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ASYMPTOTIC STABILITY OF THE WAVE EQUATION ON COMPACT SURFACES AND LOCALLY DISTRIBUTED DAMPING–A SHARP RESULT

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Abstract. This paper is concerned with the study of the wave equation on compact surfaces and locally distributed damping, described by

$$u_{tt} - \Delta_{\mathcal{M}} u + a(x)\, g(u_t) = 0 \quad \text{on } \mathcal{M} \times]0, \infty[\,,$$

where $\mathcal{M} \subset \mathbb{R}^3$ is a smooth oriented embedded compact surface without boundary. Denoting by $\mathbf{g}$ the Riemannian metric induced on $\mathcal{M}$ by $\mathbb{R}^3$, we prove that for each $\epsilon > 0$, there exist an open subset $V \subset \mathcal{M}$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$ such that $meas(V) \geq meas(\mathcal{M}) - \epsilon$, $Hess f \approx \mathbf{g}$ on V and $\inf_{x \in V} |\nabla f(x)| > 0$.

In addition, we prove that if $a(x) \geq a_0 > 0$ on an open subset $\mathcal{M}* \subset \mathcal{M}$ which contains $\mathcal{M} \backslash V$ and if g is a monotonic increasing function such that $k|s| \leq |g(s)| \leq K|s|$ for all $|s| \geq 1$, then uniform and optimal decay rates of the energy hold.

1. Introduction

Let $\mathcal{M}$ be a smooth oriented embedded compact surface without boundary in $\mathbb{R}^3$ and let $\mathbf{g}$ denote the Riemannian metric induced on $\mathcal{M}$ by $\mathbb{R}^3$. For $\epsilon > 0$ we prove that there exist an open subset $V \subset \mathcal{M}$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$ such that $meas(V) \geq meas(\mathcal{M}) - \epsilon$, $Hess f \approx \mathbf{g}$ on V and $\inf_{x \in V} |\nabla f(x)| > 0$ (see Subsection 4.4).

We denote by ∇_T the tangential gradient on $\mathcal{M}$ and by $\Delta_{\mathcal{M}}$ the Laplace-Beltrami operator on $\mathcal{M}$. This paper is devoted to the study of the uniform stabilization of solutions of the following damped problem:

$$\begin{cases} u_{tt} - \Delta_{\mathcal{M}} u + a(x)\, g(u_t) = 0 & \text{on } \mathcal{M} \times]0, \infty[\,, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & x \in \mathcal{M}, \end{cases} \tag{1.1}$$

where $a(x) \geq a_0 > 0$ on an open proper subset $\mathcal{M}_* \supset \mathcal{M} \backslash V$ of $\mathcal{M}$ and in addition g is a monotonic increasing function such that $k|s| \leq |g(s)| \leq K|s|$ for all $|s| \geq 1$.

A natural question arises in the context of the wave equation on compact surfaces: *Would it be possible to stabilize the system by considering a localized feedback acting only on a portion of the surface?* In the affirmative case, *what would be the*

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geometrical impositions we have to assume on the surface? When the damping term acts on the whole surface, the conjecture was studied by Cavalcanti and Domingos Cavalcanti in [CA-DO] and also by Andrade et al. in [An1, An2] in the context of viscoelastic problems. For linear waves, we can mention the works due to Rauch and Taylor [Ra-Ta], Hitrik [HIT] and, recently Christianson [CHR]. For the nonlinear wave equation on compact manifolds with boundary, it is important to cite the work due to Triggiani and Yao [TRI-YAO]. More recently, the authors of the present work [Ca-Do-Fu-So] studied the linear wave equation on a compact surface $\mathcal{M}$ without boundary supplemented with a nonlinear and localized dissipation. In this article the authors prove the above conjecture when the portion of $\mathcal{M}$ where the damping is effective is strategically chosen. Setting $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$, where

$$\mathcal{M}_1 := \{x \in \mathcal{M}; m(x) \cdot \nu(x) > 0\} \text{ and } \mathcal{M}_0 = \mathcal{M} \backslash \mathcal{M}_1,$$

$m(x) := x - x^0$ ($x^0 \in \mathbb{R}^3$ fixed) and ν is the exterior unit normal vector field of $\mathcal{M}$, then for $i = 1, \ldots, k$, they assume that there exist open subsets $\mathcal{M}_{0i} \subset \mathcal{M}_0$ of $\mathcal{M}$ with smooth boundary $\partial\mathcal{M}_{0i}$ such that the $\mathcal{M}_{0i}$ are umbilical. Moreover, they suppose that the *mean curvature* H of each $\mathcal{M}_{0i}$ is *nonpositive* (i.e. $H \leq 0$ on $\mathcal{M}_{0i}$ for every $i = 1, \ldots, k$) and that the damping is effective on an open subset $\mathcal{M}_* \subset \mathcal{M}$ which contains $\mathcal{M} \backslash \bigcup_{i=1}^k \mathcal{M}_{0i}$. Roughly speaking, the region which does not contain dissipative effects must be *umbilical*. This is required since the authors employ the same multipliers considered in solving the similar question for the wave equation,

$$u_{tt} - \Delta u + a(x)g(u_t) = 0 \text{ in } \Omega \times (0, +\infty),$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. They considered the well-known multiplier given by the vector field $m(x) := x - x^0$, $x^0 \in \mathbb{R}^n$ arbitrarily chosen, but fixed, taken out of the domain Ω, according to Figure 1.

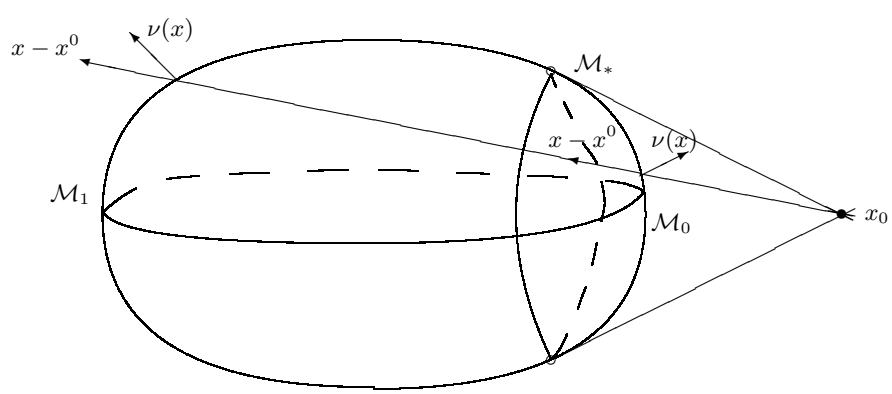

FIGURE 1. The observer is at x_0. The subset $\mathcal{M}_0$ is the "visible" part of $\mathcal{M}$ and $\mathcal{M}_1$ is its complement. The subset $\mathcal{M}_* \supset \mathcal{M} \backslash \bigcup_{i=1}^k \mathcal{M}_{0i}$ is an open set which contains $\mathcal{M} \backslash \bigcup_{i=1}^k \mathcal{M}_{0i}$ and the damping is effective there. Observe that in Figure 1, $k = 1$ and $\mathcal{M}_{0i} = \mathcal{M}_{01} = \mathcal{M}_0$.

Once the multiplier $m(x) = x - x^0$ is not *intrinsically connected* with the manifold $\mathcal{M}$ they have to impose a restriction on the part $\mathcal{M}_0$ (without damping); namely, $\mathcal{M}_0$ must be *umbilical, or umbilical by parts.*

The main goal of the present manuscript is to improve considerably the previous result due to [Ca-Do-Fu-So], *reducing arbitrarily* the volume of the region where the dissipative effect lies. For this purpose we will construct an intrinsic multiplier that will play a crucial role when establishing the desired uniform decay rates of the energy. Fix $\epsilon > 0$. This multiplier is, roughly speaking, given by the $\nabla_T f$, where $f : \mathcal{M} \to \mathbb{R}$ is a regular function which satisfies $Hess f \approx \mathbf{g}$ and $\inf_{x \in V} |\nabla f(x)| > 0$ on a subset V of $\mathcal{M}$ such that $meas(V) \geq meas(\mathcal{M}) - \epsilon$. This construction will be clarified in subsections 4.3 and 4.4.

We would like to emphasize that the proofs of [Ra-Ta, BAR, HIT], based on microlocal analysis, *do not extend* to the nonlinear problem (1.1). In addition, making use of arguments due to Cavalcanti, Domingos Cavalcanti and Lasiecka [CA-DO-LA], we obtain *explicit and optimal decay rates of the energy.* The decay rates obtained are optimal, since they are the same as the optimal rates derived in the works of Alabau-Boussouira [ALA] or Toudykov [Tou].

Our paper is organized as follows. Section 2 is concerned with the statement of the problem and we introduce some notation. Our main result is stated in Section 3. Section 4 is devoted to the proof of the main result.

2. Statement of the problem

Let $\mathcal{M}$ be a smooth oriented embedded compact surface without boundary in $\mathbb{R}^3$. For $\epsilon > 0$ we prove that there exist an open subset $V \subset \mathcal{M}$ and a smooth function $f : \mathcal{M} \to \mathbb{R}$ such that $meas(V) \geq meas(\mathcal{M}) - \epsilon$, $Hess f \approx \mathbf{g}$ on V and $\inf_{x \in V} |\nabla f(x)| > 0$ (see Subsection 4.4).

In this paper, we investigate the stability properties of the function $u(x,t)$ which solves the damped problem

$$\begin{cases} u_{tt} - \Delta_{\mathcal{M}} u + a(x)\, g(u_t) = 0 & \text{on } \mathcal{M} \times]0, \infty[\,, \\ u(0) = u^0, \quad u_t(0) = u^1, \end{cases} \tag{2.1}$$

where the feedback function g satisfies the Assumption 2.1.

Assumption 2.1.
(i) *$g(s)$ is continuous and monotone increasing,*
(ii) *$g(s)\, s > 0$ for $s \neq 0$,*
(iii) *$k\,|s| \leq g(s) \leq K\,|s|$ for $|s| > 1$,*
where k and K are two positive constants.

In addition, to obtain the stabilization of problem (2.1), we shall need the following geometrical assumption:

Assumption 2.2. *Assume that $a \in L^\infty(\mathcal{M})$ is a nonnegative function such that*

$$a(x) \geq a_0 > 0, \quad a.e. \text{ on } \mathcal{M}_*, \tag{2.2}$$

where $\mathcal{M}_*$ is an open set of $\mathcal{M}$ which contains $\mathcal{M} \backslash V$.

In the sequel, we are going to consider $\Sigma = \mathcal{M} \times]0, T[$ and the Sobolev spaces $H^s(\mathcal{M}), s \in \mathbb{R}$, as in Lions and Magenes [LiMa, section 7.3].

On the other hand, using the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ on $\mathcal{M}$, we can give a more intrinsic definition of the spaces $H^s(\mathcal{M})$. Consider

$$H^{2m}(\mathcal{M}) \quad = \quad \left\{u \in L^2(\mathcal{M}) / \Delta_{\mathcal{M}}^m u \in L^2(\mathcal{M})\right\},$$

which, equipped with the canonical norm

$$\|u\|^2_{H^{2m}(\mathcal{M})} = \|u\|^2_{L^2(\mathcal{M})} + \|\Delta^m_{\mathcal{M}} u\|^2_{L^2(\mathcal{M})}, \tag{2.3}$$

is a Hilbert space.

We set

$$V := \{v \in H^1(\mathcal{M}); \textstyle\int_{\mathcal{M}} v(x)\, d\mathcal{M} = 0\},$$

which is a Hilbert space with the topology endowed by $H^1(\mathcal{M})$.

Note that the condition $\int_{\mathcal{M}} v(x)\, d\mathcal{M} = 0$ is required in order to guarantee the validity of the Poincaré inequality,

$$||f||^2_{L^2(\mathcal{M})} \leq (\lambda_1)^{-1} ||\nabla_T f||^2_{L^2(\mathcal{M})}, \quad \text{for all } f \in V, \tag{2.4}$$

where λ_1 is the first eigenvalue of the Laplace-Beltrami operator.

Remark 2.1. It is convenient to observe that the space V may not be invariant under the flow because of the nonlinear character of the equation under consideration. In this case, it is sufficient to add an extra term $\alpha u (\alpha > 0)$ in the equation in order to control L_2 norms. However, for simplicity in the computations, we shall omit this term since it does not bring any additional difficulty or novelty.

We observe that problem (2.1) can be rewritten as

$$\frac{dU}{dt} + \mathcal{A}U = G(U),$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ -\Delta_{\mathcal{M}} & 0 \end{pmatrix}$$

is a maximal monotone operator and $G(\cdot)$ represents a locally Lipschitz perturbation. So, making use of standard semigroup arguments we have the following result:

Theorem 2.1.

- (**i**) *Under the above-mentioned conditions, problem* (2.1) *is well-posed in the space* $V \times L^2(\mathcal{M})$*; that is, for any initial data* $\left\{u^0, u^1\right\} \in V \times L^2(\mathcal{M})$*, there exists a unique weak solution of* (2.1) *in the class*

$$u \in C(\mathbb{R}_+; V) \cap C^1(\mathbb{R}_+; L^2(\mathcal{M})). \tag{2.5}$$

- (**ii**)*In addition, the velocity of the solution has the regularity given by*

$$u_t \in L^2_{loc}\left(\mathbb{R}_+; L^2(\mathcal{M})\right), \tag{2.6}$$

and, consequently, $g(u_t) \in L^2_{loc}\left(\mathbb{R}_+; L^2(\mathcal{M})\right)$ *by Assumption* 2.1.

Furthermore, if $\left\{u^0, u^1\right\} \in V \cap H^2(\mathcal{M}) \times V$*, then the solution has the following regularity:*

$$u \in L^\infty\left(\mathbb{R}_+; V \cap H^2(\mathcal{M})\right) \cap W^{1,\infty}(\mathbb{R}_+; V) \cap W^{2,\infty}\left(\mathbb{R}_+; L^2(\mathcal{M})\right).$$

Consider that u is the unique global weak solution of problem (2.1) given by Theorem 2.1. We define the corresponding energy functional by

$$E(t) = \frac{1}{2}\int_{\mathcal{M}} \left[|u_t(x,t)|^2 + |\nabla_T u(x,t)|^2\right] d\mathcal{M}. \tag{2.7}$$

For every solution of (2.1), in the class (2.5) we obtain for all $t_2 > t_1 \geq 0$,

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_{\mathcal{M}} a(x)\, g(u_t) u_t \, d\mathcal{M} dt, \tag{2.8}$$

and, therefore, the energy is a nonincreasing function of the time variable t.

3. Main result

In order to state the stability result, we need to define some functions which were first introduced in Lasiecka and Tataru [LA-TA]. For the reader's comprehension we will repeat them briefly. Let h be a concave, strictly increasing function, with $h(0) = 0$, and such that

$$h(s\, g(s)) \geq s^2 + g^2(s), \text{ for } |s| \leq 1. \tag{3.1}$$

Note that such a function can be straightforwardly constructed, considering the hypotheses on g in Assumption 2.1. In view of this function, we define

$$r(.) = h(\frac{\cdot}{meas\,(\Sigma_1)}). \tag{3.2}$$

As r is monotone increasing, then $cI + r$ is invertible for all $c \geq 0$. For L a positive constant, we set

$$p(x) = (cI + r)^{-1}(Lx), \tag{3.3}$$

where the function p is easily seen to be positive, continuous and strictly increasing with $p(0) = 0$. Finally, let

$$q(x) = x - (I + p)^{-1}(x). \tag{3.4}$$

We can now proceed to state our stability result.

Theorem 3.1. *Assume that Assumption* 2.1 *and Assumption* 2.2 *are in place. Let u be the weak solution of problem* (2.1)*. With the energy $E(t)$ defined as in* (2.7)*, there exists $T_0 > 0$ such that*

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right), \quad \forall t > T_0, \tag{3.5}$$

with $\lim_{t\to\infty} S(t) = 0$, where the contraction semigroup $S(t)$ is the solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0), \tag{3.6}$$

where q is given in (3.4)*, the constant L, which is given in* (3.3)*, depends on $meas(\Sigma)$ and the constant c is equal to $\frac{k^{-1}+K}{meas(\Sigma)(1+||a||_\infty)}$.*

Remark 3.1. If the feedback is linear, e.g., $g(s) = s$, then, under the same assumptions of Theorem 3.1, we obtain that the energy of problem (2.1) decays exponentially with respect to the initial energy; that is, there exist two positive constants $C > 0$ and $k > 0$ such that

$$E(t) \leq Ce^{-kt}E(0), \quad t > 0. \tag{3.7}$$

If we consider $g(s) = s^p$, $p > 1$ at the origin, and since the function $s^{\frac{p+1}{2}}$ is convex for $p > 1$, then, solving

$$S_t + S^{\frac{p+1}{2}} = 0, \tag{3.8}$$

we obtain the following polynomial decay rate:

$$E(t) \leq C(E(0))[E(0)^{\frac{-p+1}{2}} + t(p-1)]^{\frac{2}{-p+1}}.$$

We can find more examples of explicit decay rates in Cavalcanti, Domingos Cavalcanti and Lasiecka [CA-DO-LA].

4. Proof of main result

4.1. **Preliminaries.**

We collect, below, a few formulas to be invoked in the sequel.

Let ν be the exterior normal vector field on $\mathcal{M}$. The *Laplace- Beltrami operator* $\Delta_{\mathcal{M}}$ of a function $\varphi : \mathcal{M} \to \mathbb{R}$ of class C^2 is defined by

$$\Delta_{\mathcal{M}}\varphi := div_T \nabla_T \varphi, \tag{4.1}$$

where $div_T \nabla_T \varphi$, is the *divergent* of the vector field $\nabla_T \varphi$.

Assuming that $\varphi : \mathcal{M} \to \mathbb{R}$ is a function of class C^1 and $x \in \mathcal{M} \mapsto q(x) \in T_x(\mathcal{M})$ is a vector field of class C^1, we have

$$\int_{\mathcal{M}} q \cdot \nabla_T \varphi \, d\mathcal{M} = -\int_{\mathcal{M}} div_T q \, \varphi \, d\mathcal{M}, \tag{4.2}$$

$$2\varphi(q \cdot \nabla_T \varphi) = q \cdot \nabla_T(\varphi^2). \tag{4.3}$$

From (4.2) and (4.3), we conclude the following formula:

$$2\int_{\mathcal{M}} \varphi(q \cdot \nabla_T \varphi) \, d\mathcal{M} = \int_{\mathcal{M}} q \cdot \nabla_T(\varphi^2) \, d\mathcal{M} = -\int_{\mathcal{M}} div_T \, q|\varphi|^2 d\mathcal{M}. \tag{4.4}$$

We define a continuous linear operator $-\Delta_{\tilde{\mathcal{M}}} : H^1(\tilde{\mathcal{M}}) \to (H^1(\tilde{\mathcal{M}}))'$, where $\tilde{\mathcal{M}}$ is a nonempty open subset of $\mathcal{M}$ (sometimes the whole $\mathcal{M}$) such that

$$\langle -\Delta_{\tilde{\mathcal{M}}}\varphi, \psi\rangle = \int_{\tilde{\mathcal{M}}} \nabla_T \varphi \cdot \nabla_T \psi \, d\mathcal{M}, \quad \forall \varphi, \psi \in H^1(\tilde{\mathcal{M}}) \tag{4.5}$$

and, in particular,

$$\langle -\Delta_{\tilde{\mathcal{M}}}\varphi, \varphi\rangle = \int_{\tilde{\mathcal{M}}} |\nabla_T \varphi|^2 \, d\mathcal{M}, \quad \forall \varphi \in H^1(\tilde{\mathcal{M}}). \tag{4.6}$$

The operator $-\Delta_{\tilde{\mathcal{M}}} + I$ defines an isomorphism from $H^1(\tilde{\mathcal{M}})$ over $[H^1(\tilde{\mathcal{M}})]'$. We observe that when $\tilde{\mathcal{M}}$ is a manifold without boundary, and this is the case, for instance, if $\tilde{\mathcal{M}} = \mathcal{M}$, we have $H^1(\tilde{\mathcal{M}}) = H_0^1(\tilde{\mathcal{M}})$ and, consequently, $[H^1(\tilde{\mathcal{M}})]' = H^{-1}(\tilde{\mathcal{M}})$.

Remark 4.1. It is convenient to observe that all the above classical formulas can be extended to Sobolev spaces using density arguments.

The proof of Theorem 3.1 proceeds through several steps. In order to obtain the decay rate stated in (3.5), we will consider, initially, regular solutions of problem (2.1). Then, making use of standard density arguments, the estimate (3.5) holds for weak solutions.

4.2. **An identity.** We begin by proving the following proposition:

Proposition 4.2.1. *Let $\mathcal{M} \subset \mathbb{R}^3$ be an oriented regular compact surface without boundary and q a vector field of class C^1. Then, for every regular solution u of* (1.1) *we have the following identity:*

$$\left[\int_{\mathcal{M}} u_t\, q\cdot\nabla_T u\, d\mathcal{M}\right]_0^T + \frac{1}{2}\int_0^T\int_{\mathcal{M}} (div_T q)\left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt \\ + \int_0^T\int_{\mathcal{M}} \nabla_T u\cdot\nabla_T q\cdot\nabla_T u\, d\mathcal{M}dt + \int_0^T\int_{\mathcal{M}} a(x)\, g(u_t)(q\cdot\nabla_T u) d\mathcal{M}dt = 0. \tag{4.7}$$

Proof. Multiplying the equation (1.1) by the multiplier $q\cdot\nabla_T u$ and integrating on $\mathcal{M}\times]0,T[$, we obtain

$$0 = \int_0^T\int_{\mathcal{M}} (u_{tt} - \Delta_{\mathcal{M}} u + a(x)g(u_t))(q\cdot\nabla_T u)\, d\mathcal{M}\, dt. \tag{4.8}$$

Next, we will estimate some terms on the RHS of identity (4.8). Taking (4.2), (4.3) and (4.4) into account, we obtain

$$\begin{aligned} \int_0^T\int_{\mathcal{M}} (-\Delta_{\mathcal{M}} u)\,(q\cdot\nabla_T u)\, d\mathcal{M}dt &= \int_0^T\int_{\mathcal{M}} \nabla_T u\cdot\nabla_T(q\cdot\nabla_T u)\, d\mathcal{M}dt \\ &= \int_0^T\int_{\mathcal{M}} \nabla_T u\cdot\nabla_T q\cdot\nabla_T u\, d\mathcal{M}dt + \frac{1}{2}\int_0^T\int_{\mathcal{M}} q\cdot\nabla_T[|\nabla_T u|^2] d\mathcal{M}dt \\ &= \int_0^T\int_{\mathcal{M}} \nabla_T u\cdot\nabla_T q\cdot\nabla_T u\, d\mathcal{M}dt - \frac{1}{2}\int_0^T\int_{\mathcal{M}} |\nabla_T u|^2\, div_T q\, d\mathcal{M}dt, \end{aligned} \tag{4.9}$$

and, integrating by parts and considering (4.4), we obtain

$$\begin{aligned} &\int_0^T\int_{\mathcal{M}} (u_{tt} + a(x)\, g(u_t))\,(q\cdot\nabla_T u)\, d\mathcal{M}dt \\ = &\left[\int_{\mathcal{M}} u_t(q\cdot\nabla_T u)\right]_0^T - \int_0^T\int_{\mathcal{M}} u_t(q\cdot\nabla_T u_t) d\mathcal{M}dt \\ &+ \int_0^T\int_{\mathcal{M}} a(x)\, g\,(u_t)\,(q\cdot\nabla_T u) d\mathcal{M}dt \\ = &\left[\int_{\mathcal{M}} u_t(q\cdot\nabla_T u)\right]_0^T + \frac{1}{2}\int_0^T\int_{\mathcal{M}} (div_T q_T)\,|u_t|^2\, d\mathcal{M}dt \\ &+ \int_0^T\int_{\mathcal{M}} a(x)\, g\,(u_t)\,(q\cdot\nabla_T u) d\mathcal{M}dt. \end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we deduce (4.7), which concludes the proof of Proposition 4.2.1. □

Employing (4.7) with $q(x) = \nabla_T f$ where $f : \mathcal{M} \to \mathbb{R}$ is a C^3 function to be determined later, we infer

$$\begin{aligned}
(4.11)\qquad & \left[\int_{\mathcal{M}} u_t\, \nabla_T f \cdot \nabla_T u\, d\mathcal{M}\right]_0^T + \frac{1}{2}\int_0^T \int_{\mathcal{M}} \Delta_{\mathcal{M}} f \left\{|u_t|^2 - |\nabla_T u|^2\right\} d\mathcal{M}dt \\
& + \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot Hess(f) \cdot \nabla_T u)\, d\mathcal{M}dt \\
& + \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(\nabla_T f \cdot \nabla_T u) d\mathcal{M}dt = 0.
\end{aligned}$$

We have the following identity:

Lemma 4.2.3. *Let u be a weak solution to problem* (1.1) *and let $\xi \in C^1(\mathcal{M})$. Then*

(4.12)
$$\begin{aligned}
\left[\int_{\mathcal{M}} u_t\, \xi\, u\, d\mathcal{M}\right]_0^T &= \int_0^T \int_{\mathcal{M}} \xi |u_t|^2 d\mathcal{M}dt - \int_0^T \int_{\mathcal{M}} \xi |\nabla_T u|^2 d\mathcal{M}dt \\
&\quad - \int_0^T \int_{\mathcal{M}} (\nabla_T u \cdot \nabla_T \xi) u\, d\mathcal{M}dt - \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)\, \xi\, u\, d\mathcal{M}dt.
\end{aligned}$$

Proof. Multiplying the equation of (1.1) by $\xi\, u$ and integrating by parts we obtain the desired result. □

Substituting $\xi = \alpha > 0$ in (4.12) and combining the obtained result with identity (4.11) we deduce

$$\begin{aligned}
& \int_0^T \int_{\mathcal{M}} (\frac{\Delta_{\mathcal{M}} f}{2} - \alpha)\, |u_t|^2\, d\mathcal{M}dt \\
& + \int_0^T \int_{\mathcal{M}} \left[(\nabla_T u \cdot Hess(f) \cdot \nabla_T u) + \left(\alpha - \frac{\Delta_{\mathcal{M}} f}{2}\right) |\nabla_T u|^2\right] d\mathcal{M}dt \\
(4.13)\qquad & = -\left[\int_{\mathcal{M}} u_t\, \nabla_T f \cdot \nabla_T u\, d\mathcal{M}\right]_0^T - \alpha \left[\int_{\mathcal{M}} u_t\, u\, d\mathcal{M}\right]_0^T \\
& -\alpha \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t) u\, d\mathcal{M}dt \\
& - \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(\nabla_T f \cdot \nabla_T u) d\mathcal{M}dt.
\end{aligned}$$

Remark 4.2. This is the precise moment where the properties of the function f play an important role. Note that what we just need is to find a subset V of $\mathcal{M}$ such that

$$\begin{aligned}
(4.14)\qquad & C \int_0^T \int_V \left[u_t^2 + |\nabla_T u|^2\right] d\mathcal{M}dt \\
& \le \int_0^T \int_V (\frac{\Delta_{\mathcal{M}} f}{2} - \alpha)\, |u_t|^2\, d\mathcal{M}dt \\
& + \int_0^T \int_V \left[(\nabla_T u \cdot Hess(f) \cdot \nabla_T u) + \left(\alpha - \frac{\Delta_{\mathcal{M}} f}{2}\right) |\nabla_T u|^2\right] d\mathcal{M}dt,
\end{aligned}$$

for some positive constant C, provided that α is suitably chosen. Assuming, for a moment, that (4.14) holds, (4.13) yields

$$\begin{aligned} 2C\int_0^T E(t)dt &\leq C\int_0^T\int_{\mathcal{M}\setminus V}\left[u_t^2+|\nabla_T u|^2\right]d\mathcal{M}dt \\ &+ \left|\left[\int_{\mathcal{M}} u_t\,\nabla_T f\cdot\nabla_T u\,d\mathcal{M}\right]_0^T\right| + \alpha\left|\left[\int_{\mathcal{M}} u_t\,u\,d\mathcal{M}\right]_0^T\right| \\ &+ \left|\alpha\int_0^T\int_{\mathcal{M}} a(x)\,g(u_t)u\,d\mathcal{M}dt\right| \\ &+ \left|\int_0^T\int_{\mathcal{M}} a(x)\,g(u_t)(\nabla_T f\cdot\nabla_T u)d\mathcal{M}dt\right|. \end{aligned} \tag{4.15}$$

The inequality (4.15) is controlled by considering a standard procedure, which, for the reader's convenience, we will repeat later. The main idea behind this procedure is to consider the dissipative area, namely, $\mathcal{M}_*$, containing the set $\mathcal{M}\setminus V$ as stated in (2.2). It is important to observe that $\mathcal{M}_*$ is as small as V can be big.

The next subsections are devoted to the construction of a function f as well as a subset V of $\mathcal{M}$ such that the inequality (4.14) holds. This will be done, for simplicity, in a general setting, that is, for a Riemannian manifold (without boundary) with Riemmanian metric $\mathbf{g}$ of class C^2.

4.3. **Construction of a function such that $Hess f \approx \mathbf{g}$ and $\inf_{x\in V}|\nabla f(x)|>0$ locally.**

Throughout this subsection we are going to denote the Laplacian-Beltrami operator $\Delta_{\mathcal{M}}$ by Δ and the tangential-gradient ∇_T by ∇. Let $\mathcal{M}$ be a compact n-dimensional Riemannian manifold (without boundary) with Riemmanian metric $\mathbf{g}$ of class C^2. Let ∇ denote the Levi-Civita connection. Fix $p\in\mathcal{M}$. Our aim is to construct a function $f: V_p\to\mathbb{R}$ such that $Hess f\approx\mathbf{g}$ and $\inf\limits_{x\in V_p}|\nabla f(x)|>0$, where V_p is a neighborhood of p and the Hessian of f is seen as a bilinear form defined on the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at p.

We begin with an orthonormal basis $(e_1,\dots,e_n)$ of T_pM. Define a normal coordinate system $(x_1,\dots,x_n)$ in a neighborhood $\widetilde{V}_p$ of p such that $\partial/\partial x_i(p)=e_i(p)$ for every $i=1,\dots,n$. It is well known that in this coordinate system we have that $\Gamma_{ij}^k(p)=0$, where Γ_{ij}^k are the Christoffel symbols with respect to $(x_1,\dots,x_n)$ (see, for instance, [Do Carmo]).

The Hessian with respect to $(x_1,\dots,x_n)$ is given by

$$Hess f\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\frac{\partial^2 f}{\partial x_i\partial x_j}-\sum_{k=1}^n\Gamma_{ij}^k\frac{\partial f}{\partial x_k}.$$

The Laplacian of f is the trace of the Hessian with respect to the metric $\mathbf{g}$. If $\mathbf{g}_{ij}$ denote the components of the Riemannian metric with respect to $(x_1,\dots,x_n)$ and $\mathbf{g}^{ij}$ are the components of the inverse matrix of $\mathbf{g}_{ij}$, then the Laplacian of f is given by

$$\Delta f=\sum_{i,j}\mathbf{g}^{ij}Hess f\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right).$$

Consider the function $f : \widetilde{V}_p \to \mathbb{R}$ defined by

$$f(x) = x_1 + \frac{1}{2}\sum_{i=1}^{n} x_i^2.$$

It is immediate that $\Delta f(p) = n$ and $|\nabla f(p)| = 1$. Moreover, $Hess f(p) = \mathbf{g}(p)$, which implies that

$$Hess f(p)(v, v) = |v|_p^2.$$

We are interested in finding a neighborhood $V_p \subset \widetilde{V}_p$ of p and a strictly positive constant C such that

(4.16)

$$C\int_0^T \int_{V_p} \left(|\nabla u|^2 + u_t^2\right) d\mathcal{M}dt \leq \int_0^T \int_{V_p} \left[Hess f(\nabla u, \nabla u) + \left(\alpha - \frac{\Delta f}{2}\right)|\nabla u|^2 + \left(\frac{\Delta f}{2} - \alpha\right) u_t^2\right] d\mathcal{M}\, dt,$$

for some $\alpha \in \mathbb{R}$. We claim that if we consider $\alpha = \frac{n}{2} - \frac{1}{2}$ and $C = 1/4$ we obtain the desired inequality, which means that it is enough to prove that there exist $V_p \subset \widetilde{V}_p$ satisfying

$$\int_0^T \int_{V_p} Hess f(\nabla u, \nabla u) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right)|\nabla u|^2 d\mathcal{M}dt \geq 0 \tag{4.17}$$

and

$$\int_0^T \int_{V_p} \left(\frac{\Delta f}{2} - \frac{n}{2} + \frac{1}{4}\right) u_t^2 d\mathcal{M}dt \geq 0. \tag{4.18}$$

In order to prove the existence of a subset $V_p \subset \widetilde{V}_p$ where (4.17) holds, let θ_1 be the smooth field of symmetric bilinear forms on $\widetilde{V}_p$ defined as

$$\theta_1(X, Y) = Hess f(X, Y) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right)\mathbf{g}(X, Y),$$

where X and Y are vector fields on $\widetilde{V}_p$. It is clearly a positive definite bilinear form on p since $Hess f(p)(X, Y) = \mathbf{g}(p)(X, Y)$ and

$$\theta_1(p)(X, Y) = \frac{1}{4}\mathbf{g}(p)(X, Y).$$

Therefore, there exists a neighborhood $\widehat{V}_p$ such that θ_1 is positive definite and

$$\int_0^T \int_{\widehat{V}_p} Hess f(\nabla u, \nabla u) + \left(\frac{n}{2} - \frac{3}{4} - \frac{\Delta f}{2}\right)|\nabla u|^2 d\mathcal{M}dt \geq 0.$$

To prove the existence of $\breve{V}_p \subset \widetilde{V}_p$ such that (4.18) holds is easier. It is enough to notice that at p we have that

$$\left(\frac{\Delta f(p)}{2} - \frac{n}{2} + \frac{1}{4}\right) = \frac{1}{4}$$

and the existence of $\breve{V}_p \subset \widetilde{V}_p$ such that (4.18) holds is immediate. Furthermore we can eventually choose a smaller V_p such that $\inf_{x\in V_p} |\nabla f(x)| > 0$. Therefore the existence of $V_p \subset \widetilde{V}_p$ such that $\inf_{x\in V_p} |\nabla f(x)| > 0$ and (4.16) holds is settled.

4.4. **A function f that satisfies inequality (4.16) and $\inf_{x\in V} |\nabla f(x)| > 0$ in a wide domain.**

In what follows, $\bar{V}$ denotes the closure of V and ∂V denotes the boundary of V. When $\bar{V} \subset W$ is bounded, we say that V is compactly contained in W and we denote this relationship by $V \Subset W$.

Theorem 4.1. *Let $(\mathcal{M}, \mathbf{g})$ be a two-dimensional Riemannian manifold. Then, for every $\epsilon > 0$, there exist a finite family $\{V_i\}_{i=1\dots k}$ of open sets with smooth boundary, smooth functions $f_i : \bar{V}_i \to \mathbb{R}$ and a constant $C > 0$ such that*

(1) *the subsets $\bar{V}_i$ are pairwise disjoint;*
(2) $\mathrm{vol}(\bigcup_{i=1}^k V_i) \geq \mathrm{vol}(M) - \epsilon$*;*
(3) *inequality* (4.16) *holds for every f_i;*
(4) $\inf_{x\in V_i} |\nabla f(x)| > 0$ *for every $i = 1, \dots, k$.*

Proof. First of all, it is possible to get open subsets $\{\widetilde{W}_j\}_{j=1,\dots,s}$ with smooth boundaries and a family of smooth functions $\{\widetilde{f}_j : \widetilde{W}_j \to \mathbb{R}\}_{j=1,\dots,s}$ such that $\{\widetilde{W}_j\}_{j=1,\dots,s}$ is a cover of $\mathcal{M}$ and each $\widetilde{f}_j$ satisfies inequality (4.16). Moreover, we can choose $\widetilde{W}_j$ in such a way that their boundaries intercept themselves transversally and three or more boundaries do not intercept themselves at the same point.

Set $A := \bigcup_{j=1}^s \partial\widetilde{W}_j$. Then, $\mathcal{M}\backslash A$ is a disjoint union of connected open sets $\bigcup_{i=1}^k W_i$ such that ∂W_i is a piecewise smooth curve.

Each W_i is contained in some $\widetilde{W}_j$. Therefore, for each W_i, choose a function $\hat{f}_i := \widetilde{f}_j|_{W_i}$.

The open subsets V_i, $i = 1, \dots, k$, we are looking for are subsets of W_i. We can choose them in such a way that

(1) $V_i \Subset W_i$;
(2) ∂V_i is smooth;
(3) $\mathrm{vol}(W_i) - \mathrm{vol}(V_i) < \epsilon/k$.

Finally, if we set $f_i = \hat{f}_i|_{\bar{V}_i}$, we prove the theorem. □

Theorem 4.2. *Let $(\mathcal{M}, \mathbf{g})$ be a two-dimensional Riemannian manifold. Fix $\epsilon > 0$. Then, there exists a smooth function $f : M \to \mathbb{R}$ such that inequality* (4.16) *and the condition $\inf_{x\in V} |\nabla f(x)| > 0$ hold in a subset V with $\mathrm{vol}(V) \geq \mathrm{vol}(\mathcal{M}) - \epsilon$.*

Proof. Consider Theorem 4.1 and the constructions made in its proof. Denote $\lambda := \min_{i\neq j} \mathrm{dist}(V_i, V_j) > 0$. Consider a tubular neighborhood V^δ of $V = \bigcup_{i=1}^k V_i$ of the points whose distance is less than or equal to $\delta < \lambda/4$. Then, it is possible to define a smooth (cut-off) function $\eta : \mathcal{M} \to \mathbb{R}$ such that

$$\eta(x) \quad = \quad \begin{cases} 1 \text{ if } x \in V; \\ 0 \text{ if } x \in \mathcal{M}\backslash V^\delta; \\ \text{between } 0 \text{ and } 1 \text{ otherwise.} \end{cases}$$

Now, notice that $f : \mathcal{M} \to \mathbb{R}$ defined by

$$f(x) \quad = \quad \begin{cases} \hat{f}_i(x)\eta(x) \ \text{ if } \ x \in W_i; \\ 0 \ \text{ otherwise} \end{cases}$$

is smooth and satisfies inequality (4.16) and the condition $\inf_{x\in V} |\nabla f(x)| > 0$. In addition, the inequality $\mathrm{vol}(V) \geq \mathrm{vol}(\mathcal{M}) - \epsilon$ holds, which settles the theorem. □

We denote

$$\chi = \left[\int_{\mathcal{M}} u_t \, \nabla_T f \cdot \nabla_T u \, d\mathcal{M}\right]_0^T + \alpha \left[\int_{\mathcal{M}} u_t \, u \, d\mathcal{M}\right]_0^T. \tag{4.19}$$

Next we will estimate some terms in (4.15). Let us define

$$R := \max_{x\in\mathcal{M}} |\nabla_T f(x)|. \tag{4.20}$$

Estimate for $I_1 := \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)(\nabla_T f \cdot \nabla_T u) d\mathcal{M}dt$.

By the Cauchy-Schwarz inequality, taking (4.20) into account and considering the inequality $ab \leq \frac{a^2}{4\zeta} + \zeta b^2$, where ζ is a positive number, we obtain

$$|I_1| \leq \frac{||a||_{L^\infty(\mathcal{M})} R^2}{\zeta} \int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2 d\mathcal{M}dt + 2\zeta \int_0^T E(t)\, dt. \tag{4.21}$$

Estimate for $I_2 = \alpha \int_0^T \int_{\mathcal{M}} a(x)\, g(u_t) u \, d\mathcal{M}dt$.

Similarly we infer

$$|I_2| \leq \frac{||a||_{L^\infty(\mathcal{M})} \alpha^2 \lambda_1^{-1}}{16\zeta} \int_0^T \int_{\mathcal{M}} a(x)|g(u_t)|^2 \, d\mathcal{M}dt + 2\zeta \int_0^T E(t)\, dt, \tag{4.22}$$

where λ_1 comes from the Poincaré inequality given in (2.4).

Choosing ζ sufficiently small and inserting (4.19), (4.21) and (4.22) into (4.15) yields

$$\begin{aligned} \int_0^T E(t)\, dt \quad &\leq \quad |\chi| + C_1 \int_0^T \int_{\mathcal{M}} a(x)\, (g(u_t))^2 d\mathcal{M}dt \\ &+ \quad C_1 \int_0^T \int_{\mathcal{M}\setminus V} [|\nabla_T u|^2 + a(x)\, u_t^2] \, d\mathcal{M}dt, \end{aligned} \tag{4.23}$$

where

$$C_1 := C_1 \left(C, ||a||_{L^\infty(\mathcal{M})}, \lambda_1^{-1}, R, a_0^{-1}, n\right).$$

It remains to estimate the quantity $\int_0^T \int_{\mathcal{M}\setminus V} |\nabla_T u|^2 \, d\mathcal{M}dt$ in terms of the damping term $\int_0^T \int_{\mathcal{M}} [a(x)\, |g(u_t)|^2 + a(x)\, |u_t|^2] \, d\mathcal{M}dt$. For this purpose we have to build a "cut-off" function η_ε on a specific neighborhood of $\mathcal{M}\setminus V$. First of all, define $\tilde{\eta} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{\eta}(x) \quad = \quad \begin{cases} 1 & \text{if} \quad x \leq 0, \\ (x-1)^2 & \text{if} \quad x \in [1/2, 1], \\ 0 & \text{if} \quad x > 1, \end{cases}$$

and it is defined on $(0, 1/2)$ in such a way that $\tilde{\eta}$ is a nondecreasing function of class C^1. For $\varepsilon > 0$, set $\tilde{\eta}_\varepsilon(x) := \tilde{\eta}(x/\varepsilon)$. It is straightforward that there exists a

constant M which does not depend on ε such that

$$\frac{|\tilde{\eta}_\varepsilon'(x)|^2}{\tilde{\eta}_\varepsilon(x)} \leq \frac{M}{\varepsilon^2}$$

for every $x < \varepsilon$.

Let $\mathcal{M}_* \supset \mathcal{M}\backslash V$ be an open subset of $\mathcal{M}$ and let $\varepsilon > 0$ be such that

$$\tilde{\omega}_\varepsilon := \{x \in \mathcal{M}; \operatorname{dist}(x, \partial V) < \varepsilon\}$$

is a tubular neighborhood of ∂V and $\omega_\varepsilon := \tilde{\omega}_\varepsilon \cup \mathcal{M}\backslash V$ is contained in $\mathcal{M}_*$. Define $\eta_\varepsilon : \mathcal{M} \to \mathbb{R}$ as

$$\eta_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \mathcal{M}\backslash V, \\ \tilde{\eta}_\varepsilon(d(x, \mathcal{M}\backslash V)) & \text{if } x \in \omega_\varepsilon\backslash(\mathcal{M}\backslash V), \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward that η_ε is a function of class C^1 on $\mathcal{M}$ due to the smoothness of $\partial(\mathcal{M}\backslash V)$ and $\partial\omega_\varepsilon$. Notice also that

$$\frac{|\nabla_T \eta_\varepsilon(x)|^2}{\eta_\varepsilon(x)} = \frac{|\tilde{\eta}_\varepsilon'(d(x, \mathcal{M}_2))|^2}{\tilde{\eta}_\varepsilon(d(x, \mathcal{M}_2))} \leq \frac{M}{\varepsilon^2} \tag{4.24}$$

for every $x \in \omega_\varepsilon$. In particular, $\frac{|\nabla_T \eta_\varepsilon|^2}{\eta_\varepsilon} \in L^\infty(\omega_\varepsilon)$.

Taking $\xi = \eta_\varepsilon$ in the identity (4.12) we obtain

$$\begin{aligned} &\int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla_T u|^2 d\mathcal{M} dt \\ = &-\left[\int_{\omega_\varepsilon} u_t u \eta_\varepsilon \, d\mathcal{M}\right]_0^T + \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |u_t|^2 \, d\mathcal{M} \\ - &\int_0^T \int_{\omega_\varepsilon} u(\nabla_T u \cdot \nabla_T \eta_\varepsilon) \, d\mathcal{M} dt - \int_0^T \int_{\omega_\varepsilon} a(x) \, g(u_t) u \eta_\varepsilon \, d\mathcal{M} dt. \end{aligned} \tag{4.25}$$

Next we will estimate the terms on the RHS of (4.25).

Estimate for $K_1 := \int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |u_t|^2 \, d\mathcal{M} dt$.

From (2.2), since $\eta_\varepsilon \leq 1$ and $\omega_\varepsilon \subset \mathcal{M}_*$, where the damping lies, we deduce

$$K_1 \leq a_0^{-1} \int_0^T \int_{\mathcal{M}} a(x) \, u_t^2 \, d\mathcal{M} \, dt. \tag{4.26}$$

Estimate for $K_2 := -\int_0^T \int_{\omega_\varepsilon} a(x) \, g(u_t) u \eta_\varepsilon \, d\mathcal{M} dt$.

Taking into account the Cauchy-Schwarz inequality, the inequality $ab \leq \frac{1}{4\alpha} a^2 + \alpha b^2$ and (2.4) we obtain

$$|K_2| \leq \frac{\lambda_1^{-1} ||a||_{L^\infty(\mathcal{M})}}{4\alpha} \int_0^T \int_{\mathcal{M}} a(x) \, |g(u_t)|^2 \, d\mathcal{M} + 2\alpha \int_0^T E(t) \, dt, \tag{4.27}$$

where α is a positive constant.

Estimate for $K_3 := \int_0^T \int_{\omega_\varepsilon} u(\nabla_T u \cdot \nabla_T \eta_\varepsilon) d\mathcal{M} dt$.

Considering (4.24) and applying the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} |K_3| &\leq \frac{1}{2}\int_0^T \left[\int_{\omega_\varepsilon} \eta_\varepsilon |\nabla_T u|^2 \, d\mathcal{M} + \int_{\omega_\varepsilon} \frac{|\nabla_T \eta_\varepsilon|^2}{\eta_\varepsilon} |u|^2 \, d\mathcal{M}\right] dt \\ &\leq \frac{1}{2}\int_0^T \left[\int_{\omega_\varepsilon} \eta_\varepsilon |\nabla_T u|^2 \, d\mathcal{M} + \frac{M}{\varepsilon^2}\int_{\omega_\varepsilon} |u|^2 \, d\mathcal{M}\right] dt. \end{aligned} \tag{4.28}$$

Combining (4.25)-(4.28) we arrive at the following inequality:

$$\begin{aligned} \frac{1}{2}\int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla_T u|^2 \, d\mathcal{M} dt \leq |\mathcal{Y}| &+ \frac{\lambda_1^{-1} ||a||_{L^\infty(\mathcal{M})}}{4\alpha} \int_0^T \int_{\mathcal{M}} a(x) \, |g(u_t)|^2 \, d\mathcal{M} \\ &+ 2\alpha \int_0^T E(t) \, dt + \frac{M}{2\varepsilon^2} \int_0^T \int_{\omega_\varepsilon} |u|^2 \, d\mathcal{M} \, dt, \\ &+ a_0^{-1} \int_0^T \int_{\mathcal{M}} a(x) \, u_t^2 \, d\mathcal{M} \, dt, \end{aligned} \tag{4.29}$$

where

$$\mathcal{Y} := -\left[\int_{\omega_\varepsilon} u_t u \eta_\varepsilon \, d\mathcal{M}\right]_0^T. \tag{4.30}$$

Thus, combining (4.29) and (4.23), having in mind that

$$\frac{1}{2}\int_0^T \int_{\mathcal{M}\setminus V} |\nabla_T u|^2 \, d\mathcal{M} dt \leq \frac{1}{2}\int_0^T \int_{\omega_\varepsilon} \eta_\varepsilon |\nabla_T u|^2 \, d\mathcal{M} dt,$$

and choosing α small enough, we deduce

$$\begin{aligned} \int_0^T E(t) \, dt \leq |\chi| &+ C_1 |\mathcal{Y}| \\ &+ C_2 \int_0^T \int_{\mathcal{M}} [a(x) \, |g(u_t)|^2 + a(x) \, |u_t|^2] \, d\mathcal{M} dt \\ &+ \frac{M C_2}{\varepsilon^2} \int_0^T \int_{\omega_\varepsilon} |u|^2 \, d\mathcal{M} \, dt, \end{aligned} \tag{4.31}$$

where $C_2 = C_2(C_1, \lambda_1^{-1}, ||a||_{L^\infty(\mathcal{M})}, a_0^{-1})$.

On the other hand, from (4.19), (4.30) and (2.8) the following estimate holds:

$$\begin{aligned} |\chi| + 2C_2|\mathcal{Y}| &\leq C(E(0) + E(T)) \\ &= C\left[2\,E(T) + \int_0^T \int_{\mathcal{M}} a(x) \, g(u_t) \, u_t \, d\mathcal{M}\right], \end{aligned} \tag{4.32}$$

where C is a positive constant which depends on R.

Then, (4.31) and (4.32) yield

$$
\begin{aligned}
(4.33)\quad T\,E(T) &\leq \int_0^T E(t)\,dt \\
&\leq C\,E(T) + C\left[\int_0^T \int_{\mathcal{M}} [a(x)\,|g(u_t)|^2 + a(x)\,|u_t|^2]\,d\mathcal{M}dt\right] \\
&+ C\int_0^T \int_{\omega_\varepsilon} |u|^2\,d\mathcal{M}\,dt,
\end{aligned}
$$

where C is a positive constant which depends on $a_0, \lambda_1, R, ||a||_{L^\infty(\mathcal{M})}, n$ and $\frac{M}{\varepsilon^2}$.

Our aim is to absorb the last term on the RHS of (4.33). In order to do this, let us consider the following lemma, where T_0 is a positive constant which is sufficiently large for our purposes.

Lemma 4.3. *Under the hypotheses of Theorem* 3.1 *and for all $T > T_0$, there exists a positive constant $C(T_0, E(0))$ such that if (u, u_t) is the solution of* (1.1) *with weak initial data, we have*

(4.34)
$$
\int_0^T \int_{\mathcal{M}} |u|^2\,d\mathcal{M}\,dt \leq C(T_0, E(0))\left\{\int_0^T \int_{\mathcal{M}} \left(a(x)\,g^2(u_t)) + a(x)u_t^2\right)d\mathcal{M}\,dt\right\}.
$$

Proof. We argue by contradiction exactly as in Lasiecka and Tataru's work [LA-TA]. For simplicity we shall denote $u' := u_t$. Let us suppose that (4.34) is not satisfied and let $\{u_k(0), u_k'(0)\}$ be a sequence of initial data where the corresponding solutions $\{u_k\}_{k\in\mathbb{N}}$ of (1.1), with $E_k(0)$ assumed uniformly bounded in k, satisfy

$$
\lim_{k\to+\infty} \frac{\int_0^T ||u_k(t)||^2_{L^2(\mathcal{M})}dt}{\int_0^T \int_{\mathcal{M}} (a(x)\,g^2(u_k') + a(x)\,u_k'^2)\,d\mathcal{M}\,dt} = +\infty, \tag{4.35}
$$

that is,

$$
\lim_{k\to+\infty} \frac{\int_0^T \int_{\mathcal{M}} \left(a(x)\,g^2(u_k') + a(x)\,u_k'^2\right)d\mathcal{M}\,dt}{\int_0^T ||u_k(t)||^2_{L^2(\mathcal{M})}dt} = 0. \tag{4.36}
$$

Since $E_k(t) \leq E_k(0) \leq L$, where L is a positive constant, we obtain a subsequence, still denoted by $\{u_k\}$ from now on, which satisfies the convergence

$$
u_k \rightharpoonup u \text{ weakly in } H^1(\Sigma_T), \tag{4.37}
$$

$$
u_k \rightharpoonup u \text{ weak star in } L^\infty(0,T;V), \tag{4.38}
$$

$$
u_k' \rightharpoonup u' \text{ weak star in } L^\infty(0,T;L^2(\mathcal{M})). \tag{4.39}
$$

Employing compactness results we also deduce that

$$
u_k \to u \text{ strongly in } L^2(0,T;L^2(\mathcal{M})). \tag{4.40}
$$

At this point we will divide our proof into two cases, namely, $u \neq 0$ and $u = 0$.

(i) Case (I): $u \neq 0$.

We also observe that from (4.36) and (4.40) we have

$$
\lim_{k\to+\infty} \int_0^T \int_{\mathcal{M}} \left(a(x)\,g^2(u_k') + a(x)\,u_k'^2\right)d\mathcal{M}\,dt = 0. \tag{4.41}
$$

Passing to the limit in the equation, when $k \to +\infty$, we get

$$\begin{cases} u_{tt} - \Delta_{\mathcal{M}}\, u = 0 \text{ on } \mathcal{M} \times (0,T), \\ \qquad\qquad u_t = 0 \text{ on } \mathcal{M}_* \times (0,T), \end{cases} \tag{4.42}$$

and for $u_t = v$, we obtain, in the distributional sense,

$$\begin{cases} v_{tt} - \Delta_{\mathcal{M}}\, v = 0 \text{ on } \mathcal{M} \times (0,T), \\ \qquad\qquad v = 0 \text{ on } \mathcal{M}_* \times (0,T). \end{cases}$$

From uniqueness results due to Triggiani and Yao [TRI-YAO] we conclude that $v \equiv 0$, that is, $u_t = 0$. Indeed, let V_i be as in Theorem 4.1 and $\Gamma = \partial V_i$, which is a smooth curve contained in $\mathcal{M}_*$. Since $v \equiv 0$ on $\mathcal{M}_*$ we deduce that $v = \partial_\nu v = 0$ on Γ. Employing Triggiani and Yao's uniqueness results to the compact manifold $\bar{V}_i$ with boundary Γ we infer that $v \equiv 0$ on $\bar{V}_i$, for each $i = 1, \ldots, k$. Therefore, $v \equiv 0$ on $\mathcal{M}$ as we desired to prove. Returning to (4.42) we obtain the following elliptic equation a.e. in $(0,T)$ given by

$$\begin{cases} \Delta_{\mathcal{M}}\, u = \quad 0 \text{ on } \mathcal{M}, \\ \qquad u_t = \quad 0 \text{ on } \mathcal{M}, \end{cases}$$

which implies that $u = 0$, which is a contradiction.

(ii) Case (II): $u = 0$.

Defining

$$c_k := \left[\int_0^T \int_{\mathcal{M}} |u_k|^2 d\mathcal{M}\, dt \right]^{1/2} \tag{4.43}$$

and

$$\overline{u}_k := \frac{1}{c_k}\, u_k, \tag{4.44}$$

we obtain

$$\int_0^T \int_{\mathcal{M}} |\overline{u}_k|^2 d\mathcal{M}\, dt = \int_0^T \int_{\mathcal{M}} \frac{|u_k|^2}{c_k^2} d\mathcal{M}\, dt = \frac{1}{c_k^2} \int_0^T \int_{\mathcal{M}} |u_k|^2 d\mathcal{M}\, dt = 1. \tag{4.45}$$

Setting

$$\overline{E}_k(t) \quad := \quad \frac{1}{2} \int_{\mathcal{M}} |\overline{u}_k'|^2\, d\mathcal{M} + \frac{1}{2} \int_{\mathcal{M}} |\nabla \overline{u}_k|^2\, d\mathcal{M},$$

we deduce that

$$\overline{E}_k(t) = \frac{E_k(t)}{c_k^2}. \tag{4.46}$$

Recalling (4.33) we obtain, for T large enough, that

$$E(T) \le \hat{C} \left[\int_0^T \int_{\mathcal{M}} (a(x)\, g^2(u_t) + a(x)\, u_t^2)\, d\mathcal{M}\, dt + \int_0^T \int_{\mathcal{M}} |u|^2\, d\mathcal{M}\, dt \right],$$

and employing the identity $E(T) - E(0) = -\int_0^T \int_{\mathcal{M}} a(x)\, g(u_t)\, u_t\, d\mathcal{M}\, dt$, we get

$$E(t) \le E(0) \le \tilde{C} \left[\int_0^T \int_{\mathcal{M}} (a(x)\, g^2(u_t) + a(x)\, u_t^2)\, d\mathcal{M}\, dt + \int_0^T \int_{\mathcal{M}} |u|^2\, d\mathcal{M}\, dt \right],$$

for all $t \in (0,T)$, where T is sufficiently large. The last inequality and (4.46) yield

$$\overline{E}_k(t) := \frac{E_k(t)}{c_k^2} \leq \tilde{C}\left[\frac{\int_0^T \int_{\mathcal{M}} (a(x)\, g^2(u_k') + a(x)\, u_k'^2)}{\int_0^T \int_{\mathcal{M}} |u_k|^2\, d\mathcal{M}\, dt} + 1\right]. \tag{4.47}$$

From (4.36) and (4.47) we conclude that there exists a positive constant $\hat{M}$ such that

$$\overline{E}_k(t) := \frac{E_k(t)}{c_k^2} \leq \hat{M}, \quad \text{for all } t \in [0,T] \text{ and for all } k \in \mathbb{N},$$

that is,

$$\frac{1}{2}\int_{\mathcal{M}} |\overline{u}_k'|^2\, d\mathcal{M} + \frac{1}{2}\int_{\Omega} |\nabla \overline{u}_k|^2\, d\mathcal{M} \leq \hat{M}, \quad \text{for all } t \in [0,T] \text{ and for all } k \in \mathbb{N}. \tag{4.48}$$

For a subsequence $\{\overline{u}_k\}$, we obtain

$$\overline{u}_k \rightharpoonup \overline{u} \text{ weak star in } L^\infty(0,T;V), \tag{4.49}$$

$$\overline{u}_k' \rightharpoonup \overline{u}' \text{ weak star in } L^\infty(0,T;L^2(\mathcal{M})), \tag{4.50}$$

$$\overline{u}_k \to \overline{u} \text{ strongly in } L^2(0,T;L^2(\mathcal{M})). \tag{4.51}$$

We observe that from (4.36) we deduce

$$\lim_{k\to+\infty} \int_0^T \int_{\mathcal{M}} \frac{a(x)\, g^2(u_k')}{c_k^2}\, d\mathcal{M}\, dt = 0 \quad \text{and} \quad \lim_{k\to+\infty} \int_0^T \int_{\mathcal{M}} a(x)\, |\overline{u}_k'|^2\, d\mathcal{M}\, dt = 0. \tag{4.52}$$

In addition, $\overline{u}_k$ satisfies the equation

$$\overline{u}_k'' - \Delta_{\mathcal{M}} \overline{u}_k + a(x)\, \frac{g(u_k')}{c_k} = \quad 0 \quad \text{ on } \mathcal{M} \times (0,T).$$

Passing to the limit when $k \to +\infty$ and taking the above convergences into account, we obtain

$$\begin{cases} \overline{u}'' - \Delta_{\mathcal{M}} \overline{u} = 0 & \text{on } \mathcal{M} \times (0,T), \\ \overline{u}' = 0 & \text{on } \mathcal{M}_* \times (0,T). \end{cases} \tag{4.53}$$

Then, $v = \overline{u}_t$ satisfies, in the distributional sense,

$$\begin{cases} v_{tt} - \Delta_{\mathcal{M}}\, v = 0 \text{ on } \mathcal{M}, \\ v = 0 \text{ on } \mathcal{M}_*. \end{cases}$$

Applying, again, uniqueness results due to Triggiani and Yao [TRI-YAO], it follows that $v = \overline{u}_t = 0$. Returning to (4.53) we have a.e. in $(0,T)$ that

$$\begin{cases} \Delta_{\mathcal{M}}\, \overline{u} = 0 \text{ on } \mathcal{M}, \\ \overline{u}_t = 0 \text{ on } \mathcal{M}. \end{cases}$$

We deduce that $\overline{u} = 0$, which is a contradiction in view of (4.45) and (4.51). The lemma is settled. □

Inequalities (4.33) and (4.34) lead us to the following result.

Proposition 5.2.2. *For $T > 0$ large enough, the solution $[u, u_t]$ of (2.1) satisfies*

$$E(T) \leq C \int_0^T \int_{\mathcal{M}} \left[a(x)\, |u_t|^2 + a(x)\, |g(u_t)|^2\right] d\mathcal{M} dt, \tag{4.54}$$

where the constant $C = C(T_0, E(0), C, a_0, \lambda_1, R, ||a||_{L^\infty(\mathcal{M})}, n, \frac{M}{\varepsilon^2})$.

4.5. **Conclusion of Theorem 3.1.** In what follows we will proceed exactly as in Lasiecka and Tataru's work [LA-TA] (see Lemma 3.2 and Lemma 3.3 of the referred paper) adapted to our context. Let $\Sigma := \mathcal{M} \times (0,T)$,

$$\begin{aligned} \Sigma_\alpha &= \{(t,x) \in \Sigma \ : \quad |u_t| > 1 \ \text{ a.e.}\}, \\ \Sigma_\beta &= \Sigma \backslash \Sigma_\alpha. \end{aligned}$$

Then using hypothesis (iii) in Assumption 2.1, we obtain

$$\int_{\Sigma_\alpha} a(x) \left([g(u_t)]^2 + (u_t)^2 \right) d\Sigma_\alpha \leq \left(k^{-1} + K\right) \int_{\Sigma_\alpha} a(x) g(u_t)\, u_t d\Sigma_\alpha. \tag{4.55}$$

Moreover, from (3.1),

$$\int_{\Sigma_\beta} a(x) \left([g(u_t)]^2 + (u_t)^2 \right) d\Sigma_\beta \leq (1 + ||a||_\infty) \int_{\Sigma_\beta} h\left(a(x) g(u_t)\, u_t\right) d\Sigma_\beta. \tag{4.56}$$

Then, by Jensen's inequality,

$$\begin{aligned} &(1 + ||a||_\infty) \int_{\Sigma_\beta} h\left(g(u_t)\, u_t\right) d\Sigma_\beta \\ &\leq (1 + ||a||_\infty) meas(\Sigma)\, h\left(\frac{1}{meas(\Sigma)} \int_\Sigma a(x) g(u_t)\, u_t d\Sigma \right) \\ &= (1 + ||a||_\infty) meas(\Sigma)\, r\left(\int_\Sigma a(x) g(u_t)\, u_t d\Sigma \right), \end{aligned} \tag{4.57}$$

where $r(s) = h\left(\frac{s}{meas(\Sigma)}\right)$ is defined in (3.2). Thus

$$\begin{aligned} \int_\Sigma a(x) \left([g(u_t)]^2 + (u_t)^2 \right) d\Sigma \quad &\leq \quad \left(k^{-1} + K\right) \int_\Sigma a(x) g(u_t)\, u_t\, d\Sigma \\ &\quad + (1 + ||a||_\infty) meas(\Sigma)\, r\left(\int_\Sigma a(x) g(u_t)\, u_t\, d\Sigma \right). \end{aligned} \tag{4.58}$$

Splicing together (4.54) and (4.58), we have

$$\begin{aligned} E(T) \quad &\leq \quad (1 + ||a||_\infty) C \left[\frac{K_0}{(1 + ||a||_\infty)} \int_\Sigma a(x) g(u_t)\, u_t d\Sigma \right. \\ &\quad \left. + meas(\Sigma)\, r\left(\int_\Sigma a(x)\, g(u_t)\, u_t d\Sigma \right) \right], \end{aligned} \tag{4.59}$$

where $K_0 = k^{-1} + K$. Setting

$$\begin{aligned} L &= \frac{1}{C\, meas(\Sigma)\, (1 + ||a||_\infty)}, \\ c &= \frac{K_0}{meas(\Sigma)\, (1 + ||a||_\infty)}, \end{aligned}$$

we obtain

$$p[E(T)] \quad \leq \quad \int_\Sigma a(x)\, g(u_t)\, u_t\, d\Sigma = E(0) - E(T), \tag{4.60}$$

where the function p is as defined in (3.3). To finish the proof of Theorem 3.1, we invoke the following result from I. Lasiecka et al. [LA-TA].

Lemma B. *Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing we can define an increasing function q, $q(x) = x-(I+p)^{-1}(x)$. Consider a sequence s_n of positive numbers which satisfies*

$$s_{m+1} + p(s_{m+1}) \leq s_m.$$

Then $s_m \leq S(m)$, where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = s_0.$$

Moreover, if $p(x) > 0$ for $x > 0$, then $\lim_{t\to\infty} S(t) = 0$.

Taking into account the above result, we replace T (resp. 0) in (4.60) with $m(T+1)$ (resp. mT) in order to get

$$E(m(T+1)) + p\left(E(m(T+1))\right) \leq E(mT), \quad \text{for} \quad m = 0, 1, ... \tag{4.61}$$

Applying Lemma B with $s_m = E(mT)$ results in

$$E(mT) \leq S(m), \quad m = 0, 1, ... \tag{4.62}$$

Finally, using the inherent dissipativity of $E(t)$ given in relation (2.8), we have for $t = mT + \tau$, $0 \leq \tau \leq T$,

$$E(t) \leq E(mT) \leq S(m) \leq S\left(\frac{t-\tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right) \quad \text{for} \quad t > T, \tag{4.63}$$

where we have used the fact that $S(.)$ is dissipative. The proof of Theorem 3.1 is now completed.

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