THE TOPOLOGY OF SYMPLECTIC CIRCLE BUNDLES

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Abstract. We consider circle bundles over compact three-manifolds with symplectic total spaces. We show that the base of such a space must be irreducible or the product of the two-sphere with the circle. We then deduce that such a bundle admits a symplectic form if and only if it admits one that is invariant under the circle action in three special cases: namely, if the base is Seifert fibered, has vanishing Thurston norm, or if the total space admits a Lefschetz fibration.

1. Introduction

A conjecture due to Taubes states that if a closed, compact 4-manifold of the form \( M \times S^1 \) is symplectic, then \( M \) must fiber over \( S^1 \). A natural extension of this conjecture is to the case where \( E \to M \) is a possibly nontrivial circle bundle. In [4] it was shown that an \( S^1 \)-bundle admits an \( S^1 \)-invariant symplectic form if and only if its base fibers over \( S^1 \) and the Euler class \( e(E) \) of the total space pairs trivially with the fiber of some fibration. Thus based on the principle that an \( S^1 \)-bundle should admit a symplectic form if and only if it admits an invariant one, one arrives at the following conjecture.

Conjecture 1 (Taubes). If a circle bundle \( S^1 \to E \to M \) over a closed, oriented 3-manifold is symplectic, then there is a fibration \( \Sigma \to M \to S^1 \) such that \( e(E)([\Sigma]) = 0 \).

If an oriented 3-manifold fibers over \( S^1 \) with fiber \( \Sigma \neq S^2 \), then it follows by the long exact homotopy sequence that \( M \) is in fact aspherical. So a necessary condition for Conjecture 1 to hold is that any \( M \) that is the base of an \( S^1 \)-bundle, whose total space carries a symplectic form, must in fact be aspherical or \( S^2 \times S^1 \) in the case \( \Sigma = S^2 \). This observation provides the motivation for the following theorem, which is the main result of the first part of this paper.

Theorem 2 Let \( M \) be an oriented, closed 3-manifold so that some circle bundle \( S^1 \to E \to M \) admits a symplectic structure. Then, either \( M \) is diffeomorphic to \( S^2 \times S^1 \) and the bundle is trivial, or \( M \) is irreducible and aspherical.

A similar statement was proved by McCarthy in [20] for the case \( E = M \times S^1 \). More precisely, McCarthy showed that if \( M \times S^1 \) admits a symplectic structure,
then $M$ decomposes as a connected sum $M = A \# B$, where the first Betti number $b_1(A) \geq 1$ and $B$ has no nontrivial connected covering spaces. This can be refined quite substantially following Perelman’s proof of Thurston’s geometrisation conjecture (see [22], [23] or [21]), for one corollary of geometrisation is that the fundamental group of a closed 3-manifold is residually finite (see [11]), meaning that the $B$ in McCarthy’s theorem must have trivial fundamental group, and hence by the Poincaré Conjecture is diffeomorphic to $S^3$. Thus in fact $M$ must be prime and hence irreducible and aspherical or $S^2 \times S^1$. Theorem 2 is then a generalisation of this more refined statement to the case of nontrivial $S^1$-bundles. Our argument will rely on a vanishing result of Kronheimer-Mrowka for the Seiberg-Witten invariants of a manifold that splits into two pieces along a copy of $S^2 \times S^1$, which in itself is of independent interest (cf. Proposition 1). One may also prove Theorem 2 by following the argument of [20]; see Remark 1 below.

In the remainder of this paper we will show that Conjecture 1 holds in various special cases. First we will verify the conjecture under certain additional assumptions on the topology of the base manifold $M$. In order to be able to do this we will need to understand when a manifold fibers over $S^1$. One gains significant insight into this problem by considering the Thurston norm $|| \cdot ||_T$ on $H^1(M, \mathbb{R})$, which was introduced by Thurston in [27]. The Thurston norm enables one to see which integral classes $\alpha \in H^1(M, \mathbb{Z})$ can be represented by closed, nonvanishing 1-forms, which in turn induce fibrations of $M$ by compact surfaces.

In [5] Friedl and Vidussi showed that if $E = M \times S^1$ admits a symplectic form and $|| \cdot ||_T \equiv 0$ or $M$ is Seifert fibered, then $M$ must fiber over $S^1$. In Corollary 2 below we will show that in fact Conjecture 1 holds in these two cases. The argument will be based on understanding the Seiberg-Witten invariants of the total space $E$ given that $M$ has vanishing Thurston norm and the Seifert case will be deduced as a corollary of this. Indeed, if $M$ has vanishing Thurston norm and $S^1 \to E \to M$ is symplectic, then the canonical class of $E$ must be trivial. This combined with the restrictions on Seiberg-Witten basic classes of a symplectic manifold as proved by Taubes in [26] means that $K = 0$ is the only Seiberg-Witten basic class and the result then follows by an application of a vanishing result of Lescop (cf. [17] or [28]).

Another special case of the Taubes conjecture is when the total space $E$ admits a Lefschetz fibration, as was considered in [2] and [3] for a trivial bundle. In view of Corollary 2 we will be able to give a comparatively simple proof of the following result.

**Theorem 9.** Let $S^1 \to E \xrightarrow{\pi} M$ be a symplectic circle bundle over an irreducible base $M$. If $E$ admits a Lefschetz fibration, then $M$ fibers over $S^1$.

It then follows by considering the Kodaira classification of complex surfaces that Conjecture 1 holds under the assumption that the total space admits a complex structure.

**Outline of paper.** In Section 2 we will state the relevant vanishing result of Kronheimer-Mrowka in order to prove Theorem 2. In Section 3 we recall the definition of the Thurston norm and quote some well-known facts about it. In Section 4 we will use our knowledge of the Thurston norm to verify Conjecture 1 under the assumption that the base is Seifert fibered or has vanishing Thurston norm. Finally in Section 5 we will define Lefschetz fibrations and prove that $M$ is fibered when one has a Lefschetz fibration on the total space $E$. 
2. Asphericity of the base $M$

Throughout this article all manifolds will be closed, connected and oriented and $M$ will always denote a manifold of dimension 3. In addition we will make the convention that all (co)homology groups will be taken with integral coefficients unless otherwise stated.

In [20] it was shown that if $M \times S^1$ is symplectic, then $M$ must be irreducible and aspherical or $S^2 \times S^1$. We extend this to the case of a nontrivial $S^1$-bundle. We first collect some relevant lemmas.

**Lemma 1.** Let $M = M_1 \# M_2$ be a nontrivial connect sum decomposition with $b_1(M) \geq 1$. Then there is a finite covering $N$ of $M$ that decomposes as a direct sum $N = N_1 \# N_2$ where $b_1(N_i) \geq k$ for any given $k$.

**Proof.** It follows from Mayer-Vietoris that the Betti numbers are additive for a connect sum; hence by assumption we may assume that $b_1(M_1) \geq 1$. By the proof of geometrisation it follows that the fundamental group of a 3-manifold is residually finite (cf. [11]) and hence $M_2$ has a nontrivial $d$-fold cover $\tilde{M}_2$, with $d \geq 2$. By removing a ball from $M_2$ and its disjoint lifts from $\tilde{M}_2$ and then gluing in $d$ copies of $M_1$ we obtain a cover $\tilde{M}$ of $M = M_1 \# M_2$, and by construction $\tilde{M}$ has a connect sum decomposition as $\tilde{M} = M_1 \# P$, where $b_1(P) \geq 1$. We may now take a $k$-fold cover associated to some surjective homomorphism of $\pi_1(M_1) \to \mathbb{Z}_k$ and glue in copies of $P$ to get a cover of $\tilde{M}$ (and hence of $M$), which decomposes in two pieces one of which has first Betti number at least $k$. One more application of this procedure gives the desired result. \hfill $\square$

**Lemma 2.** Let $S^1 \to E \overset{\pi}{\to} M$ be a circle bundle, whose Euler class we denote by $e(E) \in H^2(M)$. Then

\begin{enumerate}
\item $b_2(E) = \begin{cases} 
2b_1(M) - 2 & \text{if } e(E) \text{ is not torsion}, \\
2b_1(M) & \text{if } e(E) \text{ is torsion}.
\end{cases}$
\item $b_2^+(E) = b_2^-(E) \geq b_1(M) - 1$.
\end{enumerate}

**Proof.** We consider the Gysin sequence

$$H^0(M) \overset{\pi^*}{\leftarrow} H^2(M) \overset{\pi^*}{\leftarrow} H^2(E) \overset{\pi_*}{\rightarrow} H^1(M) \overset{\cup}{\rightarrow} H^3(M),$$

where $e \in H^2(M)$ denotes the Euler class of the bundle. By Poincaré duality $H^0(M) = H^3(M) = \mathbb{Z}$ and $b_1(M) = b_2(M)$, so we conclude by exactness that $b_2(E) = 2b_1(M) - 2$ if $e$ is not torsion and $b_2(E) = 2b_1(M)$ if $e$ is torsion. Furthermore since $E$ bounds its associated disc bundle, it has zero signature and hence $b_2^+(E) = b_2^-(E) \geq b_1(M) - 1$. \hfill $\square$

We will need to appeal to a vanishing result for the Seiberg-Witten invariants of manifolds that decompose along $S^2 \times S^1$, which we take from [15]. For this we will need to define a relative notion of $b_2^+$ for an oriented 4-manifold $X$ with boundary. This is done by considering the symmetric form induced on rational cohomology that is obtained as the composition

$$H^2(X, \partial X) \times H^2(X, \partial X) \overset{i^* \times 1_d}{\rightarrow} H^2(X) \times H^2(X, \partial X) \overset{\cup}{\rightarrow} \mathbb{Q}.$$

Here the map $i^*$ is the map coming from the long exact sequence of the pair $(X, \partial X)$ and the second map is nondegenerate by Poincaré duality. This is then a symmetric,
possibly degenerate, form on \( H^2(X, \partial X) \) and we define \( b^+_2(X) \) to be the dimension of a maximal positive definite subspace.

**Theorem 1** (Kronheimer-Mrowka, [16]). Let \( X = X_1 \cup_{\partial X_1 = \partial X_2} X_2 \) where \( \partial X_1 = -\partial X_2 = S^2 \times S^1 \) and \( b^+_2(X_1), b^+_2(X_2) \geq 1 \). Then for all Spin\(c\)-structures \( \xi \),

\[
\sum_{\xi^* - \xi \in \text{Tor}} \text{SW}(\xi^*) = 0.
\]

Although it is not explicitly stated in the book [16], Theorem 1 can be deduced as follows: formula 3.27 (p. 75) allows one to compute the sum of the SW invariants of all Spin\(c\)-structures that differ by torsion as a pairing of certain Floer groups. However these groups are zero for \( S^2 \times S^1 \) by Proposition 3.10.3 in the case of an untwisted coefficient system and by Proposition 3.10.4 in the twisted case, and thus this sum must vanish.

Theorem 1 then implies certain restrictions on the decomposition of symplectic manifolds along a copy of \( S^2 \times S^1 \).

**Proposition 1.** A symplectic manifold \( X \) cannot be decomposed as \( X = X_1 \cup_{\partial X_1 = \partial X_2} X_2 \), where \( \partial X_1 = -\partial X_2 = S^2 \times S^1 \) and \( b^+_2(X_1), b^+_2(X_2) \geq 1 \).

**Proof.** By the hypotheses of the proposition, we conclude from Theorem 1 that for every Spin\(c\)-structure \( \xi \in \text{Spin}^c(X) \),

\[
\sum_{\xi^* - \xi \in \text{Tor}} \text{SW}(\xi^*) = 0.
\]

However as \( X \) is symplectic and

\[
b^+_2(X) \geq b^+_2(X_1) + b^+_2(X_2) \geq 2
\]

the nonvanishing result of Taubes implies \( \text{SW}(\xi_{\text{can}}) = \pm 1 \), where \( \xi_{\text{can}} \) denotes the canonical Spin\(c\)-structure associated to the symplectic structure on \( E \) (cf. [25]). Moreover it follows from the constraints on SW basic classes of a symplectic manifold of [26] that if \( \xi^* \) is another Spin\(c\)-structure with nontrivial SW invariant and \( \xi_{\text{can}} - \xi^* \in \text{Tor} \), then in fact \( \xi_{\text{can}} = \xi^* \). Hence

\[
\sum_{\xi^* - \xi_{\text{can}} \in \text{Tor}} \text{SW}(\xi^*) = \pm 1,
\]

which is a contradiction. \( \square \)

**Theorem 2.** Let \( M \) be an oriented, closed 3-manifold, so that some circle bundle \( S^1 \to E \overset{p}{\to} M \) admits a symplectic structure. Then \( M \) is irreducible and aspherical or \( M = S^2 \times S^1 \) and the bundle is trivial.

**Proof.** We first show that \( M \) must be prime. Since \( E \) is symplectic it follows from Lemma 2 that \( b_1(M) \geq 1 \). Assume that \( M = M_1 \# M_2 \) is a nontrivial connected sum. Then by taking a suitable covering as in Lemma 1 and pulling back \( E \) and its symplectic form we may assume without loss of generality that \( b_1(M_i) \geq 2 \). We let \( S \) denote the gluing sphere of the connected sum. Then as \( S \) is nullhomologous the bundle restricted to \( S \) is trivial. Thus the connect sum decomposition induces a decomposition \( E = E_1 \cup_{S^2 \times S^1} E_2 \). Since the bundles \( E_i \to M_i \setminus B^3 \) are trivial on the boundary we may extend them to bundles \( \tilde{E}_i \to M_i \) and as \( b_1(M_i) \geq 2 \), Lemma 2 implies that \( b^+_2(\tilde{E}_i) \geq 1 \). Further, since \( E_i \simeq \tilde{E}_i \setminus (S^1 \times pt) \) we have that

\[
b^+_2(E_i) \geq b^+_2(\tilde{E}_i) \geq 1,
\]
which then contradicts Proposition \[1\]. Hence \(M\) is prime, and thus irreducible or \(S^2 \times S^1\).

We assume that \(M\) is irreducible. Then by the sphere theorem, \(\pi_2(M) = 0\). Since \(b_1(M) \geq 1\), we have that \(\pi_1(M)\) is infinite, so the universal cover \(\tilde{M}\) of \(M\) is not compact and has \(\pi_i(\tilde{M})\) trivial for \(i = 1, 2\). The Hurewicz theorem then implies that the first nontrivial \(\pi_1(\tilde{M})\) is isomorphic to \(H_1(\tilde{M})\). But since \(M\) is not compact, \(H_3(M) = 0\) and as \(M\) is 3-dimensional, \(H_i(M) = 0\) for all \(i \geq 4\). Hence \(\pi_i(\tilde{M}) = 0\) for all \(i \geq 1\) and it follows from Whitehead’s Theorem that \(\tilde{M}\) is contractible, that is, \(M\) is aspherical.

In the case where \(M = S^2 \times S^1\), any symplectic bundle must be trivial by Lemma \[2\].

**Remark 1.** One may also give a proof of Theorem \[2\] that uses the covering construction of \[20\]. In order to do this one first takes finite coverings on each of the two pieces in the connect sum decomposition. Then one glues these together to find a covering \(\tilde{M}\) where the sphere of the connect sum lifts to a sphere that is nontrivial in real cohomology. This sphere then lifts to the total space of the pullback bundle \(\tilde{E}\) over \(\tilde{M}\). One may also assume by Lemma \[1\] that \(b_1(\tilde{M})\) is large and hence \(b_2(\tilde{E})\) is large. Then a standard vanishing theorem for the SW invariants (cf. \[13\]) implies that all invariants are zero, which then contradicts Taubes’ result if \(E\) and hence \(\tilde{E}\) is symplectic.

By considering the long exact homotopy sequence we have the following corollary that was first proved by Kotschick in \[14\].

**Corollary 1.** Let \(S^1 \to E \overset{\pi}{\to} M\) be a symplectic circle bundle over an oriented 3-manifold \(M\). Then the map \(\pi_1(S^1) \to \pi_1(E)\) induced by the inclusion of the fiber is injective. In particular a fixed point free circle action on a symplectic 4-manifold can never have contractible orbits.

### 3. The Thurston Norm

In this section we will define and collect several relevant facts about the Thurston norm. We first define the negative Euler characteristic or complexity of a possibly disconnected, orientable surface \(\Sigma = \bigsqcup \Sigma_i\) to be

\[
\chi_-(\Sigma) = \sum_{\chi(\Sigma_i) \leq 0} \chi(\Sigma_i),
\]

where \(\chi\) denotes the Euler characteristic of the surface.

Next we define the Thurston norm \(\|\cdot\|_T\) as a map on \(H^1(M)\) by

\[
\|\sigma\|_T = \min \{\chi_-(\Sigma) \mid PD(\Sigma) = \sigma\}.
\]

It is a basic fact that this map extends uniquely to a (semi)norm on \(H^1(M, \mathbb{R})\), which we will denote again by \(\|\cdot\|_T\). One particularly important property of the Thurston norm is that its unit ball, which we denote by \(B_T\), is a (possibly noncompact) convex polytope with finitely many faces. If \(B_{T^*}\) denotes the unit ball in the dual space we have the following characterisation of \(B_T\).

**Theorem 3** \([27\), p. 106\]. The unit ball \(B_{T^*}\) is a polyhedron whose vertices are integral lattice points, \(\pm \beta_1, \ldots, \pm \beta_k\) and the unit ball \(B_T\) is defined by the following inequalities:

\[
B_T = \{\alpha \mid |\beta_i(\alpha)| \leq 1, \ 1 \leq i \leq k\}.
\]
We are interested in understanding how a manifold fibers over $S^1$, and the following theorem says that the Thurston norm determines precisely which cohomology classes can be represented by fibrations.

**Theorem 4** ([27], p. 120). Let $M$ be a compact, oriented 3-manifold. The set $F$ of cohomology classes in $H^1(M, \mathbb{R})$ representable by nonsingular closed 1-forms is the union of the open cones on certain top-dimensional open faces of $B_T$, minus the origin. The set of elements in $H^1(M, \mathbb{Z})$ whose Poincaré dual is represented by the fiber of some fibration consists of the set of lattice points in $F$.

We call a top-dimensional face of the unit ball $B_T$ fibered if some integral class, and hence all, in the cone over its interior can be represented by a fibration. One also understands how the Thurston norm behaves under finite covers by the following result of Gabai.

**Theorem 5** ([7], Cor. 6.13). Let $\tilde{M} \rightarrow M$ be a finite connected $d$-sheeted covering. Then for $\sigma \in H^1(M, \mathbb{R})$ we have
\[
||\sigma||_T = \frac{1}{d}||p^*\sigma||_T.
\]

These facts then allow us to completely characterise the Thurston norm of an irreducible Seifert fibered manifold.

**Proposition 2.** If $M$ is irreducible and Seifert fibered, then either the Thurston norm of $M$ vanishes identically or $M$ fibers over $S^1$ and
\[
||\sigma||_T = m.|\sigma(\gamma)|
\]
for some class $\gamma \in H_1(M)$.

**Proof.** Since $M$ is irreducible and Seifert fibered, either $M$ has a horizontal surface, i.e. a closed surface transverse to all fibers, or every embedded surface is isotopic to a vertical surface, i.e. a surface that is a union of fibers (cf. [9], Prop 1.11) and is hence a union of tori, so the Thurston norm is identically zero. If $M$ has a horizontal surface $F$, which we may assume to be connected, then $M$ is a mapping torus with monodromy $\phi \in \text{Diff}^+(F)$ so that $\phi^n = \text{Id}$ for some $n$. This means that $M$ is covered by $\tilde{M} = F \times S^1$. If $\bar{\gamma} = pt \times S^1$, then the Thurston norm of $\tilde{M} \rightarrow M$ is given by
\[
||\sigma||_T = \chi(F)|\sigma(\bar{\gamma})|.
\]

We let $\gamma = p_*(\bar{\gamma})$. Then by Theorem 5 the norm on $M$ is given by
\[
||\sigma||_T = \frac{1}{n}||p^*\sigma||_T = \frac{\chi(F)}{n}|p^*\sigma(\bar{\gamma})| = \frac{\chi(F)}{n}|\sigma(\gamma)| = m.|\sigma(\gamma)|.
\]

\[\square\]

**Example 1** (Seifert fibered spaces with horizontal surfaces). We note that in the second case of Proposition 2 the Thurston ball $B_T$ consists of two (noncompact) faces that are both fibered and that the Thurston norm is identically zero on a codimension one subspace $K$. Thus by [4] any bundle over such an $M$ will admit an $S^1$-invariant symplectic form except possibly in the case where the Euler class $e(E)$ pairs trivially with all elements in $K$; that is, $e(E)$ pairs trivially with all tori in $M$. By taking the pullback bundle of the cover $\tilde{M} = F \times S^1 \rightarrow M$ we may assume that we have a bundle $E$ over $F \times S^1$ that is symplectic and has Euler class that again pairs trivially with embedded tori in $\tilde{M}$ and is thus a nonzero multiple of
\( PD(\tilde{\gamma}) \). This in turn has a covering \( \tilde{E} \) that is an \( S^1 \)-bundle with Euler class equal to \( PD(\tilde{\gamma}) \). Now if we let \( T = \tilde{\gamma} \times S^1 \) and \( X = \tilde{M} \times S^1 \), then the SW polynomial of \( X \) can be computed to be

\[
SW^4_X = (t_T - t_T^{-1})^{2g-2},
\]

where \( g \) is the genus of \( F \). Then by the formula of Baldridge in [1], it follows that all the SW invariants of \( \tilde{E} \) are zero, contradicting Taubes’ nonvanishing result for the SW invariants of a symplectic manifold. So in fact Conjecture [1] holds for Seifert fibered spaces that have horizontal surfaces.

4. The case of vanishing Thurston norm

In [5] Friedl and Vidussi showed that if \( E = M \times S^1 \) admits a symplectic form and \( || ||_T \equiv 0 \) or \( M \) is Seifert fibered, then \( M \) must fiber over \( S^1 \). In this section we shall extend this to the case of a nontrivial \( S^1 \)-bundle and then show that Conjecture [1] holds in both of these cases. From now on we shall assume that \( M \) is irreducible, which in view of Theorem [2] only excludes the case where \( M = S^2 \times S^1 \) and the bundle is trivial. Our argument will be based on that of [5], and we begin with the following lemma.

**Lemma 3.** If \( S^1 \to E \to M \) is a bundle over an \( M \) that has vanishing Thurston norm, then

\[
H^2(E)/\text{Tor} = V \oplus W,
\]

where \( V, W \) are isotropic subspaces that admit a basis of embedded tori.

**Proof.** We consider the Gysin sequence

\[
\mathbb{Z} \longrightarrow H^2(M) \xrightarrow{\pi^*} H^2(E) \xrightarrow{\pi_*} H^1(M) \longrightarrow \mathbb{Z}.
\]

Here \( s \) is a section defined on the image of \( \pi_* \) as follows: we represent an element \( \sigma \in H^1(M) \) by an embedded surface \( \Sigma \). By exactness, \( \sigma \) will be in \( \text{Im}(\pi_*) \) precisely when the bundle is trivial on \( \Sigma \) and in this case we may lift \( \Sigma \) to some \( \tilde{\Sigma} \) in \( E \). As \( H^1(M) \) is free, we define \( s \) on a \( \mathbb{Z} \)-basis \( \{\sigma_i\} \) by \( s(\sigma_i) = \tilde{\Sigma}_i \). We set \( V = \pi^*(H^2(M)) \) and \( W = s(H^1(M)) \). Then \( V \) is clearly spanned by embedded tori and the statement for \( W \) is precisely the assumption on the Thurston norm. □

**Proposition 3.** Let \( S^1 \to E \xrightarrow{\pi} M \) be an \( S^1 \)-bundle with torsion Euler class \( e(E) \). Then there is a finite cover \( \tilde{M} \xrightarrow{p} M \) such that the pullback bundle \( p^*E \to \tilde{M} \) is trivial.

**Proof.** We choose a splitting of \( H_1(M) = F \oplus T \), where \( T \) is the torsion subgroup and \( F \) is any free complement. We take the cover \( \tilde{M} \xrightarrow{p} M \) associated to the kernel of the composition

\[
\pi_1(M) \to H_1(M) \xrightarrow{\phi} T,
\]

where \( \phi \) is the projection with kernel \( F \). Note that the composition \( H_1(\tilde{M}) \xrightarrow{p_*} H_1(M) \xrightarrow{\phi} T \) is zero. Then by the Universal Coefficient Theorem we have the
following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}(H_1(\tilde{M}), \mathbb{Z}) & \longrightarrow & H^2(\tilde{M}) & \longrightarrow & \text{Hom}(H_2(\tilde{M}), \mathbb{Z}) & \longrightarrow & 0 \\
\uparrow{(p_\ast)}^* & & \uparrow{p^\ast} & & \uparrow{(p_\ast)}^* & & \uparrow{=} & & \\
0 & \longrightarrow & \text{Ext}(H_1(M), \mathbb{Z}) & \longrightarrow & H^2(M) & \longrightarrow & \text{Hom}(H_2(M), \mathbb{Z}) & \longrightarrow & 0.
\end{array}
\]

This implies that \( p^\ast \) is zero on torsion in \( H^2(M) \), so the pullback bundle is indeed trivial.

\[ \square \]

**Theorem 6.** If \( S^1 \to E \xrightarrow{\pi} M \) is a symplectic circle bundle over an irreducible manifold for which \( ||\cdot||_T \) is identically zero, then \( M \) fibers over \( S^1 \).

After this paper had been submitted the author was informed that Friedl and Vidussi have independently proved this result (cf. \[6\]).

**Proof.** Since \( E \) is symplectic it has an associated canonical \( \text{Spin}^c \)-structure \( \xi_{can} \) and canonical class that we denote by \( K \). We claim that our assumption on the Thurston norm of the base implies that \( K \) must be torsion. For by Taubes’ nonvanishing result \( \xi_{can} \) has nontrivial SW invariant. If \( \alpha \in H^2(E) \), the adjunction inequality (see \[15\]) and Lemma \[3\] imply that

\[ |\alpha.K| = 0. \]

This also holds in the case \( b_2^+(E) = 1 \) (cf. \[18\], Theorem E and \[12\], Theorem B). As \( M \) is irreducible and \( b_2(M) \geq 1 \) the assumption on the vanishing of the Thurston norm implies that \( \tilde{M} \) contains an embedded, incompressible torus \( T \hookrightarrow M \). Then by Proposition 7 of \[12\] either \( T \) is the fiber of some fibration or there is a finite cover \( \tilde{M} \xrightarrow{\pi} M \) with large \( b_1 \), say \( b_1(\tilde{M}) \geq 4 \). We assume that the latter holds. Then the pullback \( \tilde{E} = p^\ast E \) will be symplectic with canonical class \( \tilde{K} = p^\ast K \), with symplectic form \( \tilde{\omega} = p^\ast \omega \) and \( b_2^+(\tilde{E}) \geq 2 \). Then for any \( \text{Spin}^c \)-structure \( \xi_{can} \otimes F \) that has nontrivial SW invariant we have by \[25\],

\[ 0 \leq F.[\tilde{\omega}] \leq \tilde{K}.[\tilde{\omega}]. \]

Moreover, since \( \tilde{K} \) is torsion and equality on the left implies \( F = 0 \), we conclude that in fact \( \tilde{K} = 0 \). Thus \( \tilde{K} = 0 \), so \( \xi_{can} \) is trivial and this is the only \( \text{Spin}^c \)-structure with nonzero SW invariant. We now need to consider two cases. We first assume that \( e(E) \) and hence \( e(\tilde{E}) \) is nontorsion. In this case we compute

\[ \pm 1 = \sum_{\xi^* \in \text{Spin}^c(\tilde{E})} SW^4_{E}(\xi^*) = \sum_{\xi^* \in \text{Spin}^c(\tilde{E})} \sum_{\xi \equiv \xi^* \mod \tilde{\varepsilon}} SW^3_{M}(\xi) = \sum_{\xi \in \text{Spin}^c(\tilde{E})} SW^3_{M}(\xi), \]

where the second inequality follows from Theorem 1 in \[1\]. However as \( b_1(\tilde{M}) \geq 4 \) this sum is zero (cf. \[28\], p. 114), a contradiction. If the Euler class is torsion we may assume by Proposition \[3\] that it is indeed zero and the above calculation reduces to

\[ \pm 1 = \sum_{\xi \in \text{Spin}^c(\tilde{E})} SW^3_{E}(\xi) = \sum_{\xi \in \text{Spin}^c(\tilde{E})} SW^3_{M}(\xi) = 0. \]

In either case we obtain a contradiction and hence \( M \) must fiber over \( S^1 \). \[ \square \]
As a consequence of this theorem we conclude that Conjecture 1 holds if $M$ has vanishing Thurston norm or is Seifert fibered.

**Corollary 2.** Conjecture 1 holds if $M$ is Seifert fibered or $\|T\| = 0$.

**Proof.** If $M$ has vanishing Thurston norm, then by Theorem 4 we conclude that if one class in $H^1(M)$ can be represented by a fibration, then so can all classes and by the construction of [4] every bundle over $M$ admits an $S^1$-invariant symplectic form. If $M$ is Seifert fibered it either has vanishing Thurston norm by Proposition 2 and we proceed as in the previous case or $M$ has a horizontal surface and the claim follows by Example 1 above. □

5. **The case where $E$ admits a Lefschetz fibration**

In [2] Chen and Matveyev showed that if $S^1 \times M$ admits a symplectic Lefschetz fibration, then $M$ fibers over $S^1$. This was extended by Etgü in [3] to the case where the fibration may or may not be symplectic. In this section we shall show that the same statement holds for arbitrary $S^1$-bundles. Let us begin with some definitions and basic facts concerning Lefschetz fibrations.

**Definition 7.** Let $E$ be a compact, connected, oriented smooth 4-manifold. A Lefschetz fibration is a map $E \xrightarrow{p} B$ to an orientable surface so that any critical point has an oriented chart on which $p(z_1, z_2) = z_1^2 + z_2^2$.

We list some basic properties of Lefschetz fibrations (for proofs see [3]).

1. There are finitely many critical points, so the generic preimage of a point will be a surface and we may assume that this is connected. To each critical point one associates a vanishing cycle in the fiber.

2. A Lefschetz fibration admits a symplectic form so that the fiber is a symplectic submanifold if the class $[F]$ of the fiber is nontorsion in $H_2(E)$. Moreover this is always true if $\chi(F) \neq 0$.

3. We have a formula for the Euler characteristic given by

$$\chi(E) = \chi(B) \cdot \chi(F) + \#\{\text{critical points}\}.$$ 

We will first show that for a symplectic circle bundle any Lefschetz fibration will actually be a proper fibration, i.e. cannot have any critical points. The following lemma is essentially Lemma 3.4 of [2].

**Lemma 4.** Let $S^1 \rightarrow E \xrightarrow{\pi} B$ be a circle bundle that admits a Lefschetz fibration $E \xrightarrow{p} B$. Then $p$ has no critical points.

**Proof.** We first consider the case where $F = S^2$. Since $E$ is spin, it has an even intersection form and thus all vanishing cycles are nonseparating in the fiber $F = S^2$. However this means there cannot be any since $S^2$ is simply connected and hence $p$ has no critical points.

If $F = T^2$, the equation

$$0 = \chi(E) = \chi(B) \cdot \chi(F) + \#\{\text{critical points}\}$$

implies that $E$ has no critical points.

We now consider the case when $F$ has genus greater than 1. We know that $E$ admits a symplectic Lefschetz fibration by (2) above. Thus by the adjunction formula for symplectic surfaces we see that

$$K.F = \chi_-(F) \neq 0,$$
where $K$ is the canonical class on $E$. If $b_2^+ > 1$, then it follows from Taubes’ result that $K$ is a basic class and thus the adjunction inequality holds. In the case where $b_2^+ (E) = 1$ we may apply the adjunction inequality exactly as in the case of $b_2^+ > 1$ by ([18], Theorem E). Now we assume that our fibration has a critical point and hence a vanishing cycle $\gamma$. Then we know that this is nonseparating, so the fiber $F$ is homologous to a surface obtained by collapsing $\gamma$ to a point and this can in turn be thought of as the image of a map $F' \xrightarrow{f} E$ where $\chi_-(F') < \chi_-(F)$. Hence the image $\pi_* [F]$ may be represented by a surface of complexity at most $\chi_-(F')$ (see [2]). We know that any basic class of a circle bundle is a pullback of a class on the base (see [1]). Thus by the adjunction inequality (which still holds for $b_2^+ = 1$) and equation (1),

$$\chi_-(F) = |K.F| = |K.\pi_* F| \leq ||\pi_* F||_F \leq \chi_-(F') < \chi_-(F),$$

which is a contradiction. \hfill \Box

Our proof of Theorem 1 below, which differs from those of [2] and [3], will rely on a theorem of Stallings that characterises fibered 3-manifolds in terms of their fundamental group.

**Theorem 8 (Stallings [24]).** Let $M$ be a compact, irreducible 3-manifold and suppose there is an extension

$$1 \to G \to \pi_1 (M) \to \mathbb{Z} \to 1,$$

where $G$ is finitely generated and $G \neq \mathbb{Z}_2$. Then $M$ fibers over $S^1$.

We now come to the main result of this section.

**Theorem 9.** Let $S^1 \to E \xrightarrow{\pi} M$ be a symplectic circle bundle over an irreducible base $M$. If $E$ admits a Lefschetz fibration, then $M$ fibers over $S^1$.

**Proof.** First of all by Lemma 1 we have that $E$ actually admits a fibration $F \to E \xrightarrow{\pi} B$. In addition we note that the fiber $\gamma$ of any oriented circle bundle lies in the centre of the fundamental group of the total space. We shall have to consider two distinct cases according to whether $\gamma$ is in the kernel of $p_*$ or not.

**Case 1.** $p_*(\gamma) \neq 1$.

Since $\gamma$ was central in the fundamental group of $E$, the fact that $p_*(\gamma)$ is nontrivial in $\pi_1 (B)$ means that $B$ must be a torus. Hence the long exact homotopy sequence of the fibration gives the following short exact sequence:

$$1 \to \pi_1 (F) \to \pi_1 (E) \xrightarrow{p_*} \pi_1 (T^2) = \mathbb{Z}^2 \to 1.$$  

Since $M$ is assumed to be irreducible and hence aspherical, we also have the following exact sequence from the homotopy exact sequence of the fibration $S^1 \to E \xrightarrow{\pi} M$:

$$1 \to \pi_1 (S^1) = \langle \gamma \rangle \to \pi_1 (E) \xrightarrow{\pi_*} \pi_1 (M) \to 1.$$  

Because $\gamma$ is central in $\pi_1 (E)$, the sequence (2) gives the following exact sequence:

$$1 \to \pi_1 (F) \to \pi_1 (E)/\langle \gamma \rangle \xrightarrow{p_*} \mathbb{Z}^2 / \langle p_*(\gamma) \rangle \to 1.$$  

Moreover since $p_*(\gamma) \neq 1$ we have that $\mathbb{Z}^2 / \langle p_*(\gamma) \rangle = \mathbb{Z} \oplus \mathbb{Z}_k$ for some $k$. If we let $H = p_*^{-1} (\mathbb{Z}_k)$ we see that $H$ has $\pi_1 (F)$ as a finite index subgroup and is thus also
finitely generated. Then by taking the projection to \( \mathbb{Z} \) in the above sequence we obtain
\[
1 \to H \to \pi_1(E)/\langle \gamma \rangle = \pi_1(M) \xrightarrow{p_*} \mathbb{Z} \to 1.
\]
This is exact and \( H \neq \mathbb{Z}_2 \) since it contains \( \pi_1(F) \). As \( M \) is irreducible, the hypotheses of Theorem 8 are satisfied and we conclude that \( M \) fibers over \( S^1 \).

**Case 2.** \( p_*(\gamma) = 1 \).

In this case, \( \langle \gamma \rangle \subset \pi_1(F) \) and hence \( F = T^2 \). Thus sequence (2) above yields the following:
\[
1 \to \mathbb{Z}^2 \to \pi_1(E) \xrightarrow{p_*} \pi_1(B) \to 1
\]
and \( \langle \gamma \rangle \subset \mathbb{Z}^2 \). Again by taking the quotient by \( \langle \gamma \rangle \) we obtain the following short exact sequence:
\[
1 \to \mathbb{Z} \oplus \mathbb{Z}^k = \mathbb{Z}^2/\langle \gamma \rangle \to \pi_1(E)/\langle \gamma \rangle = \pi_1(M) \xrightarrow{p_*} \pi_1(B) \to 1.
\]
However since \( M \) is irreducible and hence prime and \( \pi_1(M) \) is infinite, it follows from ([10], Corollary 9.9) that \( \pi_1(M) \) is torsion free. Hence \( k = 0 \) and \( \pi_1(M) \) contains an infinite cyclic normal subgroup; thus by ([10], Corollary 12.8) it is in fact Seifert fibered and the result follows from Corollary 2 above.

Theorem 9 then allows us to prove Conjecture 1 under the assumption that the total space is a complex manifold.

**Corollary 3.** Conjecture 1 holds in the case that \( E \) is a complex manifold.

**Proof.** By considering the Kodaira classification and noting that \( E \) is spin, symplectic and has \( \chi(E) = 0 \) one concludes that one of the following must hold (cf. [3], Theorem 5.1):

1. \( E = S^2 \times T^2 \),
2. \( E \) is a \( T^2 \)-bundle over \( T^2 \),
3. \( E \) is a Seifert fibration over a hyperbolic orbifold.

If \( E = S^2 \times T^2 \), then \( M = S^2 \times S^1 \) and one clearly has an \( S^1 \)-invariant symplectic form. In the second case it follows from the argument above that \( M \) is a \( T^2 \)-bundle over \( S^1 \) and hence has vanishing Thurston norm. In the final case \( M \) must be Seifert fibered as in Case 2 in the proof of Theorem 9 and hence the claim holds in the latter two cases by Corollary 2.

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**References**


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