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THE RESIDUAL SPECTRUM OF AN INNER FORM OF Sp_8 SUPPORTED IN THE MINIMAL PARABOLIC SUBGROUP

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Abstract. The part of the residual spectrum of an inner form of the split group Sp_8 supported in the minimal parabolic subgroup is decomposed. Since the considered inner form is not quasi–split, the normalization of the standard intertwining operators, required for the calculation of the poles of the Eisenstein series, is out of the reach of the Langlands–Shahidi method. Hence, a normalization technique, based on the transfer of the Plancherel measure between the split group and its inner form, is applied. The obtained decomposition reveals certain features of the residual spectrum of the inner form which do not appear for the split group.

Introduction

In this paper we consider the residual spectrum of the hermitian quaternionic group H_2' defined as an algebraic group over an algebraic number field k in Section 1 below. It is a non–quasi–split inner form of the split group Sp_8. Although, in principle, the results of this paper could be obtained using the Arthur trace formula explained in [1], our strategy of the calculation is a more direct approach of the Langlands spectral theory explained in [20] and [25].

The residual spectrum of various quasi–split groups was considered by several authors. Among them are Mœglin and Walspurger [24], Mœglin [21], [22], [23], Kim [16], [17], [18], Žampera [39], and Kon–No [19]. In those papers the approach is also the Langlands spectral theory. For quasi–split groups the normalization of the intertwining operators required for the application of the Langlands spectral theory is obtained using the Langlands–Shahidi method explained in [31] and [32].

However, our situation is different. Since H_2' is not quasi–split, it is out of the scope of the Langlands–Shahidi method. Therefore, we had to develop a new technique of the normalization of the intertwining operators based on the Jacquet–Langlands correspondence explained in [5] and the transfer of the Plancherel measure based on the global idea explained in [29]. This technique was already used by the author of this paper in [6], [8] and [9], where we considered the residual spectrum of a non–quasi–split inner forms of SO_4, Sp_4, SO_8 and the parts of the

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residual spectra of non–quasi–split inner forms of SO_{4n} and Sp_{4n}. See also [7] where the residual spectrum of GL_n over a division algebra is obtained.

In this paper we decompose the part of the residual spectrum of H_2' coming from the residues of the Eisenstein series attached to cuspidal automorphic representations of the minimal standard parabolic subgroup of H_2' defined over k. The results are given as Theorems 3.2.1, 3.2.2, 3.3.1, 3.3.2, 3.3.3, 3.3.6, 3.3.8, 3.3.11, 3.3.13, and 3.3.15. When compared to the residual spectrum of the split group Sp_8, in addition to the interesting parity conditions (which appear for split groups as well) in Theorems 3.2.1, 3.3.3, and 3.3.13, the results show certain features of hermitian quaternionic groups such as the local conditions on the non–triviality of the one–dimensional representation at non–split places in Theorems 3.2.2, 3.3.2, 3.3.13, 3.3.15, and the condition on the number of non–quasi–split places of a global quaternion algebra used to define H_2' in Theorem 3.3.15. The reason for occurrence of such conditions lies in the different local normalization factors at split and non–quasi–split places which give local L–functions in the global normalizing factors. This is never the case for split groups.

The paper consists of three sections. In Section 1 we define the groups involved, review their structure and recall the Jacquet–Langlands correspondence. In Section 2 the normalizing factors for the intertwining operators are obtained. Finally, in Section 3 the considered part of the residual spectrum of H_2' is decomposed.

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1. Preliminaries

In this section we define the groups considered in this paper, review their structure and introduce the notation. Also we recall the local and global Jacquet–Langlands correspondence.

Throughout this paper, let k be an algebraic number field, k_v its completion at a place v and $\mathbb{A}$ its ring of adeles. Let D be a quaternion algebra central over k and τ the involution fixing the center of D. Then, D splits at all but finitely many places v of k, i.e. at those places where the completion $D \otimes_k k_v$ is isomorphic to the additive group $M(2, k_v)$ of 2×2 matrices with coefficients in k_v. In this paper we assume that D splits at all archimedean places. This is a technical assumption which could be removed if one had a better understanding of the local representation theory of H_2' and its Levi subgroups over the Hamilton quaternions. At finitely many non–archimedean places v of k where D is non–split, the completion $D \otimes_k k_v$ is isomorphic to the quaternion algebra D_v central over k_v. The finite non–empty set of non–archimedean places of k where D is non–split is denoted by S_D. The cardinality of S_D, denoted by $|S_D|$, is even for every D.

The algebraic group over k of invertible elements of D is denoted GL_1'. At a split place $v \notin S_D$ we have $GL_1'(k_v) \cong GL_2(k_v)$, where GL_2 is the split group over k of invertible 2×2 matrices. At a non–split place $v \in S_D$ we have $GL_1'(k_v) \cong D_v^\times$.

Let $\det'$ denote the reduced norm of the simple algebra $D \otimes_k \mathbb{A}$ and $\det_v'$ the corresponding reduced norm at a place v. If $v \notin S_D$ is split, then $\det_v' = \det_v$ is just the determinant for 2×2 matrices, while if $v \in S_D$ is non–split, then $\det_v'$ is the reduced norm of the quaternion algebra D_v. The absolute value of the reduced norms $\det'$ and $\det_v'$ is denoted by ν.

Let V be a $2n$–dimensional right vector space over D with the basis $\{e_1, \dots, e_{2n}\}$. Then

$$(e_i, e_j) = \delta_{i,2n-j+1} \quad \text{for } 1 \leqslant i \leqslant j \leqslant n$$

defines a hermitian form on V by

$$(v, v') = \tau((v', v)) \quad \text{and} \quad (vx, v'x') = \tau(x)(v, v')x'$$

for all $v, v' \in V$ and $x, x' \in D$. The group of isometries of the hermitian form $(\cdot, \cdot)$ regarded as a reductive algebraic group defined over k will be denoted by H_n'. It is an inner form of the group Sp_{4n}. Hence, $H_n'(k_v) \cong Sp_{4n}(k_v)$ for every split place $v \notin S_D$. In this paper we consider the residual spectrum of the group H_2' which is an inner form of the split group Sp_8.

Let T' be the maximal split torus in H_2'. It is isomorphic to $GL_1 \times GL_1$. Denote by Φ' the set of the roots of H_2' with respect to T'. Then

$$\Phi' = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\},$$

where $e_i(t_1, t_2) = t_i$ for all $(t_1, t_2) \in T'$. For the set of positive roots take

$$\Phi'^+ = \{e_1 \pm e_2, 2e_1, 2e_2\}.$$

The corresponding set of simple roots is

$$\Delta' = \{e_1 - e_2, 2e_2\}.$$

Let W' be the Weyl group of H_2' with respect to T'. Then

$$W' = \{1, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1, w_2w_1w_2, w_1w_2w_1w_2\},$$

where w_1 and w_2 are the simple reflections with respect to the simple root $e_1 - e_2$ and $2e_2$, respectively. The minimal parabolic subgroup $P_0' = M_0'N_0'$ of H_2' defined over k has the Levi factor $M_0' \cong GL_1' \times GL_1'$.

Let $T \cong GL_1 \times GL_1 \times GL_1 \times GL_1$ be the maximal split torus of the split Sp_8. Fix the positive roots of Sp_8 with respect to T in such a way that the split form $P_0 = M_0N_0$ of the parabolic subgroup $P_0' = M_0'N_0'$ is the standard parabolic subgroup of the split Sp_8 with the Levi factor $M_0 \cong GL_2 \times GL_2$. Let $W(M_0)$ be the subgroup of the Weyl group of Sp_8 with respect to T isomorphic to the quotient of the normalizer of M_0 modulo M_0. Then $W(M_0) \cong W'$, and we identify their elements.

For a Levi factor M of a standard parabolic subgroup of a reductive group, let $\mathfrak{a}_{M,\mathbb{C}}^* \cong X(M) \otimes_{\mathbb{Z}} \mathbb{C}$ denote the complexification of the $\mathbb{Z}$–module $X(M)$ of k–rational characters of M. Then $\mathfrak{a}_{M_0,\mathbb{C}}^* \cong \mathfrak{a}_{M_0',\mathbb{C}}^*$ are two–dimensional complex vector spaces. The isomorphisms with $\mathbb{C}^2$ are fixed by choosing for the basis the reduced norm on every copy of GL_1' in M_0' and the determinant on every copy of GL_2 in M_0. The elements of $\mathfrak{a}_{M_0,\mathbb{C}}^* \cong \mathfrak{a}_{M_0',\mathbb{C}}^*$ written in that fixed basis will be denoted

TABLE 1.1. Action of W'

w	$w(\underline{s}) = w(s_1, s_2)$	$w(\pi') \cong w(\pi'_1 \otimes \pi'_2)$
1	(s_1, s_2)	$\pi'_1 \otimes \pi'_2$
w_1	(s_2, s_1)	$\pi'_2 \otimes \pi'_1$
w_2	$(s_1, -s_2)$	$\pi'_1 \otimes \widetilde{\pi}'_2$
$w_1 w_2$	$(-s_2, s_1)$	$\widetilde{\pi}'_2 \otimes \pi'_1$
$w_2 w_1$	$(s_2, -s_1)$	$\pi'_2 \otimes \widetilde{\pi}'_1$
$w_1 w_2 w_1$	$(-s_1, s_2)$	$\widetilde{\pi}'_1 \otimes \pi'_2$
$w_2 w_1 w_2$	$(-s_2, -s_1)$	$\widetilde{\pi}'_2 \otimes \widetilde{\pi}'_1$
$w_1 w_2 w_1 w_2$	$(-s_1, -s_2)$	$\widetilde{\pi}'_1 \otimes \widetilde{\pi}'_2$

$\underline{s} = (s_1, s_2)$, where $s_1, s_2 \in \mathbb{C}$. The positive Weyl chamber is given by inequalities $Re(s_1) > Re(s_2) > 0$.

The action of the Weyl group element $w \in W'$ on $\underline{s} = (s_1, s_2) \in \mathfrak{a}_{M'_0,\mathbb{C}}$ and a cuspidal automorphic representation $\pi' \cong \pi'_1 \otimes \pi'_2$ of the Levi factor $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A}) \times GL'_1(\mathbb{A})$ is induced by the conjugation of the Levi factor. It is given in Table 1.1, where $\widetilde{\cdot}$ denotes the contragredient representation.

In this paper the parabolic induction from a standard parabolic subgroup P of a reductive group G with the Levi factor M will be denoted by Ind_M^G instead of Ind_P^G. This will not cause any confusion, since all the parabolic subgroups appearing in the paper are standard. The induction is always normalized in a sense that the representation induced from a unitary representation is again unitary.

Finally, let us recall the Jacquet–Langlands correspondence following Section 8 of [5]. In this paper we refer to the Jacquet–Langlands correspondence as the local and global lift of representations from the group GL'_1 to the split group GL_2.

Let $\pi' \cong \otimes_v \pi'_v$ be a cuspidal automorphic representation of $GL'_1(\mathbb{A})$ which is not one–dimensional. Then, at non–split places the local lift π_v of π'_v is the square–integrable representation of $GL_2(k_v)$ defined by the character relation as in Theorem (8.1) of [5]. At split places we have $GL'_1(k_v) \cong GL_2(k_v)$, and the local lift is just $\pi_v \cong \pi'_v$. The global lift of π' is defined using the local lifts as $\pi \cong \bigotimes_v \pi_v$. By Theorem (8.3) of [5] the global lift π is isomorphic to a cuspidal automorphic representation of $GL_2(\mathbb{A})$. Hence, its local components π_v are generic.

Let $\chi \circ \det' = \otimes_v \left(\chi_v \circ \det'_v\right)$ be a one–dimensional cuspidal automorphic representation of $GL'_1(\mathbb{A})$. Here χ_v are unitary characters of $k_v^\times$ and χ is a unitary character of $\mathbb{A}^\times / k^\times$. Then, in this paper, the global lift of $\chi \circ \det'$ is defined to be just the one–dimensional representation $\chi \circ \det = \otimes_v \left(\chi_v \circ \det_v\right)$ of $GL_2(\mathbb{A})$. It belongs to the residual spectrum of $GL_2(\mathbb{A})$. At non–split places the local lift of $\chi_v \circ \det'_v$ is defined by the Jacquet–Langlands correspondence as in Theorem (8.1) of [5] to be the unique irreducible subrepresentation of the induced representation $\mathrm{Ind}_{GL_1(k_v) \times GL_1(k_v)}^{GL_2(k_v)} \left(\chi_v |\cdot|^{1/2} \otimes \chi_v |\cdot|^{-1/2}\right)$. At non–archimedean places it is the Steinberg representation of $GL_2(k_v)$ twisted by χ_v, but we denote this representation by St_{χ_v} at all places. Observe that by our definition in this case the global and local lift are not consistent. The reason is that the global lift is supposed to be in the discrete spectrum of $GL_2(\mathbb{A})$, while the local lift should preserve the Plancherel measure.

In this paper a unitary character μ of $\mathbb{A}^\times/k^\times$ and μ_v of $k_v^\times$ are said to be quadratic if, respectively, μ^2 and μ_v^2 are trivial. Thus, the trivial character is among quadratic characters as well.

2. Normalization of intertwining operators

This section is devoted to the local and global normalization, using scalar meromorphic normalizing factors, of standard intertwining operators for H_2' attached to a cuspidal automorphic representation of the Levi factor $M_0'(\mathbb{A})$ of the minimal standard parabolic subgroup of H_2'. The main requirement of the normalization is that the normalized intertwining operators are holomorphic and non–vanishing in the regions required for the calculation of the residual spectrum in Section 3.

The normalizing factors are first defined locally, at every place v of k, in the first three subsections. Subsections 2.1, 2.2, and 2.3 correspond, respectively, to the possible cases: a generic representation at a split place, a non–generic representation at a split place and any unitary representation at a non–split place. Subsection 2.4 combines the results of the previous subsections to obtain the global normalizing factors as a product over all places of the local ones. All the normalizing factors are given as ratios of L–functions and ε–factors.

2.1. **Generic representation at a split place.** For the generic split case the normalization is given by the Langlands–Shahidi method of [31] and [32] for the standard intertwining operators attached to a generic irreducible representation of the Levi factor of a standard proper parabolic subgroup of any quasi–split reductive group over k_v. Of course, generic always means generic with respect to the fixed continuous non-trivial additive character ψ_v of k_v. We omit the details in this section since the proofs may be found in Section 1.1 of [8] and are based on [38].

In this subsection let G be a split classical group defined over k_v. For every subset θ of the set of simple roots Δ of G with respect to the fixed maximal split torus, let $P_\theta = M_\theta N_\theta$ be the corresponding standard parabolic subgroup of G, where M_θ is the Levi factor and N_θ the unipotent radical. Let $\mathfrak{a}^*_{M_\theta,\mathbb{C}}$ be the complexification of the $\mathbb{Z}$–module of k_v–rational characters of M_θ. It is an r–dimensional complex vector space, and its elements are denoted by $\underline{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. Let W be the Weyl group of G. Let r_θ be the adjoint representation of the Langlands dual L–group of M_θ on the Lie algebra of the L–group of N_θ.

In the special case of a maximal proper parabolic subgroup we have $\theta = \Delta \setminus \{\alpha\}$ for a simple root α. Then $\mathfrak{a}^*_{M_\theta,\mathbb{C}}$ is one-dimensional (except for $G = GL_n$ when it is one–dimensional modulo center). We fix a basis vector

$$\widetilde{\alpha} = \langle \rho_P, \alpha^\vee \rangle^{-1} \rho_P,$$

where ρ_P equals half of the sum of positive roots of G not being roots of M, and we write $s\widetilde{\alpha} = \widetilde{\alpha} \otimes s$ for $s \in \mathbb{C}$. Observe that in the maximal proper parabolic subgroup case there is at most one non-trivial element $w \in W$ such that $w(\Delta \setminus \{\alpha\}) \subset \Delta$.

For $\underline{s} \in \mathfrak{a}^*_{M_\theta,\mathbb{C}}$, an irreducible representation π_v of $M_\theta(k_v)$ and an element $w \in W$ such that $w(\theta) \subset \Delta$, we denote by $A(\underline{s}, \pi_v, w)$ the standard intertwining operator intertwining the induced representations

$$I(\underline{s}, \pi_v) = \mathrm{Ind}_{M_\theta(k_v)}^{G(k_v)} (\pi_v |\underline{s}(\cdot)|) \to I(w(\underline{s}), w(\pi_v)) = \mathrm{Ind}_{M_{w(\theta)}}^{G(k_v)} (w(\pi_v) |w(\underline{s})(\cdot)|),$$

where $|\underline{s}(\cdot)|$ and $|w(\underline{s})(\cdot)|$ are viewed as characters of $M_\theta(k_v)$. The scalar meromorphic normalizing factor for $A(\underline{s}, \pi_v, w)$, defined via the Langlands–Shahidi method

(see [32] for more details), is denoted by $r(\underline{s}, \pi_v, w)$, and the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is then defined by

$$A(\underline{s}, \pi_v, w) = r(\underline{s}, \pi_v, w) N(\underline{s}, \pi_v, w).$$

Following [38], the main result of Section 1.1 of [8] shows the holomorphy and non–vanishing of the normalized intertwining operators in a certain open set slightly bigger than the closure of the positive Weyl chamber for a generic irreducible tempered representation π_v. For the convenience we recall it here.

Proposition 2.1.1. *Let $P_\theta = M_\theta N_\theta$ be the standard proper parabolic subgroup of G corresponding to θ and w an element of the Weyl group W such that $w(\theta) \subset \Delta$. Let π_v be an irreducible generic tempered representation of $M_\theta(k_v)$. Then the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is holomorphic and non–vanishing for $\underline{s} \in \mathfrak{a}^*_{M_\theta,\mathbb{C}}$ such that*

$$\langle Re(\underline{s}), \alpha^\vee \rangle > -1/\ell_\alpha \quad \text{for all } \alpha \in \Phi^+_{w,\theta},$$

where ℓ_α is the length of the corresponding adjoint representation r_α in a decomposition of the standard intertwining operator as in Section 2.1 of [31] *and where $\Phi^+_{w,\theta}$ is the set of all positive roots α such that $w\alpha$ is a negative root.*

Next, we consider the case of any irreducible unitary generic representation but only for the parabolic subgroup $P_0 = M_0 N_0$ of the split group Sp_8. We omit the proof since it is the same as the proof of the analogous proposition in Section 1.1 of [8].

Proposition 2.1.2. *Let $P_0 = M_0 N_0$ be the standard proper parabolic subgroup of the split group Sp_8 with the Levi factor $M_0 \cong GL_2 \times GL_2$. Let $\pi_v \cong \pi_{1,v} \otimes \pi_{2,v}$ be an irreducible generic unitary representation of $M_0(k_v)$. Then, for every $w \in W(M_0)$, the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is holomorphic and non–vanishing for $\underline{s} = (s_1, s_2) \in \mathfrak{a}^*_{M_0,\mathbb{C}}$ such that*

- *$Re(s_1) \geqslant Re(s_2) \geqslant 0$, i.e. the closure of the positive Weyl chamber,*
- *$0 < s_1 < 1/2$ and $s_2 = 1/2$,*
- *$1/2 < s_1 < 1$ and $s_1 - s_2 = 1$.*

Finally, we collect in the following corollary normalizing factors for the maximal standard proper parabolic cases needed in the sequel. The normalizing factors in non–maximal cases are by definition the products of the normalizing factors of the maximal cases appearing in a decomposition of the standard intertwining operator according to the reduced decomposition of the Weyl group element as in Section 2.1 of [31].

Corollary 2.1.3. *For the case $GL_2 \times GL_2 \subset GL_4$, the normalizing factor for the standard intertwining operator $A((s_1, s_2), \pi_{1,v} \otimes \pi_{2,v}, w_1)$, where $\pi_{1,v} \otimes \pi_{2,v}$ is an irreducible generic representation of $GL_2(k_v) \times GL_2(k_v)$ and w_1 the unique non-trivial Weyl group element, equals*

(2.1)
$$r((s_1, s_2), \pi_{1,v} \otimes \pi_{2,v}, w_1) = \frac{L(s_1 - s_2, \pi_{1,v} \times \widetilde{\pi}_{2,v})}{L(1 + s_1 - s_2, \pi_{1,v} \times \widetilde{\pi}_{2,v}) \varepsilon(s_1 - s_2, \pi_{1,v} \times \widetilde{\pi}_{2,v}, \psi_v)},$$

where the L–function and ε–factor are the Rankin–Selberg ones of pairs.

For the case $GL_2 \subset Sp_4$, the normalizing factor for the intertwining operator $A(s, \pi_v, w_2)$, where π_v is an irreducible generic representation of $GL_2(k_v)$ and w_2 the unique non-trivial Weyl group element, equals

$$r(s, \pi_v, w_2) = \frac{L(s, \pi_v)}{L(1+s, \pi_v)\varepsilon(s, \pi_v, \psi_v)} \frac{L(2s, \omega_{\pi_v})}{L(1+2s, \omega_{\pi_v})\varepsilon(2s, \omega_{\pi_v}, \psi_v)}, \tag{2.2}$$

where the L–functions and ε–factors are the principal Jacquet ones and the Hecke ones of the central character ω_{π_v} of π_v.

2.2. **Non–generic representation at a split place.** A non–generic irreducible representation $\pi_v \cong \pi_{1,v} \otimes \pi_{2,v}$ of $M_0(k_v) \cong GL_2(k_v) \times GL_2(k_v)$ is the local component at a split place of a cuspidal automorphic representation $\pi' \cong \pi'_1 \otimes \pi'_2$ of $M'_0(\mathbb{A})$ if at least one of the representations π'_1 and π'_2 is one–dimensional. Then, the definition of the normalizing factor and the proof of the holomorphy and non–vanishing of the normalized intertwining operators in the closure of the positive Weyl chamber follow the proof of Lemma I.8 of [24]. It was already used in Section 1.2 of [8] for an inner form of SO_8, and hence we omit the details.

For the moment let G be any classical split group defined over k_v. Let $P_\theta = M_\theta N_\theta$ be the standard proper parabolic subgroup of G defined over k_v corresponding to a subset θ of the set of simple roots Δ with respect to the fixed maximal split torus. Let π_v be an irreducible unitary non–generic representation of $M_\theta(k_v)$. Assume that there exists a standard parabolic subgroup of M_θ with the Levi factor L, an irreducible tempered generic representation τ_v of $L(k_v)$ and $\underline{s}' \in \mathfrak{a}^*_{L,\mathbb{C}}$ such that π_v is isomorphic to the unique irreducible subrepresentation of

$$I_L^{M_\theta}(\underline{s}', \tau_v) = \mathrm{Ind}_{L(k_v)}^{M_\theta(k_v)} \left(\tau_v |\underline{s}'(\cdot)|\right).$$

Then, for every Weyl group element w such that $w(\theta) \subset \Delta$, the following diagram is commutative:

$$\begin{array}{ccc} I(\underline{s}, \pi_v) & \hookrightarrow & I(\underline{s} + \underline{s}', \tau_v) \\ A(\underline{s}, \pi_v, w)\Big\downarrow & & \Big\downarrow A(\underline{s} + \underline{s}', \tau_v, w) \\ I(w(\underline{s}), w(\pi_v)) & \hookrightarrow & I(w(\underline{s} + \underline{s}'), w(\tau_v)), \end{array}$$

where $\underline{s}$ is embedded into $\mathfrak{a}^*_{L,\mathbb{C}}$. In other words, $A(\underline{s}, \pi_v, w)$ is the restriction of $A(\underline{s} + \underline{s}', \tau_v, w)$ to $I(\underline{s}, \pi_v)$. Hence, the normalizing factor for $A(\underline{s}, \pi_v, w)$ is defined to be

$$r(\underline{s}, \pi_v, w) = r(\underline{s} + \underline{s}', \tau_v, w), \tag{2.3}$$

and the normalized operator $N(\underline{s}, \pi_v, w)$ is actually the restriction of $N(\underline{s}+\underline{s}', \tau_v, w)$ to $I(\underline{s}, \pi_v)$. The proof of the holomorphy and non–vanishing will follow from the following lemma, which we recall without a proof since it is in fact a part of the proof of Lemma I.8 in [24].

Lemma 2.2.1. *Assume that in the notation as above there exists a Weyl group element w' such that the image of the normalized intertwining operator*

$$N(w'^{-1}(\underline{s} + \underline{s}'), w'^{-1}(\tau_v), w') : \quad I(w'^{-1}(\underline{s} + \underline{s}'), w'^{-1}(\tau_v)) \to I(\underline{s} + \underline{s}', \tau_v)$$

is $I(\underline{s}, \pi_v)$. Then, for all $\underline{s} \in \mathfrak{a}^*_{M,\mathbb{C}}$ such that $w'^{-1}(\underline{s} + \underline{s}')$ satisfies the inequalities of Proposition 2.1.1 for ww', the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is holomorphic and non–vanishing at $\underline{s}$.

Now, we apply the lemma to the possible cases for $M_0 \subset Sp_8$ and omit the details since the proofs follow closely the proofs of the analogous propositions in Section 1.2 of [8]. The basic idea is always to find an appropriate w' which satisfies both conditions of the lemma. In some cases such w' does not exist, and we excluded those cases in the propositions below. Further investigation of those cases is made in Section 3, where they are treated as possible poles of the Eisenstein series when decomposing the residual spectrum. In all the propositions $P_0 = M_0N_0$ is the standard proper parabolic subgroup of the split group Sp_8 with the Levi factor $M_0 \cong GL_2 \times GL_2$.

Proposition 2.2.2. *Let $\pi_v \cong (\chi_{1,v} \circ \det_v) \otimes \pi_{2,v}$ be an irreducible non–generic unitary representation of $M_0(k_v)$, where $\chi_{1,v}$ is a unitary character of $k_v^\times$ and $\pi_{2,v}$ is a unitary generic representation of $GL_2(k_v)$. Then, for every $w \in W(M_0)$, the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is holomorphic and non–vanishing for $\underline{s} = (s_1, s_2) \in \mathfrak{a}^*_{M_0,\mathbb{C}}$ such that*

- *$Re(s_1) \geqslant Re(s_2) \geqslant 0$, i.e. the closure of the positive Weyl chamber,*
- *$0 < s_1 < 1/2$ and $s_2 = 1/2$.*

Proof. Along the same lines as the proof of the last proposition in Section 1.2 of [8]. □

Proposition 2.2.3. *Let $\pi_v \cong \pi_{1,v} \otimes (\chi_{2,v} \circ \det_v)$ be an irreducible non–generic unitary representation of $M_0(k_v)$, where $\chi_{2,v}$ is a unitary character of $k_v^\times$ and $\pi_{1,v}$ a unitary generic representation of $GL_2(k_v)$. Then, for every $w \in W(M_0)$, the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is holomorphic and non–vanishing for $\underline{s} = (s_1, s_2) \in \mathfrak{a}^*_{M_0,\mathbb{C}}$ such that*

- *$Re(s_1) \geqslant Re(s_2) \geqslant 0$, i.e. the closure of the positive Weyl chamber,*
- *$0 < s_1 < 1/2$ and $s_2 = 1/2$,*
- *$0 < s_1 < 3/2$ and $s_2 = 3/2$, except at $(s_1, s_2) = (r, 3/2)$ for some $0 < r < 1/2$; at the exceptional point, if $\pi_{1,v}$ is not a complementary series representation of the form $\pi_{1,v} \cong \mu_v|\cdot|^r \otimes \mu_v|\cdot|^{-r}$, where μ_v is a unitary character of $k_v^\times$, then it is always holomorphic and non–vanishing.*

Proof. For the last claim when verifying the surjectivity of the appropriate w' of Lemma 2.2.1, one uses the irreducibility of certain induced representations for $GL_3(k_v)$ and $GL_4(k_v)$. These are given in [37] and [33]. □

Proposition 2.2.4. *Let $\pi_v \cong (\chi_{1,v} \circ \det_v) \otimes (\chi_{2,v} \circ \det_v)$ be a one–dimensional non–generic unitary representation of $M_0(k_v)$, where $\chi_{1,v}$ and $\chi_{2,v}$ are unitary characters of $k_v^\times$. Then, for every $w \in W(M_0)$, the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is holomorphic and non–vanishing for $\underline{s} = (s_1, s_2) \in \mathfrak{a}^*_{M_0,\mathbb{C}}$ such that*

- *$Re(s_1) \geqslant Re(s_2) \geqslant 0$ except $Re(s_1) = Re(s_2) = 0$, i.e. the closure of the positive Weyl chamber except at the origin,*
- *$0 < s_1 < 1/2$ and $s_2 = 1/2$,*
- *$0 < s_1 < 3/2$ and $s_2 = 3/2$, except at $(s_1, s_2) = (1/2, 3/2)$; at the exceptional point if $\chi_{1,v} \neq \chi_{2,v}$, then it is always holomorphic and non–vanishing, while if $\chi_{1,v} = \chi_{2,v}$, then it is holomorphic and non–vanishing at least for $w \in \{1, w_2, w_1w_2, w_2w_1w_2\}$,*

- $1 < s_1 < 2$ *and* $s_1 - s_2 = 2$, *except at* $(s_1, s_2) = (3/2, -1/2)$; *at the exceptional point if* $\chi_{2,v}$ *is not quadratic, then it is always holomorphic and non–vanishing, while if* $\chi_{2,v}$ *is quadratic, then it is holomorphic and non–vanishing at least for* $w \in \{1, w_1, w_2w_1, w_1w_2w_1\}$.

Proof. For the third claim see the comment on the proof of the previous proposition and also [24]. For the last claim one uses the irreducibility of certain induced representations for $Sp_4(k_v)$ given in [30], [27] and [28]. □

At the end of this subsection we collect the normalizing factors for the maximal standard proper parabolic subgroup cases needed in the sequel.

Corollary 2.2.5. *For the case* $GL_2 \times GL_2 \subset GL_4$, *the normalizing factor for the standard intertwining operator* $A((s_1, s_2), (\chi_{1,v} \circ \det_v) \otimes \pi_{2,v}, w_1)$, *where* $\chi_{1,v}$ *is a unitary character of* $k_v^\times$, $\pi_{2,v}$ *is an irreducible unitary generic representation of* $GL_2(k_v)$ *and* w_1 *the unique nontrivial Weyl group element, equals*

(2.4)
$$r((s_1, s_2), (\chi_{1,v} \circ \det_v) \otimes \pi_{2,v}, w_1) = \frac{L(s_1 - s_2 - 1/2, \chi_{1,v}\widetilde{\pi}_{2,v})}{L(s_1 - s_2 + 3/2, \chi_{1,v}\widetilde{\pi}_{2,v})\varepsilon(s_1 - s_2 + 1/2, \chi_{1,v}\widetilde{\pi}_{2,v}, \psi_v)\varepsilon(s_1 - s_2 - 1/2, \chi_{1,v}\widetilde{\pi}_{2,v}, \psi_v)},$$

where the L–function and ε*–factor are the principal Jacquet ones. In the case of* $\pi_{1,v}$ *irreducible unitary generic and* $\pi_{2,v} \cong \chi_{2,v} \otimes \det_v$, *where* $\chi_{2,v}$ *is a unitary character of* $k_v^\times$, *the normalizing factor is of the same form with the principal Jacquet L–functions and* ε*–factors for* $\pi_{1,v}\chi_{2,v}^{-1}$ *instead of* $\chi_{1,v}\widetilde{\pi}_{2,v}$.

For the case $GL_2 \times GL_2 \subset GL_4$, *the normalizing factor for the standard intertwining operator* $A((s_1, s_2), (\chi_{1,v} \circ \det_v) \otimes (\chi_{2,v} \circ \det_v), w_1)$, *where* $\chi_{1,v}$ *and* $\chi_{2,v}$ *are unitary characters of* $k_v^\times$ *and* w_1 *is the unique non-trivial Weyl group element, equals*

$$r((s_1, s_2), (\chi_{1,v} \circ \det_v) \otimes (\chi_{2,v} \circ \det_v), w_1) = r_v(s_1 - s_2, \chi_{1,v}\chi_{2,v}^{-1}), \tag{2.5}$$

where for $s \in \mathbb{C}$ *and a unitary character* χ_v *of* $k_v^\times$,

(2.6)
$$r_v(s, \chi_v) = \frac{L(s, \chi_v)L(s-1, \chi_v)}{L(s+2, \chi_v)L(s+1, \chi_v)\varepsilon(s+1, \chi_v, \psi_v)\varepsilon(s, \chi_v, \psi_v)^2\varepsilon(s-1, \chi_v, \psi_v)},$$

and the L–function and ε*–factor are the Hecke ones.*

For the case $GL_2 \subset Sp_4$, *the normalizing factor for the intertwining operator* $A(s, \chi_v \circ \det_v, w_2)$, *where* χ_v *is a unitary character of* $k_v^\times$ *and* w_2 *the unique non-trivial Weyl group element, equals*

$$\begin{aligned} r(s, \chi_v \circ \det_v, w_2) = & \frac{L(s-1/2, \chi_v)}{L(s+3/2, \chi_v)\varepsilon(s+1/2, \chi_v, \psi_v)\varepsilon(s-1/2, \chi_v, \psi_v)} \\ & \cdot \frac{L(2s, \chi_v^2)}{L(1+2s, \chi_v^2)\varepsilon(2s, \chi_v^2, \psi_v)}, \end{aligned} \tag{2.7}$$

where the L–functions and ε*–factors are the Hecke ones of* χ_v *and of the central character* χ_v^2 *of* $\chi_v \circ \det_v$.

2.3. **Non–split place.** In this subsection let $v \in S_D$ be a place of k where D does not split. By our assumption v is non–archimedean. Let $\pi'_v \cong \pi'_{1,v} \otimes \pi'_{2,v}$ be a unitary irreducible representation of the Levi factor $M'_0(k_v) \cong GL'_1(k_v) \times GL'_1(k_v)$ of the minimal parabolic subgroup of H'_2. Observe that π'_v is supercuspidal since

M_0' has no proper parabolic subgroups defined over k_v. Let $\pi_v \cong \pi_{1,v} \otimes \pi_{2,v}$ be the local lift of π_v' from $M_0'(k_v)$ to $M_0(k_v)$ as defined in Section 1 using the Jacquet–Langlands correspondence. It is a square–integrable representation of $M_0(k_v)$.

For $\underline{s} \in \mathfrak{a}^*_{M_0',\mathbb{C}}$ and $w \in W'$, the standard intertwining operator $A(\underline{s}, \pi_v', w)$ is defined as in the split case. For the precise definition see Section 2 of [29] or Section 1.3 of [8]. It is important to choose the Haar measures on the unipotent radicals for the split group and its inner form compatibly as explained in Section 2 of [29]. See also [26]. In this case a decomposition of the standard intertwining operators according to a reduced decomposition of the Weyl group element as in Section 2.1 of [31] still holds.

The normalizing factor for the standard intertwining operator $A(\underline{s}, \pi_v', w)$ is defined to be

$$r(\underline{s}, \pi_v', w) = r(\underline{s}, \pi_v, w), \tag{2.8}$$

where the normalizing factor on the right is the generic split case normalizing factor attached to the local lift π_v which is square–integrable. Then, the normalized intertwining operator is defined by

$$A(\underline{s}, \pi_v', w) = r(\underline{s}, \pi_v', w) N(\underline{s}, \pi_v', w), \tag{2.9}$$

as usual. Here we just give a sketch of the proof of the holomorphy and non–vanishing of $N(\underline{s}, \pi_v', w)$ in the required regions since it follows closely the same proof for an inner form of SO_8 as in Section 1.3 of [8]. It is based on the comparison of the Plancherel formula of [29].

Proposition 2.3.1. *Let $\pi_v' \cong \pi_{1,v}' \otimes \pi_{2,v}'$ be an irreducible unitary representation of the Levi factor $M_0'(k_v)$. Then, for every $w \in W'$, the normalized intertwining operator $N(\underline{s}, \pi_v', w)$ is holomorphic and non–vanishing for $\underline{s} = (s_1, s_2) \in \mathfrak{a}^*_{M_0',\mathbb{C}}$ such that*

- *$Re(s_1) \geqslant Re(s_2) \geqslant 0$, i.e. the closure of the positive Weyl chamber,*
- *$0 < s_1 < 1/2$ and $s_2 = 1/2$,*
- *if $\pi_{2,v}'$ is one–dimensional, then $0 < s_1 < 3/2$ and $s_2 = 3/2$,*
- *$1/2 < s_1 < 1$ and $s_1 - s_2 = 1$,*
- *$1 < s_1 < 2$ and $s_1 - s_2 = 2$ except at $(s_1, s_2) = (3/2, -1/2)$; at the exceptional point if the central character $\omega_{\pi_{2,v}'}$ of $\pi_{2,v}'$ is non–trivial, then it is always holomorphic and non–vanishing, while if $\omega_{\pi_{2,v}'}$ is trivial, then it is holomorphic and non–vanishing at least for $w \in \{1, w_1, w_2w_1, w_1w_2w_1\}$.*

Proof. The proof goes along the same lines as in the proof of the analogous proposition in Section 1.3 of [8]. The third claim would not be true if one removed the condition of one–dimensionality. The reason lies in the reducibility points for the induced representations of $GL_2'(k_v)$ which are a special case of the results in [34]. For the last claim one needs the irreducibility of certain induced representations for $H_1'(k_v)$ obtained in [29]. □

In the following corollaries we collect the normalizing factors for the maximal standard proper parabolic subgroup cases needed in the sequel. The first is a direct consequence of equation (2.8).

Corollary 2.3.2. *For the case $GL_1' \times GL_1' \subset GL_2'$, the normalizing factor for the standard intertwining operator $A((s_1, s_2), \pi_{1,v}' \otimes \pi_{2,v}', w_1)$, where $\pi_{1,v}' \otimes \pi_{2,v}'$ is*

an irreducible unitary representation of $GL_1'(k_v) \times GL_1'(k_v)$ and w_1 the unique nontrivial Weyl group element, equals

(2.10)
$$r((s_1,s_2),\pi_{1,v}'\otimes\pi_{2,v}',w_1)=\frac{L(s_1-s_2,\pi_{1,v}\times\widetilde{\pi}_{2,v})}{L(1+s_1-s_2,\pi_{1,v}\times\widetilde{\pi}_{2,v})\varepsilon(s_1-s_2,\pi_{1,v}\times\widetilde{\pi}_{2,v},\psi_v)},$$

where the L–function and ε–factor are the Rankin–Selberg ones of pairs for the local lifts $\pi_{1,v}$ and $\pi_{2,v}$.

For the case $GL_1' \subset H_1'$, the normalizing factor for the intertwining operator $A(s,\pi_v',w_2)$, where π_v' is an irreducible unitary representation of $GL_1'(k_v)$ and w_2 the unique nontrivial Weyl group element, equals

$$r(s,\pi_v',w_2)=\frac{L(s,\pi_v)}{L(1+s,\pi_v)\varepsilon(s,\pi_v,\psi_v)}\frac{L(2s,\omega_{\pi_v})}{L(1+2s,\omega_{\pi_v})\varepsilon(2s,\omega_{\pi_v},\psi_v)}, \tag{2.11}$$

where the L–functions and ε–factors are the principal Jacquet ones of the local lift π_v and the Hecke ones of the central character ω_{π_v} of π_v. Observe that $\omega_{\pi_v'}=\omega_{\pi_v}$.

The next corollary gives the normalizing factors in a more precise form if at least one of the representations $\pi_{1,v}'$ and $\pi_{2,v}'$ is one–dimensional. The form of the normalizing factors is made suitable for obtaining the global normalizing factors in Subsection 2.4.

Recall from Section 1 that the local lift of the one–dimensional representation $\chi_v\circ\det_v'$ of $GL_1'(k_v)$, where χ_v is a unitary character of $k_v^\times$, is the Steinberg representation St_{χ_v} of $GL_2(k_v)$. Then, the corollary is obtained from the previous corollary using the expressions for the Rankin–Selberg and principal Jacquet L–functions and ε–factors involving the Steinberg representations and the fact that the central character of St_{χ_v} is χ_v^2. These expressions are given in Theorem (3.1), Sections 8 and 9 of [15] and Section (3.1) of [13].

Corollary 2.3.3. *For the case $GL_1'\times GL_1'\subset GL_2'$, the normalizing factor for the standard intertwining operator $A((s_1,s_2),(\chi_{1,v}\circ\det_v')\otimes\pi_{2,v}',w_1)$, where $\chi_{1,v}$ is a unitary character of $k_v^\times$, $\pi_{2,v}$ is the local component at v of a non–one–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$ and w_1 is the unique nontrivial Weyl group element, equals*

(2.12)
$$\begin{aligned}&r((s_1,s_2),(\chi_{1,v}\circ\det_v')\otimes\pi_{2,v}',w_1)\\&=\frac{L(s_1-s_2-1/2,\chi_{1,v}\widetilde{\pi}_{2,v})}{L(s_1-s_2+3/2,\chi_{1,v}\widetilde{\pi}_{2,v})\varepsilon(s_1-s_2+1/2,\chi_{1,v}\widetilde{\pi}_{2,v},\psi_v)\varepsilon(s_1-s_2-1/2,\chi_{1,v}\widetilde{\pi}_{2,v},\psi_v)},\end{aligned}$$

where the L–function and ε–factor are the principal Jacquet ones for $\chi_{1,v}\widetilde{\pi}_{2,v}$ and $\pi_{2,v}$ is the local lift of $\pi_{2,v}'$. In the case of $\pi_{1,v}'$ being the local component at v of a non–one–dimensional cuspidal automorphic representation of $GL_1'(\mathbb{A})$ and $\pi_{2,v}'\cong\chi_{2,v}\otimes\det_v'$, where $\chi_{2,v}$ is a unitary character of $k_v^\times$, the normalizing factor can be written in the same form with the principal Jacquet L–functions and ε–factors for $\pi_{1,v}\chi_{2,v}^{-1}$ instead of $\chi_{1,v}\widetilde{\pi}_{2,v}$.

For the case $GL_1'\times GL_1'\subset GL_2'$, the normalizing factor for the standard intertwining operator $A((s_1,s_2),(\chi_{1,v}\circ\det_v')\otimes(\chi_{2,v}\circ\det_v'),w_1)$, where $\chi_{1,v}$ and $\chi_{2,v}$

are unitary characters of $k_v^\times$ and w_1 is the unique nontrivial Weyl group element, equals

(2.13)
$$r((s_1,s_2),(\chi_{1,v}\circ\det_v')\otimes(\chi_{2,v}\circ\det_v'),w_1)$$
$$=r_v(s_1-s_2,\chi_{1,v}\chi_{2,v}^{-1})\frac{L(s_1-s_2,\chi_{1,v}\chi_{2,v}^{-1})L(s_1-s_2+1,\chi_{1,v}\chi_{2,v}^{-1})}{L(-(s_1-s_2),\chi_{1,v}^{-1}\chi_{2,v})L(1-(s_1-s_2),\chi_{1,v}^{-1}\chi_{2,v})},$$

where $r_v(s,\chi_v)$, for $s\in\mathbb{C}$ and a unitary character χ_v, is defined by equation (2.6) *in Corollary* 2.2.5 *and where the L–functions and ε–factors are the Hecke ones.*

For the case $GL_1'\subset H_1'$, the normalizing factor for the intertwining operator $A(s,\chi_v\circ\det_v',w_2)$, where χ_v is a unitary character of $k_v^\times$ and w_2 the unique nontrivial Weyl group element, equals

$$r(s,\chi_v\circ\det_v',w_2)=\frac{L(s-1/2,\chi_v)}{L(s+3/2,\chi_v)\varepsilon(s+1/2,\chi_v,\psi_v)\varepsilon(s-1/2,\chi_v,\psi_v)}\cdot\frac{L(s+1/2,\chi_v)}{L(1/2-s,\chi_v^{-1})}\frac{L(2s,\chi_v^2)}{L(1+2s,\chi_v^2)\varepsilon(2s,\chi_v^2,\psi_v)},\tag{2.14}$$

where the L–functions and ε–factors are the Hecke ones.

2.4. **Global normalization.** In this subsection we combine the local results of the previous subsections to obtain the global normalizing factors. Let $\pi'\cong\pi_1'\otimes\pi_2'$ be a cuspidal automorphic representation of the Levi factor $M_0'(\mathbb{A})\cong GL_1'(\mathbb{A})\times GL_1'(\mathbb{A})$ in $H_2'(\mathbb{A})$. In the rest of the paper we distinguish three cases depending on the type of π':

- **A.** Both π_1' and π_2' are not one–dimensional.
- **B.** One among π_1' and π_2' is one–dimensional and the other is not.
- **C.** Both π_1' and π_2' are one–dimensional.

The global lifts defined in Section 1 of π', π_1' and π_2' are denoted π, π_1 and π_2. Recall that if π_i' is not one–dimensional, then π_i is cuspidal.

Let $\pi'\cong\otimes_v\pi_v'$, where $\pi_v'\cong\pi_{1,v}'\otimes\pi_{2,v}'$, be the decomposition of a cuspidal automorphic representation into the restricted tensor product as in [4]. For $\underline{s}\in\mathfrak{a}^*_{M_0',\mathbb{C}}$ and $w\in W'$, the global standard intertwining operator denoted by $A(\underline{s},\pi',w)$ is defined by the global integral of the same form as the local integrals defining the local standard intertwining operators. For more details see Section II.1.6 of [25]. It is a tensor product of the local intertwining operators over all places. At unramified places the local standard intertwining operator sends the unique suitably normalized vector invariant for the fixed maximal compact subgroup into the invariant vector normalized in the same way multiplied by the local normalizing factor $r(\underline{s},\pi_v',w)$.

The global normalizing factor for $A(\underline{s},\pi',w)$ is defined as

$$r(\underline{s},\pi',w)=\prod_v r(\underline{s},\pi_v',w).\tag{2.15}$$

It is meromorphic in $\underline{s}\in\mathfrak{a}^*_{M_0',\mathbb{C}}$. Then, the global normalized intertwining operator is given by

$$A(\underline{s},\pi',w)=r(\underline{s},\pi',w)N(\underline{s},\pi',w).$$

It is a tensor product of the local normalized intertwining operators over all places. At unramified places it just sends the suitably normalized invariant vector for the fixed maximal compact subgroup into the invariant one normalized in the same way. The following theorem deals with the holomorphy and non–vanishing of the global

normalized intertwining operators. The standard proof, which is omitted, reduces the question to the local results of the previous subsection. The excluded points of the theorem are just the possible poles of the normalized intertwining operators, and in the calculation we regard these points as possible poles of the Eisenstein series.

Theorem 2.4.1. *Let $\pi' \cong \pi'_1 \otimes \pi'_2$ be a cuspidal automorphic representation of the Levi factor $M'_0(\mathbb{A})$ in $H'_2(\mathbb{A})$. Then, for every $w \in W'$, the global normalized operator $N(\underline{s}, \pi', w)$ is holomorphic and non–vanishing for $\underline{s} = (s_1, s_2) \in \mathfrak{a}^*_{M'_0,\mathbb{C}}$ such that*

- *$Re(s_1) \geqslant Re(s_2) \geqslant 0$ except at $Re(s_1) = Re(s_2) = 0$ in the case C, i.e. in the closure of the positive Weyl chamber except at the origin in the case C,*
- *$0 < s_1 < 1/2$ and $s_2 = 1/2$ in all the cases,*
- *in the case B with π'_2 one–dimensional, $0 < s_1 < 3/2$ and $s_2 = 3/2$, except at $(s_1, s_2) = (r, 3/2)$ for certain $0 < r < 1/2$ depending on π'; the exceptional point does not appear if the global lift π_1 of π'_1 satisfies the Ramanujan conjecture,*
- *in the case C, $0 < s_1 < 3/2$ and $s_2 = 3/2$, except at $(s_1, s_2) = (1/2, 3/2)$; at the exceptional point it is always holomorphic and non–vanishing if $w \in \{1, w_2, w_1w_2, w_2w_1w_2\}$,*
- *in the case A, $1/2 < s_1 < 1$ and $s_1 - s_2 = 1$,*
- *in the case C, $1 < s_1 < 2$ and $s_1 - s_2 = 2$, except at $(s_1, s_2) = (3/2, -1/2)$; at the exceptional point it is always holomorphic and non–vanishing if $w \in \{1, w_1, w_2w_1, w_1w_2w_1\}$.*

Finally, for Cases A, B and C, we collect the global normalizing factors for the maximal standard proper parabolic subgroups needed in the sequel. For $GL'_1 \times GL'_1 \subset GL'_2$ in Case A the local normalizing factors are given by equation (2.1) of Corollary 2.1.3 and (2.10) of Corollary 2.3.2. For $GL'_1 \subset H'_1$ in Case A the local normalizing factors are given by equation (2.2) of Corollary 2.1.3 and (2.11) of Corollary 2.3.2.

Corollary 2.4.2 (Case A). *For $GL'_1 \times GL'_1 \subset GL'_2$, the global normalizing factor for the standard intertwining operator $A((s_1, s_2), \pi'_1 \otimes \pi'_2, w_1)$, where $\pi'_1 \otimes \pi'_2$ is a case A cuspidal automorphic representation of $GL'_1(\mathbb{A}) \times GL'_1(\mathbb{A})$, equals*

$$r((s_1, s_2), \pi'_1 \otimes \pi'_2, w_1) = \frac{L(s_1 - s_2, \pi_1 \times \widetilde{\pi}_2)}{L(1 + s_1 - s_2, \pi_1 \times \widetilde{\pi}_2)\varepsilon(s_1 - s_2, \pi_1 \times \widetilde{\pi}_2)}, \tag{2.16}$$

where the L–function and ε–factor are the global Rankin–Selberg ones of pairs for the global lifts π_1 and π_2.

For $GL'_1 \subset H'_1$, the global normalizing factor for the standard intertwining operator $A(s, \pi', w_2)$, where π' is a cuspidal automorphic representation of $GL'_1(\mathbb{A})$ which is not one–dimensional, equals

$$r(s, \pi', w_2) = \frac{L(s, \pi)}{L(1 + s, \pi)\varepsilon(s, \pi)} \frac{L(2s, \omega_\pi)}{L(1 + 2s, \omega_\pi)\varepsilon(2s, \omega_\pi)}, \tag{2.17}$$

where the L–function and ε–factor are the global principal Jacquet ones for the global lift π and the global Hecke ones for the central character ω_π of the global lift π. Observe that $\omega_\pi = \omega_{\pi'}$.

For $GL_1' \times GL_1' \subset GL_2'$ in Case B the local normalizing factors are given by equation (2.4) of Corollary 2.2.5 and (2.12) of Corollary 2.3.3. For $GL_1' \subset H_1'$ in Case B the global normalizing factor is already obtained in the corollaries for either Case A or Case C.

Corollary 2.4.3 (**Case B**). *For $GL_1' \times GL_1' \subset GL_2'$, the global normalizing factor for the standard intertwining operator $A((s_1,s_2),(\chi_1 \circ \det') \otimes \pi_2', w_1)$, where χ_1 is a unitary character of $\mathbb{A}^\times/k^\times$ and π_2' a cuspidal automorphic representation of $GL_1'(\mathbb{A})$ which is not one–dimensional, equals*

(2.18)
$$r((s_1,s_2),(\chi_1 \circ \det') \otimes \pi_2', w_1) = \frac{L(s_1-s_2-1/2,\chi_1\widetilde{\pi}_2)}{L(s_1-s_2+3/2,\chi_1\widetilde{\pi}_2)\varepsilon(s_1-s_2+1/2,\chi_1\widetilde{\pi}_2)\varepsilon(s_1-s_2-1/2,\chi_1\widetilde{\pi}_2)},$$

where the L–function and ε–factor are the global principal Jacquet ones for $\chi_1\widetilde{\pi}_2$ and π_2 is the global lift of π_2'. For the intertwining operator $A((s_1,s_2),\pi_1' \otimes (\chi_2 \circ \det'), w_1)$, where now π_1' is not one–dimensional and χ_2 is a unitary character of $\mathbb{A}^\times/k^\times$, the normalizing factor is of the same form with the global principal Jacquet L–function and ε–factor for $\pi_1\chi_2^{-1}$ instead of $\chi_1\widetilde{\pi}_2$.

For $GL_1' \times GL_1' \subset GL_2'$ in Case C the local normalizing factors are given by equation (2.5) of Corollary 2.2.5 and (2.13) of Corollary 2.3.3. For $GL_1' \subset H_1'$ in Case C the local normalizing factors are given by equation (2.7) of Corollary 2.2.5 and (2.14) of Corollary 2.3.3.

Corollary 2.4.4 (**Case C**). *For $GL_1' \times GL_1' \subset GL_2'$, the global normalizing factor for the standard intertwining operator $A((s_1,s_2),(\chi_1 \circ \det') \otimes (\chi_2 \circ \det'), w_1)$, where χ_1 and χ_2 are unitary characters of $\mathbb{A}^\times/k^\times$, equals*

(2.19)
$$r((s_1,s_2),(\chi_1 \circ \det') \otimes (\chi_2 \circ \det'), w_1) = r(s_1-s_2,\chi_1\chi_2^{-1}) \prod_{v \in S_D} \frac{L(s_1-s_2,\chi_{1,v}\chi_{2,v}^{-1})L(s_1-s_2+1,\chi_{1,v}\chi_{2,v}^{-1})}{L(-(s_1-s_2),\chi_{1,v}^{-1}\chi_{2,v})L(1-(s_1-s_2),\chi_{1,v}^{-1}\chi_{2,v})},$$

where, for $s \in \mathbb{C}$ and a unitary character χ of $\mathbb{A}^\times/k^\times$, $r(s,\chi)$ is a product over all places of $r_v(s,\chi_v)$ defined in equation (2.6) of Corollary 2.2.5, i.e.

$$r(s,\chi) = \frac{L(s,\chi)L(s-1,\chi)}{L(s+2,\chi)L(s+1,\chi)\varepsilon(s+1,\chi)\varepsilon(s,\chi)^2\varepsilon(s-1,\chi)}, \tag{2.20}$$

and the L–functions and ε–factors are the global and local Hecke ones.

For $GL_1' \subset H_1'$, the global normalizing factor for the intertwining operator $A(s,\chi \circ \det', w_2)$, where χ is a unitary character of $\mathbb{A}^\times/k^\times$, equals

$$\begin{aligned} r(s,\chi \circ \det', w_2) = {} & \frac{L(s-1/2,\chi)}{L(s+3/2,\chi)\varepsilon(s+1/2,\chi)\varepsilon(s-1/2,\chi)} \\ & \cdot \prod_{v \in S_D} \frac{L(s+1/2,\chi_v)}{L(1/2-s,\chi_v^{-1})} \cdot \frac{L(2s,\chi^2)}{L(1+2s,\chi^2)\varepsilon(2s,\chi^2)}, \end{aligned} \tag{2.21}$$

where the L–functions and ε–factors are the global and local Hecke ones.

3. Calculation of the residual spectrum

Let L^2_{res} denote the residual spectrum of $H'_2(\mathbb{A})$. By definition it is the orthogonal complement of the cuspidal spectrum inside the discrete spectrum of $H'_2(\mathbb{A})$. By the Langlands spectral theory, explained in [20] and [25], the constituents of L^2_{res} are obtained by taking the iterated residues at the poles of the Eisenstein series attached to cuspidal automorphic representations of the Levi factors of standard proper parabolic subgroups of H'_2. In this paper we decompose the part L^2_{res,M'_0} of the residual spectrum coming from the poles of the Eisenstein series attached to cuspidal automorphic representations of the Levi factor $M'_0(\mathbb{A}) \cong GL'_1(\mathbb{A}) \times GL'_1(\mathbb{A})$ of the minimal parabolic subgroup $P'_0(\mathbb{A})$ of $H'_2(\mathbb{A})$. Now, very briefly, we explain the application of the Langlands spectral theory in our case. For more details see Section V of [25] or, for the low rank examples, [16] and [8].

3.1. **Brief overview of the method.** Let π' be a cuspidal automorphic representation of $M'_0(\mathbb{A})$. Let $\underline{s} \in \mathfrak{a}^*_{M'_0,\mathbb{C}}$ and $f_{\underline{s}} \in I(\underline{s}, \pi')$, where the dependency of automorphic forms $f_{\underline{s}}$ on $\underline{s}$ is analytic on $\mathfrak{a}^*_{M'_0,\mathbb{C}}$ and Paley–Wiener with values in the space of the induced representation. Then, the Eisenstein series is defined as the analytic continuation from the domain of the convergence of the series

$$E(\underline{s}, g; f_{\underline{s}}, \pi') = \sum_{\gamma \in P'_0(k)\backslash H'_2(k)} f_{\underline{s}}(\gamma g) \tag{3.1}$$

for $g \in H'_2(\mathbb{A})$. It is meromorphic as a function of $\underline{s}$. By the Langlands spectral theory, the contribution of π' to the whole space of square–integrable automorphic forms of $H'_2(\mathbb{A})$ is generated by the integrals

$$g \mapsto \frac{1}{(2\pi i)^2} \int_{Re(\underline{s})=\underline{s}_0} E(\underline{s}, g; f_{\underline{s}}, \pi') d\underline{s}, \tag{3.2}$$

where $\underline{s}_0$ is deep enough in the positive Weyl chamber so that the integral defining the global intertwining operators and the sum defining the Eisenstein series converge absolutely at $\underline{s}_0$.

During the calculation of the poles of the Eisenstein series we always assume that they are real. There is no loss of generality because that can be achieved just by twisting a cuspidal automorphic representation of a Levi factor by the appropriate imaginary power of the absolute value of the reduced norm of the determinant. Hence, this assumption is just a convenient choice of coordinates, and in the sequel we always assume that $s_1, s_2 \in \mathbb{R}$. In the figures in the following subsections only the real part of $\mathfrak{a}^*_{M'_0,\mathbb{C}}$ is presented.

When deforming the line of integration in (3.2) from $\underline{s}_0$ to the origin inside the positive Weyl chamber as in the figures below, we cross the singular hyperplanes which are the poles of the Eisenstein series. The integral at the origin gives a part of the continuous spectrum, while the residues at the singular hyperplanes are the possible contributions to the residual spectrum. Next, we take the coordinate system on the singular hyperplane such that the origin is the orthogonal projection of the origin in $\mathfrak{a}^*_{M'_0,\mathbb{C}}$ and continue with the same procedure. In such a way after taking the iterated residues at the poles of the Eisenstein series we are left with the contribution of π' to L^2_{res,M'_0}.

The analytic properties of the Eisenstein series coincide with the analytic properties of their constant terms along P'_0. Therefore, instead of the poles and square

integrability of the Eisenstein series we can study the poles and square integrability of their constant terms. The benefit lies in the fact that, by Proposition II.1.7. of [25], the constant term equals

$$\sum_{w\in W'} A(\underline{s},\pi',w)f_{\underline{s}}(g), \tag{3.3}$$

and for the standard intertwining operators $A(\underline{s},\pi',w)$ we have defined in Section 2 the scalar meromorphic normalizing factors $r(\underline{s},\pi',w)$. By Theorem 2.4.1, the normalized intertwining operators $N(\underline{s},\pi',w)$ are holomorphic and non–vanishing in the regions required in the calculations below. Thus, the calculation of the poles of (3.3) reduces to the poles of the global normalizing factors. The points excluded in Theorem 2.4.1 are also treated as possible poles during calculation.

The first step in the decomposition of L^2_{res,M'_0} is according to the type of π', i.e. Cases A, B or C. Thus,

$$L^2_{res,M'_0} \cong L^2_A \oplus L^2_B \oplus L^2_C, \tag{3.4}$$

where L^2_A, L^2_B and L^2_C are the parts of L^2_{res,M'_0} obtained as the iterated residues at the poles of the Eisenstein series attached to Cases A, B and C cuspidal automorphic representations of $M'_0(\mathbb{A})$, respectively. In the following Subsections 3.2 and 3.3 we decompose L^2_B and L^2_C. The decomposition of L^2_A is omitted because it can be rewritten line by line from the Case A decomposition in Section 2.2 of [8]. The only difference is the additional non–vanishing condition at $s = 1/2$ for the principal L–functions $L(s,\pi_i)$ attached to the global Jacquet–Langlands lifts of π'_i. This condition comes from the principal L–function in the Case A global normalizing factor for the H'_1 intertwining operator attached to w_2 given in Corollary 2.4.2. For the inner form G'_2 of SO_8 the principal L–function does not appear in the corresponding normalizing factor.

Before proceeding to the calculation we recall the Langlands square integrability criterion given in Section I.4.11 of [25] and on page 104 of [20], and the analytic properties of L–functions given for the Hecke L–functions in [36], for the principal Jacquet L–functions for GL_2 in [14] and for the Rankin–Selberg L–functions of pairs for $GL_2 \times GL_2$ in [12]. Observe that the global Hecke L–function $L(s,\mathbf{1})$ for the trivial character $\mathbf{1}$ of $\mathbb{A}^\times/k^\times$ is nothing other than the Dedekind ζ–function of the algebraic number field k. Recall that in this paper a unitary character μ of $\mathbb{A}^\times/k^\times$ is said to be quadratic if μ^2 is trivial.

Lemma 3.1.1. *The space obtained as the iterated residue at the pole $\underline{s} = (s_1,s_2) \in \mathfrak{a}^*_{M'_0,\mathbb{C}}$ of the Eisenstein series attached to a cuspidal automorphic representation π' of $M'_0(\mathbb{A})$ consists of square–integrable automorphic forms if and only if $w(\underline{s}) = (s'_1,s'_2)$ satisfies $s'_1 < 0$ and $s'_1+s'_2 < 0$, for every $w \in W'$, such that the corresponding intertwining operator in the constant term* (3.3) *gives a non-trivial contribution.*

Lemma 3.1.2. *The global Rankin–Selberg L–function of pairs $L(s,\sigma_1 \times \sigma_2)$ for cuspidal automorphic representations σ_1 and σ_2 of $GL_2(\mathbb{A})$ has simple poles at $s = 0$ and $s = 1$ if $\sigma_1 \cong \widetilde{\sigma}_2$, and it is entire otherwise. It has no zeroes for $Re(s) \geqslant 1$.*

The global principal Jacquet L–function $L(s,\sigma)$ for a cuspidal automorphic representation σ of $GL_2(\mathbb{A})$ is entire. It has no zeroes for $Re(s) \geqslant 1$.

The global Hecke L–function $L(s,\mu)$ for a unitary character μ of $\mathbb{A}^\times/k^\times$ has simple poles at $s=0$ and $s=1$ if μ is trivial, and it is entire otherwise. It has no zeroes for $Re(s)\geqslant 1$.

The local Hecke L–function $L(s,\mu_v)$ for a unitary character μ_v of a non–archimedean field $k_v^\times$ has the only real simple pole at $s=0$ if μ_v is trivial, and it is entire otherwise. It has no zeroes.

The proof of the following elementary lemma repeatedly used in the calculations is omitted.

Lemma 3.1.3. *Let $L(s)$ be a meromorphic function on $\mathbb{C}$ having only simple poles, $L(0)\neq 0$, and satisfying the functional equation $L(s)=\varepsilon(s)L(1-s)$, where $\varepsilon(s)$ is an entire non–vanishing function such that $\varepsilon(0)\varepsilon(1)=1$. Then*

$$\left.\frac{L(s)}{L(1+s)\varepsilon(s)}\right|_{s=0}=\begin{cases}-1, & \text{if } s=0 \text{ is a simple pole of } L(s),\\ 1, & \text{otherwise.}\end{cases}$$

The following simple lemma is very helpful in describing the images of the normalized intertwining operators obtained below as the residues of the constant terms of the Eisenstein series. The same lemma was used several times in our previous paper [8]. Hence, here we skip the details. When applying the lemma, w' and w'' are carefully chosen in such a way that, besides (1) and (2), $w''ww'$ is the longest Weyl group element and $w'^{-1}(\underline{s}+\underline{s}')$ is at least in the closure of the positive Weyl chamber. This enables a description of the image using the Langlands classification.

Lemma 3.1.4. *Let π_v be an irreducible unitary representation of the Levi factor $M_0(k_v)$ of the standard parabolic subgroup of Sp_8, $\underline{s}\in\mathfrak{a}^*_{M_0,\mathbb{C}}$ and $w\in W(M_0)$. Assume that there is a Levi subgroup $L\subset M_0$, $\underline{s}'\in\mathfrak{a}^*_{L,\mathbb{C}}$ and a tempered representation τ_v of $L(k_v)$ such that π_v is the unique irreducible subrepresentation of the induced representation $I_L^M(\underline{s}',\tau_v)$. Suppose that w' and w'' are the elements of the Weyl group W of Sp_8 such that the*

1. *image of the normalized intertwining operator $N(w'^{-1}(\underline{s}+\underline{s}'),w'^{-1}(\tau_v),w')$ is $I(\underline{s},\pi_v)$,*
2. *restriction of the normalized intertwining operator $N(w(\underline{s}+\underline{s}'),w(\tau_v),w'')$ to the induced representation $I(w(\underline{s}),w(\pi_v))$ is injective,*

where we identified $\underline{s}$ with an element of $\mathfrak{a}^*_{L,\mathbb{C}}$. Then the image of the normalized intertwining operator $N(\underline{s},\pi_v,w)$ is isomorphic to the image of $N(w'^{-1}(\underline{s}+\underline{s}'),w'^{-1}(\tau_v),w''ww')$.

Proof. The lemma is a simple consequence of the decomposition property of normalized intertwining operators. □

3.2. **Case B.** In this case a cuspidal automorphic representation $\pi'\cong\pi'_1\otimes\pi'_2$ of $M'_0(\mathbb{A})$ is such that one of the representations π'_1 and π'_2 is one–dimensional and the other is not. The global normalizing factors for the maximal standard proper parabolic subgroup with the Levi factor $GL'_1\times GL'_1\subset GL'_2$ are given in Corollary 2.4.3 and the Levi factor $GL'_1\subset H'_1$ in Corollary 2.4.2 for non–one–dimensional representations and Corollary 2.4.4 for one–dimensional representations. By the analytic properties of the L–functions of Lemma 3.1.2, the possible singular hyperplanes of the normalizing factors for the intertwining operators in the sum (3.3) are shown in Figure 3.1 if π'_1 is one–dimensional and in Figure 3.2 if π'_2 is one–dimensional.

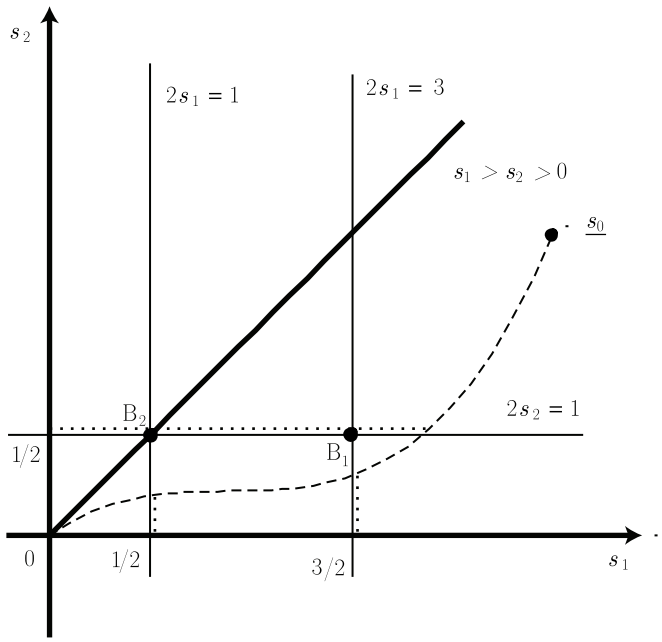

FIGURE 3.1. Case B singular hyperplanes for π_1' one–dimensional

There are four possible iterated poles at points

$$B_1(3/2, 1/2), \quad B_2(1/2, 1/2)$$

if π_1' is one–dimensional and

$$B_3(1/2, 3/2), \quad B_4(1/2, 1/2)$$

if π_2' is one–dimensional. Note that we do not consider the possible iterated pole at $B_5(r, 3/2)$ when π_2' is one–dimensional. By Theorem 2.4.1, if the Ramanujan conjecture holds for cuspidal automorphic representations of $GL_2(\mathbb{A})$, then B_5 is not a pole. Although in principle one could describe the hypothetical contribution at B_5 in the same way as for the poles at C_4 or C_6 in Section 3.3, we skip that here since it would not bring any new insight. Having that in mind, L_B^2 decomposes into

$$L_B^2 \cong L_{B_1}^2 \oplus L_{B_2}^2 \oplus L_{B_3}^2 \oplus L_{B_4}^2.$$

The cases of B_1 and B_3, as well as B_2 and B_4, are in fact the same. For a pair of points the results and the proofs can be obtained from each other just by interchanging the roles of π_1' and π_2', s_1 and s_2, etc. Therefore, we state and prove only the decomposition of $L_{B_1}^2$ and $L_{B_2}^2$.

Before giving the decomposition of $L_{B_1}^2$, consider the induced representation

$$\begin{aligned}&\mathrm{Ind}_{GL_1'(k_v)\times GL_1'(k_v)}^{GL_2'(k_v)}((\mathbf{1}_v \circ \det_v')\nu^{3/2} \otimes \pi_{2,v}'\nu^{1/2})\\ &\qquad \cong \mathrm{Ind}_{GL_1'(k_v)\times GL_1'(k_v)}^{GL_2'(k_v)}((\mathbf{1}_v \circ \det_v')\nu^{1/2} \otimes \pi_{2,v}'\nu^{-1/2}),\end{aligned}$$

where $\mathbf{1}_v$ is the trivial character of $k_v^\times$ and $\pi_{2,v}'$ is a unitary, generic at split places, irreducible representation with the trivial central character. It is irreducible as a

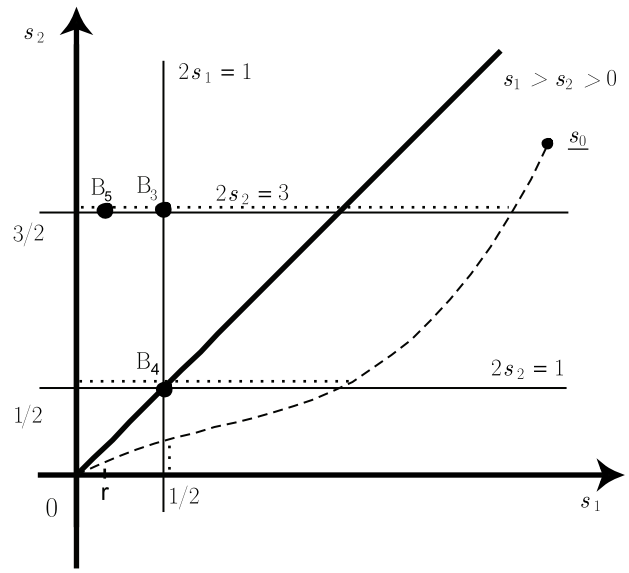

FIGURE 3.2. Case B singular hyperplanes for π_2' one–dimensional

consequence of [3], [2], and [24]. Hence, the normalized intertwining operator

$$N(1, (\mathbf{1}_v \circ \det_v') \otimes \pi_{2,v}', w_1),$$

where $1 = 1\widetilde{\alpha} = (1/2, -1/2)$, acts as Id or $-Id$. We denote the sign by η_v. Its inverse, required in the decomposition of L_{B_3}, acts by the same scalar.

Theorem 3.2.1. *The subspace $L^2_{B_1}$ of the residual spectrum of $H_2'(\mathbb{A})$ decomposes into*

$$L^2_{B_1} = \bigoplus_{\pi'} \mathcal{B}_1(\pi'),$$

where the sum is over all cuspidal automorphic representations $\pi' \cong (\mathbf{1} \circ \det') \otimes \pi_2'$ of $M_0'(\mathbb{A})$ such that $\mathbf{1}$ is the trivial character of $\mathbb{A}^\times/k^\times$, π_2' is not one–dimensional, the central character $\omega_{\pi_2'}$ of π_2' is trivial, $L(1/2, \pi_2) \neq 0$ and the parity condition

$$\frac{L(1/2, \pi_2)L(-1/2, \pi_2)}{L(5/2, \pi_2)L(3/2, \pi_2)} \prod_v \eta_v \neq -1$$

holds, where π_2 is the global lift of π_2'.

$\mathcal{B}_1(\pi')$ is the irreducible space of automorphic forms spanned by the iterated residue at $\underline{s} = (3/2, 1/2)$ of the Eisenstein series attached to π'. The constant term map gives rise to an isomorphism of $\mathcal{B}_1(\pi')$ and the image of the normalized intertwining operator $N((3/2, 1/2), \pi', w_2 w_1 w_2)$.

Proof. Let $\pi' \cong (\chi_1 \circ \det') \otimes \pi_2'$ be a Case B cuspidal automorphic representation of $M_0'(\mathbb{A})$. The iterated pole at $B_1(3/2, 1/2)$ of the Eisenstein series attached to π' is first calculated along the singular hyperplane $2s_2 = 1$ as shown in Figure 3.1. The pole along $2s_2 = 1$ occurs if only if the central character $\omega_{\pi_2'}$ is trivial and $L(1/2, \pi_2) \neq 0$. In the new variable $z = s_1$ the residues are up to a non-zero constant given in Table 3.1, where $\mathbf{1}$ is the trivial character of $\mathbb{A}^\times/k^\times$. Observe that π_2 is selfcontragredient since ω_{π_2} is trivial.

TABLE 3.1. Residues along $2s_2 = 1$ of Case B normalizing factors for $\pi_1' = \chi_1 \circ \det'$

w	$Res_{2s_2=1} r(\underline{s}, \pi', w)$
w_2	1
$w_1 w_2$	$\frac{L(z,\chi_1\pi_2)}{L(z+2,\chi_1\pi_2)\varepsilon(z,\chi_1\pi_2)\varepsilon(z+1,\chi_1\pi_2)}$
$w_2 w_1 w_2$	$\frac{L(z,\chi_1\pi_2)}{L(z+2,\chi_1\pi_2)\varepsilon(z,\chi_1\pi_2)\varepsilon(z+1,\chi_1\pi_2)}$ $\frac{L(z-1/2,\chi_1)}{L(z+3/2,\chi_1)\varepsilon(z-1/2,\chi_1)\varepsilon(z+1/2,\chi_1)} \frac{L(2z,\chi_1^2)}{L(1+2z,\chi_1^2)\varepsilon(2z,\chi_1^2)} \prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v})}{L(1/2-z,\chi_{1,v}^{-1})}$
$w_1 w_2 w_1 w_2$	$\frac{L(z,\chi_1\pi_2)}{L(z+2,\chi_1\pi_2)\varepsilon(z,\chi_1\pi_2)\varepsilon(z+1,\chi_1\pi_2)} \frac{L(z-1,\chi_1\pi_2)}{L(z+1,\chi_1\pi_2)\varepsilon(z-1,\chi_1\pi_2)\varepsilon(z,\chi_1\pi_2)}$ $\frac{L(z-1/2,\chi_1)}{L(z+3/2,\chi_1)\varepsilon(z-1/2,\chi_1)\varepsilon(z+1/2,\chi_1)} \frac{L(2z,\chi_1^2)}{L(1+2z,\chi_1^2)\varepsilon(2z,\chi_1^2)} \prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v})}{L(1/2-z,\chi_{1,v}^{-1})}$

The terms in Table 3.1 have the pole at $B_1(3/2, 1/2)$, i.e. $z = 3/2$, if and only if χ_1 is trivial. Up to a non-zero constant the residue of the term corresponding to $w_2 w_1 w_2$ equals

$$N((3/2, 1/2), \pi', w_2 w_1 w_2),$$

while after applying the global functional equation for the L–functions, the residue of the term corresponding to $w_1 w_2 w_1 w_2$ equals

$$\frac{L(1/2, \pi_2)L(-1/2, \pi_2)}{L(5/2, \pi_2)L(3/2, \pi_2)} N((3/2, 1/2), \pi', w_1 w_2 w_1 w_2).$$

The residue acting at a decomposable vector of the induced representation gives

$$N((3/2, 1/2), \pi', w_2 w_1 w_2) \left[Id + \frac{L(1/2, \pi_2)L(-1/2, \pi_2)}{L(5/2, \pi_2)L(3/2, \pi_2)} N((3/2, 1/2), \pi', w_1) \right].$$

Now, the non–vanishing condition for the square–bracket gives the parity condition of the theorem. The Langlands square–integrability criterion of Lemma 3.1.1 is satisfied. As in Case B of [8], the irreducibility of the image of the normalized operator $N((3/2, 1/2), \pi', w_2 w_1 w_2)$ is proved. □

Before giving the decomposition of $L^2_{B_2}$ we consider the induced representation

$$\mathrm{Ind}_{GL_1'(k_v)\times GL_1'(k_v)}^{GL_2'(k_v)} \left((\chi_{1,v} \circ \det_v') \otimes \pi_{2,v}' \right),$$

where $\pi_{2,v}'$ is a unitary, generic at split places, irreducible representation with the trivial central character and $\chi_{1,v}$ a quadratic character of $k_v^\times$. It is irreducible at all places by [3], [2], and [33]. Hence, the normalized intertwining operator

$$N(0, (\chi_{1,v} \circ \det_v') \otimes \pi_{2,v}', w_1)$$

acts as Id or $-Id$. We denote the sign by η_v. Its inverse required in the decomposition of $L^2_{B_4}$ acts by the same scalar.

Furthermore, consider the image of the normalized intertwining operator

$$N((1/2, 1/2), \pi_v', w_1 w_2 w_1 w_2),$$

where $\pi_v' \cong (\chi_{1,v} \circ \det_v') \otimes \pi_{2,v}'$ and χ_1 and $\pi_{2,v}'$ are as above. At non–split places the image is irreducible by the Langlands classification since π_v' is supercuspidal. At split places, the image is described in terms of the Langlands classification as in Section 2.3 of [8]. However, due to more complicated reducibilities for the

symplectic group, the image is irreducible if and only if $\chi_{1,v}$ is trivial. If $\chi_{1,v}$ is a non–trivial quadratic character, it is a sum of two irreducible constituents. In both cases, we denote the constituents by $\Pi_v^{\pm}$ and make a convention that Π_v^- is trivial if $\chi_{1,v}$ is trivial. In terms of the Langlands classification, if $\pi_{2,v}$ is tempered, then $\Pi_v^{\pm}$ is the quotient of the standard module

$$\mathrm{Ind}_{GL_1(k_v)\times GL_2(k_v)\times SL_2(k_v)}^{Sp_8(k_v)}\left(\chi_{1,v}|\cdot|\otimes\pi_{2,v}\nu^{1/2}\otimes\tau_v^{\pm}\right),$$

while if $\pi_{2,v}$ is a complementary series attached to a unitary character μ_v of $k_v^\times$ and an exponent $0<r<1/2$, then it is the quotient of the standard module

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)\times GL_1(k_v)\times SL_2(k_v)}^{Sp_8(k_v)}\left(\chi_{1,v}|\cdot|\otimes\mu_v\nu^{1/2+r}\otimes\mu_v\nu^{1/2-r}\otimes\tau_v^{\pm}\right).$$

Here $\tau_v^{\pm}$ are irreducible tempered representations of $SL_2(k_v)$ defined by

$$\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)}\chi_{1,v}\cong\tau_v^+\oplus\tau_v^-,$$

where τ_v^- is trivial if $\chi_{1,v}$ is trivial.

Theorem 3.2.2. *The subspace $L^2_{B_2}$ of the residual spectrum of $H_2'(\mathbb{A})$ decomposes into*

$$L^2_{B_2}=\bigoplus_{\pi'}\mathcal{B}_2(\pi'),$$

where the sum is over all cuspidal automorphic representations $\pi'\cong(\chi_1\circ\det')\otimes\pi_2'$ of $M_0'(\mathbb{A})$ such that χ_1 is a non-trivial quadratic character, $\chi_{1,v}$ is non-trivial for all $v\in S_D$, π_2' is not one–dimensional, the central character $\omega_{\pi_2'}$ of π_2' is trivial, $L(1/2,\pi_2)\neq 0$ and $L(1/2,\chi_1\pi_2)\neq 0$, where π_2 is the global lift of π_2', and the parity condition $\prod_v\eta_v=1$ holds.

$\mathcal{B}_2(\pi')$ is the space of automorphic forms spanned by the iterated residue at $\underline{s}=(1/2,1/2)$ of the Eisenstein series attached to π'. The constant term map gives rise to an isomorphism of $\mathcal{B}_2(\pi')$ and the sum of the irreducible representations of the form $\otimes_v\Pi_v'$, where Π_v' is one of at most two irreducible components of the image of $N((1/2,1/2),\pi_v',w_1w_2w_1w_2)$ and at almost all split places it is Π_v^+.

Proof. The proof goes along the same lines as the proof of the previous theorem. The residues along $2s_2=1$ are already given in Table 3.1. Now, the pole at $z=1/2$ of the terms in Table 3.1 is obtained if and only if χ_1 is a quadratic character such that $\chi_{1,v}$ is non-trivial at all $v\in S_D$ and $L(1/2,\chi_1\pi_2)\neq 0$. The local condition comes from the local Hecke L–function in the denominator of the global normalizing factors which would otherwise cancel the pole. Again, using the global functional equation and decomposing, the iterated residue at $B_2(1/2,1/2)$ equals

$$N((1/2,1/2),\pi',w_2w_1w_2)\left[Id+N((1/2,1/2),\pi',w_1)\right].$$

The non–vanishing of the square–bracket gives the parity condition. The square–integrability criterion of Lemma 3.1.1 is satisfied. Since $N((1/2,1/2),\pi',w_1)$ is an isomorphism, the image of $N((1/2,1/2),\pi',w_2w_1w_2)$ is isomorphic to the image of $N((1/2,1/2),\pi',w_1w_2w_1w_2)$, which was decomposed at every place just before the statement of the theorem. Since an automorphic representation is unramified at almost all places, $\Pi_v'=\Pi_v^+$ at almost all split places. $\square$

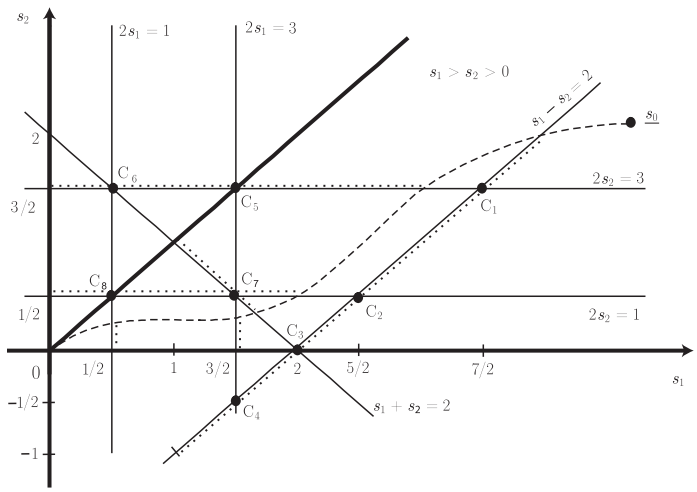

FIGURE 3.3. Case C singular hyperplanes

3.3. **Case C.** In this case $\pi' \cong \pi_1' \otimes \pi_2'$ is a cuspidal automorphic representation of $M_0'(\mathbb{A})$ such that $\pi_i' = \chi_i \circ \det'$, for $i = 1, 2$, where χ_i is a unitary character of $\mathbb{A}^\times/k^\times$. The global normalizing factors of the standard intertwining operators in the sum (3.3) are the products of the maximal proper parabolic subgroup cases given in Corollary 2.4.4. By the analytic properties of the L–functions of Lemma 3.1.2, the possible singular hyperplanes of the terms in the sum (3.3) are given in Figure 3.3. There are eight possible iterated poles denoted as in Figure 3.3 by

$$C_1(7/2, 3/2), \quad C_2(5/2, 1/2), \quad C_3(2, 0), \quad C_4(3/2, -1/2),$$
$$C_5(3/2, 3/2), \quad C_6(1/2, 3/2), \quad C_7(3/2, 1/2), \quad C_8(1/2, 1/2).$$

Hence, L^2_C decomposes accordingly into

$$L^2_C \cong L^2_{C_1} \oplus L^2_{C_2} \oplus L^2_{C_3} \oplus L^2_{C_4} \oplus L^2_{C_5} \oplus L^2_{C_6} \oplus L^2_{C_7} \oplus L^2_{C_8}.$$

Theorem 3.3.1. *The subspace $L^2_{C_1}$ of the residual spectrum of $H_2'(\mathbb{A})$ is the irreducible space of automorphic forms consisting only of constant functions on $H_2'(\mathbb{A})$.*

Proof. Let $\pi' \cong (\chi_1 \circ \det') \otimes (\chi_2 \circ \det')$ be a Case C cuspidal automorphic representation of $M_0'(\mathbb{A})$. As shown in Figure 3.3, for the contribution of π' to the residual spectrum at $C_1(7/2, 3/2)$ the iterated pole of the sum (3.3) is first considered along $s_1 - s_2 = 2$. It occurs if and only if $\chi_1 = \chi_2$. Let $\chi = \chi_1 = \chi_2$. The residues, written in a new variable z on $s_1 - s_2 = 2$ given by

$$s_1 = z + 1 \quad \text{and} \quad s_2 = z - 1$$

up to a non-zero constant, are given in Table 3.2.

Point C_1 corresponds to $z = 5/2$. The pole of the terms in Table 3.2 at $z = 5/2$ may occur only if χ is trivial. Then, only the term corresponding to the Weyl group element $w_1w_2w_1w_2$ has a pole. It is simple. Hence, up to a non-zero constant, the iterated residue at C_1 of the sum (3.3) equals

$$N((7/2, 3/2), (\mathbf{1} \circ \det') \otimes (\mathbf{1} \circ \det'), w_1w_2w_1w_2).$$

TABLE 3.2. Residues along $s_1 - s_2 = 2$ of Case C normalizing factors

w	$Res_{s_1-s_2=2} r(\underline{s}, (\chi \circ \det') \otimes (\chi \circ \det'), w)$
w_1	1
$w_2 w_1$	$\frac{L(z+1/2,\chi)}{L(z+5/2,\chi)\varepsilon(z+3/2,\chi)\varepsilon(z+1/2,\chi)} \prod_{v\in S_D} \frac{L(z+3/2,\chi_v)}{L(-1/2-z,\chi_v^{-1})} \cdot \frac{L(2z+2,\chi^2)}{L(2z+3,\chi^2)\varepsilon(2z+2,\chi^2)}$
$w_1 w_2 w_1$	$\frac{L(z+1/2,\chi)}{L(z+5/2,\chi)\varepsilon(z+3/2,\chi)\varepsilon(z+1/2,\chi)} \prod_{v\in S_D} \frac{L(z+3/2,\chi_v)}{L(-1/2-z,\chi_v^{-1})} \cdot \frac{L(2z+2,\chi^2)}{L(2z+3,\chi^2)\varepsilon(2z+2,\chi^2)} \frac{L(2z-1,\chi^2)L(2z,\chi^2)}{L(2z+1,\chi^2)L(2z+2,\chi^2)\varepsilon(2z-1,\chi^2)\varepsilon(2z,\chi^2)^2\varepsilon(2z+1,\chi^2)} \cdot \prod_{v\in S_D} \frac{L(2z,\chi_v^2)L(2z+1,\chi_v^2)}{L(-2z,\chi_v^{-2})L(1-2z,\chi_v^{-2})}$
$w_1 w_2 w_1 w_2$	$\frac{L(z+1/2,\chi)}{L(z+5/2,\chi)\varepsilon(z+3/2,\chi)\varepsilon(z+1/2,\chi)} \prod_{v\in S_D} \frac{L(z+3/2,\chi_v)}{L(-1/2-z,\chi_v^{-1})} \frac{L(2z+2,\chi^2)}{L(2z+3,\chi^2)\varepsilon(2z+2,\chi^2)} \frac{L(2z-1,\chi^2)L(2z,\chi^2)}{L(2z+1,\chi^2)L(2z+2,\chi^2)\varepsilon(2z-1,\chi^2)\varepsilon(2z,\chi^2)^2\varepsilon(2z+1,\chi^2)} \cdot \prod_{v\in S_D} \frac{L(2z,\chi_v^2)L(2z+1,\chi_v^2)}{L(-2z,\chi_v^{-2})L(1-2z,\chi_v^{-2})} \frac{L(z-3/2,\chi)}{L(z+1/2,\chi)\varepsilon(z-1/2,\chi)\varepsilon(z-3/2,\chi)} \prod_{v\in S_D} \frac{L(z-1/2,\chi_v)}{L(3/2-z,\chi_v^{-1})} \cdot \frac{L(2z-2,\chi^2)}{L(2z-1,\chi^2)\varepsilon(2z-2,\chi^2)}$

The square–integrability criterion of Lemma 3.1.1 is satisfied, and the image of that operator is the trivial representation of $Sp_8(k_v)$ at every place. Hence, $L^2_{C_1}$ consists only of the constant functions on $H_2'(\mathbb{A})$. □

Before giving the decomposition of $L^2_{C_2}$ consider the image of local normalized operator

$$N((5/2,1/2),(\chi_v \circ \det_v') \otimes (\chi_v \circ \det_v'), w_1 w_2 w_1 w_2),$$

where χ_v is a quadratic character of $k_v^\times$. It is irreducible at non–split places by the Langlands classification. At split places its image can be described as in the case of B_2 in Section 3.2. It is the sum of two irreducible representations if χ_v is non-trivial, and it is irreducible if χ_v is trivial. As before, we denote the irreducible components by Π_v^+ and Π_v^-, where Π_v^- is trivial if χ_v is trivial, and at unramified places Π_v^+ is the unramified component.

Theorem 3.3.2. *The subspace $L^2_{C_2}$ of the residual spectrum of $H_2'(\mathbb{A})$ is isomorphic to*

$$L^2_{C_2} = \bigoplus_{\pi'} \mathcal{C}_2(\pi'),$$

where the sum is over all one–dimensional cuspidal automorphic representations $\pi' \cong (\chi \circ \det') \otimes (\chi \circ \det')$ of $M_0'(\mathbb{A})$ such that χ is a non-trivial quadratic character and χ_v is non-trivial for all $v \in S_D$.

$\mathcal{C}_2(\pi')$ is the space of automorphic forms spanned by the iterated residue at $\underline{s} = (5/2,1/2)$ of the Eisenstein series attached to π'. The constant term map gives rise to an isomorphism of $\mathcal{C}_2(\pi')$ and the sum of the irreducible representations of the form $\bigotimes_v \Pi_v'$, where Π_v' is one of at most two irreducible components of the image of $N((5/2,1/2),\pi_v', w_1 w_2 w_1 w_2)$ and where it is Π_v^+ at almost all split places.

Proof. We skip the proof since it is the same as the proof of Theorem 3.3.1. The local condition of non–triviality of the local component χ_v at all places $v \in S_D$ comes from the local L–functions in the global normalizing factors. $\square$

Before decomposing $L^2_{C_3}$ consider the induced representation

$$\mathrm{Ind}_{GL_1'(k_v)}^{H_1'(k_v)} \left(\chi_v \circ \det_v'\right),$$

where χ_v is a quadratic character of $k_v^\times$. It is irreducible by [29], [30], [27], and [28]. Hence, the $H_1'(k_v)$ normalized intertwining operator

$$N(0, \chi_v \circ \det_v', w_2)$$

acts as Id or $-Id$, and we denote the sign by η_v.

Theorem 3.3.3. *The subspace $L^2_{C_3}$ of the residual spectrum of $H_2'(\mathbb{A})$ decomposes into*

$$L^2_{C_3} = \bigoplus_{\pi'} \mathcal{C}_3(\pi'),$$

where the sum is over all one–dimensional cuspidal automorphic representations $\pi' \cong (\chi \circ \det') \otimes (\chi \circ \det')$ of $M_0'(\mathbb{A})$ such that χ is a quadratic character and the parity condition $\prod_v \eta_v = -\varepsilon(1/2, \chi)$ holds.

$\mathcal{C}_3(\pi')$ is the irreducible space of automorphic forms spanned by the iterated residue at $\underline{s} = (2, 0)$ of the Eisenstein series attached to π'. The constant term map gives rise to an isomorphism of $\mathcal{C}_3(\pi')$ and the image of the normalized intertwining operator $N((2, 0), \pi', w_1w_2w_1)$. At non–split places it is the Langlands quotient of the induced representation

$$\mathrm{Ind}_{GL_1'(k_v)\times H_1'(k_v)}^{H_2'(k_v)} \left((\chi_v \circ \det_v') \otimes \tau_v\right),$$

where $\tau_v \cong \mathrm{Ind}_{GL_1'(k_v)}^{H_1'(k_v)} \left(\chi_v \circ \det_v'\right)$ is irreducible and tempered. At split places it is the Langlands quotient of the induced representation

$$\mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)} \left(\chi_v|\cdot|^{5/2} \otimes \chi_v|\cdot|^{3/2} \otimes \chi_v|\cdot|^{1/2} \otimes \chi_v|\cdot|^{1/2}\right),$$

where $T \cong GL_1 \times GL_1 \times GL_1 \times GL_1$ is the maximal split torus of Sp_8.

Proof. Calculating the residue at $z = 1$ (which corresponds to C_3) of the terms in Table 3.2, using Lemma 3.1.2, Lemma 3.1.3 and the global functional equation for L–functions shows that the pole occurs if and only if $\chi = \chi_1 = \chi_2$ is a quadratic character. The residue is non–zero only for terms corresponding to $w_1w_2w_1$ and $w_1w_2w_1w_2$. Their sum acting on the decomposable vector gives

$$N((2, 0), \pi', w_1w_2w_1) \left[Id - \varepsilon(1/2, \chi)N((2, 0), \pi', w_2)\right].$$

The parity condition comes from the non–vanishing of the square bracket. The square–integrability criterion of Lemma 3.1.1 is satisfied. The description in terms of the Langlands classification of the image of the normalized intertwining operator $N((2, 0), \pi', w_1w_2w_1w_2)$, which is isomorphic to the image of $N((2, 0), \pi', w_1w_2w_1)$, comes at a non–split place from the fact that $\tau_v' \cong \mathrm{Ind}_{GL_1'(k_v)}^{H_2'(k_v)} \left(\chi_v \circ \det_v'\right)$ is irreducible and tempered. At a split place one observes that the induced representation $\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)} \left(\chi_v|\cdot|^{-1/2}\right)$ is irreducible. $\square$

As in the decomposition of the corresponding space in Section 2.4 of [8], before decomposing $L^2_{C_4}$ we describe the images of certain local normalized intertwining operators. We study the behavior of the normalized intertwining operator $N(s, \chi_v \circ \det'_v, w_2)$ at $s = -1/2$, where χ_v is a unitary character of $k_v^\times$. If χ_v is quadratic, let X_v be the image of

$$N(1/2, \chi_v \circ \det'_v, w_2).$$

It is a subrepresentation of the induced representation

$$I_v = \mathrm{Ind}_{GL'_1(k_v)}^{H'_1(k_v)} \left((\chi_v \circ \det'_v)\nu^{-1/2} \right).$$

As a simple consequence of the Langlands classification, X_v is irreducible unless v is split and χ_v is a non-trivial quadratic character. If reducible, it is a direct sum of two non–isomorphic irreducible representations. Let $Y_v \cong I_v/X_v$ denote the quotient.

Lemma 3.3.4. *If χ_v is not quadratic, then the normalized intertwining operator*

$$N(s, \chi_v \circ \det'_v, w_2)$$

is holomorphic and non–vanishing at $s = -1/2$. Moreover, it is an isomorphism.

If χ_v is quadratic, then it has a pole at $s = -1/2$. The operator

$$\widetilde{N}(-1/2, \chi_v \circ \det'_v, w_2) = \lim_{s \to -1/2} (s + 1/2) N(s, \chi_v \circ \det'_v, w_2)$$

is holomorphic and non–vanishing. In the notation as above, its kernel is X_v and its image is isomorphic to Y_v. Thus, $N(s, \chi_v \circ \det'_v, w_2)$ at $s = -1/2$ restricted to X_v is holomorphic and non–vanishing.

Proof. The same as the proof of the corresponding Lemma in Section 2.4 of [8]. □

Corollary 3.3.5. *Let $\pi'_v \cong (\chi_v \circ \det'_v) \otimes (\chi_v \circ \det'_v)$. If χ_v is not quadratic, the image of the normalized intertwining operator $N((3/2, -1/2), \pi'_v, w_1 w_2 w_1 w_2)$, denoted by W_v, is non-trivial and isomorphic to the image of $N((3/2, 1/2), \pi'_v, w_1 w_2 w_1)$.*

The image of $N((3/2, 1/2), \pi'_v, w_1 w_2 w_1)\widetilde{N}(-1/2, \chi_v \circ \det'_v, w_2)$, denoted by W'_v, is non-trivial if χ_v is quadratic. Furthermore, if χ_v is quadratic, then the image of $N((3/2, 1/2), \pi'_v, w_1 w_2 w_1)$, again denoted by W_v, is non-trivial and contains W'_v as a subrepresentation.

Proof. Although we have not specified the irreducible constituents of Y_v in terms of the Langlands classification, the exponents are certainly at most 1, and the proof goes along the same lines as the proof of the corresponding Corollary in Section 2.4 of [8]. □

For a unitary character μ of $k^\times \backslash \mathbb{A}^\times$, let $S_1(\mu)$ denote the set of places of k such that μ_v is trivial. For a unitary character χ of $k^\times \backslash \mathbb{A}^\times$, let

$$m(\chi) = |S_1(\chi^2) \cap S_D| - |S_1(\chi) \cap S_D|.$$

Note that $m(\chi) \geq 0$ since $S_1(\chi) \subset S_1(\chi^2)$.

Theorem 3.3.6. *The subspace $L^2_{C_4}$ of the residual spectrum of $H'_2(\mathbb{A})$ decomposes into*

$$L^2_{C_4} = \left(\bigoplus_{\pi'} \mathcal{C}_4^{(1)}(\pi') \right) \oplus \left(\bigoplus_{\pi'} \mathcal{C}_4^{(2)}(\pi') \right).$$

The former sum is over all one–dimensional cuspidal automorphic representations $\pi' \cong (\chi \circ \det') \otimes (\chi \circ \det')$ *of* $M_0'(\mathbb{A})$ *such that* χ *is quadratic. The latter sum is over all one–dimensional cuspidal automorphic representations* $\pi' \cong (\chi \circ \det') \otimes (\chi \circ \det')$ *of* $M_0'(\mathbb{A})$ *such that* χ *is not quadratic but there is either at least one non–split place* $v \in S_D$ *where* χ_v *is trivial or at least one split place* $v \notin S_D$ *where* χ_v^2 *is trivial.*

The spaces $\mathcal{C}_4^{(1)}(\pi')$ *and* $\mathcal{C}_4^{(2)}(\pi')$ *are the spaces of automorphic forms spanned by the residues*

$$\lim_{z \to 1/2} (z - 1/2)^n Res_{s_1 - s_2 = 2} E(\underline{s}, g; f_{\underline{s}}, \pi'),$$

where n *is the order of the pole at* $z = 1/2$*. Here* z *is the new variable on* $s_1 - s_2 = 2$ *given by* $s_1 = z + 1$ *and* $s_2 = z - 1$*.*

If χ *is a non-trivial quadratic character, the constant term map gives rise to an isomorphism between* $\mathcal{C}_4^{(1)}(\pi')$ *and*

$$\bigoplus_V \left[\left(\bigotimes_{v \in V} W_v' \right) \otimes \left(\bigotimes_{v \notin V} W_v \right) \right],$$

where the sum is over all finite sets of places V *such that* $|V| = m(\chi)$ *and* W_v*,* W_v' *are defined in Corollary* 3.3.5*. If* χ *is trivial, then the constant term map implies that* $\mathcal{C}_4^{(1)}(\pi')$ *contains a space isomorphic to* $\bigoplus_w [W_w' \otimes (\bigotimes_{v \neq w} W_v)]$*, where the sum is over all places.*

If χ *is not quadratic, the constant term map gives rise to an isomorphism between* $\mathcal{C}_4^{(2)}(\pi')$ *and*

$$\bigoplus_V \left[\left(\bigotimes_{v \in V} W_v' \right) \otimes \left(\bigotimes_{v \notin V} W_v \right) \right],$$

where the sum is over all finite sets of places $V \subset S_1(\chi^2)$ *such that* $|V| = m(\chi) + 1$ *and* W_v*,* W_v' *are defined in Corollary* 3.3.5*.*

Proof. Along the same lines as the proof of the corresponding Theorem in Section 2.4 of [8]. The more complicated description is only due to the more complicated normalizing factors. □

Before passing to $L^2_{C_5}$ consider the normalized intertwining operator

$$N(0, (\mathbf{1}_v \circ \det_v') \otimes (\mathbf{1}_v \circ \det_v'), w_1)$$

acting on the induced representation

$$\mathrm{Ind}_{GL_1'(k_v) \times GL_1'(k_v)}^{GL_2'(k_v)} \left((\mathbf{1}_v \circ \det_v') \otimes (\mathbf{1}_v \circ \det_v') \right),$$

where $\mathbf{1}_v$ is the trivial character of $k_v^\times$. Since the induced representation is irreducible by [34], [2], and [33], the normalized intertwining operator acts as Id or $-Id$. We denote the sign by η_v.

The irreducibility of the spaces of automorphic forms appearing in the decomposition of $L^2_{C_5}$ follows from the following lemma. Using Lemma 3.1.4, it is a consequence of the Langlands classification.

Lemma 3.3.7. *Let* $\pi_v' \cong (\mathbf{1}_v \circ \det_v') \otimes (\mathbf{1}_v \circ \det_v')$ *be the trivial representation of* $M_0'(k_v)$*, where* $\mathbf{1}_v$ *is the trivial character of* $k_v^\times$*. Then, the images of the normalized intertwining operators*

$$N((3/2, 3/2), \pi_v', w_2 w_1 w_2) \quad \textit{and} \quad N((3/2, 3/2), \pi_v', w_1 w_2 w_1 w_2)$$

are isomorphic and irreducible. At non–split places it is isomorphic to the Langlands quotient of the induced representation

$$\mathrm{Ind}_{GL_1(k_v)'\times GL_1'(k_v)}^{H_2'(k_v)}\left((\mathbf{1}_v\circ\det_v')\nu^{3/2}\otimes(\mathbf{1}_v\circ\det_v')\nu^{3/2}\right),$$

while at non–split places it is isomorphic to the Langlands quotient of the induced representation

$$\mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^2\otimes|\cdot|^2\otimes|\cdot|^1\otimes|\cdot|^1\right),$$

where $T\cong GL_1\times GL_1\times GL_1\times GL_1$ is the maximal split torus of Sp_8.

Proof. The images are isomorphic because the $GL_4(k_v)$ normalized operator $N(0,\pi_v',w_1)$ is an isomorphism. At non–split places the image is irreducible by the Langlands classification since π_v' is supercuspidal and $w_1w_2w_1w_2$ is the longest Weyl group element. Let v be a split place and, in the notation of Lemma 3.1.4, $w=w_1w_2w_1w_2$ and $\underline{s}=(3/2,3/2)$. Furthermore, L is the maximal split torus $T\cong GL_1\times GL_1\times GL_1\times GL_1$,

$$\underline{s}+\underline{s}'=(1,2,1,2)\quad\text{and}\quad\tau_v\cong\mathbf{1}_v\otimes\mathbf{1}_v\otimes\mathbf{1}_v\otimes\mathbf{1}_v.$$

For w' we take the Weyl group element corresponding to the permutation

$$w'=(1,4,3)(2),$$

where $(i_1,i_2,\ldots,i_l)$ denotes the cycle mapping $i_1\mapsto i_2\mapsto\ldots\mapsto i_l\mapsto i_1$. The permutation p of m letters acts on $\underline{s}=(s_1,\ldots,s_m)\in\mathbb{C}^m$ by $p(\underline{s})=(s_{p^{-1}(1)},\ldots,s_{p^{-1}(m)})$ and on a representation $\sigma\cong\sigma_1\otimes\ldots\otimes\sigma_m$ of $GL_{n_1}(k_v)\times\ldots\times GL_{n_m}(k_v)$ by $p(\sigma)=\sigma_{p^{-1}(1)}\otimes\ldots\otimes\sigma_{p^{-1}(m)}$. Then

$$w'^{-1}(\underline{s}+\underline{s}')=(2,2,1,1),$$

and the normalized intertwining operator $N(w'^{-1}(\underline{s}+\underline{s}'),w'^{-1}(\tau_v),w')$ is surjective onto $I(\underline{s},\pi_v)$ since it can be decomposed into

$$\begin{aligned}&\mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^2\otimes|\cdot|^2\otimes|\cdot|^1\otimes|\cdot|^1\right)\\&\quad\to\mathrm{Ind}_{GL_1(k_v)\times GL_2(k_v)\times GL_1(k_v)}^{Sp_8(k_v)}\left(|\cdot|^2\otimes(\mathbf{1}_v\circ\det_v)\nu^{3/2}\otimes|\cdot|^1\right)\\&\quad\to\mathrm{Ind}_{GL_2(k_v)\times GL_1(k_v)\times GL_1(k_v)}^{Sp_8(k_v)}\left((\mathbf{1}_v\circ\det_v)\nu^{3/2}\otimes|\cdot|^2\otimes|\cdot|^1\right)\\&\quad\to\mathrm{Ind}_{GL_2(k_v)\times GL_2(k_v)}^{Sp_8(k_v)}\left((\mathbf{1}_v\circ\det_v)\nu^{3/2}\otimes(\mathbf{1}_v\circ\det_v)\nu^{3/2}\right),\end{aligned}$$

where the first and the third arrows are surjective by the Langlands classification, while the second one is an isomorphism by the results of [2] at non–archimedean places and Lemma I.7 of [24] at archimedean places. Thus condition (1) of Lemma 3.1.4 is satisfied.

In the notation of Lemma 3.1.4 we take the Weyl group element $w''=(1,2,3)(4)$. Then $w''ww'$ is the longest Weyl group element with respect to T. Now, we verify condition (2) of Lemma 3.1.4. The normalized intertwining operator $N(w(\underline{s}+\underline{s}'),w(\tau_v),w'')$ acts on the induced representation

$$\mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^{-2}\otimes|\cdot|^{-1}\otimes|\cdot|^{-2}\otimes|\cdot|^{-1}\right)$$

containing $I(w(\underline{s}), w(\pi_v))$ as a subrepresentation. If its restriction to $I(w(\underline{s}), w(\pi_v))$ were not injective, then its kernel would have non-trivial intersection with $I(w(\underline{s}), w(\pi_v))$. Decomposing the normalized intertwining operator $N(w(\underline{s}+\underline{s}'), w(\tau_v), w'')$ according to $w'' = (1,2)(3)(4) \circ (1)(2,3)(4)$ into

$$\begin{aligned}&\mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^{-2}\otimes|\cdot|^{-1}\otimes|\cdot|^{-2}\otimes|\cdot|^{-1}\right)\\ &\quad\to \mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^{-2}\otimes|\cdot|^{-2}\otimes|\cdot|^{-1}\otimes|\cdot|^{-1}\right)\\ &\quad\to \mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^{-2}\otimes|\cdot|^{-2}\otimes|\cdot|^{-1}\otimes|\cdot|^{-1}\right),\end{aligned}$$

where the second arrow is an isomorphism, we obtain that its kernel is isomorphic to the kernel of the first arrow, which is

$$\mathrm{Ind}_{GL_1(k_v)\times GL_2(k_v)\times GL_1(k_v)}^{Sp_8(k_v)}\left(|\cdot|^{-2}\otimes St_{\mathbf{1}_v}\nu^{-3/2}\otimes|\cdot|^{-1}\right),$$

where, abusing the non–archimedean notation, $St_{\mathbf{1}_v}$ at archimedean places denotes the unique irreducible subrepresentation of the induced representation $\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{GL_2(k_v)}\left(|\cdot|^{1/2}\otimes|\cdot|^{-1/2}\right)$. Since by the Langlands classification this kernel contains the Langlands quotient as the unique irreducible subrepresentation, if the intersection with $I(w(\underline{s}), w(\pi_v))$ were non-trivial, it would contain this Langlands quotient as a subrepresentation. However, such a subrepresentation would be the irreducible quotient of $I(\underline{s}, \pi_v)$, which is the quotient of

$$\mathrm{Ind}_{T(k_v)}^{Sp_8(k_v)}\left(|\cdot|^{2}\otimes|\cdot|^{2}\otimes|\cdot|^{1}\otimes|\cdot|^{1}\right)$$

by the first part of the proof. But the last induced representation has its own unique irreducible Langlands quotient, which is not isomorphic to the one in the kernel. This proves condition (2) of Lemma 3.1.4.

Applying Lemma 3.1.4 shows that the image of the normalized intertwining operator $N(\underline{s}, \pi_v, w)$ is isomorphic to the image of

$$N((2,2,1,1), \mathbf{1}_v\otimes\mathbf{1}_v\otimes\mathbf{1}_v\otimes\mathbf{1}_v, w''ww').$$

Since $w''ww'$ is the longest Weyl group element and $(2,2,1,1)\in\mathfrak{a}_{T,\mathbb{C}}^*$ satisfies the conditions of the Langlands classification, the image is irreducible as claimed. □

Theorem 3.3.8. *The subspace $L^2_{C_5}$ of the residual spectrum of $H_2'(\mathbb{A})$ is*

$$L^2_{C_5} = \begin{cases} \{0\}, & \text{if } \prod_v \eta_v = 1,\\ \mathcal{C}_5\left((\mathbf{1}\circ\det')\otimes(\mathbf{1}\circ\det')\right), & \text{if } \prod_v \eta_v = -1.\end{cases}$$

Here $\mathcal{C}_5\left((\mathbf{1}\circ\det')\otimes(\mathbf{1}\circ\det')\right)$ is the irreducible space of automorphic forms spanned by the iterated residue at $\underline{s}=(3/2,3/2)$ of the Eisenstein series attached to the trivial representation $\pi'\cong(\mathbf{1}\circ\det')\otimes(\mathbf{1}\circ\det')$ of $M_0'(\mathbb{A})$. The constant term map gives rise to an isomorphism of $\mathcal{C}_5\left((\mathbf{1}\circ\det')\otimes(\mathbf{1}\circ\det')\right)$ and the image of the normalized operator $N((3/2,3/2),\pi',w_2w_1w_2)$ described in the previous Lemma 3.3.7.

Proof. In order to find the contribution to the residual spectrum at $C_5(3/2,3/2)$ we study the iterated pole of the Eisenstein series attached to a Case C cuspidal automorphic representation $\pi'\cong(\chi_1\circ\det')\otimes(\chi_2\circ\det')$. As shown in Figure 3.3, we first look at the pole of the normalizing factors along $2s_2=3$. It occurs if and only if χ_2 is trivial. The residues, up to a non–zero constant, are given in Table 3.3, where $z=s_1$.

TABLE 3.3. Residues along $2s_2 = 3$ of Case C normalizing factors

w	$Res_{2s_2=3} r(\underline{s}, (\chi_1 \circ \det') \otimes (\mathbf{1} \circ \det'), w)$
w_2	1
$w_1 w_2$	$\frac{L(z+3/2,\chi_1)L(z+1/2,\chi_1)}{L(z+7/2,\chi_1)L(z+5/2,\chi_1)\varepsilon(z+5/2,\chi_1)\varepsilon(z+3/2,\chi_1)^2\varepsilon(z+1/2,\chi_1)} \cdot \prod_{v\in S_D} \frac{L(z+3/2,\chi_{1,v})L(z+5/2,\chi_{1,v})}{L(-z-3/2,\chi_{1,v}^{-1})L(-z-1/2,\chi_{1,v}^{-1})}$
$w_2 w_1 w_2$	$\frac{L(z+3/2,\chi_1)L(z+1/2,\chi_1)}{L(z+7/2,\chi_1)L(z+5/2,\chi_1)\varepsilon(z+5/2,\chi_1)\varepsilon(z+3/2,\chi_1)^2\varepsilon(z+1/2,\chi_1)} \cdot \prod_{v\in S_D} \frac{L(z+3/2,\chi_{1,v})L(z+5/2,\chi_{1,v})}{L(-z-3/2,\chi_{1,v}^{-1})L(-z-1/2,\chi_{1,v}^{-1})} \frac{L(z-1/2,\chi_1)}{L(z+3/2,\chi_1)\varepsilon(z+1/2,\chi_1)\varepsilon(z-1/2,\chi_1)} \prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v})}{L(1/2-z,\chi_{1,v}^{-1})} \cdot \frac{L(2z,\chi_1^2)}{L(2z+1,\chi_1^2)\varepsilon(2z,\chi_1^2)}$
$w_1 w_2 w_1 w_2$	$\frac{L(z+3/2,\chi_1)L(z+1/2,\chi_1)}{L(z+7/2,\chi_1)L(z+5/2,\chi_1)\varepsilon(z+5/2,\chi_1)\varepsilon(z+3/2,\chi_1)^2\varepsilon(z+1/2,\chi_1)} \cdot \prod_{v\in S_D} \frac{L(z+3/2,\chi_{1,v})L(z+5/2,\chi_{1,v})}{L(-z-3/2,\chi_{1,v}^{-1})L(-z-1/2,\chi_{1,v}^{-1})} \frac{L(z-1/2,\chi_1)}{L(z+3/2,\chi_1)\varepsilon(z+1/2,\chi_1)\varepsilon(z-1/2,\chi_1)} \prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v})}{L(1/2-z,\chi_{1,v}^{-1})} \cdot \frac{L(2z,\chi_1^2)}{L(2z+1,\chi_1^2)\varepsilon(2z,\chi_1^2)} \frac{L(z-5/2,\chi_1)L(z-3/2,\chi_1)}{L(z+1/2,\chi_1)L(z-1/2,\chi_1)\varepsilon(z-1/2,\chi_1)\varepsilon(z-3/2,\chi_1)^2\varepsilon(z-5/2,\chi_1)} \cdot \prod_{v\in S_D} \frac{L(z-3/2,\chi_{1,v})L(z-1/2,\chi_{1,v})}{L(3/2-z,\chi_{1,v}^{-1})L(5/2-z,\chi_{1,v}^{-1})}$

Point C_5 corresponds to $z = 3/2$. By Lemma 3.1.2, the pole of terms in Table 3.3 at $z = 3/2$ occurs if and only if χ_1 is trivial. Only the terms corresponding to the Weyl group elements $w_2w_1w_2$ and $w_1w_2w_1w_2$ have the pole, and it is simple. Up to a non–zero constant, using the global functional equation and Lemma 3.1.3, the sum of its residues acting on a decomposable vector gives

$$N((3/2,3/2),\pi',w_2w_1w_2)\left[Id - N((3/2,3/2),\pi',w_1)\right].$$

The parity condition is obtained from the non–vanishing of the square brackets. The square–integrability criterion of Lemma 3.1.1 is satisfied, and the irreducibility of the image of the normalized intertwining operator

$$N((3/2,3/2),(\mathbf{1}\circ\det')\otimes(\mathbf{1}\circ\det'),w_2w_1w_2)$$

follows from the previous Lemma 3.3.7. □

Decomposing $L^2_{C_6}$ is quite similar to $L^2_{C_4}$. We use the same notation to emphasize the analogy. For a split place v consider the behavior of the normalized intertwining operator $N((s,3/2),\pi_v,w_1)$ at $s = 1/2$, where $\pi_v \cong (\chi_{1,v}\otimes\det_v)\otimes(\mathbf{1}_v\otimes\det_v)$. Here $\mathbf{1}_v$ is the trivial and $\chi_{1,v}$ is a unitary character of $k_v^\times$. If $\chi_{1,v}$ is trivial, let X_v denote the image of $N((3/2,1/2),\pi_v,w_1)$. By the Langlands classification it is an irreducible subrepresentation of

$$I_v = \mathrm{Ind}_{GL_2(k_v)\times GL_2(k_v)}^{GL_4(k_v)}\left((\chi_{1,v}\otimes\det_v)\nu^{1/2}\otimes(\mathbf{1}_v\otimes\det_v)\nu^{3/2}\right).$$

Let $Y_v \cong I_v/X_v$ denote the quotient. The proofs of the following lemma and its corollary are the same as the corresponding proofs in Section 2.4 of [8] and the corresponding proofs in the decomposition of $L^2_{C_4}$ above.

Lemma 3.3.9. *If either $v \in S_D$ or $\chi_{1,v}$ is non-trivial, then $N((s,3/2),\pi_v,w_1)$ at $s=1/2$ is an isomorphism. Thus, it is holomorphic and non–vanishing.*

If $v \notin S_D$ and $\chi_{1,v}$ is trivial, then $N((s,3/2),\pi_v,w_1)$ has a pole at $s=1/2$. The operator

$$\widetilde{N}((1/2,3/2),\pi_v,w_1) = \lim_{s\to 1/2}(s-1/2)N((s,3/2),\pi_v,w_1)$$

is holomorphic, its image is isomorphic to Y_v, and its kernel is X_v. Thus, the restriction of $N((s,3/2),\pi_v,w_1)$ at $s=1/2$ to X_v is holomorphic and non–vanishing.

Corollary 3.3.10. *Let $\pi'_v \cong (\chi_{1,v}\circ\det'_v)\otimes(\mathbf{1}_v\circ\det'_v)$. If $v\in S_D$ or $\chi_{1,v}$ is non-trivial, then the image of $N((1/2,3/2),\pi'_v,w_1w_2w_1w_2)$, denoted by W_v, is non-trivial and isomorphic to the image of $N((3/2,1/2),\pi'_v,w_2w_1w_2)$.*

If $v\notin S_D$ and $\chi_{1,v}$ is trivial, then the image of

$$N((3/2,1/2),\pi'_v,w_2w_1w_2)\widetilde{N}((1/2,3/2),\pi'_v,w_1),$$

denoted by W'_v, is non-trivial. Furthermore, in this case the image of

$$N((1/2,3/2),\pi'_v,w_2w_1w_2),$$

again denoted by W_v, is non-trivial and contains W'_v as a subrepresentation.

As before, let $S_1(\mu)$ denote the set of places where a local component μ_v of a unitary character μ of $k^\times\backslash\mathbb{A}^\times$ is trivial. Let η_v be the sign of $N((1/2,3/2),\pi'_v,w_1)$ acting on X_v. For χ_1 a non-trivial quadratic character of $k^\times\backslash\mathbb{A}^\times$ such that $\chi_{1,v}$ is non-trivial for all $v\in S_D$, let

$$C = \frac{L(-2,\chi_1)L(-1,\chi_1)}{L(1,\chi_1)L(0,\chi_1)\varepsilon(0,\chi_1)\varepsilon(-1,\chi_1)^2\varepsilon(-2,\chi_1)}\prod_{v\in S_D}\frac{L(-1,\chi_{1,v})L(0,\chi_{1,v})}{L(1,\chi_{1,v})L(2,\chi_{1,v})}$$

be the non–zero constant appearing in the parity conditions of the theorem below.

Theorem 3.3.11. *The subspace $L^2_{C_6}$ of the residual spectrum of $H'_2(\mathbb{A})$ decomposes into*

$$L^2_{C_6} = \left(\bigoplus_{\pi'}\mathcal{C}_6^{(1)}(\pi')\right)\oplus\left(\bigoplus_{\pi'}\mathcal{C}_6^{(2)}(\pi')\right).$$

The former sum is over all one–dimensional cuspidal automorphic representations $\pi'\cong(\chi_1\circ\det')\otimes(\mathbf{1}\circ\det')$ of $M'_0(\mathbb{A})$ such that χ_1 is a non-trivial quadratic character and either $\chi_{1,v}$ is non-trivial for all $v\in S_D$ and the parity condition $C\cdot\prod_v\eta_v\neq -1$ holds, or there is a non–split place $v\in S_D$ where $\chi_{1,v}$ is trivial. The latter sum is over all one–dimensional cuspidal automorphic representations $\pi'\cong(\chi_1\circ\det')\otimes(\mathbf{1}\circ\det')$ of $M'_0(\mathbb{A})$ such that there is a split place $v\notin S_D$ where $\chi_{1,v}$ is trivial, and if χ_1 is a nontrivial quadratic character, then the parity condition $C\cdot\prod_v\eta_v=-1$ holds.

The spaces $\mathcal{C}_6^{(1)}(\pi')$ and $\mathcal{C}_6^{(2)}(\pi')$ are the spaces of automorphic forms spanned by the residues

$$\lim_{s_1\to 1/2}(s_1-1/2)^n Res_{2s_2=3}E(\underline{s},g;f_{\underline{s}},\pi'),$$

where n is the order of the pole at $s_1=1/2$.

In the notation of Corollary 3.3.10, the constant term map gives rise to an isomorphism between $\mathcal{C}_6^{(1)}(\pi')$ and $\bigotimes_v W_v$. Unless χ_1 is trivial, the constant term

map gives rise to an isomorphism between $\mathcal{C}_6^{(2)}(\pi')$ *and*

$$\bigoplus_{w\in S_1(\chi_1)\backslash S_D}\left[W'_w\otimes\left(\bigotimes_{v\neq w}W_v\right)\right].$$

Finally, if χ_1 *is trivial the constant term map implies that* $\mathcal{C}_6^{(2)}(\pi')$ *contains a space isomorphic to* $\bigoplus_{w\notin S_D}[W'_w\otimes(\bigotimes_{v\neq w}W_v)]$.

Proof. The proof is quite similar to the proof of Theorem 3.3.6 and the corresponding Theorem in Section 2.4 of [8]. The parity condition comes from the fact that there is a case in which the pole occurs for the normalizing factors of operators attached to both $w_2w_1w_2$ and $w_1w_2w_1w_2$. An argument similar to the proof of Theorem 3.3.13 shows that the cancellation of the pole is precisely the parity condition of the theorem. □

For the description of the irreducible constituents of $L^2_{C_7}$ we need the following lemma.

Lemma 3.3.12. *Let* $\pi'_v\cong(\mathbf{1}_v\circ\det'_v)\otimes(\chi_{2,v}\circ\det'_v)$ *be a representation of* $M'_0(k_v)$, *where* $\chi_{2,v}$ *is a quadratic character of* $k_v^\times$. *Then the images of the normalized intertwining operators*

$$N((3/2,1/2),\pi'_v,w_2w_1w_2)\quad\textit{and}\quad N((3/2,1/2),\pi'_v,w_1w_2w_1w_2)$$

are isomorphic. At non–split places the image is irreducible and isomorphic to the Langlands quotient of the induced representation

$$\mathrm{Ind}_{GL'_1(k_v)\times GL'_1(k_v)}^{H'_2(k_v)}\left((\mathbf{1}_v\circ\det'_v)\nu^{3/2}\otimes(\chi_{2,v}\circ\det'_v)\nu^{1/2}\right).$$

At the split places where $\chi_{2,v}=\mathbf{1}_v$ *is trivial it is irreducible and isomorphic to the Langlands quotient of the induced representation*

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)\times GL_1(k_v)\times SL_2(k_v)}^{Sp_8(k_v)}\left(|\cdot|^2\otimes|\cdot|^1\otimes|\cdot|^1\otimes\tau_{1,v}\right),$$

where $\tau_{1,v}\cong\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)}\mathbf{1}_v$ *is irreducible and tempered. At the split places where* $\chi_{2,v}$ *is non-trivial, it is the direct sum of two irreducible representations isomorphic to the Langlands quotients of the induced representations*

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)\times GL_1(k_v)\times SL_2(k_v)}^{Sp_8(k_v)}\left(|\cdot|^2\otimes|\cdot|^1\otimes\chi_{2,v}|\cdot|^1\otimes\tau_{i,v}\right),$$

for $i=1,2$, *where* $\tau_{1,v}\oplus\tau_{2,v}\cong\mathrm{Ind}_{GL_1(k_v)}^{SL_2(k_v)}\chi_{2,v}$ *and* $\tau_{i,v}$ *are irreducible and tempered.*

Proof. Another application of Lemma 3.1.4, similar to Lemma 3.3.7. Thus we omit the proof. □

By the lemma the normalized intertwining operator

$$N((-1/2,-3/2),(\chi_{2,v}\circ\det'_v)\otimes(\mathbf{1}_v\circ\det'_v),w_1)$$

restricted to the image of $N((3/2,1/2),\pi'_v,w_2w_1w_2)$ is an isomorphism of the two images described in the lemma. Since those images are at all places completely reducible, let $\Pi'^{\pm}_v$ denote the ±1–eigenspaces. It is possible that one of the spaces is trivial, and the unramified component is always Π'^{+}_v.

TABLE 3.4. Residues along $2s_2 = 1$ of Case C normalizing factors

w	$Res_{2s_2=1} r(\underline{s}, \pi', w)$
w_2	1
$w_1 w_2$	$\frac{L(z-1/2,\chi_1\chi_2)L(z+1/2,\chi_1\chi_2)}{L(z+3/2,\chi_1\chi_2)L(z+5/2,\chi_1\chi_2)\varepsilon(z-1/2,\chi_1\chi_2)\varepsilon(z+1/2,\chi_1\chi_2)^2\varepsilon(z+3/2,\chi_1\chi_2)}$ $\prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v}\chi_{2,v})L(z+3/2,\chi_{1,v}\chi_{2,v})}{L(-z-1/2,\chi_{1,v}^{-1}\chi_{2,v})L(1/2-z,\chi_{1,v}^{-1}\chi_{2,v})}$
$w_2 w_1 w_2$	$\frac{L(z-1/2,\chi_1\chi_2)L(z+1/2,\chi_1\chi_2)}{L(z+3/2,\chi_1\chi_2)L(z+5/2,\chi_1\chi_2)\varepsilon(z-1/2,\chi_1\chi_2)\varepsilon(z+1/2,\chi_1\chi_2)^2\varepsilon(z+3/2,\chi_1\chi_2)}$ $\prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v}\chi_{2,v})L(z+3/2,\chi_{1,v}\chi_{2,v})}{L(-z-1/2,\chi_{1,v}^{-1}\chi_{2,v})L(1/2-z,\chi_{1,v}^{-1}\chi_{2,v})}$ $\frac{L(z-1/2,\chi_1)}{L(z+3/2,\chi_1)\varepsilon(z+1/2,\chi_1)\varepsilon(z-1/2,\chi_1)} \prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v})}{L(1/2-z,\chi_{1,v}^{-1})}$ $\cdot \frac{L(2z,\chi_1^2)}{L(2z+1,\chi_1^2)\varepsilon(2z,\chi_1^2)}$
$w_1 w_2 w_1 w_2$	$\frac{L(z-1/2,\chi_1\chi_2)L(z+1/2,\chi_1\chi_2)}{L(z+3/2,\chi_1\chi_2)L(z+5/2,\chi_1\chi_2)\varepsilon(z-1/2,\chi_1\chi_2)\varepsilon(z+1/2,\chi_1\chi_2)^2\varepsilon(z+3/2,\chi_1\chi_2)}$ $\prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v}\chi_{2,v})L(z+3/2,\chi_{1,v}\chi_{2,v})}{L(-z-1/2,\chi_{1,v}^{-1}\chi_{2,v})L(1/2-z,\chi_{1,v}^{-1}\chi_{2,v})}$ $\frac{L(z-1/2,\chi_1)}{L(z+3/2,\chi_1)\varepsilon(z+1/2,\chi_1)\varepsilon(z-1/2,\chi_1)} \prod_{v\in S_D} \frac{L(z+1/2,\chi_{1,v})}{L(1/2-z,\chi_{1,v}^{-1})}$ $\cdot \frac{L(2z,\chi_1^2)}{L(2z+1,\chi_1^2)\varepsilon(2z,\chi_1^2)}$ $\frac{L(z-3/2,\chi_1\chi_2)L(z-1/2,\chi_1\chi_2)}{L(z+1/2,\chi_1\chi_2)L(z+3/2,\chi_1\chi_2)\varepsilon(z-3/2,\chi_1\chi_2)\varepsilon(z-1/2,\chi_1\chi_2)^2\varepsilon(z+1/2,\chi_1\chi_2)}$ $\prod_{v\in S_D} \frac{L(z-1/2,\chi_{1,v}\chi_{2,v})L(z+1/2,\chi_{1,v}\chi_{2,v})}{L(-z+1/2,\chi_{1,v}^{-1}\chi_{2,v})L(3/2-z,\chi_{1,v}^{-1}\chi_{2,v})}$

Theorem 3.3.13. *The subspace $L^2_{C_7}$ of the residual spectrum of $H_2'(\mathbb{A})$ decomposes into*

$$L^2_{C_7} = \bigoplus_{\pi'} \mathcal{C}_7(\pi'),$$

where the sum is over all cuspidal automorphic representations of the form $\pi' \cong (\mathbf{1} \otimes \det') \otimes (\chi_2 \otimes \det')$ of $M_0'(\mathbb{A})$ such that χ_2 is a non-trivial quadratic character and $\chi_{2,v}$ is non-trivial for all $v \in S_D$.

$\mathcal{C}_7(\pi')$ is the space of automorphic forms spanned by the iterated residue at $\underline{s} = (3/2, 1/2)$ of the Eisenstein series attached to π'. The constant term map gives rise to an isomorphism of $\mathcal{C}_7(\pi')$ and the direct sum of the spaces of the form $\bigotimes_v \Pi_v^{\eta_v}$, where $\eta_v \in \{+,-\}$, $\eta_v = +$ for almost all v, and the parity condition

$$\prod_v \eta_v \cdot \frac{L(0,\chi_2)L(1,\chi_2)}{L(2,\chi_2)L(3,\chi_2)\varepsilon(0,\chi_2)\varepsilon(1,\chi_2)^2\varepsilon(2,\chi_2)} \prod_{v\in S_D} \frac{L(1,\chi_{2,v})L(2,\chi_{2,v})}{L(-1,\chi_{2,v})L(0,\chi_{2,v})} \neq -1$$

holds.

Proof. At C_7 the iterated residue of the constant term (3.3) of the Eisenstein series attached to a Case C cuspidal automorphic representation $\pi' \cong (\chi_1 \circ \det') \otimes (\chi_2 \circ \det')$ of $M_0'(\mathbb{A})$ is first calculated along $2s_2 = 1$ as shown in Figure 3.3. By the analytic properties of the L–functions in Lemma 3.1.2, the pole of the normalizing factors occurs if and only if χ_2 is a non-trivial quadratic character such that $\chi_{2,v}$ is non-trivial at all places $v \in S_D$. Then, the terms corresponding to the Weyl group elements w_2, w_1w_2, $w_2w_1w_2$ and $w_1w_2w_1w_2$ have poles, and they are simple. The residues, up to a non-zero constant are given in Table 3.4, where $z = s_1$.

Point C_7 corresponds to $z = 3/2$. There are two possibilities for obtaining the pole of the terms in Table 3.4. First, the pole occurs if $\chi_1\chi_2$ is trivial, i.e.

$\chi_1 = \chi_2$ is a non-trivial quadratic character such that $\chi_{1,v} = \chi_{2,v}$ is non-trivial at all $v \in S_D$. Then, the terms corresponding to the Weyl group elements w_1w_2, $w_2w_1w_2$ and $w_1w_2w_1w_2$ have poles at $z = 3/2$, and they are all simple. However, since $w_1w_2(3/2, 1/2) = (-1/2, 3/2)$ does not satisfy the square–integrability criterion of Lemma 3.1.1, the contribution of the iterated pole can be square–integrable only for automorphic forms f such that

$$N((3/2, 1/2), \pi', w_1w_2)f = 0.$$

But then, the remaining two residues also vanish on f by the decomposition property of the intertwining operators, and there is no contribution to $L^2_{C_7}$ in this case.

The other possibility for the pole of the terms in Table 3.4 at $z = 3/2$ is for χ_1 trivial. Then $\chi_1\chi_2 = \chi_2$. Hence, by the analytic properties of the L–functions of Lemma 3.1.2, the terms corresponding to the Weyl group elements $w_2w_1w_2$ and $w_1w_2w_1w_2$ have poles, and they are simple. Up to a non-zero constant, the sum of the residues acing on a decomposable vector gives

$$[Id + C \cdot N((-1/2, -3/2), w_2w_1w_2(\pi'), w_1)]\, N((3/2, 1/2), \pi', w_2w_1w_2),$$

where the constant C is given by

$$\frac{L(0, \chi_2)L(1, \chi_2)}{L(2, \chi_2)L(3, \chi_2)\varepsilon(0, \chi_2)\varepsilon(1, \chi_2)^2\varepsilon(2, \chi_2)} \prod_{v \in S_D} \frac{L(1, \chi_{2,v})L(2, \chi_{2,v})}{L(-1, \chi_{2,v})L(0, \chi_{2,v})}.$$

The parity condition is just the non–vanishing condition for the square–bracket acting on the image of $N((3/2, 1/2), \pi', w_2w_1w_2)$. The square–integrability criterion of Lemma 3.1.1 is satisfied, and the contribution of the iterated residue is isomorphic to the part of the image of the normalized intertwining operator $N((3/2, 1/2), \pi', w_2w_1w_2)$ satisfying the parity condition. □

Before decomposing $L^2_{C_8}$ consider the normalized intertwining operator

$$N(0, \chi_{1,v} \circ \det' \otimes \chi'_{2,v} \circ \det', w_1)$$

acting on the induced representation

$$\mathrm{Ind}^{GL'_2(k_v)}_{GL'_1(k_v)\times GL'_1(k_v)} \left((\chi_{1,v} \circ \det'_v) \otimes (\chi_{2,v} \circ \det'_v)\right).$$

The induced representation is irreducible by [34], [2], and[33]. Hence, the normalized operator acts as Id or $-Id$, and we denote the sign by η_v. For the description of the irreducible components of $L^2_{C_8}$ we need the following lemma.

Lemma 3.3.14. *Let $\pi'_v \cong (\chi_{1,v} \circ \det'_v) \otimes (\chi_{2,v} \circ \det'_v)$ be a representation of $M'_0(\mathbb{A})$, where $\chi_{i,v}$ are quadratic characters of $k_v^\times$. Then the images of the normalized intertwining operators*

$$N((1/2, 1/2), \pi'_v, w_2w_1w_2) \quad \textit{and} \quad N((1/2, 1/2), \pi'_v, w_1w_2w_1w_2)$$

are isomorphic. At non–split places the image is irreducible as the Langlands quotient of the induced representation

$$\mathrm{Ind}^{H'_2(k_v)}_{GL'_1(k_v)\times GL'_1(k_v)} \left((\chi_{1,v} \circ \det'_v)\nu^{1/2} \otimes (\chi_{2,v} \circ \det'_v)\nu^{1/2}\right).$$

At split places, it is either irreducible or the direct sum of two or four irreducible constituents, where the irreducible constituents are isomorphic to the Langlands quotients of the induced representations of the form

$$\mathrm{Ind}^{Sp_8(k_v)}_{GL_1(k_v)\times GL_1(k_v)\times Sp_4(k_v)} \left(\chi_{2,v}|\cdot|^1 \otimes \chi_{1,v}|\cdot|^1 \otimes \sigma_v\right),$$

where σ_v is one of the irreducible tempered constituents of the induced representation

$$\mathrm{Ind}_{GL_1(k_v)\times GL_1(k_v)}^{Sp_4(k_v)}\left(\chi_{1,v}\otimes\chi_{2,v}\right),$$

whose decomposition is given in [30] *at non–archimedean places and* [27] *and* [28] *at archimedean places.*

Proof. Again Lemma 3.1.4 applies similarly as in 3.3.7. Hence, we omit the proof. □

Theorem 3.3.15. *The subspace $L^2_{C_8}$ of the residual spectrum of $H_2'(\mathbb{A})$ decomposes into*

$$L^2_{C_8}=\left(\bigoplus_{\pi'}\mathcal{C}_8^{(1)}(\pi')\right)\oplus\left(\bigoplus_{\pi'}\mathcal{C}_8^{(2)}(\pi')\right).$$

The former sum is over all one–dimensional cuspidal automorphic representations $\pi'\cong(\chi\circ\det')\otimes(\chi\circ\det')$ of $M_0'(\mathbb{A})$ such that χ is a non-trivial quadratic character, χ_v is non-trivial for all $v\in S_D$, the cardinality $|S_D|=2$ and the parity condition $\prod_v\eta_v=-1$ holds. The latter sum is over all one–dimensional cuspidal automorphic representations $\pi'\cong(\chi_1\circ\det')\otimes(\chi_2\circ\det')$ of $M_0'(\mathbb{A})$ such that $\chi_1\neq\chi_2$ are both non-trivial quadratic characters, $\chi_{1,v}$ and $\chi_{2,v}$ are non-trivial for all $v\in S_D$, $\chi_{1,v}\neq\chi_{2,v}$ for all $v\in S_D$ and the parity condition $\prod_v\eta_v=1$ holds.

Both $\mathcal{C}_8^{(1)}(\pi')$ and $\mathcal{C}_8^{(2)}(\pi')$ are the spaces of automorphic forms spanned by the iterated residues at $\underline{s}=(1/2,1/2)$ of the Eisenstein series attached to π'. The constant term map gives rise to isomorphisms of both spaces and the sum of the irreducible representations of the form $\bigotimes_v\Pi_v'$, where Π_v' is one of the irreducible constituents of the image of the normalized intertwining operator $N((1/2,1/2),\pi',w_2w_1w_2)$ described in the previous lemma, Lemma 3.3.14*, and it is the unramified one at almost all places.*

Proof. The first step in calculating the iterated pole at $C_8(1/2,1/2)$ is along $2s_2=1$, as in the proof of the previous theorem. Thus, the residues are given in Table 3.4, and the pole appears if and only if χ_2 is a non-trivial quadratic character with $\chi_{2,v}$ non-trivial at all places $v\in S_D$. Point C_8 corresponds to $z=1/2$. By the analytic properties of the L–functions of Lemma 3.1.2, the pole at $z=1/2$ of the terms in Table 3.4 does not occur unless χ_1 is a quadratic character. Indeed, if χ_1 were not quadratic, then both χ_1 and $\chi_1\chi_2$ would be non-trivial. Therefore, let χ_1 be a quadratic character. Now, we distinguish two cases.

First, assume $\chi_1=\chi_2$, i.e. $\chi_1\chi_2$ is trivial. Due to the local L–functions in the denominator, the term corresponding to w_1w_2 has a zero of order $|S_D|-2\geqslant 0$. Recall that $|S_D|$ is always even. The terms corresponding to $w_2w_1w_2$ and $w_1w_2w_1w_2$ have a simple pole only if $|S_D|=2$. Otherwise, the order of the pole in the denominator is $|S_D|\geqslant 4$ and cancels the pole in the numerator, which is of order 3. Moreover, up to a constant which is non–zero due to the sum of the residues acting on a decomposable vector gives

$$N((1/2,1/2),\pi',w_2w_1w_2)\left[Id-N((1/2,1/2),\pi',w_1)\right].$$

The non–vanishing of the square bracket implies the parity condition. The square–integrability criterion of Lemma 3.1.1 is satisfied, and the contribution to the resid-

ual spectrum is isomorphic to the image of the normalized intertwining operator

$$N((1/2,1/2),\pi',w_2w_1w_2),$$

which is described in the previous Lemma 3.3.14.

Now, assume $\chi_1 \neq \chi_2$, i.e. $\chi_1\chi_2$ is non-trivial. If χ_1 is trivial, then the double pole in the numerator is cancelled by the pole of the local L–functions in the denominator. If χ_1 is non-trivial, then the numerator has only a simple pole, but it is not cancelled if $\chi_{1,v}$ is non-trivial and $\chi_{1,v} \neq \chi_{2,v}$ for all $v \in S_D$. Therefore, the pole occurs in this case if and only if χ_1 is a non-trivial quadratic character, $\chi_{1,v}$ is non-trivial and $\chi_{1,v} \neq \chi_{2,v}$ for all $v \in S_D$. Then, the terms corresponding to the Weyl group elements $w_2w_1w_2$ and $w_1w_2w_1w_2$ have simple poles. Up to a non-zero constant the sum of the residues acting on a decomposable vector is of the form

$$N((1/2,1/2),\pi',w_2w_1w_2)\left[Id + N((1/2,1/2),\pi',w_1w_2w_1w_2)\right].$$

The non–vanishing of the square bracket is the parity condition in this case. The square–integrability criterion of Lemma 3.1.1 is satisfied, and the contribution of this case to the residual spectrum is isomorphic to the image of the normalized intertwining operator

$$N((1/2,1/2),\pi',w_2w_1w_2),$$

which is described in the previous Lemma 3.3.14. □

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