathfrak{U},F). \tag{4.5}$$

Indeed, by (4.3), (4.4), and since $\delta|Z^q(\mathfrak{W},F)=0$,

$$\begin{aligned}\Phi^q\beta&=\Phi^q\Psi^q\varepsilon-\Phi^q\mu\delta\Psi^q\varepsilon\\&\equiv\mathrm{id}_{Z^q(\mathfrak{U},F)}+\kappa\varepsilon+\Phi^q\mu(\Psi^{q+1}\delta-\delta\Psi^q)\varepsilon\qquad\mod\delta C^{q-1}(\mathfrak{U},F).\end{aligned}$$

Hence by Lemma 3.1 and the analog of Proposition 4.3 for Ψ, for any $S\subset Z^q(\mathfrak{U},F)$ we have

$$S\subset\Phi^q\beta S+H+\delta C^{q-1}(\mathfrak{U},F), \tag{4.6}$$

with some σ–compact $H\subset C^q(\mathfrak{U},F)$. If now S stands for the left hand side of (4.5), then

$$\begin{aligned}\Phi^q\beta S&=\Phi^q(K+\delta C^{q-1}(\mathfrak{V},E))\\&\subset\Phi^qK+(\Phi^q\delta-\delta\Phi^{q-1})C^{q-1}(\mathfrak{V},E)+\delta C^{q-1}(\mathfrak{U},F).\end{aligned}$$

The first term in the last line is σ–compact, and by Lemma 3.1 and Proposition 4.3, the second is contained in a σ–compact set. Combining this with (4.6) we obtain a σ–compact $L\subset C^q(\mathfrak{U},F)$ such that

$$S\subset L+\delta C^{q-1}(\mathfrak{U},F).$$

Since L can be replaced by $L\cap Z^q(\mathfrak{U},F)$, (4.5) is verified.

By hypothesis, $H^q(M,E)\approx H^q(\mathfrak{V},E)$ is finite dimensional. We take for $K\subset Z^q(\mathfrak{V},E)$ a finite dimensional complementary subspace to $\delta C^{q-1}(\mathfrak{V},E)$. In this case (4.5) gives

$$Z^q(\mathfrak{U},F)=L+\delta C^{q-1}(\mathfrak{U},F)$$

with some σ–compact L. Hence Corollary 2.6 and Lemma 3.2 imply

$$H^q(M,F)\approx H^q(\mathfrak{U},F)=Z^q(\mathfrak{U},F)/\delta C^{q-1}(\mathfrak{U},F)$$

is indeed Hausdorff and finite dimensional.

Theorem 1.2 also holds on compact manifolds with strongly pseudoconvex boundaries, by essentially the same proof, and, as said, it should have a sheaf version as well. But in the setting of this paper there does not seem to be much room to improve on it. Without the assumption that $H^{q+1}(M,E)$ is Hausdorff it would not hold. The reasoning to follow was inspired by an idea that the author learned from

Leiterer, who attributed it to Vâjâitu. Suppose M is Kähler and $H^1(M,\mathcal{O}) \neq 0$, so that there is a sequence of nontrivial line bundles $\Lambda_k \to M$ converging to the trivial line bundle. If E is (a suitable Hilbertian completion of) $\bigoplus_1^\infty \Lambda_k$, and F the trivial Hilbert bundle $l^2 \times M \to M$, then E, F are compact perturbations of one another, $\dim H^0(M,E) < \infty$ but $\dim H^0(M,F) = \infty$. In more detail, let $\mathfrak{U}$ be a Stein cover of M and $g = (g_{UV}) \in Z^1(\mathfrak{U},\mathcal{O})$ not exact. The multiplicative cocycles $(e^{g_{UV}/k}) \in Z^1(\mathfrak{U},\mathcal{O}^*)$ are not exact either for large k; by rescaling g we can assume they are not exact for any $k \in \mathbb{N}$. Thus the line bundles Λ_k they determine are holomorphically nontrivial. Since topologically they nevertheless are trivial, $H^0(M,\Lambda_k) = 0$. The infinite diagonal matrices h_{UV} with diagonal entries $e^{g_{UV}/k}, k \in \mathbb{N}$, define a cocycle $h = (h_{UV}) \in Z^1(\mathfrak{U}, \mathrm{GL}(l^2))$ and so a Hilbert space bundle $E \to M$. Each Λ_k canonically embeds in E, $\bigoplus \Lambda_k$ is dense in E, and there are holomorphic projections $E \to \Lambda_k$. It follows that $H^0(M,E) = 0$. However, the trivial bundle $F = l^2 \times M \to M$, a compact perturbation of E, has $\dim H^0(M,F) = \infty$.

By Theorem 1.2 we conclude that $H^1(M,E)$ is not Hausdorff. This example reveals one more notable thing: while all groups $H^q(M,F)$ are Hausdorff, a compact perturbation E has a non-Hausdorff cohomology group—non-Hausdorff cohomology groups of holomorphic Hilbert bundles over compact manifolds were first constructed by Erat in [E]. His construction is similar, being based on a finite rank vector bundle over $\mathbb{P}_1$ that can be deformed into a nonisomorphic bundle.

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