LOCAL WELL-POSEDNESS FOR THE MODIFIED KDV EQUATION IN ALMOST CRITICAL $\dot{H}^{s}$-SPACES

AXEL GRÜNROCK AND LUIS VEGA

ABSTRACT. We study the Cauchy problem for the modified KdV equation

$$u_t + u_{xxx} + (u^3)_x = 0, \quad u(0) = u_0$$

for data $u_0$ in the space $\dot{H}^s_x$ defined by the norm

$$\|u_0\|_{\dot{H}^s_x} := \|\langle \xi \rangle^s \hat{u}_0\|_{L^r_x}.$$ 

Local well-posedness of this problem is established in the parameter range $2 \geq r > 1$, $s \geq 1 \frac{1}{2} - \frac{1}{2r}$, so the case $(s,r) = (0,1)$, which is critical in view of scaling considerations, is almost reached. To show this result, we use an appropriate variant of the Fourier restriction norm method as well as bi- and trilinear estimates for solutions of the Airy equation.

1. Introduction and main result

In this paper we study the local well-posedness (LWP) of the Cauchy problem for the modified KdV equation

$$(1) \quad u_t + u_{xxx} + (u^3)_x = 0, \quad u(0) = u_0, \quad x \in \mathbb{R}.$$ 

As long as data $u_0$ in the classical Sobolev spaces $H^s_x$ are considered, this problem is known to be well-posed for $s \geq \frac{1}{4}$ and ill-posed (in the $C^0$-uniform sense) for $s < \frac{1}{4}$. Both the positive and the negative results were shown by Kenig, Ponce, and the second author; see [KPV93, Theorem 2.4] and [KPV01, Theorem 1.3], respectively. The situation remains the same when the defocusing modified KdV equation, i.e. (1) with a negative sign in front of the nonlinearity, is considered. In this case the proof of the well-posedness result remains identically valid, while the ill-posedness result here is due to Christ, Colliander and Tao; cf. [CCT03, Theorem 4]. In both cases the standard scaling argument suggests LWP for $s > -\frac{1}{2}$, so - on the $H^s_x$-scale - there is a considerable gap of $\frac{1}{2}$ derivatives between the scaling prediction and the optimal LWP result.

This gap could be closed partially by the first author in [G04], where data in the spaces $\dot{H}^s_x$ are considered, which are defined by the norms

$$\|u_0\|_{\dot{H}^s_x} := \|\langle \xi \rangle^s \hat{u}_0\|_{L^r_x},$$

where $\hat{u}_0$ denotes the Fourier transform of $u_0$, $\langle \xi \rangle = (1 + \xi^2)^{\frac{1}{2}}$ and $\frac{1}{r} + \frac{1}{r'} = 1$. The choice of these norms was motivated by earlier work of Cazenave, Vilela and
the second author on nonlinear Schrödinger equations (see [VV01]), yet another alternative class of data spaces has been considered in [VV01].

The main result in [G04] was LWP for (1) in the parameter range \(2 \geq r > \frac{4}{3}, s \geq s(r) := \frac{1}{2} - \frac{1}{2r},\) which coincides for \(r = 2\) with the optimal result on the \(H^s\)-scale. The proof used an appropriate variant of Bourgain’s Fourier restriction norm method; cf. [B93]. Especially the function spaces \(X^r_{s,b}\), defined by

\[
\|f\|_{X^r_{s,b}} := \left( \int d\xi d\tau (\xi)^{sr'} |\tau - \xi^3|^{br'} |\hat{f}(\xi,\tau)|^{r'} \right)^{\frac{1}{r'}} , \quad \frac{1}{r} + \frac{1}{r'} = 1,
\]

were utilised, as well as the time restriction norm spaces

\[
X^r_{s,b}(\delta) := \{ f = \hat{f} \in [-\delta,\delta] \times \mathbb{R} : \hat{f} \in X^r_{s,b} \}
\]

with norm

\[
\|f\|_{X^r_{s,b}(\delta)} := \inf \{ \|\hat{f}\|_{X^r_{s,b}} : \hat{f} \in X^r_{s,b} \} = f \}. 
\]

A key estimate in [G04] was the following Airy-version of the Fefferman-Stein-estimate (cf. [F70] and [G04, Corollary 3.6]):

\[
\| e^{-\frac{r}{3} \partial^3} u_0 \|_{L^2_{x,t}} \leq c \| I^r u_0 \|_{L^2_x} , \quad r > \frac{4}{3}. 
\]

Here and below \(I (J)\) denotes the Riesz (Bessel) potential operator of order \(-1\) and \(\tilde{L}_x = \tilde{H}^s_x\). This estimate fails to be true for \(r \leq \frac{4}{3}\), which explains the restriction \(r > \frac{4}{3}\) in [G04].

It is the aim of the present paper to show how this difficulty can be overcome by using bi- and trilinear estimates for solutions of the Airy equation (instead of linear and bilinear ones). This allows us to extend the LWP result for (1) to the parameter range \(2 \geq r > 1, s \geq s(r)\). More precisely, the following theorem is the main result of this paper.

**Theorem 1.** Let \(2 \geq r > 1, s \geq s(r) = \frac{1}{2} - \frac{1}{2r}\) and \(u_0 \in \tilde{H}^s_x\). Then there exist \(b > \frac{1}{r}, \delta = \delta(\|u_0\|_{\tilde{H}^s_x}) > 0\) and a unique solution \(u \in X^r_{s,b}(\delta)\) of (1). This solution is persistent, and the flow map \(S : u_0 \mapsto u, \tilde{H}^s_x \rightarrow X^r_{s,b}(\delta_0)\) is locally Lipschitz continuous for any \(\delta_0 \in (0, \delta)\).

Theorem 1 is sharp in the sense that, for given \(r \in (1,2]\), we have ill-posedness in the \(C^0\)-uniform sense for \(\frac{1}{r} - 1 < s < s(r)\). This can be seen by using the counterexample from [KPV01], as it was discussed in [G04] section 5]. Combined with scaling considerations - observe that \(\tilde{H}^s_x\) scales like \(H^s\) if \(s - \frac{1}{r} = \sigma - \frac{1}{r}\) - this shows that the case \((s,r) = (0,1)\) becomes critical in our setting and that our result covers the whole subcritical range. Unfortunately, our argument breaks down - even for small data - in the critical case, and we must leave this as an open problem. Notice, however, that for specific data

\[
u_0 = a \delta + \mu p.v. \frac{1}{x} \quad (a, \mu \text{ small})
\]

of critical regularity the existence of global solutions of (1) was shown in [PV05 Theorem 1.2]. By the general LWP Theorem [G04] Theorem 2.3] the proof of the following estimate is sufficient to establish Theorem 1.
Theorem 2. Let $2 \geq r > 1$ and $s \geq s(r) = \frac{1}{2} - \frac{1}{2r}$. Then for all $b' < 0$ and $b > \frac{1}{r}$ the estimate

$$
\left\| \partial_x \left( \prod_{i=1}^{3} u_i \right) \right\|_{X^r_{s,b'}} \leq c \prod_{i=1}^{3} \| u_i \|_{X^r_{s,b}}
$$

holds true.

Remarks.

i) (On the lifespan of local solutions) Using [G05] Lemma 5.2, we have for $u_1$, $u_2$, $u_3$ supported in $[-\delta, \delta] \times \mathbb{R}$ $(0 < \delta \leq 1)$ the estimate

$$
\left\| \partial_x \left( \prod_{i=1}^{3} u_i \right) \right\|_{X^r_{s,b-1}} \leq c \delta^{1 - \frac{1}{q} - \varepsilon} \prod_{i=1}^{3} \| u_i \|_{X^r_{s,b}}
$$

provided $2 \geq r > 1, s \geq s(r), b > \frac{1}{r}, \varepsilon > 0$. Inserting this estimate, especially the specific power of $\delta$, into the proof of the local result, we obtain a lifespan of size $\delta \sim \| u_0 \|_{\dot{H}^s_x}^{\frac{-2q}{2q-1} - \varepsilon'}$. For $r = 2$ this coincides up to $\varepsilon'$ - with the result in [KPV93] (see also [FLP99, Theorem 1.1]).

ii) Concerning related results for the one-dimensional cubic NLS and DNLS equations we refer to [Z09].

2. Bi- and trilinear Airy estimates

Throughout this section we consider solutions $u(t) = e^{-i\alpha x} u_0$, $v(t) = e^{-i\alpha x} v_0$ and $w(t) = e^{-i\alpha x} w_0$ of the Airy equation with data $u_0$, $v_0$ and $w_0$, respectively. Certain bi- and trilinear expressions involving these solutions will be estimated in the spaces $\mathcal{L}_x^q(\mathcal{L}_x^p)$ and $\mathcal{L}_{x,t}^s := \mathcal{L}_x^q(\mathcal{L}_t^p)$, where

$$
\| f \|_{\mathcal{L}_x^q(\mathcal{L}_t^p)} := \left( \int \left( \int |f(\xi, \tau)|^{p'} \, d\tau \right)^{\frac{p}{p'}} \, d\xi \right)^{\frac{1}{q}}, \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{p'} = 1.
$$

(Below we will always write $p'$, $q'$, etc. to indicate conjugate Hölder exponents, $\hat{f}$ or $\mathcal{F} f$ will denote the Fourier transform of $f$, while for the partial Fourier transform in the space variable the symbol $\mathcal{F}_x$ will be used.) We begin with the following bilinear estimate, which we state and prove in a slightly more general version than actually needed.

Lemma 1. Let $I^s$ denote the Riesz potential of order $-s$ and let $I^s_x(f, g)$ be defined by its Fourier transform (in the space variable):

$$
\mathcal{F}_x I^s_x(f, g)(\xi) := \int d\xi_1 |\xi_1 - \xi_2|^s \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2),
$$

where $\int_*$ is shorthand for $\int_{\xi_1 + \xi_2 = \xi}$. Then we have

$$
\| I^\frac{s}{2} f^\frac{1}{2} (u, v) \|_{\mathcal{L}_x^q(\mathcal{L}_t^p)} \leq c \| u_0 \|_{\mathcal{L}_x^q} \| v_0 \|_{\mathcal{L}_x^q},
$$

provided $1 \leq q \leq r_{1,2} \leq p < \infty$ and $\frac{1}{q} + \frac{1}{q'} = \frac{1}{r_1} + \frac{1}{r_2}$. 
Proof. Taking the Fourier transform first in space and then in time we obtain
\[
\mathcal{F}_x I_{\tau}^{\frac{1}{2}} (u,v)(\xi,t) = c|\xi|^{\frac{1}{2}} \int_s d\xi_1 |\xi_1 \xi_2 \xi_3|^\frac{1}{2} \hat{h}(\xi_1 + \xi_2 + \xi_3) \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2)
\]
and
\[
\mathcal{F} I_{\tau}^{\frac{1}{2}} (u,v)(\xi,t) = c|\xi|^{\frac{1}{2}} \int_s d\xi_1 |\xi_1 \xi_2 \xi_3|^\frac{1}{2} \delta(\tau - \xi_1 - \xi_2 - \xi_3) \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\xi_2),
\]
respectively. We use \( \delta(g(x)) = \sum_n \frac{1}{g(x_n)} \delta(x - x_n) \), where the sum is taken over all simple zeros of \( g \), which in our case is
\[
g(\xi_1) = \tau - \xi_1 - \xi_2 = \tau - \xi^3 + 3\xi_1(\xi - \xi_1)
\]
with the zeros
\[
\xi_1^\pm = \frac{\xi \pm y}{2}, \quad y := 2\sqrt{\frac{\tau - \xi^3}{12}}
\]
and the derivative
\[
g'(\xi_1^\pm) = 3\xi(\xi - 2\xi_1^\pm) = \mp 3\xi y.
\]
Hence
\[
(4) \quad \mathcal{F}_x I_{\tau}^{\frac{1}{2}} (u,v)(\xi,t)
\]
\[
= c|\xi|^{\frac{1}{2}} y \hat{h} \left( \mathcal{F}_x u_0(\frac{\xi + y}{2}) \mathcal{F}_x v_0(\frac{\xi - y}{2}) + \mathcal{F}_x u_0(\frac{\xi - y}{2}) \mathcal{F}_x v_0(\frac{\xi + y}{2}) \right).
\]
Using \( d\tau = 3|\xi|y dy \), we see that the \( L^p_x \)-norm of the first contribution equals
\[
\left( \int d\xi |\mathcal{F}_x u_0(\frac{\xi + y}{2}) \mathcal{F}_x v_0(\frac{\xi - y}{2})|^p \right)^{\frac{1}{p}} = c \left( |\mathcal{F}_x u_0|^p \ast |\mathcal{F}_x v_0|^p(\xi) \right)^{\frac{1}{p}}.
\]
Now Young’s inequality is applied to see that
\[
\left( \int d\xi (|\mathcal{F}_x u_0|^p \ast |\mathcal{F}_x v_0|^p(\xi)) \right)^{\frac{1}{p}} \leq c\|u_0\|_{L^2_x} \|v_0\|_{L^2_x}
\]
(cf. the proof of [G03, Lemma 1]), which is the desired bound. Finally we observe that the second contribution in (4) can be treated in precisely the same manner with \( r_1 \) and \( r_2 \) interchanged. \qed

Arguing similarly as in the proof of Lemma 2.1 in [G04] we obtain:

Corollary 1. For \( p, q, r_{1,2} \) as in the previous lemma and \( b_i > \frac{1}{r_i} \), the estimate
\[
\| I_{\tau}^{\frac{1}{2}} (u_1, u_2) \|_{L^2_x(L^r)} \leq c\|u_1\|_{X^0_{b_1}} \|u_2\|_{X^0_{b_2}}
\]
is valid.

The next step is to dualize the preceding corollary. For that purpose we recall the bilinear operator \( I^+_\frac{1}{2} \), defined by
\[
\mathcal{F}_x I^+_\frac{1}{2} (f,g)(\xi) := \int_{\xi_1 + \xi_2 = \xi} d\xi_1 \xi + \xi_2 \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2),
\]
and the linear operators
\[
M^+ \varphi := I^+_\frac{1}{2} (u,v) \quad \text{and} \quad N^+ \varphi := I^+_\frac{1}{2} (w, \varphi),
\]
Corollary 2.\ If the phase function $\phi(\xi) = \xi^3$ is odd, we have $\|u_1\|_X^{r_1} = \|\pi_1\|_X^{r_1}$, and we may replace $\pi_1$ by $u_1$ in the left-hand side of (5).

The special case in (5), where $p = q = r_{1,2}$, will be sufficient for our purposes. In this case, (5) can be written as

\begin{align}
(6) \quad \|I_+^{\frac{1}{r_{1,2}}}(I_+^{\frac{1}{r_{1,2}}}u_2, \pi_1)\|_{X_{0,-b_2}^{r_{1,2}}} &\leq c\|u_1\|_{X_{0,0}^{r_{1}}} \|u_2\|_{L_x^2(\mathbb{R})}, \\
\end{align}

provided $1 < r < \infty$, $b' < -\frac{2}{r}$. Combining this with the trivial endpoint of the Hausdorff-Young inequality, i.e.

\begin{align}
\|u_2u_1\|_{L_x^2(\mathbb{T})} &\leq c\|u_1\|_{L_x^2(\mathbb{T})} \|u_2\|_{L_x^2(\mathbb{T})}, \\
\end{align}

we obtain by elementary Hölder estimates

\begin{align}
(7) \quad \|I_+^{\frac{1}{r_{1,2}}}(I_+^{\frac{1}{r_{1,2}}}u_2, u_1)\|_{X_{0,0}^{r_{1,2}}} &\leq c\|u_1\|_{X_{0,-b_{1}}^{r_{1}}} \|u_2\|_{L_x^2(\mathbb{T})}, \\
\end{align}

where $0 \leq \frac{1}{r'} \leq \frac{1}{r_2}$ and $\beta < -\frac{1}{r'}$. In this form we shall actually make use of Corollary 2.

Now we turn to the trilinear estimates. Again we take the Fourier transform first in $x$ and then in $t$ to obtain

\begin{align}
\hat{F}_x(uvw)(\xi, t) = c \int_s d\xi_1 d\xi_2 e^{i(\xi^3 + \xi_2 \xi_3 \xi_1)} F_x u_0(\xi_1) F_x v_0(\xi_2) F_x w_0(\xi_3) \\
\end{align}

(where now $f_s = f_{\xi_1+\xi_2+\xi_3} = \xi$) and

\begin{align}
\hat{F}(uvw)(\xi, \tau) = c \int_s d\xi_1 d\xi_2 \delta(\xi^3 + \xi_2^3 + \xi_3^3 - \tau) F_x u_0(\xi_1) F_x v_0(\xi_2) F_x w_0(\xi_3). \\
\end{align}

Now the argument of $\delta$, that is,

\begin{align}
g(\xi_2) = 3(\xi - \xi_1)\xi_2^2 - 3(\xi - \xi_1)^2 \xi_2 - 3\xi_1(\xi - \xi_1) + \xi^3 - \tau, \\
\end{align}

has exactly two zeros

\begin{align}
(8) \quad \xi_2^\pm = \frac{\xi - \xi_1}{2} \pm \sqrt{\frac{(\xi + \xi_1)^2}{4} + \frac{\tau - \xi^3}{3(\xi - \xi_1)}} : \frac{\xi - \xi_1}{2} \pm y.
\end{align}
with
\[ |g'(\xi_2^\pm)| = 6|\xi - \xi_1| \sqrt{\frac{(\xi + \xi_1)^2}{4} + \frac{\tau - \xi_3^3}{3(\xi - \xi_1)}} = 6|\xi - \xi_1| y. \]

Using \( \delta(g(\xi_2)) = \sum_{g(x_n) = 0} \frac{\delta(x_2-x_n)}{g(x_n)} \), where the sum is taken over all simple zeros of \( g \), we see that
\[ \mathcal{F}(uvw)(\xi, \tau) = c(K_+((\xi, \tau) + K_-(\xi, \tau)), \]

where
\[ K_\pm(\xi, \tau) = \int d\xi_1 \frac{1}{|\xi - \xi_1| y} \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\frac{\xi - \xi_1}{2} \pm y) \mathcal{F}_x w_0(\frac{\xi - \xi_1}{2} \mp y) \]

with \( y \) as defined in (9).

In order to estimate \( \|uvw\|_{L_t^s} \) we distinguish between three cases depending on the relative size of the frequencies \( \xi_1, \xi_2 \) and \( \xi_3 \):

i) \( |\xi_1| \sim |\xi_2| \gg (\xi_3) \),

ii) \( |\xi_2 - \xi_3| \geq |\xi_2 + \xi_3| \),

iii) \( 1 \leq |\xi_2 - \xi_3| \leq |\xi_2 + \xi_3| \).

To treat the first case we define the trilinear operator \( T \) by
\[ \mathcal{F}_x T(f, g, h) := \int_x d\xi_1 d\xi_2 \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2) \mathcal{F}_x h(\xi_3) \chi_{\{|\xi_1| \sim |\xi_2| \gg (\xi_3)\}}, \]

where again \( \int_x = \int_{\xi_1 + \xi_2 + \xi_3 = \xi} \). In this case we have:

**Lemma 2.** Let \( 1 \leq r \leq 2 \) and \( s_1 > \frac{1}{2r} - \frac{1}{2}, \ s_2 \geq \frac{1}{2r}. \) Then
\[ \|T(u, v, w)\|_{L_t^s} \leq c\|u_0\|_{\dot{H}^{r_1}_x} \|v_0\|_{\dot{H}^{r_2}_x} \|w_0\|_{\dot{H}^{r_3}_x}. \]

**Proof.** By the above computation we have
\[ \mathcal{F}T(u, v, w)(\xi, \tau) = c(K^+(\xi, \tau) + K^-(\xi, \tau)) \]

with
\[ K^\pm(\xi, \tau) = \int_{A_\pm} d\xi_1 \frac{1}{|\xi - \xi_1| y} \mathcal{F}_x u_0(\xi_1) \mathcal{F}_x v_0(\frac{\xi - \xi_1}{2} \pm y) \mathcal{F}_x w_0(\frac{\xi - \xi_1}{2} \mp y), \]

where \( A_\pm = \{|\xi_1| \sim \frac{|\xi_2 - \xi_3|}{2} \pm y \gg (\frac{|\xi_2 + \xi_3|}{2} \mp y)\} \) and \( y \) is defined by (9). Since in \( A_\pm \) the inequality \( |\xi_1| \frac{|\xi_2 - \xi_3|}{2} \pm y \leq c|\xi - \xi_1| y \) holds true, we get the upper bound
\[ K^\pm(\xi, \tau) \leq c \int d\xi_1 \mathcal{F}_x J^{-1} u_0(\xi_1) \mathcal{F}_x J^{-1} v_0(\frac{\xi - \xi_1}{2} \pm y) \mathcal{F}_x w_0(\frac{\xi - \xi_1}{2} \mp y), \]

leading to
\[ \|T(u, v, w)\|_{L_t^s} \leq c\|J^{-1} u_0\|_{L_t^{r_1}} \|J^{-1} v_0\|_{L_t^{r_2}} \|w_0\|_{L_t^{r_3}}. \]

By symmetry between the first two factors and multilinear interpolation we obtain
\[ (9) \quad \|T(u, v, w)\|_{L_t^s} \leq c\|J^{-1} u_0\|_{L_t^s} \|J^{-1} v_0\|_{L_t^s} \|w_0\|_{L_t^s}. \]

On the other hand we have
\[ \|uvw\|_{L_t^s} \leq c\|u\|_{L_t^s(L_t^s)} \|v\|_{L_t^s(L_t^s)} \|w\|_{L_t^s(L_t^s)}, \]

with
\[ (10) \quad \|w\|_{L_t^s(L_t^s)} \leq c\|J^\dagger u_0\|_{L_t^s}, \]
which is the maximal function estimate from [SS7] Thm. 3. Concerning the first two factors we interpolate between the sharp version of Kato’s smoothing effect, i.e. \( \|Iu\|_{L^p_T(L^2_x)} = c\|u_0\|_{L^2_x} \) (see [KPV91] Thm. 4.1) and (10) to obtain
\[
\|I\hat{v}\|_{L^p_T(L^1_x)} \leq c\|u_0\|_{L^2_x},
\]
such that
\[
\|T(u, v, w)\|_{L^p_t(L^2_x)} \leq c\|J^{\frac{s}{2} - 1}u_0\|_{L^2_x}\|J^{\frac{\rho}{2} - 1}v_0\|_{L^2_x}\|J^{\frac{\sigma}{2}}w_0\|_{L^2_x}.
\]
Using multilinear interpolation again, now between (9) and (11) we finally see that, for \( 1 \leq r \leq 2 \),
\[
\|T(u, v, w)\|_{L^p_t(L^2_x)} \leq c\|J^{\frac{s}{2} - 1}u_0\|_{L^2_x}\|J^{\frac{\rho}{2} - 1}v_0\|_{L^2_x}\|J^{\frac{\sigma}{2}}w_0\|_{L^2_x}
\]
where in the last step we have used the Sobolev type embedding \( \hat{H}^s \subset \hat{H}^\sigma \), which holds true for \( s - \frac{\rho}{2} > \sigma - \frac{\rho}{2} \), \( r \leq \rho \).

**Corollary 3.** For \( r, s_{1,2} \) as in the previous lemma and \( b > \frac{1}{r} \) the estimate
\[
\|T(u_1, u_2, u_3)\|_{L^p_t(L^2_x)} \leq c\|u_1\|_{X^{s_1,b}_T}\|u_2\|_{X^{s_2,b}_T}\|u_3\|_{X^{s_2,b}_T}
\]
holds true.

Next we introduce \( T_{\geq} (T_{\leq}) \) by
\[
\mathcal{F}_xT_{\geq}(f, g, h) := \int d\xi_1d\xi_2\mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2)\mathcal{F}_x h(\xi_3)\chi_{\{\xi_2 - \xi_1 \geq |\xi_2 + \xi_3|\}}
\]
and
\[
\mathcal{F}_xT_{\leq}(f, g, h) := \int d\xi_1d\xi_2\mathcal{F}_x f(\xi_1)\mathcal{F}_x g(\xi_2)\mathcal{F}_x h(\xi_3)\chi_{\{|\xi_2 - \xi_1| \leq |\xi_2 + \xi_3|\}}.
\]

**Lemma 3.** Let \( 1 < p_1 < p < p_0 < \infty \), \( p < p'_0 \), \( \frac{\rho}{p} = \frac{1}{p_0} + \frac{\sigma}{p_1} \) and \( \frac{\rho}{p} < 1 + \frac{1}{r} \). Then the estimate
\[
\|T_{\geq}(u, v, w)\|_{L^p_t(L^2_x)} \leq c\|u_0\|_{L^p_t(L^2_x)}\|I^{-\frac{\rho}{2}}v_0\|_{L^p_t(L^2_x)}\|I^{-\frac{\sigma}{2}}w_0\|_{L^p_t(L^2_x)}
\]
is valid.

**Proof.** For the Fourier transform of \( T_{\geq}(u, v, w) \) in both variables we obtain
\[
\mathcal{F}T_{\geq}(u, v, w)(\xi, \tau) = c(K^+_{\geq}(\xi, \tau) + K^-_{\geq}(\xi, \tau)),
\]
where
\[
K^\pm_{\geq}(\xi, \tau) = \int_{\{y\leq|\xi_2 + \xi_3|\}} d\xi_1\frac{\chi_{\{\xi_1 \leq |\xi_2 - \xi_1|\}}}{|\xi_2 - \xi_1|}\mathcal{F}_x u_0(\xi_1)\mathcal{F}_x v_0(\frac{\xi - \xi_1}{2} \pm y)\mathcal{F}_x w_0(\frac{\xi - \xi_1}{2} \mp y),
\]
with \( y \) as in (11) again. By symmetry we may restrict ourselves to the estimation of \( K^+_{\geq} \). Using \( |\xi_1| \leq 2y \) and Hölder’s inequality, we see that
\[
K^+_{\geq}(\xi, \tau) \leq c\left(\int d\xi_1\frac{|\mathcal{F}_x u_0(\xi_1)|^p}{|\xi_1|^{1-\theta}}\right)^{\frac{1}{p}} \times \left(\int d\xi_1\frac{|\mathcal{F}_x I^{-\frac{\rho}{2}}v_0(\frac{\xi - \xi_1}{2} + y)\mathcal{F}_x I^{-\frac{\sigma}{2}}w_0(\frac{\xi - \xi_1}{2} - y)|^{p'}}{|\xi_1|^{1-\theta}y^{p'}}\right)^{\frac{1}{p'}},
\]
where \( \theta = \frac{3}{p} - \frac{2}{p_1} \) (\( \in (0, 1) \) by our assumptions). Taking the \( L^{p_1}_\xi \)-norm of both sides and using \( d\tau = 6|\xi - \xi_1|dyd\xi \) we arrive at

\[
\| F T_{\geq}(u, v, w)(\xi, \cdot) \|_{L^{p_1}_\xi} \leq c(\| F_x u_0 \|^p \ast \| \xi \|^{(\theta - 1)p})^{\frac{1}{p}}
\times \left( \int \frac{d\xi_1 dy}{|\xi - \xi_1|^{p_1 - 1}} |F_x I^{-\frac{1}{p_1}} v_0(\frac{\xi - \xi_1}{2} + y) F_x I^{-\frac{1}{p_1}} w_0(\frac{\xi - \xi_1}{2} - y)|^{p_1'} \right)^{\frac{1}{p_1'}}.
\]

Changing variables \( (z_\pm := \frac{\xi - \xi_1}{2} \pm y) \) we see that the second factor equals

\[
\left( \int \frac{dz d\xi}{|z_+ + z_-|^{p_1 - 1}} |F_x I^{-\frac{1}{p_1}} v_0(z_+) F_x I^{-\frac{1}{p_1}} w_0(z_-)|^{p_1'} \right)^{\frac{1}{p_1'}} \leq c\| I^{-\frac{1}{p_1}} v_0 \|_{L^{p_1}_\xi} \| I^{-\frac{1}{p_1}} w_0 \|_{L^{p_1}_\xi},
\]

by the Hardy-Littlewood-Sobolev-inequality, requiring \( \theta \) to be chosen as above and \( 1 < \theta p' < 2 \), which follows from our assumptions. It remains to estimate the \( L^{p_1}_\xi \)-norm of the first factor, that is,

\[
\| F_x u_0 \|^p \ast \| \xi \|^{(\theta - 1)p} \|_{L^{p_1}_\xi} \leq c(\| F_x u_0 \|^p \ast \| \xi \|^{(\theta - 1)p})^{\frac{1}{p}} \leq c\| u_0 \|_{L^{p_1}_\xi},
\]

where the HLS inequality was used again. For its application we need

\[
0 < (1 - \theta)p < 1; \quad 1 < \frac{p_0}{p} < \frac{1}{1 - (1 - \theta)p} \quad \text{and} \quad \theta = \frac{1}{p_0},
\]

which follows from the assumptions, too. \( \square \)

**Corollary 4.** For \( 1 < r < 2 \) there exist \( s_{0,1} \geq 0 \) with \( s_0 + 2s_1 = \frac{1}{2} \), such that

\[
\| T_{\geq}(u, v, w) \|_{L^r} \leq c\| I^{-s_0} u_0 \|_{L^2_\xi} \| I^{-s_1} v_0 \|_{L^2_\xi} \| I^{-s_1} w_0 \|_{L^2_\xi}.
\]

In addition, for \( b > \frac{1}{3} \) we have

\[
\| T_{\geq}(u_1, u_2, u_3) \|_{L^r} \leq c\| I^{-s_0} u_1 \|_{X^b_0} \| I^{-s_1} u_2 \|_{X^b_0} \| I^{-s_1} u_3 \|_{X^b_0}.
\]

**Proof of (12).** Using Hölder’s inequality and the Airy-version of the Fefferman-Stein-estimate, that is,

\[
\| u \|_{L^q_\xi} \leq c\| I^{-\frac{1}{q_0}} u_0 \|_{L^2_\xi}, \quad q > \frac{4}{3}
\]

(see [G04 Corollary 3.6]), we get for

\[
\frac{4}{3} < q_0 < 2 < q_1 \quad \text{with} \quad \frac{3}{2} = \frac{1}{q_0} + \frac{2}{q_1},
\]

that

\[
\| T_{\geq}(u, v, w) \|_{L^2_\xi} \leq \| uvw \|_{L^2_\xi} \leq c\| I^{-\frac{1}{q_0}} u_0 \|_{L^2_\xi} \| I^{-\frac{1}{q_1}} v_0 \|_{L^2_\xi} \| I^{-\frac{1}{q_1}} w_0 \|_{L^2_\xi}.
\]

Multilinear interpolation of (15) with Lemma 3 yields (12), provided \( p, p_0, p_1; q_0, q_1 \), defined by the interpolation conditions

\[
\frac{1}{r} = \frac{1 - \theta}{p} + \frac{\theta}{2} = \frac{1 - \theta}{p_0} + \frac{\theta}{q_0} = \frac{1 - \theta}{p_1} + \frac{\theta}{q_1},
\]
fulfill the assumptions of Lemmas 3 and 13, respectively, which can be guaranteed by choosing \( \theta \) sufficiently small. Now \( s_{0,1} \) are obtained from
\[
 s_0 = \frac{\theta}{3q_0} \quad \text{and} \quad s_1 = \frac{1 - \theta}{2p} + \frac{\theta}{3q_1},
\]
which gives
\[
 s_0 + 2s_1 = \frac{1 - \theta}{p} + \frac{\theta}{3q_1} + \frac{2}{q_1} = \frac{1}{r},
\]
as desired. \( \square \)

**Remark.** By 13, Corollary 4 still holds true for \( r \geq 2 \) (with \( s_0 = s_1 = \frac{1}{3p} \)).

**Lemma 4.** Let \( 1 \leq r < \rho \leq \infty \). Then
\[
 \| T_\leq (u,v,w) \|_{L^r_x} \leq c \| u_0 \|_{L^\rho_x} \| I^{-\frac{3}{2}} v_0 \|_{L^2_y} \| I^{-\frac{3}{2}} w_0 \|_{L^2_y}.
\]

**Proof.** We have
\[
 FT_\leq (u,v,w)(\xi,\tau) = c(K_\leq^+(\xi,\tau) + K_\leq^-(\xi,\tau)),
\]
where
\[
 K_\leq^\pm(\xi,\tau) = \int_{\{1 \leq 2y \leq |\xi - \xi_1| \}} \frac{d\xi_1}{|\xi - \xi_1|} F_x u_0(\xi_1) F_x v_0(\frac{\xi - \xi_1}{2} \pm y) F_x w_0(\frac{\xi - \xi_1}{2} \mp y)
\]
with \( y \) as defined in 3. By symmetry between \( v \) and \( w \) it suffices to treat \( K_\leq^+ \), which we decompose dyadically with respect to \( y \) to obtain the upper bound:
\[
 c \sum_{j=0}^{\infty} \int_{\{1 \leq 2y \leq |\xi - \xi_1|, y \sim 2^j\}} \frac{d\xi_1}{|\xi - \xi_1|} F_x u_0(\xi_1) F_x v_0(\frac{\xi - \xi_1}{2} + y) F_x w_0(\frac{\xi - \xi_1}{2} - y)
\]
\[
 \leq c \sum_{j=0}^{\infty} 2^{-j} \int_{\{y \sim 2^j\}} d\xi_1 F_x u_0(\xi_1) F_x I^{-\frac{3}{2}} v_0(\frac{\xi - \xi_1}{2} + y) F_x I^{-\frac{3}{2}} w_0(\frac{\xi - \xi_1}{2} - y)
\]
\[
 \leq c \sum_{j=0}^{\infty} 2^{-j} \| u_0 \|_{L^\rho_x} \lambda(\{y \sim 2^j\}) \| I^{-\frac{3}{2}} v_0 \|_{L^2_y} \| I^{-\frac{3}{2}} w_0 \|_{L^2_y},
\]
where \( \lambda(\{y \sim 2^j\}) \) denotes the Lebesgue measure of \( \{\xi_1 : y(\xi_1) \sim 2^j\} \), which is bounded by \( c2^j \). Hence, for any \( p > 1 \),
\[
 \| K_\leq^\pm \|_{L^\rho_x} \leq c \sum_{j=0}^{\infty} 2^{-\frac{j}{p}} \| u_0 \|_{L^\rho_x} \| I^{-\frac{3}{2}} v_0 \|_{L^2_y} \| I^{-\frac{3}{2}} w_0 \|_{L^2_y}
\]
\[
 \leq c \| u_0 \|_{L^\rho_x} \| I^{-\frac{3}{2}} v_0 \|_{L^2_y} \| I^{-\frac{3}{2}} w_0 \|_{L^2_y}.
\]

\[\text{To see this, we write } \{\xi_1 : y(\xi_1) \sim 2^j\} = S_1 \cup S_2, \text{ where in } S_1 \text{ we assume that } |\xi - \xi_1| \leq 2^j, |\xi + \xi_1| \leq 2^j \text{ or } |\xi - 3\xi_1| \leq 2^j. \text{ Then } S_1 \text{ consists of a finite number of intervals of total length bounded by } c2^j. \text{ For } S_2 \text{ we have } |\xi - \xi_1| \gg 2^j, |\xi + \xi_1| \gg 2^j \text{ and } |\xi - 3\xi_1| \gg 2^j, \text{ implying that}
\]
\[
 \left| \frac{dy}{d\xi_1} \right| = \frac{1}{2y |\xi - \xi_1|} \left( |\xi + \xi_1| (|\xi - 3\xi_1|) \right) \geq \frac{|\xi + \xi_1| |\xi - 3\xi_1|}{y |\xi - \xi_1|} \geq 1,
\]
which gives
\[
 \lambda(S_2) = \int_{S_2} d\xi_1 \leq \int \frac{d\xi_1}{dy} \chi(\{y \sim 2^j\}) dy \leq c2^j.
\]
On the other hand, by integration with respect first to \( d\tau = 6y(\xi - \xi_1)dy \), then to \( d\xi \), and finally to \( d\xi_1 \), we see that
\[
\|K^{\pm}_{\xi}\|_{L^1_t} \leq c\|u_0\|_{L^\infty_x}^\beta\|v_0\|_{L^\infty_x}\|w_0\|_{L^\infty_x}.
\]
Now multilinear interpolation between (16) and (17) leads to
\[
\|K^{\pm}_{\xi}\|_{L^1_t} \leq c\|u_0\|_{L^\infty_x}\|I^{-\frac{\alpha}{2}}v_0\|_{L^\infty_x}\|I^{-\frac{\alpha}{2}}w_0\|_{L^\infty_x},
\]
which gives the desired result. \(\square\)

**Corollary 5.** Let \( 1 \leq r < \rho \leq \infty, \beta > \frac{1}{\rho}, b > \frac{1}{r} \) and \( \varepsilon > 0 \). Then
\[
\|T_{\xi}(u_1, u_2, u_3)\|_{L^r_{\xi\tau}} \leq c\|u_1\|_{X^\rho_{0,\beta}}\|I^{-\frac{\alpha}{2}}u_2\|_{X^\rho_{0,b}}\|I^{-\frac{\alpha}{2}}u_3\|_{X^\rho_{0,b}}
\]
and
\[
\|T_{\xi}(u_1, u_2, u_3)\|_{L^r_{\xi\tau}} \leq c\|u_1\|_{X^\rho_{r,\beta}}\|I^{-\frac{\alpha}{2}}u_2\|_{X^\rho_{r,b}}\|I^{-\frac{\alpha}{2}}u_3\|_{X^\rho_{r,b}}
\]
are valid.

### 3. Proof of Theorem 2

Without loss of generality we may assume that \( s = s(r) \). Then we rewrite the left hand side of (14) as
\[
\left\| \left( \tau - \xi^3 \right)^{\frac{1}{3}} \xi^s \xi \int \sum_{i=1}^3 \tilde{u}_i(\xi, \tau_i) \right\|_{L^r_{\xi\tau}},
\]
where \( dv = d\xi d\xi_2 d\tau_1 d\tau_2 \) and \( \sum_{i=1}^3 (\xi, \tau_i) = (\xi, \tau) \).

In the sequel, we shall use the following notation:
- \( \xi_{\text{max}}, \xi_{\text{med}}, \xi_{\text{min}} \) are defined by \( |\xi_{\text{max}}| \geq |\xi_{\text{med}}| \geq |\xi_{\text{min}}| \),
- \( p \) denotes the projection on low frequencies, i.e. \( pf(\xi) = \chi_{\{|\xi| \leq 1\}} \hat{f}(\xi) \),
- \( f \preceq g \) is shorthand for \( |\hat{f}| \leq c|\hat{g}| \),
- for the mixed weights coming from the \( X^r_{s,b} \)-norms we shall write \( \sigma_0 := \tau - \xi^3 \) and \( \sigma_i := \tau_i - \xi^3_i, 1 \leq i \leq 3 \), respectively,
- the Fourier multiplier associated with these weights is denoted by \( \Lambda^b := \mathcal{F}^{-1}(r - \xi^3)^{b}\mathcal{F} \),
- for a real number \( x \) we write \( x \pm \varepsilon \) to denote \( x \pm \varepsilon \) for arbitrarily small \( \varepsilon > 0 \); \( \infty \) stands for an arbitrarily large real number.

Apart from the trivial region where \( |\xi_{\text{max}}| \leq 1 \), whose contribution can be estimated by
\[
\left\| \prod_{i=1}^3 pu_i \right\|_{L^r_{\xi\tau}} \leq c\prod_{i=1}^3 \|pu_i\|_{L^\infty_{\xi\tau}} \leq c\prod_{i=1}^3 \|pu_i\|_{X^r_{s,b}} \leq c\prod_{i=1}^3 \|u_i\|_{X^r_{s,b}},
\]
we consider three cases:
1. the nonresonant case, where \( |\xi_{\text{max}}| \gg |\xi_{\text{med}}| \),
2. the semiresonant case with \( |\xi_{\text{max}}| \sim |\xi_{\text{med}}| \gg |\xi_{\text{min}}| \) and, finally,
3. the resonant case, where \( |\xi_{\text{max}}| \sim |\xi_{\text{min}}| \).
1. In the nonresonant case we assume without loss of generality that $|\xi_1| \geq |\xi_2| \geq |\xi_3|$. Then we have for this region
\[ J^s \partial_x (u_1 u_2 u_3) \leq \partial_x (J^s u_1 J^s u_2 J^{-s} u_3) \leq I^+ I^+ (J^s u_1, J^s u_2, J^{1-s-\frac{1}{2}} u_3) \leq I^+_+ (I^+ I^+ (J^s u_1, J^s u_2, J^{1-s-\frac{1}{2}} u_3)). \]

Now the dual version (7) of the bilinear estimate is applied to obtain
\[ \| I^+_+ (I^+ I^+ (J^s u_1, J^s u_2, J^{1-s-\frac{1}{2}} u_3)) \|_{X^{r,b}_{0,0}} \leq c \| I^+_+ I^+ (J^s u_1, J^s u_2) \|_{L^\infty_t} \| J^{1-s-\frac{1}{2}} u_3 \|_{X^{0,0}_{\infty}} \leq c \prod_{i=1}^3 \| u_i \|_{X^{r,b}_s}, \]
where in the last step we have used the bilinear estimate itself (Corollary 1) for the first and Sobolev-type embeddings for the second factor.

2. In the semiresonant case we assume again $|\xi_1| \geq |\xi_2| \geq |\xi_3|$ and consider two subcases: If, in addition, $|\xi_1 + \xi_2| \leq 1$ (so that $|^1| \leq c(|3|))$, we can argue as in case 1, with $u_1$ and $u_3$ interchanged:
\[ J^s \partial_x (u_1 u_2 u_3) \leq \partial_x (J^{-s} u_1 J^s u_2 J^s u_3) \leq I^+_+ (I^+ I^+ (J^s u_1, J^s u_2, J^{1-s-\frac{1}{2}} u_1)), \]
which can be treated as above by applying (7), Sobolev-type embeddings and Corollary 1. On the other hand, if $|\xi_1 + \xi_2| \geq 1$, we have
\[ |\sigma_0 - \sigma_1 - \sigma_2 - \sigma_3| = 3|\xi_1 + \xi_2||\xi_2 + \xi_3||\xi_3 + \xi_1| \gtrsim |\xi_1| |\xi_2|, \]
and hence, for any $\varepsilon > 0$,
\[ |\xi_1|\varepsilon |\xi_2| \varepsilon \leq c \prod_{i=0}^3 |\sigma_i|^\varepsilon. \]
So, in this subcase, we have the upper bound
\[ \| T(J^{\frac{1}{2}+} - \Lambda^{0+} u_1, J^{\frac{1}{2}+} - \Lambda^{0+} u_2, \Lambda^{0+} u_3) \|_{L^0_t} \leq c \prod_{i=1}^3 \| u_i \|_{X^{r,b}_s}, \]
by Corollary 3.

3. In the resonant case we distinguish several subcases:

3.1. At least for one pair $(i, j)$ we have $|\xi_i - \xi_j| \geq |\xi_i + \xi_j|$. Here we may assume by symmetry that $|\xi_2 - \xi_3| \geq |\xi_2 + \xi_3|$. Then we have for non-negative $s_{0,1}$ with $s_0 + 2s_1 = \frac{1}{r}$
\[ \partial_x J^s (u_1 u_2 u_3) \leq T_{\geq} (J^{s+s} u_1, J^{s+s} u_2, J^{s+s} u_3), \]
so that Corollary 4 leads to the desired bound.
3.2. \(|\xi_1 - \xi_2| \leq |\xi_1 + \xi_2|, |\xi_2 - \xi_3| \leq |\xi_2 + \xi_3|\) and \(|\xi_3 - \xi_1| \leq |\xi_3 + \xi_1|\), so that all the \(\xi_i\) have the same sign, which implies
\[
|\xi_1|^3 \sim |\xi_2|^3 \sim |\xi_3|^3 \leq \prod_{i=0}^{3} |\sigma_i|.
\]

3.2.1. At least one of the \(|\xi_i - \xi_j| \geq 1\).

By symmetry we may assume that \(|\xi_2 - \xi_3| \geq 1\). Gaining a \(\langle \xi \rangle^c\) from the \(\sigma\)'s we obtain an upper bound for this subcase
\[
\|T_{\xi} (J^{s-} - \Lambda^{0+} u_1, J^{\frac{1}{2}} \Lambda^{0+} u_2, J^{\frac{1}{2}} \Lambda^{0+} u_3)\|_{L_T^\infty} \leq c \prod_{i=1}^{3} \|u_i\|_{X^{s,b}_{\xi,\tau}},
\]
where we have used the second part of Corollary 5.

3.2.2. \(|\xi_i - \xi_j| \leq 1\) for all \(1 \leq i \neq j \leq 3\).

Again, we can gain a \(\langle \xi \rangle^c\) from the \(\sigma\)'s. Now, writing
\[
f_i(\xi, \tau) = \langle \xi \rangle^{4\beta} (\tau - \xi^3)^b F_{ui}(\xi, \tau), \quad 1 \leq i \leq 3,
\]
such that \(\|f_i\|_{L_t^\infty} = \|u_i\|_{X^{s,b}_{\xi,\tau}}\), it suffices to show
\[
(18) \quad \left\| \langle \xi \rangle^{1-2s} \langle \xi \rangle \int_A d\nu \prod_{i=1}^{3} \langle \xi_i \rangle^{-\frac{1}{3}} \langle \tau_i - \xi_i^3 \rangle^{-\frac{1}{3}} f_i(\xi_i, \tau_i) \right\|_{L_t^\infty} \leq c \prod_{i=1}^{3} \|f_i\|_{L_t^\infty},
\]
where in \(A\) all the differences \(|\xi_k - \xi_j|, 1 \leq k \neq j \leq 3\), are bounded by 1 and \(\|\xi\| \sim |\xi_i| \sim |\xi_j|\) for all \(1 \leq i \leq 3\). By Hölder’s inequality and Fubini’s Theorem the proof of (18) is reduced to show that
\[
(19) \quad \sup_{\xi, \tau} \langle \xi \rangle^{1-2s} \left( \int_A d\nu \prod_{i=1}^{3} \langle \tau_i - \xi_i^3 \rangle^{-1} \right)^{\frac{1}{3}} < \infty.
\]

Using [GTV97] Lemma 4.2 twice, we see that
\[
\int_A d\nu \prod_{i=1}^{3} \langle \tau_i - \xi_i^3 \rangle^{-1} \leq c \int_{A'} d\xi_1 d\xi_2 \langle \tau - \xi^3 + 3(\xi_1 + \xi_2)(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1},
\]
where \(A'\) is simply the projection of \(A\) onto \(\mathbb{R}^2\). We decompose
\[
A' = A_0 \cup A_1 \cup \bigcup_{0 \leq k,j \leq c \ln (|\xi|)} A_{kj},
\]
where in \(A_0 (A_1)\) we have that \(|\xi_1 + \xi_2 - \frac{2\xi}{3}| \leq \frac{100}{11} (|\xi_1 + \xi_3 - \frac{2\xi}{3}| \leq \frac{100}{11})\), so that the contributions of these subregions are bounded by \(\frac{100}{11}\), while in \(A_{kj}\) it should hold that \(|\xi_1 + \xi_2 - \frac{2\xi}{3}| \sim 2^{-k}\) and \(|\xi_1 + \xi_3 - \frac{2\xi}{3}| \sim 2^{-j}\). By symmetry we may assume \(k \leq j\). To estimate the integral over \(A_{kj}\), we introduce new variables \(x_1 := \xi_1 + \xi_2 - \frac{2\xi}{3}\) and \(x_2 := \xi_1 - \xi_2\), such that
\[
|x_1| \sim 2^{-k} \quad \text{and} \quad |x_2| = |\xi_1 + \xi_2 - \frac{2\xi}{3} + 2(\xi_1 + \xi_3 - \frac{2\xi}{3})| \leq 2^{-k}.
\]
Then
\[ \int_{A_{k}} d\xi_1 d\xi_2 \langle \tau - \xi^3 + 3(\xi_1 + \xi_2)(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1} \]
\[ \leq \int_{|x_2| \leq 2^{-k}} dx_2 \int_{|x_1| \sim 2^{-k}} dx_1 \langle \tau - \xi^3 + 3(1 + \frac{2d}{3})\left(\frac{x_1 + x_2}{2} - \frac{2d}{3}\right)^2 \rangle^{-1}. \]
Substituting \( z := (1 + \frac{2d}{3})\left(\frac{x_1 + x_2}{2} - \frac{2d}{3}\right)^2 \), so that
\[ \frac{dz}{dx_1} = \frac{3x_1^2 - x_2^2}{4} - x_1 \xi \sim |x_1 \xi| \sim |\xi| 2^{-k}, \]
we see that the latter is bounded by
\[ \int_{|x_2| \leq 2^{-k}} dx_2 \int_{|x_1| \sim 2^{-k}} dx_1 \langle 3\xi^3 + 3z \rangle^{-1} \leq \frac{c}{|\xi|}. \]
Finally, summing up over \( j \) and \( k \), we have
\[ \int_{A'} d\xi_1 d\xi_2 \langle \tau - \xi^3 + 3(\xi_1 + \xi_2)(\xi - \xi_1)(\xi - \xi_2) \rangle^{-1} \leq c \left( \frac{\ln |\xi|}{|\xi|} \right)^2 \leq c|\xi|^{-1}, \]
which gives \( \square \).

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References


Fachbereich C: Mathematik/Naturwissenschaften, Bergische Universität Wuppertal, D-42097 Wuppertal, Germany
E-mail address: Axel.Gruenrock@math.uni-wuppertal.de
Current address: Mathematisches Institut, Universität Bonn, Beringstrasse 4, D-53115 Bonn, Germany
E-mail address: gruenroc@math.uni-bonn.de

Departamento de Matemáticas, Universidad del País Vasco, 48080 Bilbao, Spain
E-mail address: luis.vega@ehu.es