ISOMETRIC IMMERSIONS INTO $S^n \times \mathbb{R}$ AND $H^n \times \mathbb{R}$ AND APPLICATIONS TO MINIMAL SURFACES

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Abstract. We give a necessary and sufficient condition for an $n$-dimensional Riemannian manifold to be isometrically immersed in $S^n \times \mathbb{R}$ or $H^n \times \mathbb{R}$ in terms of its first and second fundamental forms and of the projection of the vertical vector field on its tangent plane. We deduce the existence of a one-parameter family of isometric minimal deformations of a given minimal surface in $S^2 \times \mathbb{R}$ or $H^2 \times \mathbb{R}$, obtained by rotating the shape operator.

1. Introduction

It is well known that the first and second fundamental forms of a hypersurface of a Riemannian manifold satisfy two compatibility equations called the Gauss and Codazzi equations. More precisely, let $\bar{V}$ be an orientable Riemannian manifold of dimension $n + 1$ and $V$ a submanifold of $\bar{V}$ of dimension $n$. Let $\nabla$ (respectively, $\bar{\nabla}$) be the Riemannian connection of $V$ (respectively, $\bar{V}$), $R$ (respectively, $\bar{R}$) be the Riemann curvature tensor of $V$ (respectively, $\bar{V}$), i.e.,

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z,$$

and $S$ be the shape operator of $V$ associated to its unit normal $N$, i.e., $SX = -\bar{\nabla}_X N$.

Then the following equations hold for all vector fields $X, Y, Z, W$ on $V$:

$$\langle R(X, Y)Z, W \rangle - \langle \bar{R}(X, Y)Z, W \rangle = \langle SX, Z \rangle \langle SY, W \rangle - \langle SY, Z \rangle \langle SX, W \rangle,$$

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = R(X, Y)N.$$

These are respectively the Gauss and Codazzi equations.

In the case where $\bar{V}$ is a space form, i.e., the sphere $S^{n+1}$, the Euclidean space $\mathbb{R}^{n+1}$ or the hyperbolic space $H^{n+1}$, these equations become the following:

$$\langle R(X, Y)Z, W \rangle - \kappa \langle (X, Z) \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle \rangle = \langle SX, Z \rangle \langle SY, W \rangle - \langle SY, Z \rangle \langle SX, W \rangle,$$

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = R(X, Y)N,$$

where $\kappa$ is the sectional curvature of $\bar{V}$, i.e., $\kappa = 1, 0, -1$ for $S^{n+1}$, $\mathbb{R}^{n+1}$ and $H^{n+1}$ respectively. Thus the Gauss and Codazzi equations only involve the first and second fundamental forms of $V$; they are defined intrinsically on $V$ (as soon as we know $S$). This comes from the fact that these ambient spaces are isotropic. Moreover, in this case the Gauss and Codazzi equations are also sufficient conditions...
for an \( n \)-dimensional simply connected manifold to be immersed into \( \bar{\mathcal{V}} \) with given first and second fundamental forms: if \( \mathcal{V} \) is a Riemannian manifold endowed with a field \( S_y : T_y \mathcal{V} \to T_y \mathcal{V} \) such that (1) and (2) hold (where \( R \) denotes the Riemann curvature tensor of \( \mathcal{V} \)), then there exists an isometric immersion from \( \mathcal{V} \) into \( \bar{\mathcal{V}} \) with \( S \) as the shape operator. The reader can refer to [Car92] and also to [Ten71] for a proof in the case of \( \mathbb{R}^{n+1} \).

In the case of a general manifold \( \bar{\mathcal{V}} \), the Gauss and Codazzi equations are not defined intrinsically on \( \mathcal{V} \), since the Riemann curvature tensor of the ambient space \( \bar{\mathcal{V}} \) is involved. Yet, in the case where \( \bar{\mathcal{V}} = \mathbb{S}^n \times \mathbb{R} \) or \( \bar{\mathcal{V}} = \mathbb{H}^n \times \mathbb{R} \), these equations are well defined as soon as we know:

1. the projection \( T \) of the vertical vector \( \frac{\partial}{\partial t} \) (corresponding to the factor \( \mathbb{R} \)) onto the tangent space of \( \mathcal{V} \),
2. the normal component \( \nu \) of \( \frac{\partial}{\partial t} \), i.e., \( \nu = \langle N, \frac{\partial}{\partial t} \rangle \).

Indeed, the Gauss and Codazzi equations become the following:

\[
R(X,Y)Z = \langle SX, Z \rangle SY - \langle SY, Z \rangle SX \\
+ \kappa(\langle X, Z \rangle Y - \langle Y, Z \rangle X - \langle Y, T \rangle \langle X, Z \rangle T \\
- \langle X, T \rangle \langle Z, T \rangle Y + \langle X, T \rangle \langle Y, Z \rangle T + \langle Y, T \rangle \langle X, Z \rangle T, X \rangle X + \langle X, T \rangle \langle Y, Z \rangle T + \langle Y, T \rangle \langle X, Z \rangle T, X \rangle X,
\]

\[
\nabla_X SY - \nabla_Y SX - S[X,Y] = \kappa \nu (\langle Y, T \rangle X - \langle X, T \rangle Y),
\]

where \( \kappa = 1 \) and \( \kappa = -1 \) for \( \mathbb{S}^n \times \mathbb{R} \) and \( \mathbb{H}^n \times \mathbb{R} \) respectively.

The Gauss equation can be formulated in the following equivalent way: the sectional curvature \( K(P) \) (for the metric of \( \mathcal{V} \)) of every plane \( P \subset \mathcal{V} \) satisfies

\[
K(P) = \det S_P + \kappa (1 - ||T_P||^2),
\]

where \( S_P \) is the restriction of \( S \) on \( P \) and \( T_P \) is the orthogonal projection of \( T \) on \( P \).

The first aim of this paper is to give a necessary and sufficient condition in order for a Riemannian manifold with a symmetric operator \( S \) to be isometrically immersed into \( \mathbb{S}^n \times \mathbb{R} \) or \( \mathbb{H}^n \times \mathbb{R} \) with \( S \) as shape operator. More precisely, we prove the following Theorem.

**Theorem** (Theorem 3.3). Let \( \mathcal{V} \) be a simply connected Riemannian manifold of dimension \( n \), \( ds^2 \) its metric (which we also denote by \( \langle \cdot, \cdot \rangle \)) and \( \nabla \) its Riemannian connection. Let \( S \) be a field of symmetric operators \( S_y : T_y \mathcal{V} \to T_y \mathcal{V} \), \( T \) a vector field on \( \mathcal{V} \) and \( \nu \) a smooth function on \( \mathcal{V} \) such that \( ||T||^2 + \nu^2 = 1 \).

Let \( \mathcal{M}^n = \mathbb{S}^n \) or \( \mathcal{M}^n = \mathbb{H}^n \). Assume that \( (ds^2, S, T, \nu) \) satisfies the Gauss and Codazzi equations for \( \mathcal{M}^n \times \mathbb{R} \) and the following equations:

\[
\nabla_X T = \nu S X, \quad d\nu(X) = -(SX, T).
\]

Then there exists an isometric immersion \( f : \mathcal{V} \to \mathcal{M}^n \times \mathbb{R} \) such that the shape operator with respect to the normal \( N \) associated to \( f \) is

\[
df \circ S \circ df^{-1}
\]

and such that

\[
\frac{\partial}{\partial t} = df(T) + \nu N.
\]

Moreover the immersion is unique up to a global isometry of \( \mathcal{M}^n \times \mathbb{R} \) preserving the orientations of both \( \mathcal{M}^n \) and \( \mathbb{R} \).


The two additional conditions come from the fact that the vertical vector field \( \frac{\partial}{\partial t} \) is parallel.

The method to prove this theorem is similar to that of Tenenblat (\cite{Ten71}): it is based on differential forms, moving frames and integrable distributions.

This work was motivated by the study of minimal surfaces in \( S^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \). There were many recent developments in the theory of these surfaces. Rosenberg (\cite{Ros02}) studied minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) and proved a Jenkins-Serrin theorem. Hauswirth (\cite{Hau06}) constructed many examples in \( \mathbb{H}^2 \times \mathbb{R} \). Meeks and Rosenberg (\cite{MR05}) initiated the theory of minimal surfaces in \( M \times \mathbb{R} \) where \( M \) is a compact surface. Recently, Abresch and Rosenberg (\cite{AR04}) extended the notion of a holomorphic Hopf differential to constant mean curvature surfaces in \( S^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \); using this holomorphic differential, they proved that all immersed constant mean curvature spheres are embedded and rotational.

In this paper, we use our Theorem 3.3 to prove the existence of a one-parameter family of isometric minimal deformations of a given minimal surface in \( S^2 \times \mathbb{R} \) or \( \mathbb{H}^2 \times \mathbb{R} \). This family is obtained by rotating the shape operator; hence it is the analog of the associate family of a minimal surface in \( \mathbb{R}^3 \). This is the following theorem.

**Theorem (Theorem 3.3).** Let \( \Sigma \) be a simply connected Riemann surface and \( x : \Sigma \to M^2 \times \mathbb{R} \) a conformal minimal immersion. Let \( N \) be the induced normal. Let \( S \) be the symmetric operator on \( \Sigma \) induced by the shape operator of \( x(\Sigma) \). Let \( T \) be the vector field on \( \Sigma \) such that \( dx(T) \) is the projection of \( \frac{\partial}{\partial t} \) onto \( T(x(\Sigma)) \). Let \( \nu = \langle N, \frac{\partial}{\partial t} \rangle \).

Let \( z_0 \in \Sigma \). Then there exists a unique family \( (x_\theta)_{\theta \in \mathbb{R}} \) of conformal minimal immersions \( x_\theta : \Sigma \to M^2 \times \mathbb{R} \) such that:

1. \( x_\theta(z_0) = x(z_0) \) and \( (dx_\theta)_{z_0} = (dx)_{z_0} \),
2. the metrics induced on \( \Sigma \) by \( x \) and \( x_\theta \) are the same,
3. the symmetric operator on \( \Sigma \) induced by the shape operator of \( x_\theta(\Sigma) \) is \( e^{\theta}S \),
4. \( \frac{\partial}{\partial t} = dx_\theta(e^{\theta}T) + \nu N_\theta \), where \( N_\theta \) is the unit normal to \( x_\theta \).

Moreover we have \( x_0 = x \), and the family \( (x_\theta) \) is continuous with respect to \( \theta \).

In particular taking \( \theta = \frac{\pi}{2} \) defines a conjugate surface; the geometric properties of conjugate surfaces in \( M^2 \times \mathbb{R} \) and in \( \mathbb{R}^3 \) are similar. Finally, we give examples of conjugate surfaces. In \( S^2 \times \mathbb{R} \), we show that helicoids and unduloids are conjugate. In \( \mathbb{H}^2 \times \mathbb{R} \), we show that helicoids are conjugated to catenoids or to minimal surfaces foliated by horizontal curves of constant curvature belonging to the Hauswirth family (see \cite{Hau06}).

### 2. Preliminaries

**Notation.** In this paper we will use the following index conventions: Latin letters \( i, j, \) etc., denote integers between 1 and \( n \), and Greek letters \( \alpha, \beta, \) etc., denote integers between 0 and \( n + 1 \). For example, the notation \( A_i^j = B_j^i \) means that this relation holds for all integers \( i, j \) between 1 and \( n \), and the notation \( \sum_{\alpha} C_{\alpha} \) means \( C_0 + C_1 + \cdots + C_{n+1} \).

The set of vector fields on a Riemannian manifold \( \mathcal{V} \) will be denoted by \( \mathfrak{X}(\mathcal{V}) \).
We denote by $\frac{\partial}{\partial t}$ the unit vector giving the orientation of $\mathbb{R}$ in $\mathbb{M}^n \times \mathbb{R}$; we call it the vertical vector.

2.1. The compatibility equations in $\mathbb{M}^n \times \mathbb{R}$. Let $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$; in the first case we set $\kappa = 1$ and in the second case we set $\kappa = -1$. Let $\bar{R}$ be the Riemann curvature tensor of $\mathbb{M}^n \times \mathbb{R}$. Let $\mathcal{V}$ be an oriented hypersurface of $\mathbb{M}^n \times \mathbb{R}$ and $N$ the unit normal to $\mathcal{V}$.

**Proposition 2.1.** For $X, Y, Z, W \in \mathfrak{x}(\mathcal{V})$ we have
\[
\langle \bar{R}(X, Y)Z, W \rangle = \kappa(\langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle - \langle Y, T \rangle \langle W, T \rangle \langle X, Z \rangle - \langle X, T \rangle \langle Z, T \rangle \langle Y, W \rangle + \langle X, T \rangle \langle W, T \rangle \langle Y, Z \rangle + \langle Y, T \rangle \langle Z, T \rangle \langle X, W \rangle),
\]
where
\[
\nu = \langle N, \frac{\partial}{\partial t} \rangle
\]
and $T$ is the projection of $\frac{\partial}{\partial t}$ on $\mathcal{V}$, i.e.,
\[
T = \frac{\partial}{\partial t} - \nu N.
\]

**Proof.** Any vector field on $\mathbb{M}^n \times \mathbb{R}$ can be written $X(m, t) = (X^t_m(m), X^m_t)$, where, for each $t \in \mathbb{R}$, $X^t_m$ is a vector field on $\mathbb{M}^n$ and, for each $m \in \mathbb{M}^n$, $X^m_t$ is a vector field on $\mathbb{R}$. Then for $X, Y, Z, W \in \mathfrak{x}(\mathbb{M}^n \times \mathbb{R})$ we have
\[
\langle \bar{R}(X, Y)Z, W \rangle = \kappa(\langle X^t_m, Z^t_n \rangle \langle Y^m_t, W^m_n \rangle - \langle X^m_t, Z^m_n \rangle \langle Y^t_m, W^t_n \rangle).
\]
We have $X^t_m = X - \langle X, \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t}$, thus, if $X \in TV$, we have $X^t_m = X - \langle X, T \rangle \frac{\partial}{\partial t}$, and similar expressions for $Y, Z, W \in TV$. A computation gives the expected formula for $\langle \bar{R}(X, Y)Z, W \rangle$.

Finally we have $N^t_m = N - \nu \frac{\partial}{\partial t}$, so a computation gives the expected formula for $\langle \bar{R}(X, Y)N, Z \rangle$.

Using the fact that the vector field $\frac{\partial}{\partial t}$ is parallel, we obtain the following equations.

**Proposition 2.2.** For $X \in \mathfrak{x}(\mathcal{V})$ we have
\[
\nabla_X T = \nu SX, \quad d\nu(X) = -\langle SX, T \rangle.
\]

**Proof.** We have $\frac{\partial}{\partial t} = T + \nu N$ and $\nabla_X \frac{\partial}{\partial t} = 0$. Thus we get
\[
0 = \nabla_X T + (d\nu(X))N + \nu \nabla_X N = \nabla_X T + \langle SX, T \rangle N + (d\nu(X))N - \nu SX.
\]
Taking the tangential and the normal components in this equality, we obtain the expected formulas.

**Remark 2.3.** In the case of an orthonormal pair $(X, Y)$ we get
\[
\langle \bar{R}(X, Y)X, Y \rangle = \kappa(1 - \langle X, T \rangle^2 - \langle X, T \rangle^2).
\]
The reader can also refer to section 3.2 in [AR04].
2.2. Moving frames. In this section we introduce some material about the technique of moving frames. The reader can also refer to [Ros02a].

Let \( V \) be a Riemannian manifold of dimension \( n \), \( \nabla \) its Levi-Civita connection, and \( R \) the Riemannian curvature tensor. Let \( S \) be a field of symmetric operators \( S_y : T_y V \to T_y V \). Let \((e_1, \ldots, e_n)\) be a local orthonormal frame on \( V \) and \((\omega^1, \ldots, \omega^n)\) the dual basis of \((e_1, \ldots, e_n)\), i.e.,

\[
\omega^i(e_k) = \delta^i_k.
\]

We also set

\[
\omega^{n+1} = 0.
\]

We define the forms \( \omega^1, \omega^1_{n+1}, \omega^1_{n+1} \) and \( \omega^{n+1}_{n+1} \) on \( V \) by

\[
\omega^j(e_k) = \langle \nabla_{e_k} e_j, e_i \rangle, \quad \omega^j_{n+1}(e_k) = \langle Se_k, e_j \rangle, \quad \omega^j_{n+1} = -\omega^j_{n+1}, \quad \omega^{n+1}_{n+1} = 0.
\]

Then we have

\[
\nabla_{e_k} e_j = \sum_i \omega^i_j(e_k)e_i, \quad Se_k = \sum_j \omega^j_{n+1}(e_k)e_j.
\]

Finally we set \( R^i_{klj} = \langle R(e_k, e_l)e_j, e_i \rangle \).

**Proposition 2.4.** We have the following formulas:

(3) \[
\frac{d}{dt} \omega^i + \sum_p \omega^i_p \wedge \omega^p = 0,
\]

(4) \[
\sum_p \omega^i_{n+1} \wedge \omega^p = 0,
\]

(5) \[
\frac{d}{dt} \omega^i_j + \sum_p \omega^i_p \wedge \omega^p_j = -\frac{1}{2} \sum_k \sum_l R^i_{klj} \omega^j \wedge \omega^j,
\]

(6) \[
\frac{d}{dt} \omega^{n+1}_j + \sum_p \omega^{n+1}_p \wedge \omega^j_p = \frac{1}{2} \sum_k \sum_l (\nabla_{e_k} Se_l - \nabla_{e_l} Se_k - S[e_k, e_l], e_j) \omega^j \wedge \omega^j.
\]

**Proof.** These are well known formulas. However, since our conventions slightly differ from those of [Ten71] and [Ros02a], we give a proof for sake of clarity.

We have \( \frac{d}{dt}(e_p, e_q) = -\omega^i(e_p, e_q) = -\omega^l(e_p, e_q) = -\omega^l(e_p, e_q) + \omega^l(e_p) \) and \( \sum_k \omega^k \wedge \omega^k(e_p, e_q) = \omega^k(e_p) - \omega^k(e_q) \), so (3) is proved. Also, we have \( \sum_k (\omega^{n+1}_k \wedge \omega^k(e_p, e_q) = \omega^{n+1}_k(e_p) - \omega^{n+1}_k(e_q)) = \langle Se_p, e_q \rangle - \langle Se_q, e_p \rangle = 0 \), so (4) is proved.

We have \( \omega^j = \sum_k \langle e_i, \nabla_{e_k} e_j \rangle \omega^k \), so

\[
\frac{d}{dt} \omega^j = \sum_k \sum_l \langle e_i, \nabla_{e_k} e_j \rangle \omega^l \wedge \omega^k + \sum_k \langle e_i, \nabla_{e_k} e_j \rangle d\omega^k,
\]

\[
= \sum_k \sum_l (\langle \nabla_{e_k} e_i, \nabla_{e_k} e_j \rangle + \langle e_i, \nabla_{e_k} e_j \rangle) \omega^l \wedge \omega^k - \sum_k \sum_l (e_i, \nabla_{e_k} e_j) \omega^l_k \wedge \omega^l.
\]
Moreover we have
\[ \sum_k \sum_l \langle e_i, \nabla e_k e_j \rangle \omega^k_l \wedge \omega^l_j = \sum_k \sum_l \sum_q \langle e_i, \nabla e_q e_l \rangle \langle e_k, \nabla e_q e_l \rangle \omega^q_l \wedge \omega^l_j \]
\[ = \sum_k \sum_l \langle e_i, \nabla e_q e_l \rangle \omega^q_l \wedge \omega^l_j. \]

On the other hand we have
\[ \sum_p \omega^j_p \wedge \omega^l_j = \sum_k \sum_l \sum_p \langle e_i, \nabla e_k e_j \rangle \langle e_p, \nabla e_k e_j \rangle \omega^k_l \wedge \omega^i_j \]
\[ = - \sum_k \sum_l \sum_p \langle \nabla e_k e_j, e_p \rangle \langle e_p, \nabla e_k e_j \rangle \omega^l_j \wedge \omega^k_l \]
\[ = - \sum_k \sum_l \langle \nabla e_k e_j, \nabla e_k e_j \rangle \omega^l_j \wedge \omega^k_l. \]

Thus we conclude that
\[ d\omega^l_j + \sum_p \omega^i_p \wedge \omega^l_p = \sum_k \sum_l \langle e_i, \nabla e_k e_j \rangle \omega^i_l \wedge \omega^k_j. \]

Adding this equality with itself after exchanging \(k\) and \(l\) and using the fact that \(\omega^k \wedge \omega^l = -\omega^l \wedge \omega^k\), we get
\[ 2 \left( d\omega^l_j + \sum_p \omega^i_p \wedge \omega^l_p \right) = \sum_k \sum_l \langle e_i, R(e_k, e_l) e_j \rangle \omega^i_l \wedge \omega^k_j, \]

and finally we get (5).

We have \(\omega^{n+1}_j = \sum_k \langle Se_k, e_j \rangle \omega^k_j\), so
\[ d\omega^{n+1}_j = \sum_k \sum_l e_l \langle Se_k, e_j \rangle \omega^l_j \wedge \omega^k_j + \sum_k \langle Se_k, e_j \rangle d\omega^k_j \]
\[ = \sum_k \sum_l ((\nabla e_l Se_k, e_j) + \langle Se_k, \nabla e_l e_j \rangle) \omega^l_j \wedge \omega^k_j - \sum_k \sum_l \langle Se_k, e_j \rangle \omega^l_k \wedge \omega^i_j. \]

Moreover we have
\[ \sum_k \sum_l \langle Se_k, e_j \rangle \omega^k_l \wedge \omega^l_j = \sum_k \sum_l \sum_q \langle Se_k, e_j \rangle \langle e_k, \nabla e_q e_l \rangle \omega^q_l \wedge \omega^l_j \]
\[ = \sum_l \sum_q \langle Se_j, \nabla e_q e_l \rangle \omega^q_l \wedge \omega^l_j. \]

On the other hand we have
\[ \sum_p \omega^p_n \wedge \omega^l_p = \sum_k \sum_p \langle Se_k, e_p \rangle \omega^k_p \wedge \omega^l_p \]
\[ = \sum_k \sum_p \langle Se_k, e_p \rangle \langle e_p, \nabla e_l e_j \rangle \omega^k \wedge \omega^l \]
\[ = \sum_k \sum_l \langle Se_k, \nabla e_l e_j \rangle \omega^k \wedge \omega^l. \]
Thus we conclude that
\[
\begin{align*}
d\omega_{j+1} + \sum_p \omega_p^{n+1} \wedge \omega_j &= \sum_k \sum_l ((\nabla_{e_l} S e_k, e_j) - \langle S e_j, \nabla_{e_l} e_k \rangle) \omega^l \wedge \omega^k \\
&= \sum_k \sum_l (e_j, \nabla_{e_l} S e_k - S \nabla_{e_l} e_k) \omega^l \wedge \omega^k.
\end{align*}
\]

Adding this equality with itself after exchanging \(k\) and \(l\) and using the fact that \(\omega^k \wedge \omega^l = -\omega^l \wedge \omega^k\), we get
\[
2 \left( d\omega_{j+1} + \sum_p \omega_p^{n+1} \wedge \omega_j \right) = \sum_k \sum_l \langle e_j, \nabla_{e_l} S e_k - S \nabla_{e_l} e_k \rangle \omega^l \wedge \omega^k,
\]
and finally we get \(\square\).

2.3. Some facts about hypersurfaces of \(\mathbb{S}^n \times \mathbb{R}\) and \(\mathbb{H}^n \times \mathbb{R}\). In this section we consider an orientable hypersurface \(V\) of \(M^n \times \mathbb{R}\) with \(M^n = \mathbb{S}^n\) or \(M^n = \mathbb{H}^n\).

We denote by \(L^p\) the \(p\)-dimensional Lorentz space, i.e., \(\mathbb{R}^p\) endowed with the quadratic form
\[
-(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{p-1})^2.
\]

We will use the following inclusions: we have
\[
\mathbb{S}^n = \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1}; (x^0)^2 + \sum_i (x^i)^2 = 1\},
\]
and so
\[
\mathbb{S}^n \times \mathbb{R} \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2},
\]
and we have
\[
\mathbb{H}^n = \{(x^0, \ldots, x^n) \in \mathbb{L}^{n+1}; -(x^0)^2 + \sum_i (x^i)^2 = -1, x^0 > 0\},
\]
and so
\[
\mathbb{H}^n \times \mathbb{R} \subset \mathbb{L}^{n+1} \times \mathbb{R} = \mathbb{L}^{n+2}.
\]

In the case of \(\mathbb{S}^n \times \mathbb{R}\) we set \(\kappa = 1\) and \(\mathbb{E}^{n+2} = \mathbb{R}^{n+2}\). In the case of \(\mathbb{H}^n \times \mathbb{R}\) we set \(\kappa = -1\) and \(\mathbb{E}^{n+2} = \mathbb{L}^{n+2}\).

We denote by \(\nabla\), \(\overline{\nabla}\) and \(\overline{\nabla}\) the connections of \(V\), \(M^n \times \mathbb{R}\) and \(\mathbb{E}^{n+2}\) respectively, by \(\bar{N}(x)\) the normal to \(M^n \times \mathbb{R}\) in \(\mathbb{E}^{n+2}\) at a point \(x \in M^n \times \mathbb{R}\), i.e.,
\[
\bar{N}(x) = (x^0, \ldots, x^n, 0),
\]
and by \(N(x)\) the normal to \(V\) in \(M^n \times \mathbb{R}\) at a point \(x \in V\). We denote by \(S\) the shape operator of \(V\) in \(M^n \times \mathbb{R}\). The shape operator of \(M^n \times \mathbb{R}\) is \(S X = -\kappa d\bar{N}(X) = \kappa (-X + \langle X, \frac{\partial}{\partial x^0} \rangle \frac{\partial}{\partial x^0})\). We should be careful with the sign convention in the definition of the shape operator: here we have chosen
\[
\bar{\nabla}_XY = \nabla_X Y + \langle \bar{S}X, Y \rangle \bar{N},
\]
i.e.,
\[
\langle \bar{S}X, Y \rangle = \kappa \langle \bar{\nabla}_XY, \bar{N} \rangle,
\]
because in the case of \(\mathbb{S}^n \times \mathbb{R}\) we have \(\langle \bar{N}, \bar{N} \rangle = 1\), whereas in the case of \(\mathbb{H}^n \times \mathbb{R}\) we have \(\langle \bar{N}, \bar{N} \rangle = -1\).
Let \((e_1, \ldots, e_n)\) be a local orthonormal frame on \(V\), \(e_{n+1} = N\) and \(e_0 = \tilde{N}\) (on \(V\)). We define the forms \(\omega^i, \omega^{n+1}, \omega^i_{n+1}\) and \(\omega^{n+1}_{n+1}\) as in Section [2.2]. Moreover we set

\[
\omega^i_0(e_k) = \langle \tilde{S}e_k, e_\gamma \rangle = -\kappa \langle e_k, e_\gamma \rangle + \kappa \left( e_k, \frac{\partial}{\partial t} \right) \left( e_\gamma, \frac{\partial}{\partial t} \right),
\]

\[
\omega^i_{n+1} = -\kappa \omega^i_0.
\]

With these definitions we have

\[
\mathring{\nabla}_{e_k}e_\beta = \sum_\alpha \omega^\alpha_\beta(e_k)e_\alpha.
\]

Let \((E_0, \ldots, E_{n+1})\) be the canonical frame of \(\mathbb{E}^{n+2}\) (with \(\langle E_0, E_0 \rangle = \kappa\) and \(E_{n+1} = \frac{\partial}{\partial t}\)). Let \(A \in M_{n+2}(\mathbb{R})\) be the matrix (the indices going from 0 to \(n+1\)) whose columns are the coordinates of the \(e_\beta\) in the frame \((E_\alpha)\), i.e.,

\[
e_\beta = \sum_\alpha A^\alpha_\beta E_\alpha.
\]

Then, on the one hand we have

\[
\mathring{\nabla}_{e_k}e_\beta = \sum_\alpha dA^\alpha_\beta(e_k)E_\alpha,
\]

and on the other hand we have

\[
\mathring{\nabla}_{e_k}e_\beta = \sum_\alpha \sum_\gamma \omega^\gamma_\beta(e_k)A^\gamma_\alpha E_\alpha.
\]

Thus we have

\[
A^{-1}dA = \Omega
\]

with \(\Omega = (\omega^\alpha_\beta) \in M_{n+2}(\mathbb{R})\), the indices going from 0 to \(n+1\).

Setting \(G = \text{diag}(\kappa, 1, \ldots, 1) \in M_{n+2}(\mathbb{R})\), we have

\[
A \in \text{SO}^+(\mathbb{E}^{n+2}), \quad \Omega \in \mathfrak{so}(\mathbb{E}^{n+2}),
\]

where \(\text{SO}^+(\mathbb{E}^{n+2})\) is the connected component of \(I_{n+2}\) in

\[
\text{SO}(\mathbb{E}^{n+2}) = \{ Z \in M_{n+2}(\mathbb{R}) ; \ Z^T G Z = G, \ \det Z = 1 \}
\]

and where

\[
\mathfrak{so}(\mathbb{E}^{n+2}) = \{ H \in M_{n+2}(\mathbb{R}) ; \ H^T G + G H = 0 \}.
\]

In the case of \(S^n \times \mathbb{R}\) we have \(\text{SO}^+(\mathbb{E}^{n+2}) = \text{SO}(\mathbb{R}^{n+2})\).

### 3. ISOMETRIC IMMERSIONS INTO \(S^n \times \mathbb{R}\) AND \(H^n \times \mathbb{R}\)

#### 3.1. The compatibility equations

We consider a simply connected Riemannian manifold \(V\) of dimension \(n\). Let \(ds^2\) be the metric on \(V\) (we will also denote it by \(\langle \cdot, \cdot \rangle\)), \(\nabla\) the Riemannian connection of \(V\) and \(R\) its Riemann curvature tensor. Let \(S\) be a field of symmetric operators \(Sonds : T_yV \to T_yV\), \(T\) a vector field on \(V\) such that \(\|T\| \leq 1\) and \(\nu\) a smooth function on \(V\) such that \(\nu^2 \leq 1\).

The compatibility equations for hypersurfaces in \(S^n \times \mathbb{R}\) and \(H^n \times \mathbb{R}\) established in Section [2.1] suggest we introduce the following definition.
**Definition 3.1.** We say that \((ds^2, S, T, \nu)\) satisfies the compatibility equations respectively for \(S^n \times \mathbb{R}\) and \(H^n \times \mathbb{R}\) if

\[
||T||^2 + \nu^2 = 1
\]

and, for all \(X, Y, Z \in \mathfrak{X}(\mathcal{V})\),

\[
R(X, Y)Z = (SX, Z)SY - (SY, Z)SX + \kappa(\langle X, Z \rangle Y - \langle Y, Z \rangle X - \langle X, Z \rangle T)
\]

\[
-\langle X, T \rangle \langle Z, T \rangle Y + \langle X, T \rangle \langle Y, Z \rangle T + \langle Y, T \rangle \langle Z, T \rangle X,
\]

\[
\nabla_X SY - \nabla_Y SX - S[X, Y] = \kappa \nu \langle Y, T \rangle X - \langle X, T \rangle Y,
\]

\[
\nabla_X T = \nu SX,
\]

\[
d\nu(X) = -\langle SX, T \rangle,
\]

where \(\kappa = 1\) and \(\kappa = -1\) for \(S^n \times \mathbb{R}\) and \(H^n \times \mathbb{R}\) respectively.

**Remark 3.2.** We notice that (9) implies (10) except when \(\nu = 0\) (by differentiating the identity \(\langle T, T \rangle + \nu^2 = 1\) with respect to \(X\)).

3.2. **Codimension 1 isometric immersions into \(S^n \times \mathbb{R}\) and \(H^n \times \mathbb{R}\).** In this section we will prove the following theorem.

**Theorem 3.3.** Let \(\mathcal{V}\) be a simply connected Riemannian manifold of dimension \(n\), \(ds^2\) its metric and \(\nabla\) its Riemannian connection. Let \(S\) be a field of symmetric operators \(S_g : T_g \mathcal{V} \to T_g \mathcal{V}\), \(T\) a vector field on \(\mathcal{V}\) and \(\nu\) a smooth function on \(\mathcal{V}\) such that \(||T||^2 + \nu^2 = 1\).

Let \(\mathbb{M}^n = S^n\) or \(\mathbb{M}^n = H^n\). Assume that \((ds^2, S, T, \nu)\) satisfies the compatibility equations for \(\mathbb{M}^n \times \mathbb{R}\). Then there exists an isometric immersion \(f : \mathcal{V} \to \mathbb{M}^n \times \mathbb{R}\) such that the shape operator with respect to the normal \(N\) associated to \(f\) is

\[
df \circ S \circ df^{-1}
\]

and such that

\[
\frac{\partial}{\partial t} = df(T) + \nu N.
\]

Moreover the immersion is unique up to a global isometry of \(\mathbb{M}^n \times \mathbb{R}\) preserving the orientations of both \(\mathbb{M}^n\) and \(\mathbb{R}\).

To prove this theorem, we consider a local orthonormal frame \((e_1, \ldots, e_n)\) on \(\mathcal{V}\) and the forms \(\omega^1, \omega^{n+1}, \omega^j, \omega_j^{n+1}, \omega_{n+1}^j\) and \(\omega_{n+1}^{n+1}\) as in Section 2.2. We set \(E^{n+2} = \mathbb{R}^{n+2}\) or \(E^{n+2} = L^{n+2}\) (according to \(\mathbb{M}^n\)). We denote by \((E_0, \ldots, E_{n+1})\) the canonical frame of \(E^{n+2}\) (with \(\langle E_0, E_n \rangle = -1\) in the case of \(L^{n+2}\)); in particular we have \(E_{n+1} = \frac{\partial}{\partial t}\). We set

\[
T^k = \langle T, e_k \rangle, \quad T^{n+1} = \nu, \quad T^0 = 0.
\]

Moreover we set

\[
\omega_j^0(e_k) = \kappa(T^j T^k - S_j^k), \quad \omega_{n+1}^0(e_k) = \kappa \nu T^k,
\]

\[
\omega_0^j = -\kappa \omega_j^0, \quad \omega_0^{n+1} = -\kappa \omega_{n+1}^0, \quad \omega_0^0 = 0.
\]

We define the one-form \(\eta\) on \(\mathcal{V}\) by

\[
\eta(X) = \langle T, X \rangle.
\]
In the frame \((e_1, \ldots, e_n)\) we have \(\eta = \sum_k T^k \omega^k\). Finally we define the following matrix of one-forms:

\[ \Omega = (\omega_j^i) \in \mathcal{M}_{n+2}(\mathbb{R}), \]
the indices going from 0 to \(n+1\).

From now on we assume that the hypotheses of Theorem 3.3 are satisfied. We first prove some technical lemmas that are consequences of the compatibility equations.

**Lemma 3.4.** We have

\[ d\eta = 0. \]

**Proof.** We have

\[
d\eta(X,Y) = X \cdot \eta(Y) - Y \cdot \eta(X) - \eta([X,Y])
= \langle \nabla_X T, Y \rangle - \langle \nabla_Y T, X \rangle
= \langle \nu S X, Y \rangle - \langle \nu S Y, X \rangle
= 0,
\]
where we have used condition (9).

**Lemma 3.5.** We have

\[ dT^\alpha = \sum_\gamma T^\gamma \omega_\alpha^\gamma. \]

**Proof.** This is a consequence of condition (9) for \(\alpha = j\), of condition (10) for \(\alpha = n+1\), and of the definitions for \(\alpha = 0\).

**Lemma 3.6.** We have

\[ d\Omega + \Omega \wedge \Omega = 0. \]

**Proof.** We set \(\Psi = d\Omega + \Omega \wedge \Omega\) and \(R_{klj}^i = \langle R(e_k, e_l)e_j, e_i \rangle\).

By Proposition 2.4 we have

\[ \Psi_j^n = -\frac{1}{2} \sum_k \sum_l R^k_{klj} \omega^k \wedge \omega^l + \omega_{n+1}^i \wedge \omega_j^{n+1} + \omega_0^i \wedge \omega_j^0. \]

Since the Gauss equation (9) is satisfied, we have

\[ R_{klj}^i = \bar{R}_{klj}^i + \omega_{n+1}^i \wedge \omega_j^{n+1}(e_k, e_l). \]

with

\[ \bar{R}_{klj}^i = \kappa(\delta_j^k \delta_i^l - \delta_j^l \delta_i^k - T^i T^j \delta_k^l - T^k T^j \delta_i^l + T^k T^i \delta_j^l + T^l T^i \delta_j^k). \]

On the other hand, a computation shows that \(\omega_0^i \wedge \omega_j^0(e_k, e_l) = \bar{R}_{klj}^i\). Thus we have

\[ R_{klj}^i = \omega_{n+1}^i \wedge \omega_j^{n+1}(e_k, e_l) + \omega_0^i \wedge \omega_j^0(e_k, e_l). \]

We conclude that \(\Psi_j^n = 0\).

By Proposition 2.4 we have

\[ \Psi_j^{n+1} = \frac{1}{2} \sum_k \sum_l \langle \nabla_{e_k} S e_l - \nabla_{e_l} S e_k - S[e_k, e_l], e_j \rangle \omega^k \wedge \omega^l + \omega_0^{n+1} \wedge \omega_j^0. \]

Since the Codazzi equation (8) is satisfied, we have

\[ \langle \nabla_{e_k} S e_l - \nabla_{e_l} S e_k - S[e_k, e_l], e_j \rangle = \kappa(T^l T^{n+1} \delta_j^k - T^k T^{n+1} \delta_j^l). \]

On the other hand, a computation shows that

\[ \omega_0^{n+1} \wedge \omega_j^0(e_k, e_l) = \kappa(T^k T^{n+1} \delta_j^l - T^l T^{n+1} \delta_j^k). \]

We conclude that \(\Psi_j^{n+1} = 0\).
We have $\omega_0^j = \kappa(T^j(\eta - \omega^j))$. Since $d\eta = 0$ (by Lemma 2.3) we get
\[
d\omega_0^j = \kappa(dT^j \wedge \eta - d\omega^j) = \kappa dT^j \wedge \eta + \kappa \sum_k \omega_k^j \wedge \omega^k
\]
by Proposition 2.4. Thus by a straightforward computation we get
\[
\Psi_j^0(e_p, e_q) = \kappa(dT^j(e_p))\eta(e_q) - dT^j(e_q)\eta(e_p) + \omega_j^0(e_p, e_q)
= \kappa(dT^j(e_p))\eta(e_q) - dT^j(e_q)\eta(e_p) + \omega_j^0(e_p, e_q)
= \kappa(T^p \sum_k T^k \omega_j^k(e_q) - T^q \sum_k T^k \omega_j^k(e_q) - \omega_j^p(e_q) + \omega_j^q(e_q))
+ \kappa(T^p T^{n+1} \omega_j^{n+1}(e_q) - T^q T^{n+1} \omega_j^{n+1}(e_p)).
\]
Using the definition of $\eta$ and Lemma 3.5 for $\alpha = j$, we conclude that $\hat{\Psi}_j^{n+2} = 0$.

We have $\omega_0^{n+1} = \kappa T^{n+1} \eta$, and so $d\omega_0^{n+1} = \kappa dT^{n+1} \wedge \eta$ by Lemma 2.3. Thus by a straightforward computation we get
\[
\Psi_{n+1}^0(e_p, e_q) = \kappa(T^p dT^{n+1}(e_p) - T^p dT^{n+1}(e_q))
= \kappa(T^p \sum_k T^k \omega_{n+1}(e_q) - T^q \sum_k T^k \omega_{n+1}(e_q) - \omega_{n+1}^p(e_q) + \omega_{n+1}^q(e_q))
+ \kappa(-\omega_{n+1}^p(e_q) + \omega_{n+1}^q(e_q)).
\]
The last two terms cancel because $S$ is symmetric. Using Lemma 3.5 for $\alpha = n + 1$, we conclude that $\Psi_{n+1}^0 = 0$.

The fact that $\Psi_{n+2}^0 = 0$ and $\Psi_{n+1}^{n+1} = 0$ is clear. We conclude by noticing that $\Psi_{n+1}^n = -\Psi_{n+1}^{n+1} = 0$.

For $y \in \mathring{V}$, let $z(y)$ be the set of matrices $Z \in SO^+(\mathbb{E}^{n+2})$ such that the coefficients of the last line of $Z$ are the $T^\beta(y)$. It is a manifold of dimension $\frac{n(n+1)}{2}$ (since the map $F : SO^+(\mathbb{E}^{n+2}) \to S(\mathbb{E}^{n+2}), Z \mapsto (Z^\beta_{\beta})_\beta$ (i.e., $F(Z)$ is the last line of $Z$), where $S(\mathbb{E}^{n+2}) = \{x \in \mathbb{E}^{n+2}; \langle E, E \rangle = 1\}$ is a submersion).

We now prove the following proposition.

**Proposition 3.7.** Assume that the compatibility equations for $M^n \times \mathbb{R}$ are satisfied. Let $y_0 \in \mathring{V}$ and $A_0 \in z(y_0)$. Then there exist a neighbourhood $U_1$ of $y_0$ in $\mathring{V}$ and a unique map $A : U_1 \to SO^+(\mathbb{E}^{n+2})$ such that
\[
A^{-1} dA = \Omega,
\forall y \in U_1, A(y) \in z(y),
A(y_0) = A_0.
\]

**Proof.** Let $U$ be a coordinate neighbourhood in $\mathring{V}$. The set
\[
\mathcal{T} = \{(y, Z) \in U \times SO^+(\mathbb{E}^{n+2}); Z \in z(y)\}
\]
is a manifold of dimension $n + \frac{n(n+1)}{2}$, and
\[
T_{(y, Z)} \mathcal{T} = \{(u, \zeta) \in T_y U \oplus T_Z SO^+(\mathbb{E}^{n+2}); \zeta_{\beta}^{n+1} = (dT^\beta)_y(u)\}.
\]
Indeed, in the neighbourhood of point of $U$ there exists a map $y \mapsto M(y) \in \text{SO}^+(\mathbb{E}^{n+2})$ such that the last line of $M(y)$ is $(T^\beta(y))_\beta$, and we have $Z \in \mathcal{Z}(y)$ if and only if

$$ZM(y)^{-1} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $B \in \text{SO}^+(\mathbb{E}^{n+1})$. Then, if $\varphi$ is a local parametrization of the set of such matrices, the map $(y, v) \mapsto (y, \varphi(v)M(y))$ is a local parametrization of $\mathcal{F}$.

Let $Z$ denote the projection $U \times \text{SO}^+(\mathbb{E}^{n+2}) \to \text{SO}^+(\mathbb{E}^{n+2}) \subset M_{n+2}(\mathbb{R})$. We consider on $\mathcal{F}$ the following matrix of 1-forms:

$$\Theta = Z^{-1}dZ - \Omega,$$

namely for $(y, Z) \in \mathcal{F}$ we have

$$\Theta_{(y,Z)} : T_{(y,Z)}\mathcal{F} \to M_{n+2}(\mathbb{R}),$$

$$\Theta_{(y,Z)}(u, \zeta) = Z^{-1}\zeta - \Omega_y(u).$$

We claim that, for each $(y, Z) \in \mathcal{F}$, the space

$$\mathcal{D}(y, Z) = \ker \Theta_{(y,Z)}$$

has dimension $n$. We first notice that the matrix $\Theta$ belongs to $\mathfrak{so}(\mathbb{E}^{n+2})$ since $\Omega$ and $Z^{-1}dZ$ do as well. Moreover we have

$$(Z\Theta)_{\beta}^{n+1} = dZ_{\beta}^{n+1} - \sum_\gamma Z_{\gamma}^{n+1}\omega_\beta^{\gamma} = d\gamma - \sum_\gamma \gamma^{\gamma}\omega_\beta^{\gamma} = 0$$

by Lemma 3.5. Thus the values of $\Theta_{(y,Z)}$ lie in the space

$$\mathcal{H} = \{ H \in \mathfrak{so}(\mathbb{E}^{n+2}); (ZH)^{n+1}_\beta = 0 \},$$

which has dimension $\frac{n(n+1)}{2}$ (indeed, the map $F : \text{SO}^+(\mathbb{E}^{n+2}) \to \mathbb{S}(\mathbb{E}^{n+2}), Z \mapsto (Z^\beta)^{n+1} \beta$ is a submersion, and we have $H \in \mathcal{H}$ if and only if $ZH \in \ker(dF)_Z$).

Moreover, the space $T_{(y,Z)}\mathcal{F}$ contains the subspace $\{(0, ZH); H \in \mathcal{H}\}$, and the restriction of $\Theta_{(y,Z)}$ on this subspace is the map $(0, ZH) \mapsto H$. Thus $\Theta_{(y,Z)}$ is onto $\mathcal{H}$, and consequently the linear map $\Theta_{(y,Z)}$ has rank $\frac{n(n+1)}{2}$. This finishes proving the claim.

We now prove that the distribution $\mathcal{D}$ is involutive. Using Lemma 3.6 we get

$$d\Theta = -Z^{-1}dZ \wedge Z^{-1}dZ - d\Omega$$

$$= -(\Theta + \Omega) \wedge (\Theta + \Omega) - d\Omega$$

$$= -\Theta \wedge \Theta - \Theta \wedge \Omega - \Omega \wedge \Theta.$$

From this formula we deduce that if $\xi_1, \xi_2 \in \mathcal{D}$, then $d\Theta(\xi_1, \xi_2) = 0$, and so $\Theta(\xi_1, \xi_2) = \xi_1 \cdot \Theta(\xi_2) - \xi_2 \cdot \Theta(\xi_1) - d\Theta(\xi_1, \xi_2) = 0$, i.e., $\xi_1, \xi_2 \in \mathcal{D}$. Thus the distribution $\mathcal{D}$ is involutive, and so, by the theorem of Frobenius, it is integrable.

Let $\mathcal{A}$ be the integral manifold through $(y_0, A_0)$. If $\zeta \in T_{A_0}\text{SO}^+(\mathbb{E}^{n+2})$ is such that $(0, \zeta) \in T_{(y_0, A_0)}\mathcal{A}$, then we have $0 = \Theta_{(y_0, A_0)}(0, \zeta) = A_0^{-1}\zeta$. This proves that

$$T_{(y_0, A_0)}\mathcal{A} \cap \{(0) \times T_{A_0}\text{SO}^+(\mathbb{E}^{n+2})\} = \{0\}.$$

Thus the manifold $\mathcal{A}$ is locally the graph of a function $A : U_1 \to \text{SO}^+(\mathbb{E}^{n+2})$, where $U_1$ is a neighbourhood of $y_0$ in $U$. By construction, this map satisfies the properties of Proposition 3.7 and is unique. \qed
Proof of Theorem 3.3. Let \( y_0 \in \mathcal{V} \), \( A \in \mathcal{Z}(y_0) \) and \( t_0 \in \mathbb{R} \). We consider on \( \mathcal{V} \) a local orthonormal frame \((e_1, \ldots, e_n)\) in the neighbourhood of \( y_0 \), and we keep the same notation. Then by Proposition 3.4 there exists a unique map \( A : U_1 \to \text{SO}^+(\mathbb{E}^{n+2}) \) such that
\[
A^{-1}dA = \Omega, \quad \forall y \in U_1, \quad A(y) \in Z(y),
\]
where \( U_1 \) is a neighbourhood of \( y_0 \), which we can assume is simply connected.

We set \( f^0 = A_0^0 \), \( f^i = A_0^i \) and we call \( f^{n+1} \) the unique function on \( U_1 \) such that \( df^{n+1} = \eta \) and \( f^{n+1}(y_0) = t_0 \) (this function exists since \( U_1 \) is simply connected and \( d\eta = 0 \)). Thus we defined a map \( f : U_1 \to \mathbb{E}^{n+2}. \) Since \( A_0^{n+1} = T^0 = 0 \) and \( A \in \text{SO}^+(\mathbb{E}^{n+2}), \) in the case of \( \mathbb{S}^n \times \mathbb{R} \) we have \((f^0)^2 + \sum_i (f^i)^2 = \sum \alpha (A_0^\alpha)^2 = 1 ,\) and in the case of \( \mathbb{H}^n \times \mathbb{R} \) we have \( -(f^0)^2 + \sum_i (f^i)^2 = -(A_0^0)^2 + \sum (A_0^\alpha)^2 + (A_0^{n+1})^2 = -1 \) and \( f^0 > A_0^0 > 0. \) Thus in both cases we have \((f^0, \ldots, f^n) \in \mathcal{M}^n, i.e., the values of \( f \) lie in \( \mathcal{M}^n \times \mathbb{R}. \)

Since \( dA = A\Omega, \) we have, for \( \alpha < n+1, \)
\[
df^\alpha(e_k) = \sum_j A_\alpha^j \omega_0^j(e_k) + A_\alpha^{n+1} \omega_0^{n+1}(e_k) = \sum_j A_\alpha^j (S_j - T^j T^k) - A_\alpha^{n+1} T^{n+1} T^k = A_\alpha^k - T^k \sum_\beta A_\beta^\alpha A_\beta^{n+1} = A_\alpha^k
\]
and
\[
df^{n+1}(e_k) = \eta(e_k) = T^k = A_\alpha^{n+1}.
\]
This means that \( \eta(e_k) \) is given by the column \( k \) of the matrix \( A. \)

Since \( A \) is an invertible matrix, \( df \) has rank \( n, \) and so \( f \) is an immersion. Also, since \( A \in \text{SO}^+(\mathbb{E}^{n+2}), \) we have \( \langle df(e_p), df(e_q) \rangle = \delta_{pq}, \) and so \( f \) is an isometry.

The columns of \( A(y) \) form a direct orthonormal frame of \( \mathbb{E}^{n+2}. \) Columns 1 to \( n \) form a direct orthonormal frame of \( T_f(y) \mathcal{V} \) and column 0 is the projection of \( f(y) \) on \( \mathbb{M}^n \times \{0\}, i.e., the unit normal \( N(f(y)) \) to \( \mathbb{M}^n \times \mathbb{R} \) at the point \( f(y). \) Thus column \( (n+1) \) is the unit normal \( N(f(y)) \) to \( f(\mathcal{V}) \) in \( \mathbb{M}^n \times \mathbb{R} \) at the point \( f(y). \)

We set \( X_j = df(e_j). \) Then we have
\[
\langle dX_j(X_k), N \rangle = \sum_\alpha dA_\alpha^0 (e_k) A_\alpha^{n+1} = \sum_\alpha A_\alpha^0 A_\alpha^{n+1} \omega_0^{n+1}(e_k) = \omega_j^{n+1}(e_k) = \langle S e_k, e_j \rangle.
\]
This means that the shape operator of \( f(\mathcal{V}) \) in \( \mathbb{M}^n \times \mathbb{R} \) is \( df \circ S \circ df^{-1}. \)

Finally, the coefficients of the vertical vector \( \frac{\partial}{\partial l} = E_{n+1} \) in the orthonormal frame \((\tilde{N}, X_1, \ldots, X_n, N)\) are given by the last line of \( A. \) Since \( A(y) \in \mathcal{Z}(y) \) for all \( y \in U_2 \) we get
\[
\frac{\partial}{\partial l} = \sum_j T^j X_j + T^{n+1} N = df(T) + \nu N.
\]
We now prove that the local immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$. Let $\hat{f} : U_3 \to \mathbb{M}^n \times \mathbb{R}$ be another immersion satisfying the conclusion of the theorem, where $U_3$ is a simply connected neighbourhood of $y_0$ included in $U_1$, let $\tilde{X}_\beta$ be the associated frame (i.e., $\tilde{X}_j = df(e_j)$, $\tilde{X}_{n+1}$ is the normal of $\hat{f}(V)$ in $\mathbb{M}^n \times \mathbb{R}$ and $\tilde{X}_0$ is the normal to $\mathbb{M}^n \times \mathbb{R}$ in $\mathbb{E}^{n+2}$) and let $\hat{A}$ be the matrix of the coordinates of the frame $(\tilde{X}_\beta)$ in the frame $(E_\alpha)$. Up to a direct isometry of $\mathbb{M}^n \times \mathbb{R}$, we can assume that $f(y_0) = \hat{f}(y_0)$ and that the frames $(X_\beta(y_0))$ and $(\tilde{X}_\beta(y_0))$ coincide, i.e., $A(y_0) = \hat{A}(y_0)$. We notice that this isometry necessarily fixes $\frac{\partial}{\partial \xi}$ since the $T^\alpha$ are the same for $x$ and $\hat{x}$. The matrices $A$ and $\hat{A}$ satisfy $A^{-1}dA = \Omega$ and $\hat{A}^{-1}d\hat{A} = \Omega$ (see Section 2.3), $A(y), \hat{A}(y) \in \mathcal{Z}(y)$ and $A(y_0) = \hat{A}(y_0)$. Thus by the uniqueness of the solution of the equation in Proposition 3.7, we get $A(y) = \hat{A}(y)$. Considering the columns 0 of these matrices, we get $f^i = \hat{f}^i$ and $f^0 = \hat{f}^0$. Finally we have $df^{n+1} = \eta = d\hat{f}^{n+1}$ and $f^{n+1}(y_0) = \hat{f}^{n+1}(y_0)$; thus we have $f^{n+1} = \hat{f}^{n+1}$. This finishes proving that $f = \hat{f}$ on $U_3$.

Finally we prove that this local immersion $f$ can be extended to $\mathcal{V}$ in a unique way. Let $y_1 \in \mathcal{V}$. Then there exists a curve $\Gamma : [0,1] \to \mathcal{V}$ such that $\Gamma(0) = y_0$ and $\Gamma(1) = y_1$. Each point of $\Gamma$ has a neighbourhood such that there exists an isometric immersion (unique up to an isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of $\mathbb{M}^n$ and $\mathbb{R}$) of this neighbourhood satisfying the properties of the theorem. From this family of neighbourhoods we can extract a finite family $(W_1, \ldots, W_p)$ covering $\Gamma$ with $W_1 = U_1$. Then the above uniqueness argument shows that we can extend successively the immersion $f$ to the $W_k$ in a unique way. In particular $f(y_1)$ is defined. Moreover, this value $f(y_1)$ does not depend on the choice of the curve $\Gamma$ joining $y_0$ to $y_1$ because $\mathcal{V}$ is simply connected. \hfill $\square$

**Proposition 3.8.** If $\left(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1\right)$ satisfies the compatibility equations and corresponds to an immersion $f : \Sigma \to \mathbb{M}^n \times \mathbb{R}$, then $\left(\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2, \mathcal{D}_2\right)$ and $\left(\mathcal{A}_3, \mathcal{B}_3, \mathcal{C}_3, \mathcal{D}_3\right)$ also satisfy the compatibility equations and correspond to the immersion $\mathcal{E}_0$ where $\mathcal{E}$ is an isometry of $\mathbb{M}^n \times \mathbb{R}$:

1. reversing the orientation of $\mathbb{M}^n$ and preserving the orientation of $\mathbb{R}$ in the case of $\left(\mathcal{A}_2, \mathcal{B}_2, \mathcal{C}_2, \mathcal{D}_2\right)$,
2. preserving the orientation of $\mathbb{M}^n$ and reversing the orientation of $\mathbb{R}$ in the case of $\left(\mathcal{A}_3, \mathcal{B}_3, \mathcal{C}_3, \mathcal{D}_3\right)$,
3. reversing the orientations of both $\mathbb{M}^n$ and $\mathbb{R}$ in the case of $\left(\mathcal{A}_3, \mathcal{B}_3, \mathcal{C}_3, \mathcal{D}_3\right)$.

**Proof.** We deal with the first case (the two others are similar). Let $\hat{f} = \sigma \circ f$. Then the normal to $\mathbb{M}^n \times \mathbb{R}$ is $\sigma \circ N$, and since $\sigma$ reverses the orientation of $\mathbb{M}^n \times \mathbb{R}$ the normal to $\hat{f}(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ is $\hat{N} = -\sigma \circ N$. From this we deduce that $\hat{S} = -S$. Moreover we have $\frac{\partial}{\partial t} = df(T) + \nu N$, and so, since $\sigma$ preserves the orientation of $\mathbb{R}$, we have

$$\frac{\partial}{\partial t} = \sigma \circ df(T) + \nu \sigma \circ N = df(T) - \nu \hat{N}.$$ 

We conclude that $\hat{T} = T$ and $\hat{\nu} = -\nu$. \hfill $\square$

### 3.3. Remark: Another proof in the case of $\mathbb{H}^n \times \mathbb{R}$.

In this section we outline another proof of Theorem 3.3 in the case of $\mathbb{H}^n \times \mathbb{R}$ that does not involve the Lorentz space. Greek letters will denote indices between 1 and $n + 1$.

We first consider an orientable hypersurface $\mathcal{V}$ of an $(n + 1)$-dimensional Riemannian manifold $\mathcal{V}$. Let $(e_1, \ldots, e_n)$ be a local orthonormal frame on $\mathcal{V}$, $e_{n+1}$ the
normal to \( \mathcal{V} \), and \((E_1, \ldots, E_{n+1})\) a local orthonormal frame on \( \mathcal{V} \). We denote by \( \nabla \) and \( \bar{\nabla} \) the Riemannian connections on \( \mathcal{V} \) and \( \mathcal{V} \) respectively, and by \( S \) the shape operator of \( \mathcal{V} \) (with respect to the normal \( e_{n+1} \)). We define the forms \( \omega^\alpha, \omega^\beta_\alpha \) on \( \mathcal{V} \) as in Section 2.2. Then we have

\[
\nabla e_\beta = \sum_\gamma \omega^\gamma_\beta (e_k) e_\gamma.
\]

Let \( A \in SO_{n+1}(\mathbb{R}) \) be the matrix whose columns are the coordinates of the \( e_\beta \) in the frame \((E_\alpha)\), namely \( A^\beta_\alpha = (e_\beta, E_\alpha) \). Let \( \Omega = (\omega^\alpha_\beta) \in \mathcal{M}_{n+1}(\mathbb{R}) \). The matrix \( A \) satisfies the following equation:

\[
A^{-1} dA = \Omega + L(A)
\]

with

\[
L(A)_{\beta}^\alpha = \sum_k \left( \sum_{\gamma, \delta, \epsilon} A^\gamma_k A^\delta_k \Gamma_{\gamma \delta \epsilon}^\beta \right) \omega^k = \sum_k \left( \sum_{\gamma, \delta, \epsilon} A^\gamma_k A^\delta_k \Gamma_{\gamma \delta \epsilon}^\beta \right) \omega^k,
\]

where the \( \Gamma_{\gamma \delta \epsilon}^\beta \) are the Christoffel symbols of the frame \((E_\alpha)\). Notice that these matrices have size \( n+1 \), whereas those of Section 2.3 have size \( n+2 \).

We now assume that \( \mathcal{V} = \mathbb{H}^n \times \mathbb{R} \) and that \( \mathcal{V} \) is a Riemannian manifold of dimension \( n \) endowed with \( S, T, \nu \) satisfying the compatibility equations for \( \mathbb{H}^n \times \mathbb{R} \). We consider a local orthonormal frame \((e_1, \ldots, e_n)\) on \( U \subset \mathcal{V} \), the associated one-forms \( \omega^\alpha, \omega^\alpha_\beta \) and the matrix of one-forms \( \Omega \in \mathcal{M}_{n+1}(\mathbb{R}) \).

We use the fact that there exists an orthonormal frame on \( \mathbb{H}^n \) whose Christoffel symbols are constant. More precisely, we can choose the frame \((E_\alpha)\) on \( \mathbb{H}^n \times \mathbb{R} \) such that \( \Gamma_{ij} = -\Gamma_{ji} = \frac{1}{\sqrt{n}} \) for \( i \neq j, i, j \leq n \) and all the other Christoffel symbols vanish.

The first step is to prove the following proposition, which is analogous to Proposition 3.7.

**Proposition 3.9.** Let \( y_0 \in \mathcal{V} \) and \( A_0 \in \mathcal{Z}(y_0) \). Then there exist a neighbourhood \( U_1 \) of \( y_0 \) in \( \mathcal{V} \) and a unique map \( A : U_1 \to SO_{n+1}(\mathbb{R}) \) such that

\[
A^{-1} dA = \Omega + L(A),
\]

\[
\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),
\]

\[
A(y_0) = A_0,
\]

where \( \mathcal{Z}(y) \) is defined in a way analogous to that of Section 2.2.

To prove this proposition, we introduce the form \( \Theta = Z^{-1} dZ - \Omega - L(Z) \) on \( \mathcal{F} = \{(y, Z) \in U \times SO_{n+1}(\mathbb{R}); Z \in \mathcal{Z}(y)\} \); this is well defined since the Christoffel symbols are constant. A long calculation shows that the distribution \( \mathcal{D}(y, Z) = \ker \Theta_{(y,Z)} \) is involutive. We conclude as in the proof of Proposition 3.7.

The second step is to prove the following proposition.

**Proposition 3.10.** Let \( x_0 \in \mathbb{H}^n \times \mathbb{R} \). There exist a neighbourhood \( U_2 \) of \( y_0 \) contained in \( U_1 \) and a function \( f : U_2 \to \mathbb{H}^n \times \mathbb{R} \) such that

\[
df = (B \circ f) A \omega,
\]

\[
f(y_0) = x_0,
\]

where \( B \) is the matrix whose columns are the coordinates of the \( e_\beta \) in the frame \((E_\alpha)\) on \( \mathcal{V} \).
where \( \omega \) is the column \((\omega^1, \ldots, \omega^n, 0)\) and, for \( x \in \mathbb{H}^n \times \mathbb{R} \), \( B(x) \in \mathcal{M}_{n+1}(\mathbb{R}) \) is the matrix of the coordinates of the frame \((E_\alpha(x))\) in the frame \((\frac{\partial}{\partial x^\alpha})\) (we choose the upper half-space model for \( \mathbb{H}^n \)).

To prove it, we consider the form \( B^{-1} dx - A\omega \) on \( U_1 \times \bar{V} \), and we show that its kernel again defines an involutive distribution.

The last step is to check that this map \( f \) satisfies the conclusions of Theorem 

4. Applications to minimal surfaces in \( \mathbb{M}^2 \times \mathbb{R} \)

4.1. The associate family. Let \( \mathbb{M}^2 = S^2 \) or \( \mathbb{M}^2 = \mathbb{H}^2 \). Let \( \Sigma \) be a Riemann surface with a metric \( ds^2 \) (which we also denote by \( \langle \cdot, \cdot \rangle \)), \( \nabla \) its Riemannian connection, and \( J \) the rotation of angle \( \frac{\pi}{2} \) on \( T\Sigma \). Let \( S \) be a field of symmetric operators \( S_y : T_y \Sigma \to T_y \Sigma \). Let \( T \) be a vector field on \( \Sigma \) and \( \nu \) a smooth function on \( \Sigma \) such that \(||T||^2 + \nu^2 = 1\).

**Proposition 4.1.** Assume that \( S \) is trace-free and that \((ds^2, S, T, \nu)\) satisfies the compatibility equations for \( \mathbb{M}^2 \times \mathbb{R} \). For \( \theta \in \mathbb{R} \) we set
\[
S_\theta = e^{\theta J}S = (\cos \theta)S + (\sin \theta)JS,
\]
\[
T_\theta = e^{\theta J}T = (\cos \theta)T + (\sin \theta)JT,
\]
i.e., \( S_\theta \) and \( T_\theta \) are obtained by rotating \( S \) and \( T \) by the angle \( \theta \).

Then \( S_\theta \) is symmetric and trace-free, \(||T_\theta||^2 + \nu^2 = 1\) and \((ds^2, S_\theta, T_\theta, \nu)\) satisfies the compatibility equations for \( \mathbb{M}^2 \times \mathbb{R} \).

**Proof.** The fact that \( S_\theta \) is symmetric and trace-free comes from an elementary computation. Moreover we have \(||T_\theta|| = ||T||\). We notice that, since \( \dim \Sigma = 2 \), the Gauss equation \( (\nabla X \cdot Y) \) is equivalent to
\[
K = \det S + \kappa(1 - ||T||^2),
\]
where \( K \) is the Gauss curvature of \( ds^2 \). Since \( \det(e^{\theta J}) = 1 \), we have \( \det S_\theta = \det S \), and so the Gauss equation is satisfied for \((ds^2, S_\theta, T_\theta, \nu)\).

Since \( e^{\theta J} \) commutes with \( \nabla X \) (see [AR04], section 3.2) and preserves the metric, equations \( (\nabla X \cdot Y) \) and \((\nabla T, \nu) = 0 \) are also satisfied for \((ds^2, S_\theta, T_\theta, \nu)\).

To prove that the Codazzi equation \( (\nabla X \cdot Y) \) is satisfied by \((ds^2, S_\theta, T_\theta, \nu)\), we first notice that, since
\[
\nabla_X e^{\theta J}S_Y - \nabla_Y e^{\theta J}S_X - e^{\theta J}S[X, Y] = e^{\theta J}(\nabla_X S_Y - \nabla_Y S_X - S[X, Y]),
\]
it suffices to prove that
\[
\langle e^{\theta J}T, Y \rangle X - \langle e^{\theta J}T, X \rangle Y = e^{\theta J}(\langle T, Y \rangle X - \langle T, X \rangle Y)
\]
This is obvious at a point where \( X = 0 \). At a point where \( X \neq 0 \), we can write \( Y = \lambda X + \mu JX \), and a computation shows that both expressions are equal to
\[
\mu \cos \theta (\langle T, X \rangle JX) X + \mu \sin \theta (\langle T, X \rangle JX) X - \mu \cos \theta (\langle T, JX \rangle JX) + \mu \sin \theta (\langle T, JX \rangle JX)JX.
\]

**Theorem 4.2.** Let \( \Sigma \) be a simply connected Riemann surface and \( x : \Sigma \to \mathbb{M}^2 \times \mathbb{R} \) a conformal minimal immersion. Let \( N \) be the induced normal. Let \( S \) be the symmetric operator on \( \Sigma \) induced by the shape operator of \( x(\Sigma) \). Also, let \( T \) be the vector field on \( \Sigma \) such that \( dx(T) \) is the projection of \( \frac{\partial}{\partial t} \) onto \( T(x(\Sigma)) \) and let \( \nu = \langle N, \frac{\partial}{\partial t} \rangle \).
Let \( z_0 \in \Sigma \). Then there exists a unique family \((x_\theta)_{\theta \in \mathbb{R}}\) of conformal minimal immersions \( x_\theta : \Sigma \to M^2 \times \mathbb{R} \) such that:

1. \( x_\theta(z_0) = x(z_0) \) and \((dx_\theta)_{z_0} = (dx)_{z_0}\),
2. the metrics induced on \( \Sigma \) by \( x \) and \( x_\theta \) are the same,
3. the symmetric operator on \( \Sigma \) induced by the shape operator of \( x_\theta(\Sigma) \) is \( e^{\theta J} S \),
4. \( \frac{\partial}{\partial z} = dx_\theta(e^{\theta J} T) + \nu N_\theta \), where \( N_\theta \) is the unit normal to \( x_\theta \).

Moreover we have \( x_0 = x \), and the family \((x_\theta)\) is continuous with respect to \( \theta \).

The family of immersions \((x_\theta)_{\theta \in \mathbb{R}}\) is called the associate family of the immersion \( x \), and the immersion \( x_{\bar{\theta}} \) is called the conjugate immersion of the immersion \( x \), and the immersion \( x_{-\theta} \) is called the opposite immersion of the immersion \( x \).

**Proof.** Let \( ds^2 \) be the metric on \( \Sigma \) induced by \( x \). Then \((ds^2, S, T, \nu)\) satisfies the compatibility equations for \( M^2 \times \mathbb{R} \). Thus, by Proposition 4.1 \((ds^2, e^{\theta J} S, e^{\theta J} T, \nu)\) does as well. Thus by Theorem 3.3 there exists a unique immersion \( x_\theta \) satisfying the properties of the theorem. The fact that \( x_0 = x \) is clear.

Finally, \((ds^2, e^{\theta J} S, e^{\theta J} T, \nu)\) defines a matrix of one-forms \( \Omega_\theta \) and a matrix of functions \( A_\theta \) satisfying \( A_\theta^{-1} dA_\theta = \Omega_\theta \) (by Proposition 3.7). By continuity of \( \Omega_\theta \) with respect to \( \theta \), we obtain the continuity of \( A_\theta \) with respect to \( \theta \) and then the continuity of \( x_\theta \) with respect to \( \theta \).

**Remark 4.3.** Let \( \tau : \Sigma' \to \Sigma \) be a conformal diffeomorphism. If \( \tau \) preserves the orientation, then \((x \circ \tau)_\theta = x_\theta \circ \tau \); if \( \tau \) reverses the orientation, then \((x \circ \tau)_\theta = x_{-\theta} \circ \tau \).

In the sequel, we will speak of associate and conjugate immersions even if condition 1 is not satisfied; i.e., we will consider these notions up to isometries of \( M^2 \times \mathbb{R} \) preserving the orientations of both \( M^2 \) and \( \mathbb{R} \).

**Remark 4.4.** The opposite immersion is \( x_{-\theta} = \sigma \circ x \), where \( \sigma \) is an isometry of \( M^2 \times \mathbb{R} \) preserving the orientation of \( M^2 \) and reversing the orientation of \( \mathbb{R} \) (see Proposition 3.3, case (2)).

**Remark 4.5.** This associate family for minimal immersions in \( M^2 \times \mathbb{R} \) is analogous to the associate family for minimal immersions in \( \mathbb{R}^3 \). Conformal minimal immersions in \( \mathbb{R}^3 \) are given by the Weierstrass representation

\[
x(z) = x(z_0) + \text{Re} \int_{z_0}^{z} (1 - g^2, i(1 + g^2), 2g) \omega,
\]

where \( g \) is a meromorphic function on \( \Sigma \) (the Gauss map) and \( \omega \) a holomorphic one-form. Then the associate immersions are

\[
x_\theta(z) = x(z_0) + \text{Re} \int_{z_0}^{z} (1 - g^2, i(1 + g^2), 2g) e^{-i\theta} \omega.
\]

Let \( x = (\varphi, h) : \Sigma \to M^2 \times \mathbb{R} \) be a conformal minimal immersion. Then \( h \) is a real harmonic function and \( \varphi \) is a harmonic map to \( M^2 \). We set

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).
\]

The Hopf differential of \( \varphi \) is the following 2-form (see [Ros02b]):

\[
Q\varphi = 4 \left( \frac{\partial \varphi}{\partial z} \frac{\partial \bar{\varphi}}{\partial \bar{z}} \right) dz^2 = \left( \left\| \frac{\partial \varphi}{\partial u} \right\|^2 - \left\| \frac{\partial \varphi}{\partial v} \right\|^2 - 2i \left( \frac{\partial \varphi}{\partial u} \frac{\partial \bar{\varphi}}{\partial v} \right) \right) dz^2.
\]
It is a holomorphic 2-form on \( \Sigma \), and since \( x \) is conformal we have
\[
Q\varphi = -4 \left( \frac{\partial h}{\partial z} \right)^2 \, dz^2 = -(d(h + ih^*))^2 = -4 \left( T, \frac{\partial x}{\partial z} \right) \, dz^2,
\]
where \( h^* \) is the harmonic conjugate function of \( h \) (i.e., \( \frac{\partial h^*}{\partial u} = -\frac{\partial h}{\partial v} \) and \( \frac{\partial h^*}{\partial v} = \frac{\partial h}{\partial u} \)).

The reader can refer to [SY97] for harmonic maps.

**Proposition 4.6.** Let \( x = (\varphi, h) : \Sigma \to \mathbb{H}^2 \times \mathbb{R} \) be a conformal minimal immersion, and \( (x_\theta) = (\varphi_\theta, h_\theta) \) its associate family of conformal minimal immersions. Let \( h^* \) be the harmonic conjugate of \( h \). Then we have
\[
h_\theta = (\cos \theta) h + (\sin \theta) h^*, \quad Q\varphi_\theta = e^{-2i\theta} Q\varphi.
\]

**Proof.** We have
\[
\frac{\partial h_\theta}{\partial u} = \left( \frac{\partial x_\theta}{\partial u}, \frac{\partial x_\theta}{\partial u} \right) = \left( \frac{\partial x}{\partial u}, T_\theta \right) = \cos \theta \left( \frac{\partial x}{\partial u}, T \right) + \sin \theta \left( \frac{\partial x}{\partial u}, JT \right)
\]
\[
= \cos \theta \left( \frac{\partial x}{\partial u}, T \right) - \sin \theta \left( \frac{\partial x}{\partial u}, T \right)
\]
\[
= \cos \theta \partial h \frac{\partial h}{\partial u} - \sin \theta \frac{\partial h}{\partial v}.
\]

In the same way we have \( \frac{\partial h_\theta}{\partial v} = \cos \theta \left( \frac{\partial x}{\partial v}, T \right) + \sin \theta \left( \frac{\partial x}{\partial v}, JT \right) = \cos \theta \frac{\partial h}{\partial u} + \sin \theta \frac{\partial h}{\partial u} \).

This proves that \( h_\theta = (\cos \theta) h + (\sin \theta) h^* \). The expression of \( Q\varphi_\theta \) follows immediately. \( \square \)

**Remark 4.7.** Recently, Hauswirth, Sá Earp and Toubiana ([HSET08]) defined the following notion of associated immersions in \( \mathbb{H}^2 \times \mathbb{R} \): two isometric conformal minimal immersions in \( \mathbb{H}^2 \times \mathbb{R} \) are said to be associated if their Hopf differential differ by the multiplication by some constant \( e^{i\theta} \). Moreover, they proved that two isometric conformal minimal immersions in \( \mathbb{H}^2 \times \mathbb{R} \) having the same Hopf differential are equal up to an isometry of \( \mathbb{H}^2 \times \mathbb{R} \). Thus the notions of associated immersions in the sense of this paper and in the sense of [HSET08] are equivalent.

In [SET05], Sá Earp and Toubiana ask the following question: if two conformal minimal immersions \( x, \tilde{x} : \Sigma \to \mathbb{M}^2 \times \mathbb{R} \) are isometric, are they associated? (This result holds for \( \mathbb{R}^3 \).)

**Remark 4.8.** Abresch and Rosenberg ([AR04]) defined a holomorphic Hopf differential for constant mean curvature surfaces in \( \mathbb{M}^2 \times \mathbb{R} \). For minimal surfaces in \( \mathbb{M}^2 \times \mathbb{R} \), this Hopf differential is
\[
Q(X, Y) = -\frac{\kappa}{2} \langle (T, X) \langle T, Y \rangle - \langle T, JX \rangle \langle T, JY \rangle \rangle
\]
\[
+ \frac{\kappa}{2} \langle (T, JX) \langle T, Y \rangle + \langle T, X \rangle \langle T, JY \rangle \rangle.
\]

A computation shows that
\[
Q = \frac{\kappa}{2} Q\varphi.
\]

**Proposition 4.9.** Let \( x : \Sigma \to \mathbb{M}^2 \times \mathbb{R} \) be a conformal minimal immersion. If \( x \) does not define a horizontal \( \mathbb{M}^2 \times \{t\} \), then the zeros of \( T \) are isolated.

**Proof.** The height function \( h = \langle x, \frac{\partial}{\partial t} \rangle \) satisfies \( dh(X) = \langle T, X \rangle \); thus the zeroes of \( T \) are the zeroes of \( dh \). Since \( h \) is harmonic, either the zeroes of \( dh \) are isolated or \( h \) is constant. The latter case is excluded by hypothesis. \( \square \)
Remark 4.10. Umbilic points (i.e., zeroes of the shape operator) may be non-isolated: for example, helicoids and unduloids in $\mathbb{S}^2 \times \mathbb{R}$ have curves of umbilic points (see Section 4.2).

We now give some geometric properties of conjugate surfaces.

The transformation $S \mapsto J_S$ implies that curvature lines and asymptotic lines are exchanged by conjugation (as in $\mathbb{R}^3$). (More generally the normal curvature and the normal torsion of a curve are swapped up to a sign.) The reader can refer to [Kar05] for geometric properties of conjugate surfaces in $\mathbb{R}^3$.

Moreover, the transformation $T \mapsto J_T$ implies the following transformation: a horizontal curve $\gamma$ along which the surface is vertical (i.e., $\nu = 0$ along $\gamma$ and $\gamma'$ is orthogonal to $T$) is mapped to a vertical curve (i.e., $\nu = 0$ along $\gamma$ and $\gamma'$ is proportional to $T$), and vice versa. We also notice that a minimal surface cannot be horizontal along a horizontal curve unless the minimal surface is a horizontal surface $M^2 \times \{t\}$ (indeed, this would imply that $T = 0$ along this curve).

Hence conjugation swaps two pairs of Schwarz reflections:

1. the symmetry with respect to a vertical plane containing a curvature line becomes the rotation with respect to a horizontal geodesic of $M^2$, and vice versa,
2. the symmetry with respect to a horizontal plane containing a curvature line becomes the rotation with respect to a vertical straight line, and vice versa.

The first case is illustrated by a generatrix curve of an unduloid or a catenoid and a horizontal line of a helicoid; the second case is illustrated by the waist circle of an unduloid or a catenoid and the axis of a helicoid. These examples are detailed in Sections 4.2 and 4.3.

4.2. Helicoids and unduloids in $\mathbb{S}^2 \times \mathbb{R}$. Apart from the horizontal spheres $\mathbb{S}^2 \times \{t\}$ and the vertical cylinders $\mathbb{S}^1 \times \mathbb{R}$ ($\mathbb{S}^1$ being a great circle in $\mathbb{S}^2$), the most simple examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are helicoids and unduloids. These surfaces are described in [PR99] and [Ros02a]. They are properly embedded and foliated by circles. Unduloids are rotational and vertically periodic; helicoids are invariant by a screw motion.

Helicoids. For $\beta \neq 0$, the helicoid $\mathcal{H}_\beta$ is given by the following conformal immersion:

$$x(u, v) = \begin{pmatrix} \sin \varphi(u) \cos \beta v \\ \sin \varphi(u) \sin \beta v \\ \cos \varphi(u) \\ v \end{pmatrix},$$

where the function $\varphi$ satisfies

$$\varphi'(u)^2 = 1 + \beta^2 \sin^2 \varphi(u), \quad \varphi''(u) = \beta^2 \sin \varphi(u) \cos \varphi(u).$$

We can assume that $\varphi(0) = 0$ and $\varphi'(u) > 0$. When $\beta > 0$ we say that $\mathcal{H}_\beta$ is a right helicoid; when $\beta < 0$ we say that $\mathcal{H}_\beta$ is a left helicoid.

The normal to $\mathbb{S}^2 \times \mathbb{R}$ in $\mathbb{R}^4$ is

$$\bar{N}(u, v) = \begin{pmatrix} \sin \varphi(u) \cos \beta v \\ \sin \varphi(u) \sin \beta v \\ \cos \varphi(u) \\ 0 \end{pmatrix}. $$
The normal to $\mathcal{H}_\beta$ in $\mathbb{S}^2 \times \mathbb{R}$ is

$$N(u, v) = \frac{1}{\varphi'(u)} \begin{pmatrix} \sin \beta v \\ -\cos \beta v \\ 0 \\ \beta \sin \varphi(u) \end{pmatrix}. $$

We compute

$$\left\langle \frac{\partial^2 x}{\partial u^2}, N \right\rangle = \left\langle \frac{\partial^2 x}{\partial v^2}, N \right\rangle = 0, \quad \left\langle \frac{\partial^2 x}{\partial u \partial v}, N \right\rangle = -\beta \cos \varphi(u).$$

Using the fact that $\langle SX, Y \rangle = \langle dY(X), N \rangle$, we compute that the matrix of $S$ in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\beta \cos \varphi(u)}{\varphi'(u)^2} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

In particular the points where $\cos \varphi(u) = 0$ are umbilic points. We also have

$$T = \frac{1}{\varphi'(u)^2} \frac{\partial}{\partial v}, \quad \nu = \frac{\beta \sin \varphi(u)}{\varphi'(u)}.$$

**Remark 4.11.** When $\beta = 0$, the formula defines a vertical cylinder $\mathbb{S}^1 \times \mathbb{R}$. When $\beta \to \infty$, the surface converges to the foliation by horizontal spheres $\mathbb{S}^2 \times \{t\}$.

**Unduloids.** For $\alpha > 1$ or $\alpha < -1$, the unduloid $\mathcal{U}_\alpha$ is given by the following conformal immersion:

$$x(u, v) = \begin{pmatrix} \sin \psi(u) \cos \alpha v \\ \sin \psi(u) \sin \alpha v \\ \cos \psi(u) \\ u \end{pmatrix},$$

where the function $\psi$ satisfies

$$1 + \psi'(u)^2 = \alpha^2 \sin^2 \psi(u), \quad \psi''(u) = \alpha^2 \sin \psi(u) \cos \psi(u).$$

We can assume that $\psi'(0) = 0$, $\psi(u) \in (0, \pi)$ and $\cos \psi(0) > 0$.

The normal to $\mathcal{U}_\alpha$ in $\mathbb{S}^2 \times \mathbb{R}$ is

$$N(u, v) = \frac{1}{\alpha \sin \psi(u)} \begin{pmatrix} -\cos \psi(u) \cos \alpha v \\ -\cos \psi(u) \sin \alpha v \\ \sin \psi(u) \\ \psi'(u) \end{pmatrix}. $$

We compute that the matrix of $S$ in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\alpha \cos \psi(u)}{1 + \psi'(u)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular the points where $\cos \psi(u) = 0$ are umbilic points. We also have

$$T = \frac{1}{1 + \psi'(u)^2} \frac{\partial}{\partial u}, \quad \nu = \frac{\psi'(u)}{\alpha \sin \psi(u)}.$$

**Remark 4.12.** When $\alpha = \pm 1$, the formula defines a vertical cylinder $\mathbb{S}^1 \times \mathbb{R}$. When $\alpha \to \infty$, the surface converges to the foliation by horizontal spheres $\mathbb{S}^2 \times \{t\}$.

**Proposition 4.13.** The conjugate surface of the unduloid $\mathcal{U}_\alpha$ is the helicoid $\mathcal{H}_\beta$ with $\alpha^2 = 1 + \beta^2$ and $\alpha$, $\beta$ having the same sign.
Proof. We set \( y_1(u) = \alpha \cos \psi(u) \) and \( y_2(u) = \beta \cos \varphi(u) \). A computation shows that both \( y_1 \) and \( y_2 \) are solutions of the equation

\[
(y')^2 = (y^2 - \alpha^2)(y^2 - \beta^2),
\]

and hence of the equation

\[
y'' = y(2y^2 - \alpha^2 - \beta^2).
\]

We have \( \psi'(0) = 0 \), and so by (12) we have \( y_1(0)^2 = \beta^2 \) and thus \( y_1'(0) = 0 \); also, \( \varphi(0) = 0 \), so \( y_2(0) = \beta \) and thus \( y_2'(0) = 0 \). Moreover, \( \cos \psi(0) > 0 \), so \( y_1(0) \) has the sign of \( \alpha \); since \( \alpha \) and \( \beta \) have the same sign, we have \( y_1(0) = \beta \). By the Cauchy-Lipschitz theorem we conclude that \( y_1 = y_2 \). From this we deduce using (12) and (11) that \( \varphi'(u)^2 = 1 + \psi'(u)^2 \); thus \( \mathcal{U}_\alpha \) and \( \mathcal{H}_\beta \) are locally isometric, and \( S_{\mathcal{H}_\beta} = JS_{\mathcal{U}_\alpha} \) and \( T_{\mathcal{H}_\beta} = JT_{\mathcal{U}_\alpha} \). Finally we have \( \nu_{\mathcal{U}_u} = -\frac{u_1}{\alpha^2 - u_1^2} \) and \( \nu_{\mathcal{H}_\beta} = -\frac{y_2}{\alpha^2 - y_2^2} \), so we get \( \nu_{T_{\mathcal{H}_\beta}} = \nu_{U_{\mathcal{U}_u}} \).

**Remark 4.14.** The vertical cylinder \( S^1 \times \mathbb{R} \) is globally invariant by conjugation, but the vertical lines and the horizontal circles are exchanged. For example, a rectangle of height \( t \) and whose basis is an arc of angle \( \theta \) becomes a rectangle of height \( \theta \) and whose basis is an arc of angle \( t \).

The horizontal sphere \( S^2 \times \{0\} \) is pointwise invariant by conjugation (since it satisfies \( S = 0 \) and \( T = 0 \)).

**Remark 4.15.** The horizontal projections of helicoids and unduloids are the Gauss maps of constant mean curvature Delaunay surfaces in \( \mathbb{H}^3 \): helicoids in \( S^2 \times \mathbb{R} \) come from nodoids in \( \mathbb{R}^3 \), and unduloids in \( S^2 \times \mathbb{R} \) come from unduloids in \( \mathbb{R}^3 \). This correspondence is described in [Ros03].

### 4.3. Helicoids and generalized catenoids in \( \mathbb{H}^2 \times \mathbb{R} \)

Apart from the horizontal planes \( \mathbb{H}^2 \times \{t\} \) and the vertical planes \( \mathbb{H}^1 \times \mathbb{R} \) (\( \mathbb{H}^1 \) being a geodesic of \( \mathbb{H}^2 \)), the most simple examples of minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) are helicoids and catenoids. These surfaces are described in [PR99] and [NR02]. They are properly embedded. Catenoids are rotational; helicoids are invariant by a screw motion and foliated by geodesics of \( \mathbb{H}^2 \).

More generally, Hauswirth classified minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) foliated by horizontal curves of constant curvature in \( \mathbb{H}^2 \) ([Hau06]). These surfaces form a two-parameter family. This family includes, among others, catenoids, helicoids and Riemann-type examples. All the surfaces described in this section belong to the Hauswirth family.

**Helicoids.** For \( \beta \neq 0 \), the helicoid \( \mathcal{H}_\beta \) is given by the following conformal immersion:

\[
x(u, v) = \begin{pmatrix}
cosh \varphi(u) \\
\sinh \varphi(u) \cos \beta v \\
\sinh \varphi(u) \sin \beta v \\
v
\end{pmatrix},
\]

where the function \( \varphi \) satisfies

\[
\varphi'(u)^2 = 1 + \beta^2 \sinh^2 \varphi(u), \quad \varphi''(u) = \beta^2 \sinh \varphi(u) \cosh \varphi(u).
\]

We can assume that \( \varphi(0) = 0 \) and \( \varphi'(u) > 0 \). The function \( \varphi \) is defined on a bounded interval. When \( \beta > 0 \) we say that \( \mathcal{H}_\beta \) is a right helicoid; when \( \beta < 0 \) we say that \( \mathcal{H}_\beta \) is a left helicoid.
The normal to \( H_\beta \) in \( \mathbb{H}^2 \times \mathbb{R} \) is
\[
N(u, v) = \frac{1}{\varphi'(u)} \begin{pmatrix} 0 & \sin \beta v & -\cos \beta v \\ -\cos \beta v & \beta \sinh \varphi(u) & \end{pmatrix}.
\]

Now \( \beta > 0 \). We compute that the matrix of \( S \) in the frame \( \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \) is the following:
\[
-\frac{\beta \cosh \varphi(u)}{\varphi'(u)^2} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
\]

We also have
\[
T = \frac{1}{\varphi'(u)^2} \frac{\partial}{\partial v}, \quad \nu = \frac{\beta \sinh \varphi(u)}{\varphi'(u)}.
\]

**Remark 4.16.** When \( \beta = 0 \), the formula defines a vertical plane \( \mathbb{H}^3 \times \mathbb{R} \). When \( \beta \to \infty \), the surface converges to the foliation by horizontal planes \( \mathbb{H}^2 \times \{t\} \).

**Catenoids.** For \( \alpha \neq 0 \), the catenoid \( \mathcal{C}_\alpha \) is given by the following conformal immersion:
\[
x(u, v) = \begin{pmatrix} \cosh \psi(u) \\ \sinh \psi(u) \cos \alpha v \\ \sinh \psi(u) \sin \alpha v \\ u \end{pmatrix},
\]
where the function \( \psi \) satisfies
\[
1 + \psi'(u)^2 = \alpha^2 \sinh^2 \psi(u), \quad \psi''(u) = \alpha^2 \sinh \psi(u) \cosh \psi(u).
\]

We can assume that \( \psi'(0) = 0 \) and \( \psi(u) > 0 \). The function \( \psi \) is defined on the interval \( (-u_0, u_0) \) with
\[
u_0 = \int_{\psi(0)}^\infty \frac{d\psi}{\sqrt{\alpha^2 \sinh^2 \psi - 1}} = \int_1^\infty \frac{dx}{\sqrt{(x^2 + \alpha^2)(x^2 - 1)}},
\]
Thus we have
\[
u_0 < \int_1^\infty \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{2}.
\]

This proves that the height of the catenoid \( \mathcal{C}_\alpha \) is smaller than \( \pi \); moreover the height tends to 0 when \( \alpha \to \infty \) and to \( \pi \) when \( \alpha \to 0 \) (theorem 1 in [NR02] holds for \( t \in (0, \frac{\pi}{2}) \)). The function \( \psi \) is decreasing on \( (-u_0, 0) \) and increasing on \( (0, u_0) \).

The waist circle is given by \( u = 0 \).

The normal to \( \mathcal{C}_\alpha \) in \( \mathbb{H}^2 \times \mathbb{R} \) is
\[
N(u, v) = \frac{1}{\alpha \sinh \psi(u)} \begin{pmatrix} -\sinh \psi(u) & -\cosh \psi(u) \cos \alpha v \\ -\cosh \psi(u) \sin \alpha v & \psi'(u) \end{pmatrix}.
\]

We compute that the matrix of \( S \) in the frame \( \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \) is the following:
\[
-\frac{\alpha \cosh \psi(u)}{1 + \psi'(u)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We also have
\[
T = \frac{1}{1 + \psi'(u)^2} \frac{\partial}{\partial u}, \quad \nu = \frac{\psi'(u) \alpha \sinh \psi(u)}{\alpha \sinh \psi(u)}.
\]
A minimal surface foliated by horocycles. We search a minimal surface such that each horizontal curve is a horocycle in \( \mathbb{H}^2 \) and such that all the horocycles have the same asymptotic point. Such a surface can be parametrized in the following way:

\[
x(u,v) = \begin{pmatrix}
\frac{\lambda(u)}{2} + \frac{1+f(u,v)^2}{2\lambda(u)} \\
f(u,v) \\
-\frac{\lambda(u)}{2} + \frac{1+f(u,v)^2}{2\lambda(u)} \\
u
\end{pmatrix}
\]

with \( \lambda > 0 \) and \( \frac{\partial f}{\partial v} > 0 \). This immersion is conformal if and only if

\[
\frac{\partial f}{\partial u} = f\lambda' \lambda \quad \text{and} \quad \left( \frac{\partial f}{\partial v} \right)^2 = 1 + \left( \frac{\lambda'}{\lambda} \right)^2.
\]

We deduce from the second relation that \( \frac{\partial^2 f}{\partial v^2} = 0 \), and so

\[
f(u,v) = \alpha(u)v + \beta(u).
\]

Reporting in the first relation we get

\[
\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} = \frac{\lambda'}{\lambda}.
\]

The immersion is minimal if and only if \( \Delta x \) is proportional to the normal \( \bar{N} \) to \( \mathbb{H}^2 \times \mathbb{R} \); a computation shows that this happens if and only if \( (\lambda')^2 + \alpha^2\lambda^2 = \lambda\lambda'' \), i.e., if and only if \( 2(\lambda')^2 + \lambda^2 = \lambda\lambda'' \), or, equivalently,

\[
\left( \frac{1}{\lambda} \right)'' = -\frac{1}{\lambda}.
\]

Up to a reparametrization and an isometry of \( \mathbb{H}^2 \) we can choose \( \lambda(u) = \alpha(u) = \frac{1}{\cos u} \) for \( u \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \beta(u) = 0 \). Thus we get the following proposition.

**Proposition 4.17.** The map

\[
x(u,v) = \begin{pmatrix}
\frac{v^2+1}{2\cos u} + \frac{\cos u}{2} \\
\frac{1}{2} - \frac{\cos u}{2} + \frac{\cos u}{u}
\end{pmatrix}
\]

defined for \( (u,v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \) is a conformal minimal embedding such that the curves \( u = u_0 \) are horocycles in \( \mathbb{H}^2 \) having the same asymptotic point. We will denote this surface by \( \mathcal{C}_0 \).

Moreover, the surface \( \mathcal{C}_0 \) is the unique one (up to isometries of \( \mathbb{H}^2 \times \mathbb{R} \)) having this property.

In the upper half-plane model for \( \mathbb{H}^2 \), the curve at height \( u \) of \( \mathcal{C}_0 \) is the horizontal Euclidean line \( x_2 = \cos u \). Figure 1 is a picture of \( \mathcal{C}_0 \) (in this picture the model for \( \mathbb{H}^2 \) is the Poincaré unit disk model). The surface \( \mathcal{C}_0 \) has height \( \pi \). It is symmetric with respect to the horizontal plane \( \mathbb{H}^2 \times \{0\} \), and it is invariant by a one-parameter family of horizontal parabolic isometries.
The normal to \( C_0 \) in \( \mathbb{H}^2 \times \mathbb{R} \) is

\[
N(u, v) = \begin{pmatrix}
-\frac{v^2 + 1}{2} + \frac{\cos^2 u}{2} \\
-v \\
\frac{1 - v^2}{2} + \frac{\cos^2 u}{2} \\
\sin u
\end{pmatrix}.
\]

We compute that the matrix of \( S \) in the frame \( \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \) is the following:

\[
- \cos u \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

We also have

\[
T = \cos^2 u \frac{\partial}{\partial u}, \quad \nu = \sin u.
\]
Minimal surfaces foliated by equidistants. For $\gamma \in (0, 1)$ or $\gamma \in (-1, 0)$, we consider the following immersion:

$$x(u, v) = \left( \begin{array}{c} \cosh \chi(u) \cosh \gamma v \\ \sinh \chi(u) \\ \cosh \chi(u) \sinh \gamma v \\ u \end{array} \right)$$

with

$$1 + \chi'(u)^2 = \gamma^2 \cosh^2 \chi(u), \quad \chi''(u) = \gamma^2 \cosh \chi(u) \sinh \chi(u).$$

It is a conformal minimal immersion.

We choose $\chi$ such that $\chi'(0) = 0$ and $\chi(u) > 0$. The function $\chi$ is defined on the interval $(-u_0, u_0)$ with

$$u_0 = \frac{\int_{\chi(0)}^{\infty} \frac{d\chi}{\sqrt{\gamma^2 \cosh^2 \chi - 1}}} = \frac{\int_{1}^{\infty} \frac{dx}{\sqrt{(x^2 - \gamma^2)(x^2 - 1)}}}.$$

Thus we have

$$u_0 > \frac{\int_{1}^{\infty} \frac{dx}{x \sqrt{x^2 - 1}}} = \frac{\pi}{2}.$$

We have defined a minimal surface $\mathcal{S}_\gamma$, which we call a generalized catenoid. Its height is greater than $\pi$, and tends to $\pi$ when $\gamma \to 0$ and to $+\infty$ when $\gamma \to 1$. The function $\chi$ is decreasing on $(-u_0, 0)$ and increasing on $(0, u_0)$. The surface is symmetric with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$, and it is invariant by a one-parameter family of horizontal hyperbolic isometries. The horizontal curves are equidistants to a geodesic in $\mathbb{H}^2$.

The normal to $\mathcal{S}_\gamma$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u, v) = -\frac{1}{\gamma \cosh \chi(u)} \left( \begin{array}{c} \sinh \chi(u) \cosh \gamma v \\ \cosh \chi(u) \\ \sinh \chi(u) \sinh \gamma v \\ -\chi'(u) \end{array} \right).$$

We compute that the matrix of $S$ in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\gamma \sinh \chi(u)}{1 + \chi'(u)^2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$

We also have

$$T = \frac{1}{1 + \chi'(u)^2} \frac{\partial}{\partial u}, \quad \nu = \frac{\chi'(u)}{\gamma \cosh \chi(u)}.$$

Remark 4.18. When $\gamma = \pm 1$, the formula defines a vertical plane $\mathbb{H}^1 \times \mathbb{R}$.

Proposition 4.19. The conjugate surface of the catenoid $\mathcal{C}_\alpha$ is the helicoid $\mathcal{H}_\beta$ with $\beta^2 = 1 + \alpha^2$ and $\alpha, \beta$ having the same sign.

Proof. We set $y_1(u) = \alpha \cosh \psi(u)$ and $y_2(u) = \beta \cosh \varphi(u)$. A computation shows that both $y_1$ and $y_2$ are solutions of the equation

$$(y')^2 = (y^2 - \alpha^2)(y^2 - \beta^2),$$

and hence of the equation

$$y'' = y(2y^2 - \alpha^2 - \beta^2).$$
We have \( \psi'(0) = 0 \), and so by (14) we have \( y_1(0)^2 = \beta^2 \) and thus \( y'_1(0) = 0 \); also, \( \varphi'(0) = 0 \), so \( y_2(0) = \beta \) and thus \( y'_2(0) = 0 \). Moreover, \( y_1(0) \) has the sign of \( \alpha \), i.e., the sign of \( \beta \), so we get \( y_1(0) = \beta \). By the Cauchy-Lipschitz theorem we conclude that \( y_1 = y_2 \) (and in particular they have the same domain of definition). From this we deduce using (13) and (14) that \( \varphi'(u)^2 = 1 + \psi'(u)^2 \), and thus \( \mathcal{C}_\alpha \) and \( \mathcal{H}_\beta \) are locally isometric, \( S_{\mathcal{H}_\beta} = JS_{\mathcal{C}_\alpha} \) and \( T_{\mathcal{H}_\beta} = JT_{\mathcal{C}_\alpha} \). Finally we have \( \nu_{\mathcal{C}_\alpha} = \frac{\beta y_2'}{y_2^2 - \alpha^2} \) and \( \nu_{\mathcal{H}_\beta} = \frac{\beta y_2'}{y_2^2 + \gamma^2} \), so we get \( \nu_{\mathcal{H}_\beta} = \nu_{\mathcal{C}_\alpha} \). \( \square \)

**Proposition 4.20.** The conjugate surface of the surface \( \mathcal{C}_0 \) is the helicoid \( \mathcal{H}_1 \).

**Proof.** In the case where \( \beta = 1 \), the function \( \varphi \) satisfies \( \varphi' = \cosh \varphi \), and thus we have \( \varphi(u) = \ln(\tan(\frac{u}{2} + \frac{\pi}{4})) \), \( \varphi'(u) = \frac{1}{\cos u} \) and \( \sinh \varphi(u) = \tan u \). Then, using the above calculations, we easily check that \( \mathcal{C}_0 \) and \( \mathcal{H}_1 \) are locally isometric, and that \( S_{\mathcal{H}_1} = JS_{\mathcal{C}_0} \), \( T_{\mathcal{H}_1} = JT_{\mathcal{C}_0} \), \( \nu_{\mathcal{H}_1} = \nu_{\mathcal{C}_0} \). \( \square \)

**Remark 4.21.** The conjugate surface of the surface \( \mathcal{C}_0 \) with the opposite orientation is the helicoid \( \mathcal{H}_{-1} \).

**Proposition 4.22.** The conjugate surface of the generalized catenoid \( \mathcal{G}_\gamma \) is the helicoid \( \mathcal{H}_\beta \) with \( \beta^2 + \gamma^2 = 1 \) and \( \beta, \gamma \) having the same sign.

**Proof.** We set \( y_1(u) = \gamma \sinh \chi(u) \) and \( y_2(u) = \beta \cosh \varphi(u) \). A computation shows that both \( y_1 \) and \( y_2 \) are solutions of the equation

\[
(y')^2 = (y^2 + \gamma^2)(y^2 - \beta^2),
\]

and hence of the equation

\[
y'' = y(2y^2 + \gamma^2 - \beta^2).
\]

We have \( \chi'(0) = 0 \), and so by (13) we have \( y_1(0)^2 = \beta^2 \) and thus \( y'_1(0) = 0 \); also, \( \varphi'(0) = 0 \), so \( y_2(0) = \beta \) and thus \( y'_2(0) = 0 \). Moreover, \( y_1(0) \) has the sign of \( \gamma \), i.e., the sign of \( \beta \), so we get \( y_1(0) = \beta \). By the Cauchy-Lipschitz theorem we conclude that \( y_1 = y_2 \) (and in particular they have the same domain of definition). From this we deduce using (13) and (15) that \( \varphi'(u)^2 = 1 + \chi'(u)^2 \), and thus \( \mathcal{G}_\gamma \) and \( \mathcal{H}_\beta \) are locally isometric, \( S_{\mathcal{H}_\beta} = JS_{\mathcal{G}_\gamma} \) and \( T_{\mathcal{H}_\beta} = JT_{\mathcal{G}_\gamma} \). Finally we have \( \nu_{\mathcal{G}_\gamma} = \nu_{\mathcal{C}_\alpha} \) and \( \nu_{\mathcal{H}_\beta} = \nu_{\mathcal{C}_\alpha} \). \( \square \)

**Remark 4.23.** This study shows that there are three types of helicoid conjugates according to the parameter of the screw-motion associated to the helicoid: the first type ones are the catenoids, which are rotational surfaces, the second type one is \( \mathcal{C}_0 \), which is invariant by a one-parameter family of horizontal parabolic isometries and which corresponds to a critical value of the parameter, the third type ones are the generalized catenoids, which are invariant by a one-parameter family of horizontal hyperbolic isometries.

This phenomenon is very similar to what happens to the conjugate cousins in \( \mathbb{H}^3 \) of the helicoids in \( \mathbb{R}^3 \). There exists an isometric correspondence between minimal surfaces in \( \mathbb{R}^3 \) and constant mean curvature one surfaces in \( \mathbb{H}^3 \) called the cousin relation (see [Bry87] and [UY93]). Starting from a helicoid in \( \mathbb{R}^3 \), we consider its conjugate surface, which is a catenoid in \( \mathbb{R}^3 \), and then the cousin surface in \( \mathbb{H}^3 \), which is a catenoid cousin. Catenoid cousins are of three types according to the parameter of the minimal helicoid: some are rotational surfaces, one is invariant by
a one-parameter family of parabolic isometries (and corresponds to a critical value of the parameter), and some are invariant by a one-parameter family of hyperbolic isometries. These surfaces are described in detail in [SET01] and [Ros02a].

**Remark 4.24.** All the above surfaces belong to the Hauswirth family: with the notation of [Hau06], helicoids correspond to \( d = 0, c > 0, c \neq 1 \); catenoids correspond to \( c = 0, d > 1 \); the surface \( C_0 \) corresponds to \( c = 0, d = 1 \); the surfaces \( G_{\gamma} \) correspond to \( c = 0, d \in (0, 1) \).

**Remark 4.25.** The vertical plane \( \mathbb{H}^1 \times \mathbb{R} \) is globally invariant by conjugation, but the vertical lines and the horizontal geodesics of \( \mathbb{H}^2 \) are exchanged. The horizontal plane \( \mathbb{H}^2 \times \{0\} \) is pointwise invariant by conjugation (since it satisfies \( S = 0 \) and \( T = 0 \)). This is similar to what happens in \( S^2 \times \mathbb{R} \).

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