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A COMPARISON PRINCIPLE FOR THE COMPLEX MONGE-AMPÈRE OPERATOR IN CEGRELL'S CLASSES AND APPLICATIONS

NGUYEN VAN KHUE AND PHAM HOANG HIEP

ABSTRACT. In this article we will first prove a result about the convergence in capacity. Next we will obtain a general decomposition theorem for complex Monge-Ampère measures which will be used to prove a comparison principle for the complex Monge-Ampère operator.

1. INTRODUCTION

Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$. Denote by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic (psh) functions on Ω and by $\mathrm{PSH}^-(\Omega)$ the subclass of negative functions. In [4], [5] the authors established and used the comparison principle to study the Dirichlet problem of the complex Monge-Ampère operator in $\mathrm{PSH} \cap L^\infty_{loc}(\Omega)$. Recently, Cegrell introduced a general class $\mathcal{E}$ of psh functions on which the complex Monge-Ampère operator $(dd^c.)^n$ is well defined. He obtained many important results of pluripotential theory in the class $\mathcal{E}$ (see [7]–[9]).

The main results of our paper are Theorem 4.1 and some Xing-type comparision principles. Theorem 4.1 generalizes Lemma 5.4 in [7], Lemma 7.2 in [1] and Lemma 3.4 in [9]. For definitions of Cegrell's classes, see Section 2. After giving some preliminaries, we start in Proposition 3.1 with a comparison principle, which is analogous to a comparison principle due to Xing (Lemma 1 in [19]). It should be observed that our proof is quite different from Xing's proof, and the inequality we obtain is slightly stronger than Xing's inequality, even in the case of bounded psh functions. Using Proposition 3.1, we give in Theorem 3.5 a sufficient condition for convergence in C_n-capacity of a sequence of psh functions in the class $\mathcal{F}$. This result should be compared to Theorem 3 of [19] in which bounded psh functions were studied. Applying Theorem 3.5 we give generalizations of recent results in [11] and [12] about convergences of multipole Green functions and a criterion for pluripolarity, respectively. Section 4 focuses on Theorem 4.1 and Theorem 4.9. By applying Theorem 4.1 we give some results on Cegrell's classes. We prove in Proposition 4.4 a local estimate for the Monge-Ampère measures in terms of the Bedford-Taylor relative capacity. As an application, we give in Theorem 4.5 a decomposition result for Monge-Ampère measures, which is similar in spirit to Theorem 6.3 in [7]. From Proposition 3.1 and Theorem 4.1 we obtain easily a Xing-type comparison principle for functions in classes $\mathcal{F}$ and $\mathcal{E}$.

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2. Preliminaries

First we recall some elements of pluripotential theory that will be used throughout the paper. See also [5], [7], [8], [19], [20].

2.1. We will always denote by Ω a bounded hyperconvex domain in $\mathbf{C}^n$ unless otherwise stated. The C_n-capacity in the sense of Bedford and Taylor on Ω is the set function given by

$$C_n(E) = C_n(E,\Omega) = \sup\{\int_E (dd^c u)^n \ : \ u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0\}$$

for every Borel set E in Ω. It is proved in [5] that

$$C_n(E) = \int_E (dd^c h^*_{E,\Omega})^n,$$

where $h^*_{E,\Omega}$ is the upper regularization of the relative extremal function $h_{E,\Omega}$ for E (relative to Ω) i.e.,

$$h_{E,\Omega}(z) = \sup\{u(z) \ : \ u \in \mathrm{PSH}^-(\Omega),\ u \le -1 \text{ on } E\}.$$

The following concepts are taken from [19] and [20].
$*$A sequence of functions u_j on Ω is said to converge to a function u in C_n-capacity on a set $E \subset \Omega$ if for every $\delta > 0$ we have $C_n(\{z \in E \ : \ |u_j(z) - u(z)| > \delta\}) \to 0$ as $j \to \infty$.
$*$A family of positive measures $\{\mu_\alpha\}$ on Ω is said to be uniformly absolutely continuous with respect to C_n-capacity in a set $E \subset \Omega$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each Borel subset $F \subset E$ with $C_n(\mathrm{F}) < \delta$ the inequality $\mu_\alpha(F) < \epsilon$ holds for all α. We write $\mu_\alpha \ll C_n$ in E uniformly for α.

2.2. The following classes of psh functions were introduced by Cegrell in [7] and [8]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \ \lim_{z \to \partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c \varphi)^n < +\infty\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \ \exists\ \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j \ge 1} \int_\Omega (dd^c \varphi_j)^n < +\infty\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \ \exists\ \varphi_K \in \mathcal{F}(\Omega) \text{ such that } \varphi_K = \varphi \text{ on } K,\ \forall K \Subset \Omega\},$$

$$\mathcal{E}^a = \mathcal{E}^a(\Omega) = \{u \in \mathcal{E}(\Omega) \ : \ (dd^c u)^n(E) = 0\ \forall\ E \text{ is pluripolar in } \Omega\}.$$

2.3. Let $A = \{(w_j, \nu_j)\}_{j=1,\dots,p}$ be a finite subset of $\Omega \times \mathbf{R}^+$. According to Lelong (see [18]), the pluricomplex Green function with poles in A is defined by

$$g(A)(z) = \sup\{u(z) : \ u \in \mathcal{L}_A\},$$

where

$$\mathcal{L}_A = \{u \in \mathrm{PSH}^-(\Omega) : \ u(z) - \nu_j \log|z - w_j| \le O(1) \text{ as } z \to w_j,\ j = 1, ..., p\}.$$

Set

$$\nu(A) = \sum_{j=1}^p \nu_j^n,\ \ \hat{A} = \{w_j\}_{j=1,\dots,p}.$$

2.4. We write $\underline{\lim}_{z\to\partial\Omega}[u(z)-v(z)]\geq a$ if for every $\epsilon>0$ there exists a compact set K in Ω such that

$$u(z)-v(z)\geq a-\epsilon \text{ for } z\in(\Omega\backslash K)\cap\{u>-\infty\}$$

and

$$v(z)=-\infty \text{ for } z\in(\Omega\backslash K)\cap\{u=-\infty\}.$$

2.5. Xing's comparison principle (see Lemma 1 in [19]). Let Ω be a bounded open subset in $\mathbf{C}^n$ and $u,v\in \mathrm{PSH}\cap L^\infty(\Omega)$ satisfy $\underline{\lim}_{z\to\partial\Omega}[u(z)-v(z)]\geq 0$. Then for any constant $r\geq 1$ and all $w_j\in\mathrm{PSH}(\Omega)$ with $0\leq w_j\leq 1$, $j=1,2,...,n$ we have

$$\frac{1}{(n!)^2}\int\limits_{\{u<v\}}(v-u)^n dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_{\{u<v\}}(r-w_1)(dd^c v)^n\leq\int\limits_{\{u<v\}}(r-w_1)(dd^c u)^n.$$

3. Some convergence theorems

In order to study the convergence of a sequence of psh functions in C_n-capacity, we start with the following.

Proposition 3.1. *a) Let $u,v\in\mathcal{F}$ be such that $u\leq v$ on Ω. Then for $1\leq k\leq n$,*

$$\frac{1}{k!}\int\limits_\Omega(v-u)^k dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_\Omega(r-w_1)(dd^c v)^k\wedge dd^c w_{k+1}\wedge ...\wedge dd^c w_n$$
$$\leq\int\limits_\Omega(r-w_1)(dd^c u)^k\wedge dd^c w_{k+1}\wedge ...\wedge dd^c w_n,$$

for all $w_j\in PSH(\Omega)$, $0\leq w_j\leq 1$, $j=1,...,k$, $w_{k+1},...,w_n\in\mathcal{F}$ and all $r\geq 1$.
b) Let $u,v\in\mathcal{E}$ be such that $u\leq v$ on Ω and $u=v$ on $\Omega\backslash K$ for some $K\Subset\Omega$. Then for $1\leq k\leq n$,

$$\frac{1}{k!}\int\limits_\Omega(v-u)^k dd^c w_1\wedge ...\wedge dd^c w_n+\int\limits_\Omega(r-w_1)(dd^c v)^k\wedge dd^c w_{k+1}\wedge ...\wedge dd^c w_n$$
$$\leq\int\limits_\Omega(r-w_1)(dd^c u)^k\wedge dd^c w_{k+1}\wedge ...\wedge dd^c w_n,$$

for all $w_j\in PSH(\Omega)$, $0\leq w_j\leq 1$, $j=1,...,k$, $w_{k+1},...,w_n\in\mathcal{E}$ and all $r\geq 1$.

We proceed through some lemmas.

Lemma 3.2. *Let $u,v\in PSH\cap L^\infty(\Omega)$ be such that $u\leq v$ on Ω and $\lim_{z\to\partial\Omega}[u(z)-v(z)]=0$. Then*

$$\int\limits_\Omega(v-u)^k dd^c w\wedge T\leq k\int\limits_\Omega(1-w)(v-u)^{k-1}dd^c u\wedge T,$$

for all $w\in PSH(\Omega)$, $0\leq w\leq 1$ and all positive closed currents T.

Proof. First, assume $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$, $u \le v$ on Ω and $u = v$ on $\Omega \backslash K$, $K \Subset \Omega$. Then, using the Stokes formula we obtain

$$\begin{aligned}
\int\limits_\Omega (v-u)^k dd^c w \wedge T &= \int\limits_\Omega (v-u)^k dd^c (w-1) \wedge T \\
&= \int\limits_\Omega (w-1) dd^c (v-u)^k \wedge T \\
&= -k(k-1) \int\limits_\Omega (1-w) d(v-u) \wedge d^c (v-u) \wedge T \\
&\quad + k \int\limits_\Omega (1-w)(v-u)^{k-1} dd^c (u-v) \wedge T \\
&\le k \int\limits_\Omega (1-w)(v-u)^{k-1} dd^c (u-v) \wedge T \\
&\le k \int\limits_\Omega (1-w)(v-u)^{k-1} dd^c u \wedge T.
\end{aligned}$$

In the general case, for each $\epsilon > 0$ we set $v_\epsilon = \max(u, v-\epsilon)$. Then $v_\epsilon \nearrow v$ on Ω, $v_\epsilon \ge u$ on Ω and $v_\epsilon = u$ on $\Omega \backslash K$ for some $K \Subset \Omega$. Hence

$$\int\limits_\Omega (v_\epsilon - u)^k dd^c w \wedge T \le k \int\limits_\Omega (1-w)(v_\epsilon - u)^{k-1} dd^c u \wedge T.$$

Since $0 \le v_\epsilon - u \nearrow v - u$ as $\epsilon \searrow 0$, letting $\epsilon \searrow 0$ we get

$$\int\limits_\Omega (v-u)^k dd^c w \wedge T \le k \int\limits_\Omega (1-w)(v-u)^{k-1} dd^c u \wedge T.$$

□

Lemma 3.3. *Let $u, v \in PSH \cap L^\infty(\Omega)$ be such that $u \le v$ on Ω and $\lim\limits_{z \to \partial\Omega} [u(z) - v(z)] = 0$. Then for $1 \le k \le n$,*

$$\begin{aligned}
&\frac{1}{k!} \int\limits_\Omega (v-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_\Omega (r - w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n \\
&\quad \le \int\limits_\Omega (r - w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n
\end{aligned}$$

for all $w_1, ..., w_k \in PSH(\Omega)$, $0 \le w_j \le 1$ $\forall$ $j = 1, ..., k$, $w_{k+1}, ..., w_n \in \mathcal{E}$ and all $r \ge 1$.

Proof. To simplify the notation we set

$$T = dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

First, assume that $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$, $u \le v$ on Ω, and $u = v$ on $\Omega \backslash K$, $K \Subset \Omega$. Using Lemma 3.2 we get

$$\begin{aligned}
\int\limits_\Omega (v-u)^k dd^c w_1 \wedge \ldots \wedge dd^c w_n &\le k \int\limits_\Omega (v-u)^{k-1} dd^c w_1 \wedge \ldots \wedge dd^c w_{k-1} \wedge dd^c u \wedge T \\
&\le k! \int\limits_\Omega (v-u) dd^c w_1 \wedge (dd^c u)^{k-1} \wedge T \\
&\le k! \int\limits_\Omega (v-u) dd^c w_1 \wedge [\sum_{i=0}^{k-1} (dd^c u)^i \wedge (dd^c v)^{k-i-1}] \wedge T \\
&= k! \int\limits_\Omega (w_1 - r) dd^c (v-u) \wedge [\sum_{i=0}^{k-1} (dd^c u)^i \wedge (dd^c v)^{k-i-1}] \wedge T \\
&= k! \int\limits_\Omega (r - w_1) dd^c (u-v) \wedge [\sum_{i=0}^{k-1} (dd^c u)^i \wedge (dd^c v)^{k-i-1}] \wedge T \\
&= k! \int\limits_\Omega (r - w_1) [(dd^c u)^k - (dd^c v)^k] \wedge T.
\end{aligned}$$

In the general case, for each $\epsilon > 0$ we put $v_\epsilon = \max(u, v - \epsilon)$. Then $v_\epsilon \nearrow v$ on Ω, $v_\epsilon \ge u$ on Ω and $v_\epsilon = u$ on $\Omega \backslash K$ for some $K \Subset \Omega$. Hence

$$\begin{aligned}
&\frac{1}{k!} \int\limits_\Omega (v_\epsilon - u)^k dd^c w_1 \wedge \ldots \wedge dd^c w_n + \int\limits_\Omega (r - w_1)(dd^c v_\epsilon)^k \wedge T \\
&\quad \le \int\limits_\Omega (r - w_1)(dd^c u)^k \wedge T.
\end{aligned}$$

Observe that $0 \le v_\epsilon - u \nearrow v - u$ and $(dd^c v_\epsilon)^k \wedge T \to (dd^c v)^k \wedge T$ weakly as $\epsilon \searrow 0$, $r - w_1$ is lower semicontinuous, and by letting $\epsilon \searrow 0$ we have

$$\begin{aligned}
&\frac{1}{k!} \int\limits_\Omega (v - u)^k dd^c w_1 \wedge \ldots \wedge dd^c w_n + \int\limits_\Omega (r - w_1)(dd^c v)^k \wedge T \\
&\quad \le \int\limits_\Omega (r - w_1)(dd^c u)^k \wedge T.
\end{aligned}$$

The proof is finished. □

Proof of Proposition 3.1. a) Let $\mathcal{E}_0 \ni u_j \searrow u$ and $\mathcal{E}_0 \ni v_j \searrow v$ as in the definition of $\mathcal{F}$. Replacing v_j by $\max(u_j, v_j)$ we may assume that $u_j \le v_j$ for $j \ge 1$. By Lemma 3.3 we have

$$\begin{aligned}
&\frac{1}{k!} \int\limits_\Omega (v_j - u_t)^k dd^c w_1 \wedge \ldots \wedge dd^c w_n + \int\limits_\Omega (r - w_1)(dd^c v_j)^k \wedge dd^c w_{k+1} \wedge \ldots \wedge dd^c w_n \\
&\le \int\limits_\Omega (r - w_1)(dd^c u_t)^k \wedge dd^c w_{k+1} \wedge \ldots \wedge dd^c w_n
\end{aligned}$$

for $t \geq j \geq 1$. By Proposition 5.1 in [8], letting $t \to \infty$ in the above inequality, we have

$$\frac{1}{k!}\int\limits_{\Omega}(v_j-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\Omega}(r-w_1)(dd^c v_j)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\Omega}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$

for $j \geq 1$. Next letting $j \to \infty$ again by Proposition 5.1 in [8] we get the desired conclusion.

b) Let G, W be open sets such that $K \Subset G \Subset W \Subset \Omega$. According to the remark after Definition 4.6 in [8] we can choose a function $\tilde{v} \in \mathcal{F}$ such that $\tilde{v} \geq v$ and $\tilde{v} = v$ on W. Set

$$\tilde{u} = \begin{cases} u & \text{on } G, \\ \tilde{v} & \text{on } \Omega \backslash G. \end{cases}$$

Since $u = v = \tilde{v}$ on $W \backslash K$ we have $\tilde{u} \in \mathrm{PSH}^-(\Omega)$. It is easy to see that $\tilde{u} \in \mathcal{F}$, $\tilde{u} \leq \tilde{v}$ and $\tilde{u} = u$ on W. By a) we have

$$\frac{1}{k!}\int\limits_{\Omega}(\tilde{v}-\tilde{u})^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\Omega}(r-w_1)(dd^c \tilde{v})^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\Omega}(r-w_1)(dd^c \tilde{u})^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

Since $\tilde{u} = \tilde{v}$ on $\Omega \backslash G$ we have

$$\frac{1}{k!}\int\limits_{W}(\tilde{v}-\tilde{u})^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{W}(r-w_1)(dd^c \tilde{v})^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{W}(r-w_1)(dd^c \tilde{u})^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

Since $\tilde{u} = u$, $\tilde{v} = v$ on W and $u = v$ on $\Omega \backslash K$ we obtain

$$\frac{1}{k!}\int\limits_{\Omega}(v-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\Omega}(r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\Omega}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

□

Proposition 3.4. *Let $u, v \in \mathcal{F}$ be such that $u \leq v$ on Ω. Then*

$$\frac{1}{n!}\int\limits_{\Omega}(v-u)^n dd^c w_1 \wedge ... \wedge dd^c w_n \leq \int\limits_{\Omega}(-w_1)[(dd^c u)^n - (dd^c v)^n]$$

for all $w_j \in PSH(\Omega)$, $-1 \leq w_j \leq 0$, $j = 1, ..., n$.

Proof. The proposition follows from Proposition 3.1 with $k = n$, $r = 1$ and with w_j replaced by $w_j + 1$. □

Theorem 3.5. *Let $u, u_j \in \mathcal{F}$ be such that $u_j \leq u$ for $j \geq 1$. Assume that*

$$\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n < +\infty,$$

and $||(dd^c u_j)^n - (dd^c u)^n||_E \to 0$ as $j \to \infty$ for all $E \Subset \Omega$. Then $u_j \to u$ in C_n-capacity on every $E \Subset \Omega$ as $j \to \infty$.

Proof. Let $\Omega' \Subset \Omega$ and $\delta > 0$. Put

$$A_j = \{z \in \overline{\Omega'} \ : \ |u_j - u| \geq \delta\} = \{z \in \overline{\Omega'} \ : \ u - u_j \geq \delta\}.$$

We prove that $C_n(A_j) \to 0$ as $j \to \infty$. Given $\epsilon > 0$, by quasicontinuity of u and u_j, there is an open set G in Ω such that $C_n(G) < \epsilon$, and $u_j|_{\Omega \setminus G},\ u|_{\Omega\setminus G}$ are continuous. We have

$$A_j = B_j \cup \{z \in G \ : \ u - u_j \geq \delta\},$$

where $B_j = \{z \in \overline{\Omega'} \setminus G \ : \ u - u_j \geq \delta\}$ are compact sets in Ω and

$$\overline{\lim_{j\to\infty}}\, C_n(A_j) \leq \overline{\lim_{j\to\infty}}\, C_n(B_j) + \epsilon.$$

We claim that $\lim\limits_{j\to\infty} C_n(B_j) = 0$. By Proposition 3.4 we have

$$\begin{aligned}
C_n(B_j) &= \int_{B_j} (dd^c h^*_{B_j})^n \\
&\leq \frac{1}{\delta^n} \int_{B_j} (u - u_j)^n (dd^c h^*_{B_j})^n \\
&\leq \frac{n!}{\delta^n} \int_\Omega (-h^*_{B_j})[(dd^c u_j)^n - (dd^c u)^n] \\
&\leq \frac{n!}{\delta^n} \{||(dd^c u_j)^n - (dd^c u)^n||_K + \int_{\Omega\setminus K} (-h_{\Omega'})[(dd^c u_j)^n + (dd^c u)^n]\} \\
&\leq \frac{n!}{\delta^n} \{||(dd^c u_j)^n - (dd^c u)^n||_K + \sup_{\Omega\setminus K} |h_{\Omega'}| [\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n + \int_\Omega (dd^c u)^n]\}.
\end{aligned}$$

As $\lim\limits_{z\to\partial\Omega} h_{\Omega'}(z) = 0$ there exists $K \Subset \Omega$ such that

$$\frac{n!}{\delta^n} \sup_{\Omega\setminus K} |h_{\Omega'}| [\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n + \int_\Omega (dd^c u)^n] < \epsilon.$$

By the hypothesis,

$$\frac{n!}{\delta^n} ||(dd^c u_j)^n - (dd^c u)^n||_K < \epsilon \text{ for } j > j_0.$$

Thus

$$C_n(B_j) < 2\epsilon \text{ for } j > j_0.$$

This proves the claim and hence the theorem. ∎

As an application of Theorem 3.5 we have the following.

Proposition 3.6. *Let $g(A_j)$ be multipolar Green functions on Ω such that*

$$\hat{A}_j = \{w_1^j, ..., w_{p_j}^j\} \to \partial\Omega \text{ and } \sup_{j\geq 1} \nu(A_j) = \sup_{j\geq 1} \sum_{k=1}^{p_j} (\nu_k^j)^n < +\infty.$$

Then $g(A_j) \to 0$ as $j \to \infty$ in C_n-capacity.

Proof. By the hypothesis we have

$$\sup_{j\geq 1} (dd^c g(A_j))^n(\Omega) = \sup_{j\geq 1} \nu(A_j) < +\infty$$

and

$$||(dd^c g(A_j))^n||_K \to 0 \text{ as } j \to \infty \text{ for all } K \Subset \Omega.$$

Theorem 3.5 implies that $g(A_j) \to 0$ as $j \to \infty$ in C_n-capacity. □

This section ends with a criterion for pluripolarity.

Theorem 3.7. *Let $u_j \in \mathcal{F}$ be such that $\sup_{j\geq 1} \int_\Omega (dd^c u_j)^n < +\infty$.*
Then there is a constant $A > 0$ such that
i) $(\overline{\lim}_{j\to\infty} u_j)^* \in \mathcal{F}$,
ii) $C_n(\{z \in \Omega : (\overline{\lim}_{j\to\infty} u_j)^*(z) < -t\}) \leq \frac{A}{t^n}$,
iii) $\{z \in \Omega : \lim_{j\to\infty} u_j(z) = -\infty\}$ is pluripolar.

Proof. i) For each $j \geq 1$ put $v_j = \sup\{u_j, u_{j+1}, ...\}$. By [8], $v_j^* \in \mathcal{F}$ and

$$\sup_{j\geq 1} \int_\Omega (dd^c v_j^*)^n \leq \sup_{j\geq 1} \int_\Omega (dd^c u_j)^n < +\infty.$$

By [8] we have $v_j^* \searrow v \in \mathcal{F}$.
ii) By Proposition 3.1 in [10] we have

$$C_n\{z \in \Omega : (\overline{\lim}_{j\to\infty} u_j)^*(z) < -t\} = C_n\{z \in \Omega : v(z) < -t\} \leq \frac{2^n \int_\Omega (dd^c v)^n}{t^n} = \frac{A}{t^n},$$

where $A = 2^n \int_\Omega (dd^c v)^n$.
iii) According to [5] we have

$$C_n\{z \in \Omega : \lim_{j\to\infty} u_j(z) = -\infty\} = C_n\{z \in \Omega : v(z) = -\infty\} = 0. \qquad \square$$

Remark. Theorem 3.7 in the case where the u_j are multipole Green functions was proved by D. Coman, N. Levenberg and A. Poletsky in Theorem 4.1 of [12].

4. Some properties of Cegrell's classes and applications

In this section, first we prove the following.

Theorem 4.1. *Let $u, u_1, ..., u_{n-1} \in \mathcal{E}$, $v \in PSH^-(\Omega)$ and $T = dd^c u_1 \wedge ... \wedge dd^c u_{n-1}$. Then*

$$dd^c \max(u, v) \wedge T|_{\{u>v\}} = dd^c u \wedge T|_{\{u>v\}}.$$

We need the following well-known fact.

Lemma 4.2. *Let μ be a Borel measure on Ω and $f : \Omega \to \mathbf{R}$ a measurable function on Ω. The following are equivalent:*
i) $\mu(E) = 0$ for all Borel sets $E \subset \{f \neq 0\}$,
ii) $\int\limits_E f d\mu = 0$ for all Borel sets E in Ω.

Proof. i)$\Rightarrow$ii) follows from:

$$\int\limits_E f d\mu = \int\limits_{E\setminus\{f=0\}} f d\mu + \int\limits_{E\cap\{f=0\}} f d\mu = 0.$$

ii)$\Rightarrow$i). It suffices to show that $\mu = 0$ on every $X_\delta = \{f > \delta > 0\}$. By the Hahn decomposition theorem, there exist measurable subsets X_δ^+ and X_δ^- of X_δ such that $X_\delta = X_\delta^+ \cup X_\delta^-$, $X_\delta^+ \cap X_\delta^- = \emptyset$ and $\mu \geq 0$ on X_δ^+, $\mu \leq 0$ on X_δ^-. We have

$$\begin{cases} \delta\mu(X_\delta^+) \leq \int\limits_{X_\delta^+} f d\mu = 0, \\ \delta\mu(X_\delta^-) \geq \int\limits_{X_\delta^-} f d\mu = 0. \end{cases}$$

Hence, $\mu(X_\delta^+) = \mu(X_\delta^-) = 0$. Therefore, we have $\mu = 0$ on X_δ. $\square$

Proof of Theorem 4.1. a) First we prove the proposition for $v \equiv a < 0$. According to the remark following Definition 4.6 in [8], without loss of generality we may assume that $u, u_1, ..., u_{n-1} \in \mathcal{F}$. Using Theorem 2.1 in [8] we can find

$$\mathcal{E}_0 \cap C(\bar{\Omega}) \ni u^j \searrow u,\ \mathcal{E}_0 \cap C(\bar{\Omega}) \ni u_k^j \searrow u_k,\ k = 1, ..., n-1.$$

Since $\{u^j > a\}$ is open we have

$$dd^c \max(u^j, a) \wedge T_j|_{\{u^j>a\}} = dd^c u^j \wedge T_j|_{\{u^j>a\}}.$$

Thus from the inclusion $\{u > a\} \subset \{u^j > a\}$ we obtain

$$dd^c \max(u^j, a) \wedge T_j|_{\{u>a\}} = dd^c u^j \wedge T_j|_{\{u>a\}},$$

where $T_j = dd^c u_1^j \wedge ... \wedge dd^c u_{n-1}^j$. By Corollary 5.2 in [8], it follows that

$$\max(u-a, 0) dd^c \max(u^j, a) \wedge T_j \to \max(u-a, 0) dd^c \max(u, a) \wedge T,$$
$$\max(u-a, 0) dd^c u^j \wedge T_j \to \max(u-a, 0) dd^c u \wedge T.$$

Hence

$$\max(u-a, 0)[dd^c \max(u, a) \wedge T - dd^c u \wedge T] = 0.$$

Using Lemma 4.2 we have

$$dd^c \max(u, a) \wedge T = dd^c u \wedge T \text{ on } \{u > a\}.$$

b) Assume that $v \in PSH^-(\Omega)$. Since $\{u > v\} = \bigcup\limits_{a\in\mathbf{Q}^-} \{u > a > v\}$, it suffices to show that

$$dd^c \max(u, v) \wedge T = dd^c u \wedge T \text{ on } \{u > a > v\}$$

for all $a \in \mathbf{Q}^-$, the set of negative rational numbers. Since $\max(u, v) \in \mathcal{E}$, by a) we have

$$dd^c \max(u, v) \wedge T|_{\{\max(u,v)>a\}} = dd^c \max(\max(u, v), a) \wedge T|_{\{\max(u,v)>a\}} \tag{1}$$
$$= dd^c \max(u, v, a) \wedge T|_{\{\max(u,v)>a\}}, \tag{2}$$
$$dd^c u \wedge T|_{\{u>a\}} = dd^c \max(u, a) \wedge T|_{\{u>a\}}. \tag{3}$$

Since $\max(u,v,a)=\max(u,a)$ on the open set $\{a>v\}$, we have

$$dd^c\max(u,v,a)\wedge T|_{\{a>v\}} = dd^c\max(u,a)\wedge T|_{\{a>v\}}. \tag{4}$$

Since $\{u>a>v\}\subset\{u>a\},\ \{a>v\},\ \{\max(u,v)>a\}$ and (1), (2), (3) we have

$$dd^c\max(u,v)\wedge T|_{\{u>a>v\}} = dd^c u\wedge T|_{\{u>a>v\}}.$$

The next result is an analogue of an inequality due to Demailly in [14]. □

Proposition 4.3. *a) Let $u,v\in\mathcal{E}$ be such that $(dd^cu)^n(\{u=v=-\infty\})=0$. Then*

$$(dd^c\max(u,v))^n \geq 1_{\{u\geq v\}}(dd^cu)^n + 1_{\{u<v\}}(dd^cv)^n,$$

where 1_E denotes the characteristic function of E.
b) Let μ be a positive measure which vanishes on all pluripolar subsets of Ω. Suppose $u,v\in\mathcal{E}$ such that $(dd^cu)^n\geq\mu,(dd^cv)^n\geq\mu$. Then $(dd^c\max(u,v))^n\geq\mu$.

Proof. a) For each $\epsilon>0$ put $A_\epsilon=\{u=v-\epsilon\}\backslash\{u=v=-\infty\}$. Since $A_\epsilon\cap A_\delta=\emptyset$ for $\epsilon\neq\delta$ there exists $\epsilon_j\searrow 0$ such that $(dd^cu)^n(A_{\epsilon_j})=0$ for $j\geq 1$. On the other hand, since $(dd^cu)^n(\{u=v=-\infty\})=0$ we have $(dd^cu)^n(\{u=v-\epsilon_j\})=0$ for $j\geq 1$. By Theorem 4.1 it follows that

$$\begin{aligned}(dd^c\max(u,v-\epsilon_j))^n &\geq (dd^c\max(u,v-\epsilon_j))^n|_{\{u>v-\epsilon_j\}}\\ &\quad + (dd^c\max(u,v-\epsilon_j))^n|_{\{u<v-\epsilon_j\}}\\ &= (dd^cu)^n|_{\{u\geq v-\epsilon_j\}} + (dd^cv)^n|_{\{u<v-\epsilon_j\}}\\ &= 1_{\{u\geq v-\epsilon_j\}}(dd^cu)^n + 1_{\{u<v-\epsilon_j\}}(dd^cv)^n\\ &\geq 1_{\{u\geq v\}}(dd^cu)^n + 1_{\{u<v-\epsilon_j\}}(dd^cv)^n.\end{aligned}$$

Letting $j\to\infty$ and by the remark following Theorem 5.15 in [8] we get

$$(dd^c\max(u,v))^n \geq 1_{\{u\geq v\}}(dd^cu)^n + 1_{\{u<v\}}(dd^cv)^n,$$

because $\max(u,v-\epsilon_j)\nearrow\max(u,v)$ and $1_{\{u<v-\epsilon_j\}}\nearrow 1_{\{u<v\}}$ as $j\to\infty$.
b) The argument is the same as a). □

Proposition 4.4. *Let $u_1,...,u_k\in PSH(\Omega)\cap L^\infty(\Omega)$ and $u_{k+1},...,u_n\in\mathcal{E}$. Then*
i) $\int\limits_B dd^cu_1\wedge...\wedge dd^cu_n = O((C_n(B))^{\frac{k}{n}})$ for all Borel sets $B\subset\Omega'\Subset\Omega$.
ii) $\int\limits_{B(a,r)} dd^cu_1\wedge...\wedge dd^cu_n = o((C_n(B(a,r)))^{\frac{k}{n}})$ as $r\to 0$ for all $a\in\Omega$,
where $B(a,r)=\{z\in\mathbf{C}^n\ :\ |z-a|<r\}$.

Proof. We may assume that $0\leq u_j\leq 1$ for $j=1,...,k$. On the other hand, by the remark following Definition 4.6 in [8] we again may assume that $u_{k+1},...,u_n\in\mathcal{F}$.

i) For each open set $B \Subset \Omega$, applying Proposition 3.1 and Corollary 5.6 in [8] we get

$$\begin{aligned}
\int_B dd^c u_1 \wedge ... \wedge dd^c u_n &= \int_B (-h_B^*)^k dd^c u_1 \wedge ... \wedge dd^c u_n \\
&\leq \int_\Omega (-h_B^*)^k dd^c u_1 \wedge ... \wedge dd^c u_n \\
&\leq k! \int_\Omega (1-u_1)(dd^c h_B^*)^k \wedge dd^c u_{k+1} \wedge ... \wedge dd^c u_n \\
&\leq k! \int_\Omega (dd^c h_B^*)^k \wedge dd^c u_{k+1} \wedge ... \wedge dd^c u_n \\
&\leq k! [\int_\Omega (dd^c h_B^*)^n]^{\frac{k}{n}} [\int_\Omega (dd^c u_{k+1})^n]^{\frac{1}{n}} ... [\int_\Omega (dd^c u_n)^n]^{\frac{1}{n}} \\
&\leq k! [\int_\Omega (dd^c u_{k+1})^n]^{\frac{1}{n}} ... [\int_\Omega (dd^c u_n)^n]^{\frac{1}{n}} . [C_n(B)]^{\frac{k}{n}} \\
&\leq \text{constants}. [C_n(B)]^{\frac{k}{n}}.
\end{aligned}$$

Hence

$$\int_B dd^c u_1 \wedge ... \wedge dd^c u_n \leq \text{constants}. [C_n(B)]^{\frac{k}{n}}$$

for all Borel sets $B \subset \Omega$.

ii) By Proposition 3.1 we have

$$\begin{aligned}
\int_\Omega (-\varphi)^k dd^c u_1 \wedge ... \wedge dd^c u_n &\leq k! \int_\Omega (1-u_1)(dd^c \varphi)^k \wedge dd^c u_{k+1} \wedge ... \wedge dd^c u_n \\
&\leq k! \int_\Omega (dd^c \varphi)^k \wedge dd^c u_{k+1} \wedge ... \wedge dd^c u_n < +\infty.
\end{aligned}$$

Hence $(-\varphi)^k \in L_1(dd^c u_1 \wedge ... \wedge dd^c u_n)$ for all $\varphi \in \mathcal{F}(\Omega)$. Given $a \in \Omega$ let r_0, R_0 be such that $B(a, r_0) \Subset \Omega \Subset B(a, R_0)$. Then

$$\log \frac{|z-a|}{R_0} \leq g_a(z) \leq \log \frac{|z-a|}{r_0},$$

for all $z \in \Omega$, where g_a denotes the Green function of Ω with pole at a. Since $(-g_a)^k \in L_1(dd^c u_1 \wedge ... \wedge dd^c u_n)$, it follows that

$$\int_{B(a,r)} (-g_a)^k dd^c u_1 \wedge ... \wedge dd^c u_n \to 0 \text{ as } r \to 0.$$

Hence

$$(\log r_0 - \log r)^k \int_{B(a,r)} dd^c u_1 \wedge ... \wedge dd^c u_n \leq \int_{B(a,r)} (-g_a)^k dd^c u_1 \wedge ... \wedge dd^c u_n \to 0$$

as $r \to 0$. This means that

$$\int\limits_{B(a,r)} dd^c u_1 \wedge ... \wedge dd^c u_n = o((\frac{1}{\log r_0 - \log r})^k) \text{ as } r \to 0.$$

Combining this with the inequality

$$C_n(B(a,r),\Omega) \geq C_n(B(a,r),B(a,R_0)) = (\frac{1}{\log R_0 - \log r})^n = O((\frac{1}{\log r_0 - \log r)^n}),$$

we get

$$\int\limits_{B(a,r)} dd^c u_1 \wedge ... \wedge dd^c u_n = o((C_n(B(a,r)))^{\frac{k}{n}}).$$

The next result should be compared with Theorem 6.3 in [7]. □

Theorem 4.5. *Let $u_1, ..., u_n \in \mathcal{E}$. Then there exists $\tilde{u} \in \mathcal{E}^a$ such that*

$$dd^c u_1 \wedge ... \wedge dd^c u_n = (dd^c \tilde{u})^n + dd^c u_1 \wedge ... \wedge dd^c u_n|_{\{u_1 = ... = u_n = -\infty\}}.$$

Proof. First, we write

$$dd^c u_1 \wedge ... \wedge dd^c u_n = \mu + dd^c u_1 \wedge ... \wedge dd^c u_n|_{\{u_1 = ... = u_n = -\infty\}},$$

where

$$\mu = dd^c u_1 \wedge ... \wedge dd^c u_n|_{\{u_1 > -\infty\} \cup ... \cup \{u_n > -\infty\}}.$$

It is easy to see that $\mu \ll C_n$ in every $E \Subset \Omega$. Indeed, by Theorem 4.1 we have

$$dd^c u_1 \wedge ... \wedge dd^c u_n|_{\{u_1 > -j\}} = dd^c \max(u_1, -j) \wedge ... \wedge dd^c u_n|_{\{u_1 > -j\}}.$$

Hence, by Proposition 4.4 (i) it follows that $dd^c u_1 \wedge ... \wedge dd^c u_n|_{\{u_1 > -j\}} \ll C_n$ in every $E \Subset \Omega$. Next, it remains to show that there exists $\tilde{u} \in \mathcal{E}^a$ such that $\mu = (dd^c \tilde{u})^n$. Let $\{\Omega_j\}$ be an increasing exhaustion sequence of Ω. For each $j \geq 1$ put $\mu_j = \mu|_{\Omega_j}$. By [1] there exists $\tilde{u}_j \in \mathcal{F}$ such that $(dd^c \tilde{u}_j)^n = \mu_j$. Notice that $\mu_j \nearrow \mu$ and

$$(dd^c \tilde{u}_j)^n \leq \mu \leq (dd^c(u_1 + ... + u_n))^n.$$

Applying the comparison principle we obtain

$$\tilde{u}_j \searrow \tilde{u} \geq u_1 + ... + u_n \in \mathcal{E}.$$

Hence, $\tilde{u} \in \mathcal{E}^a$ and $(dd^c \tilde{u})^n = \lim\limits_{j \to \infty} (dd^c \tilde{u}_j)^n = \mu$. The proof is thereby completed. □

Corollary 4.6. $u_1, ..., u_n \in \mathcal{E}$. *Then the following are equivalent:*
i) $dd^c u_1 \wedge ... \wedge dd^c u_n \ll C_n$ *in every* $E \Subset \Omega$,
ii) $\int\limits_{\{u_1 = ... = u_n = -\infty\}} dd^c u_1 \wedge ... \wedge dd^c u_n = 0$,
iii) $\int\limits_{\{u_1 < -s, ..., u_n < -s\} \cap E} dd^c u_1 \wedge ... \wedge dd^c u_n \to 0$ *as* $s \to +\infty$ *for all* $E \Subset \Omega$.

Proof. The proof is a direct application of Theorem 4.5.

The comparison principle for class $\mathcal{F}$ was studied in [9] and [16], [17]. By using Proposition 3.1 and Theorem 4.1 we prove a Xing-type comparison principle for $\mathcal{F}$. □

Theorem 4.7. *Let $u \in \mathcal{F}$, $v \in \mathcal{E}$ and $1 \leq k \leq n$. Then*

$$\frac{1}{k!}\int\limits_{\{u<v\}} (v-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\{u<v\}\cup\{u=v=-\infty\}} (r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$

for all $w_j \in PSH(\Omega)$, $0 \leq w_j \leq 1$, $j = 1, ..., k$, $w_{k+1}, ..., w_n \in \mathcal{F}$ and all $r \geq 1$.

Proof. Let $\epsilon > 0$. We set $\tilde{v} = \max(u, v-\epsilon)$. By a) in Proposition 3.1 we have

$$\frac{1}{k!}\int\limits_{\Omega} (\tilde{v}-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\Omega} (r-w_1)(dd^c \tilde{v})^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\Omega} (r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

Since $\{u < \tilde{v}\} = \{u < v - \epsilon\}$ and using Theorem 4.1 we have

$$\frac{1}{k!}\int\limits_{\{u<v-\epsilon\}} (v-\epsilon-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n$$
$$+ \int\limits_{\{u<v-\epsilon\}} (r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\{u\leq v-\epsilon\}} (r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\{u<v\}\cup\{u=v=-\infty\}} (r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

Letting $\epsilon \searrow 0$ we obtain

$$\frac{1}{k!}\int\limits_{\{u<v\}} (v-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\{u<v\}\cup\{u=v=-\infty\}} (r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

$\square$

Corollary 4.8. *Let $u \in \mathcal{E}^a$ be such that $u \geq v$ for all functions $v \in \mathcal{E}$ satisfying $(dd^c u)^n \leq (dd^c v)^n$. Then*

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^n dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^n$$
$$\leq \int\limits_{\{u<v\}} (r-w_1)(dd^c u)^n$$

for all $v \in \mathcal{E}$, $r \geq 1$ and all $w_1, ..., w_n \in PSH(\Omega)$, $0 \leq w_1, ..., w_n \leq 1$.

Proof. Let $\{\Omega_j\}$ be an increasing exhaustion sequence of relatively compact subdomains of Ω. Set $\mu_j = 1_{\Omega_j} 1_{\{u>-j\}}(dd^c u)^n$, where 1_E denotes the characteristic function of $E \subset \Omega$. Applying Theorem 4.1 we have

$$\mu_j = 1_{\Omega_j} 1_{\{u>-j\}}(dd^c \max(u,-j))^n \leq 1_{\Omega_j}(dd^c \max(u,-j))^n.$$

Take $\phi \in \mathcal{E}_0(\Omega) \cap C(\bar{\Omega})$. Put

$$\phi_j = \max(u, -j, a_j\phi),$$

where $a_j = \frac{-j}{\sup\limits_{\Omega_{j+1}} \phi}$. Then $\phi_j = \max(u,-j)$ on Ω_{j+1}, $\phi_j \in \mathcal{E}_0$ and

$$\mu_j \leq 1_{\Omega_j}(dd^c \max(u,-j))^n = 1_{\Omega_j}(dd^c\phi_j)^n \leq (dd^c\phi_j)^n.$$

By Kołodziej's theorem (see [15]) there exists $u_j \in \mathcal{E}_0$ such that

$$(dd^c u_j)^n = \mu_j = 1_{\Omega_j} 1_{\{u>-j\}}(dd^c u)^n$$

for all $j \geq 1$. By the comparison principle we have $u_j \searrow \tilde{u} \geq u$. On the other hand, since $(dd^c u)^n(\{u = -\infty\}) = 0$, it follows that

$$(dd^c u_j)^n = 1_{\Omega_j} 1_{\{u>-j\}}(dd^c u)^n \to (dd^c u)^n$$

weakly as $j \to \infty$. Thus $(dd^c\tilde{u})^n = \lim\limits_{j\to\infty}(dd^c u_j)^n = (dd^c u)^n$. By the hypothesis we have $\tilde{u} = u$. Applying Theorem 4.7 we get

$$\begin{aligned}
&\frac{1}{n!}\int\limits_{\{u_j<v\}} (v-u_j)^n dd^c w_1 \wedge \ldots \wedge dd^c w_n + \int\limits_{\{u_j<v\}} (r-w_1)(dd^c v)^n \\
&\quad \leq \int\limits_{\{u_j<v\}} (r-w_1)(dd^c u_j)^n \\
&\quad \leq \int\limits_{\{u_j<v\}} (r-w_1)(dd^c u)^n.
\end{aligned}$$

Letting $j \to \infty$ we obtain

$$\frac{1}{n!}\int\limits_{\{u<v\}} (v-u)^n dd^c w_1 \wedge \ldots \wedge dd^c w_n + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^n.$$

Arguing as in Theorem 4.7 we prove a Xing-type comparison principle for $\mathcal{E}$. $\square$

Theorem 4.9. *Let $u, v \in \mathcal{E}$ and $1 \leq k \leq n$ be such that $\varliminf\limits_{z\to\partial\Omega}[u(z) - v(z)] \geq 0$. Then*

$$\begin{aligned}
&\frac{1}{k!}\int\limits_{\{u<v\}} (v-u)^k dd^c w_1 \wedge \ldots \wedge dd^c w_n \\
&\qquad + \int\limits_{\{u<v\}} (r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge \ldots \wedge dd^c w_n \\
&\quad \leq \int\limits_{\{u<v\}\cup\{u=v=-\infty\}} (r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge \ldots \wedge dd^c w_n
\end{aligned}$$

for all $w_j \in PSH(\Omega)$, $0 \leq w_j \leq 1$, $j = 1, \ldots, k$, $w_{k+1}, \ldots, w_n \in \mathcal{E}$ and all $r \geq 1$.

Proof. Let $\epsilon > 0$. We set $\tilde{v} = \max(u, v - \epsilon)$. By b) in Proposition 3.1 we have

$$\frac{1}{k!}\int\limits_{\Omega}(\tilde{v}-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\Omega}(r-w_1)(dd^c\tilde{v})^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\Omega}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

Since $\{u < \tilde{v}\} = \{u < v - \epsilon\}$ and using Theorem 4.1 we have

$$\begin{aligned}\frac{1}{k!}&\int\limits_{\{u<v-\epsilon\}}(v-\epsilon-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n\\ &+\int\limits_{\{u<v-\epsilon\}}(r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n\\ &\leq \int\limits_{\{u\leq v-\epsilon\}}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n\\ &\leq \int\limits_{\{u<v\}\cup\{u=v=-\infty\}}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.\end{aligned}$$

Letting $\epsilon \searrow 0$ we obtain

$$\frac{1}{k!}\int\limits_{\{u<v\}}(v-u)^k dd^c w_1 \wedge ... \wedge dd^c w_n + \int\limits_{\{u<v\}}(r-w_1)(dd^c v)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n$$
$$\leq \int\limits_{\{u<v\}\cup\{u=v=-\infty\}}(r-w_1)(dd^c u)^k \wedge dd^c w_{k+1} \wedge ... \wedge dd^c w_n.$$

□

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Department of Mathematics, Hanoi University of Education, Dai hoc Su Pham Hanoi, Cau Giay, Hanoi, Vietnam

Department of Mathematics, Hanoi University of Education, Dai hoc Su Pham Hanoi, Cau Giay, Hanoi, Vietnam
E-mail address: phhiep_vn@yahoo.com