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OPERATOR-VALUED FRAMES

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Abstract. We develop a natural generalization of vector-valued frame theory, which we term operator-valued frame theory, using operator-algebraic methods. This extends work of the second author and D. Han which can be viewed as the multiplicity one case and extends to higher multiplicity their dilation approach. We prove several results for operator-valued frames concerning duality, disjointedness, complementarity, and composition of operator-valued frames and the relationship between the two types of similarity (left and right) of such frames. A key technical tool is the parametrization of Parseval operator-valued frames in terms of a class of partial isometries in the Hilbert space of the analysis operator. We apply these notions to an analysis of multiframe generators for the action of a discrete group G on a Hilbert space. One of the main results of the Han-Larson work was the parametrization of the Parseval frame generators in terms of the unitary operators in the von Neumann algebra generated by the group representation, and the resulting norm path-connectedness of the set of frame generators due to the connectedness of the group of unitary operators of an arbitrary von Neumann algebra. In this paper we generalize this multiplicity one result to operator-valued frames. However, both the parametrization and the proof of norm path-connectedness turn out to be necessarily more complicated, and this is at least in part the rationale for this paper. Our parametrization involves a class of partial isometries of a different von Neumann algebra. These partial isometries are not path-connected in the norm topology, but only in the strong operator topology. We prove that the set of operator frame generators is norm pathwise-connected precisely when the von Neumann algebra generated by the right representation of the group has no minimal projections. As in the multiplicity one theory there are analogous results for general (non-Parseval) frames.

1. Introduction

The mathematical theory of frame sequences on Hilbert space has developed rather rapidly in the past decade. This is true of both the finite dimensional and infinite dimensional aspect of the theory. The motivation has come from applications to engineering as well as from the pure mathematics of the theory.

The theory of finite frames has developed almost as a separate theory in itself, with applications to industry (cf. the recent work [4] of Balans, Casazza, and Edidin

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on signal reconstruction without noisy phase) as well as recently demonstrated connections to theoretical problems such as the Kadison-Singer problem [5].

Important examples of infinite frames are the Gabor (Weyl-Heisenberg) frames of time-frequency analysis and the wavelet frames [7]. Some papers dealing with infinite frames which relate directly or indirectly to this article are [15, 13, 9, 1, 10, 20, 19, 21].

Work on this article began in January 1999, when the first named author visited the second named author at Texas A&M University following the special session on "The functional and harmonic analysis of wavelets and frames" that took place at the annual AMS meeting at San Antonio. Our purpose was to develop operator-theoretic methods for dealing with multiwavelets and multiframes, thus extending the approach of the AMS Memoir [15]. We developed the theory of operator-valued frames to provide a framework for such problems and we tested this model by solving a problem concerning norm path-connectedness. It has been brought to our attention that a few other recent papers in the literature overlap to some extent with our approach, notably works of Casazza, Kutyniok and Li [6] on "fusion frames", and also recent work of Bodmann [2] on quantum computing and work of W. Sun [25] on G-frames. These do not deal however with the path-connectedness that we address. The papers of Kadison on the Pythagorean theorem [17, 16] are examples of works of pure mathematics that several authors have realized are both directly and indirectly relevant to frame theory. They contain theorems on the possible diagonals of positive operators both in $B(H)$ and in von Neumann algebras. This topic is closely related to the topic of rank-one decompositions and more general summation decompositions of positive operators, and resolutions of the identity operator, as investigated in [10, 20] for its relevance to frame theory.

Also, several papers in the literature deal with frames in Hilbert C^*-modules, including one by the same authors of this paper [11, 12, 19]. The problems and framework considered in this paper are of a significantly different nature and there is no essential overlap.

We note that the key idea in [15] was the observation that frames "dilate" to Riesz bases. This was proven at the beginning of [15] (see also [22, p. 145]), and was then used to obtain results on Gabor frames, more generally frames generated by the action of unitary systems, and certain group representations. The dilation result for the special case of Parseval frames can be simply deduced from Naimark's dilation theorem for positive operator-valued measures, in fact from the special case of Naimark's Theorem specific to purely atomic positive operator-valued measures. W. Czaja gives a nice account of this in [8], along with some new dilation results. V. Paulsen gives a nice proof of Naimark's theorem in [24] using the theory of completely positive mappings. Similarly, we use dilation theory in the present paper to work with operator-valued frames.

Consider a *multiframe generator* $\{\psi_1, \psi_2\}$ for a unitary system $\mathcal{U}$, that is, two vectors in a Hilbert space H for which the collection $\{U\psi_m \mid U \in \mathcal{U}, m = 1, 2\}$ forms a frame:

$$a\|x\|^2 \leq \sum_{U\in\mathcal{U}} \Big(|(x, U\psi_1)|^2 + |(x, U\psi_2)|^2 \Big) \leq b\|x\|^2 \tag{1}$$

for some positive constants a and b and all $x \in H$. Set $H_o := \mathbb{C}^2$, choose $\{e_1, e_2\}$ to be an orthonormal basis of H_o, define the rank-two operator A given for $z \in H$ by

$Az := (z, \psi_1)e_1 + (z, \psi_2)e_2$ and then denote $A_U := AU^*$. Then equation (1) holds precisely when

$$aI \leq \sum_{U \in \mathcal{U}} A_U^* A_U \leq bI,$$

where the series converges in the strong operator topology. In other words, in lieu of considering the two vectors $\{\psi_1, \psi_2\}$, we can consider the rank-two operator A.

The above is a simple example of an *operator-valued frame* generator and leads naturally to the more general Definition 2.1 below of an operator-valued frame consisting of operators with ranges in a given Hilbert space H_o, and the frame condition is expressed in terms of boundedness above and below of a series of positive operators converging in the strong operator topology. So, the usual (vector) frames can be seen as operator-valued frames of "multiplicity one".

It is easy to recover from the operator A defined above the vectors ψ_1 and ψ_2 that were used to define it, and, in general, to decompose (but not uniquely) an operator-valued frame in a (vector) multiframe (see comments after Remark 4.10). However, we expect that this paper will make clear that "assembling" a multiframe in an operator-valued frame is more than just a space-saving bookkeeping device.

Indeed, operator theory techniques allow us to obtain directly for operator-valued frames (and hence for the related vector multiframes) properties known for (vector) frames.

More importantly, however, treating multiframes as operator-valued frames permits us to parametrize them in an explicit and transparent way and thus handle the sometimes major differences that occur when the multiplicity rises above one, and in particular, when it is infinite.

A case in point, and in a sense our best "test" of the usefulness of the notion of operator-valued frames, is the analysis of frame generators for a discrete group (see [15] and Section 6 for a review of the definitions). Han and the second named author proved in [15, Theorem 6.17] that the collection of all the Parseval frame generators for a given unitary representation $\{G, \pi, H\}$ of a countable group G is (uniquely) parametrized by the unitary operators of the von Neumann algebra $\pi(G)''$ generated by the unitaries π_g of the representation. Since the unitary group of any von Neumann algebra is path-connected in the norm topology, the collection of all the Parseval frame generators is therefore also path-connected; i.e., it has a single homotopy class.

As soon as $\dim H_o > 1$, the above parametrization is no longer sufficient (see Remark 7.7), and it must be replaced by a new parametrization involving a class of partial isometries of a different von Neumann algebra (see Theorem 7.1, Proposition 7.3).

Furthermore, when $\dim H_o = \infty$, it is possible to show that the partial isometries involved in this parametrization are not path-connected in the norm topology (they are path-connected in the strong operator topology, though). Nevertheless, we prove in Theorem 8.1 that the collection of operator frame generators is still norm connected, precisely when the von Neumann algebra generated by the left (or right) regular representation of the group has no minimal projections. The key step is provided by Lemma 8.4, where the strong continuity of a certain path of partial isometries is parlayed into the norm continuity of the corresponding path of operator frame generators.

One of the main themes of this article is the analysis of one-to-one parametrizations of operator-valued frames in general, and of operator frame generators for unitary systems and groups in particular. In the process we extend to operator-valued frames many of the properties of vector frames. In more detail:

In Section 2 we define operator-valued frames, their analysis operators and their frame projections, and then prove that the dilation approach of [15] carries over to the higher multiplicity case, i.e., that Parseval operator-valued frames are the compressions of "orthonormal" operator-valued frames, namely collections of partial isometries, all with the same initial projection and with mutually orthogonal ranges spanning the space.

In Section 3 we obtain a one-to-one parametrization of all the operator-valued frames on a certain Hilbert space H, with given multiplicity and index set, in terms of a class of operators in the analysis operator Hilbert space (partial isometries if we consider only Parseval frames).

In Section 4 we study two kinds of similarities of operator-valued frames. The similarity obtained by multiplying an operator-valued frame from the right generalizes the usual one in the vector case and inherits its main properties. For higher multiplicity, however, we also have a similarity from the left which has different properties. We characterize the case when two operator-valued frames are similar both from the right and from the left, in terms of the parametrization mentioned above (Proposition 4.8). We also discuss composition of frames, when the range of the operators in one frame matches the domain of the operators in a second frame and we present this notion as the tool to decompose an operator-valued frame into a (vector) multiframe.

In Section 5 we define and parametrize the dual of operator-valued frames and extend to higher multiplicity also the notions of disjoint, strongly disjoint, and strongly complementary frames that were introduced in [15] for the vector case.

In Section 6 we start the analysis of operator frame generators for unitary systems. The notion of *local commutant* introduced in [9] has a natural analog in the higher multiplicity case (see Proposition 6.2). Unitary representations of discrete groups have an operator frame generator with values in H_o precisely when they are unitarily equivalent to a subrepresentation of the left regular representation with multiplicity $\dim H_o$, i.e., $\lambda \otimes I_o$ with I_o the identity of $B(H_o)$ (Theorem 6.5). This result was previously formulated in terms of (vector) multiframes in [15, Theorem 3.11].

In Section 7 we present parametrizations of operator frame generators for a discrete group representation (Theorem 7.1). As mentioned above, higher multiplicity brings substantial differences with the vector case, which are illustrated by Proposition 7.6.

As already mentioned, Section 8 studies the path-connectedness of the operator frame generators for a unitary representation of a discrete group using von Neumann algebra techniques.

Finally, let us notice explicitly that although, in the applications, frames are mainly indexed by finite or countable index sets and the vectors in a frame belong to finite or separable Hilbert spaces, and similarly, discrete groups are finite or countable, we found that making these assumptions provides no simplification in our proofs (with one very minor exception). Thus we decided to state and prove our results in the general case. The only thing to keep in mind, when the index set

$\mathbb{J}$ is not finite or $\mathbb{N}$, is that the convergence of $\sum_{j\in\mathbb{J}} x_j$ means the convergence of the net of the finite partial sums for all finite subsets of $\mathbb{J}$.

2. Operator-valued frames

Definition 2.1. Let H and H_o be Hilbert spaces. A collection $\{A_j\}_{j\in\mathbb{J}}$ of operators $A_j \in B(H, H_o)$ indexed by $\mathbb{J}$ is called an operator-valued frame on H with range in H_o if the series

$$S_A := \sum_{j\in\mathbb{J}} A_j^* A_j \tag{2}$$

converges in the strong operator topology to a positive bounded invertible operator S_A. The frame bounds a and b are the largest number $a > 0$ and the smallest number $b > 0$ for which $aI \leq S_A \leq bI$; i.e., $b = \|S_A\|$ and $a^{-1} = \|S_A^{-1}\|$. If $a = b$, i.e., $S_A = aI$, then the frame is called tight; if $S_A = I$, the frame is called Parseval. $\sup\{\mathrm{rank}(A_j) \mid j \in \mathbb{J}\}$ is called the multiplicity of the operator-valued frame. When the reference to H, H_o, and $\mathbb{J}$ is understood, we denote by $\mathcal{F}$ the set of all the operator-valued frames on H, with ranges in H_o and indexed by $\mathbb{J}$.

If the operator-valued frame has multiplicity one, the operators A_j can be identified through the Riesz Representation Theorem with Hilbert space vectors, and hence in this case an operator-valued frame is indeed a (vector) frame under the usual definition. Explicitly, if A_j is the rank one operator given by $A_j z = (x, x_j)e_j$ for some unit vectors $e_j \in H_o$, some vectors $x_j \in H$ and all $z \in H$, and then $S_A = \sum_{j\in\mathbb{J}} A_j^* A_j$; hence S_A is bounded and invertible if and only if $aI \leq S_A \leq bI$ for some a, $b > 0$; namely,

$$a\|x\|^2 \leq (S_A x, x) = \sum_{j\in\mathbb{J}} |(x, x_j)|^2 \leq a\|x\|^2$$

for all $x \in H$. This is precisely the condition that guarantees that $\{x_j\}_{j\in\mathbb{J}}$ is a (vector) frame.

Notice that if $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ and if $\{e_m\}_{m\in\mathbb{M}}$ is an orthonormal basis of H_o, then it is easy to see that $\{A_j^* e_m\}_{(j,m)\in\mathbb{J}\times\mathbb{M}}$ is a (vector) frame on H; i.e., operator-valued frames can be decomposed into (vector) multiframes. We will revisit this decomposition when discussing frame compositions more generally.

The advantage of treating a collection of vectors forming a multiframe as an operator-valued frame is that we can more easily apply to it the formalism of operator theory. This is already evidenced by the next example.

Example 2.2. Let K be an infinite dimensional Hilbert space and let $\{V_n\}_{n\in\mathbb{N}}$ be a collection of partial isometries with mutually orthogonal range projections $V_n V_n^*$ summing to the identity and all with the same initial projection $V_n^* V_n = E_o$. Let $P \in B(K)$ be a nonzero projection and let $A_n := V_n^* P \in B(H, H_o)$, where we set $H := PK$, $H_o := E_o K$. Then

$$\sum_{n=1}^{\infty} A_n^* A_n = P(\sum_{n=1}^{\infty} V_n V_n^*)\, P|_{PK} = P|_{PK} = I;$$

i.e., the sequence $\{A_n\}$ is a Parseval frame with range in H_o.

By introducing the *analysis operator* (also called the frame transform, e.g., [15]), we will see that this example is 'generic' (see Proposition 2.4 below).

Analysis operator. Given a Hilbert space H_o and an index set $\mathbb{J}$, define the partial isometries

$$L_j \,:\, H_o \ni h \mapsto e_j \otimes h \in \ell(\mathbb{J}) \otimes H_o, \tag{3}$$

where $\{e_j\}$ is the standard basis of $\ell^2(\mathbb{J})$. Then

$$L_j^* L_i = \begin{cases} I_o & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{4}$$

and

$$\sum_{j \in \mathbb{J}} L_j L_j^* = I \otimes I_o, \tag{5}$$

where I denotes the identity operator on $\ell^2(\mathbb{J})$ and I_o denotes the identity operator on H_o and the convergence is in the strong topology.

Proposition 2.3. *For every $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$,*

(i) *The series $\sum_{j\in\mathbb{J}} L_j A_j$ converges in the strong operator topology to an operator $\theta_A \in B(H, \ell(\mathbb{J}) \otimes H_o)$.*

(ii) $S_A = \theta_A^* \theta_A$.

(iii) *$\{A_j\}_{j\in\mathbb{J}}$ is Parseval if and only if θ_A is an isometry.*

Proof. For every $x \in H$,

$$\|\theta_A x\|^2 = \sum_{j\in\mathbb{J}} \|L_j A_j x\|^2 = \sum_{j\in\mathbb{J}} \|A_j x\|^2 = (S_A x, x),$$

where the first identity holds because the operators L_j have mutually orthogonal ranges, the second one holds because they are partial isometries, and the third one holds by the definition (2) of S_A. These identities and routine arguments prove (i)-(iii). □

Explicitly,

$$\theta_A = \sum_{j\in\mathbb{J}} L_j A_j \tag{6}$$

and for every $x \in H$, $\theta_A(x) = \sum_{j\in\mathbb{J}} (e_j \otimes A_j x)$. As a consequence of Proposition 2.3 (ii),

$$\theta_A S_A^{-1/2} \quad \text{is an isometry,} \tag{7}$$

and hence

$$P_A := \theta_A S_A^{-1} \theta_A^* \tag{8}$$

is the range projection of $\theta_A S_A^{-1/2}$ and hence of θ_A. Moreover, $\{A_j\}_{j\in\mathbb{J}}$ is Parseval if and only if $\theta_A \theta_A^*$ is a projection.

Given $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$, the operator $\theta_A \in B(H, \ell^2(\mathbb{J}) \otimes H_o)$ is called the *analysis operator* and the projection $P_A \in B(\ell^2(\mathbb{J}) \otimes H_o)$ is called the *frame projection* of $\{A_j\}_{j\in\mathbb{J}}$.

The analysis operator fully 'encodes' the information carried by the operator-valued frame; namely, the frame can be *reconstructed* from its analysis operator via the identity

$$A_j = L_j^* \theta_A \quad \text{for all } j \in \mathbb{J}. \tag{9}$$

Indeed,

$$L_j^*\theta_A = L_j^* \sum_{i\in\mathbb{J}} L_i A_i = \sum_{i\in\mathbb{J}} L_j^* L_i A_i = A_j$$

by (4). In particular, two operator-valued frames $\{A_j\}_{j\in\mathbb{J}}$ and $\{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ are identical if and only if $\theta_A = \theta_B$. Also,

$$\theta_A^* = \sum_{i\in\mathbb{J}} A_j^* L_j^*, \tag{10}$$

where the convergence is in the strong topology, because by (9), $A_j^* = \theta_A^* L_j$ and $L_j L_j^*$ are mutually orthogonal projections that sum to the identity $I \otimes I_o$ of $\ell^2(\mathbb{J}) \otimes H_o$. The same argument shows that for any two operator-valued frames $\{A_j\}_{j\in\mathbb{J}}$ and $\{B_j\}_{j\in\mathbb{J}}$,

$$\theta_B^*\theta_A = \sum_{i\in\mathbb{J}} B_j^* A_j \quad \text{in the strong operator topology.} \tag{11}$$

Recall now that Parseval (vector) frames were shown in [15, Proposition 1.1] to be compressions of orthonormal bases. The higher multiplicity analog of that result is given by the following proposition.

Proposition 2.4. *For every $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$, there is a Hilbert space K containing H and H_o, a collection of partial isometries $\{V_j\}_{j\in\mathbb{J}}$ all with the same domain H_o and with mutually orthogonal ranges spanning K, and a positive invertible operator $T \in B(H)$ such that $A_j = V_j^* T$ for all $j \in \mathbb{J}$. In particular, if $\{A_j\}_{j\in\mathbb{J}}$ is a Parseval frame, then T can be chosen to be the projection on H.*

Proof. Let $K = \ell^2(\mathbb{J}) \otimes H_o$. Identify H_o with $\mathbb{C} \otimes H_o \subset K$ and identify H with its image $P_A K$ under the isomorphism $\theta_A S_A^{-1/2}$ (see (7)). Then for all $j \in \mathbb{J}$, we identify A_j with

$$A_j(\theta_A S_A^{-1/2})^*\Big|_{P_A K} = L_J^*(\theta_A S_A^{-1/2}\theta_A^*)\Big|_{P_A K}.$$

Clearly, $T := \theta_A S_A^{-1/2}\, \theta_A^*|_{P_A K}$ is positive and invertible and it is the identity $P_A|_{P_A K}$ on H if and only if $\{A_j\}_{j\in\mathbb{J}}$ is a Parseval frame. □

In analogy with the vector case (see [15, Chapter 1]), we introduce the following terminology.

Definition 2.5. An operator-valued frame $\{A_j\}_{j\in\mathbb{J}}$ for which $P_A = I \otimes I_o$ is called a Riesz frame, and an operator-valued frame that is both Parseval and Riesz, i.e., such that $\theta_A^*\theta_A = I$ and $\theta_A\theta_A^* = I \otimes I_o$, is called an orthonormal frame.

Remark 2.6. If we identify operator-valued frames with their images in the analysis space $\ell^2(\mathbb{J}) \otimes H_o$, (i.e., up to right unitary equivalence of the frames, in the notation of Section 4 below), then

(i) General frames are the frames of the form $\{L_j^* T\}_{j\in\mathbb{J}}$ for some positive operator $T = PTP$ invertible in $B(P\ell^2(\mathbb{J}) \otimes H_o)$ and some projection $P \in B(\ell^2(\mathbb{J}) \otimes H_o)$. The operator T and the projection P are uniquely determined (up to Hilbert space isomorphism, i.e., right unitary equivalence of the operator-valued frames).

(ii) Parseval frames are the frames of the form $\{L_j^* P\}_{j\in\mathbb{J}}$ for some projection P in $B(\ell^2(\mathbb{J}) \otimes H_o)$.

(iii) Riesz frames are the frames of the form $\{L_j^*T\}_{j\in\mathbb{J}}$ for some invertible operator $T \in B(\ell^2(\mathbb{J}) \otimes H_o)$.

(iv) $\{L_j^*\}_{j\in\mathbb{J}}$ is the unique orthonormal frame.

In particular, in the notation of Section 4, Riesz operator-valued frames are the frames that are right-similar to an orthonormal frame (cf. [15, Proposition 1.5]).

Remark 2.7. Notice that in the definition of an operator-valued frame $\{A_j\}_{j\in\mathbb{J}}$ there is no request for H_o to be "minimal", i.e., for $\dim H_o$ to coincide with the multiplicity of the frame, i.e., with $\sup\{\text{rank } A_j \mid j \in \mathbb{J}\}$. In fact we can consider the operators A_j as having range in a "larger" Hilbert space, e.g., in H itself. Doing so will produce a new analysis operator into a "larger" space; however, both analysis operators will carry the same information about the original frame, which can be recovered equally well from either of them.

3. Parametrization of operator-valued frames

In this section we show that all the operator-valued frames with the same multiplicity and same index set, operating on the same Hilbert space (up to isomorphism) can be "obtained" from a single operator-valued frame. Consider first Parseval frames. Following the dilation viewpoint (cf. Proposition 2.4 above), we can immerse all these frames in the analysis space $\ell^2(\mathbb{J})\otimes H_o$ by identifying them with the compression of $\{L_j^*\}_{j\in\mathbb{J}}$ to their frame projection. Since all the frame projections are equivalent (each range is isomorphic to the original Hilbert space of the frame), the partial isometries implementing these equivalences will provide the link between the frames. To make this idea precise, and to handle at the same time frames that are not Parseval, we introduce the following notation.

Given $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$, define

$$\mathcal{M}_A := \{M \in B(\ell(\mathbb{J}) \otimes H_o) \mid M = MP_A,\ M^*M|_{P_A\ell(\mathbb{J})\otimes H_o} \text{ is invertible}\}. \tag{12}$$

Equivalently, $\mathcal{M}_A := \{M \in B(P_A\ell^2(\mathbb{J}) \otimes H_o, \ell^2(\mathbb{J}) \otimes H_o) \mid M \text{ is left invertible}\}$. If $M \in \mathcal{M}_A$, denote by $(M^*M)^{-1} \in B(P_A\ell^2(\mathbb{J})\otimes H_o)$ the inverse of $M^*M|_{P_A\ell^2(\mathbb{J})\otimes H_o}$.

Theorem 3.1. *Let $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$. For all $\{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ define*

$$\Phi_A(\{B_j\}_{j\in\mathbb{J}}) := \theta_B S_A^{-1}\theta_A^*. \tag{13}$$

Then $\Phi_A : \mathcal{F} \mapsto \mathcal{M}_A$ is one-to-one and onto and $\Phi_A^{-1}(M) = \{L_j^*M\theta_A\}_{j\in\mathbb{J}}$ for all $M \in \mathcal{M}_A$. If $\{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ and $M := \Phi_A(\{B_j\}_{j\in\mathbb{J}})$, then $\theta_B = M\theta_A$ and

$$V_M := M(M^*M)^{-1/2}$$

is a partial isometry that implements the equivalence $P_B \sim P_A$, i.e., $P_B = V_MV_M^*$ and $P_A = V_M^*V_M$.

Proof. Let $\{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ and let $M := \Phi_A(\{B_j\}_{j\in\mathbb{J}})$. Then $MP_A = M$ because P_A is the range projection of θ_A. Moreover,

$$M^*M = \theta_A S_A^{-1}\theta_B^*\theta_B S_A^{-1}\theta_A^* \geq \theta_A S_A^{-1} a I S_A^{-1}\theta_A^* \geq \frac{a}{b}\theta_A S_A^{-1}\theta_A^* = \frac{a}{b}P_A,$$

where a is a lower bound for S_B and b is an upper bound for S_A. Thus M^*M is invertible in $B(P_A\ell^2(\mathbb{J}) \otimes H_o)$ and hence $M \in \mathcal{M}_A$, i.e., Φ_A maps into $\mathcal{M}_A$. Assume now that we have $\Phi_A(\{B_j\}_{j\in\mathbb{J}}) = \Phi_A(\{C_j\}_{j\in\mathbb{J}})$ for two frames in $\mathcal{F}$. Since $\theta_B = \theta_B S_A^{-1}\theta_A^*\theta_A = M\theta_A$, it follows that $\theta_B = \theta_C$, and by (9), the two frames

coincide, i.e., Φ_A is injective. We prove now that Φ_A is onto. For any $M \in \mathcal{M}_A$, define $\{B_j := L_j^* M\theta_A\}_{j\in\mathbb{J}}$. Then

$$\sum_{j\in\mathbb{J}} B_j^* B_j = \sum_{j\in\mathbb{J}} \theta_A^* M^* L_j L_j^* M\theta_A = \theta_A^* M^* M\theta_A \leq \|M\|^2 \theta_A^*\theta_A = \|M\|^2 S_A.$$

Similarly,

$$\sum_{j\in\mathbb{J}} B_j^* B_j \geq \|(M^*M)^{-1}\|^{-1} S_A,$$

which proves that $\{B_j\}_{j\in\mathbb{J}}$ is an operator-valued frame. Moreover,

$$\theta_B = \sum_{j\in\mathbb{J}} L_j B_j = \sum_{j\in\mathbb{J}} L_j L_j^* M\theta_A = M\theta_A.$$

We have just seen that $\theta_B = \Phi_A(\{B_j\}_{j\in\mathbb{J}})\theta_A$. Thus $M\theta_A = \Phi_A(\{B_j\}_{j\in\mathbb{J}})\theta_A$ and hence

$$M = MP_A = M\theta_A S_A^{-1}\theta_A^* = \Phi_A(\{B_j\}_{j\in\mathbb{J}})\theta_A S_A^{-1}\theta_A^* = \Phi(\{B_j\}_{j\in\mathbb{J}}).$$

This proves that the map Φ_A is onto and that $\Phi_A^{-1}(M) = \{L_j^* M\theta_A\}_{j\in\mathbb{J}}$ for all M in $\mathcal{M}_A$. It remains only to compute P_B, which, by definition, is the range projection of $\theta_B = M\theta_A$. Since $M = MP_A$ and θ_A is one-to-one (recall that $\theta_A S_A^{-1/2}$ is an isometry), P_B is the range projection of M. Now

$$V_M^* V_M = (M^*M)^{-1/2} M^* M (M^*M)^{-1/2} = P_A$$

(recall that $(M^*M)^{-1}$ denotes the inverse of M^*M in $B(P_A\ell^2(\mathbb{J}) \otimes H_o)$); hence V_M is a partial isometry. Since $(M^*M)^{-1/2}$ is invertible, the range of V_M coincides with the range of M, and hence $P_B = V_M V_M^*$. □

An easy consequence of Theorem 3.1 and its proof is the following:

Corollary 3.2. *If $\{A_j\}_{j\in\mathbb{J}}, \{B_j\}_{j\in\mathbb{J}}$, and $\{C_j\}_{j\in\mathbb{J}}$ are operator-valued frames , then $\Phi_A(\{C_j\}_{j\in\mathbb{J}}) = \Phi_B(\{C_j\}_{j\in\mathbb{J}})\Phi_A(\{B_j\}_{j\in\mathbb{J}})$.*

This corollary shows that $\Phi_A(\{B_j\}_{j\in\mathbb{J}})$ behave like a partial isometry with initial projection P_A and range projection P_A. In fact, if both frames are Parseval, $\Phi_A(\{B_j\}_{j\in\mathbb{J}})$ is precisely a partial isometry with these initial and range projections.

Since every operator-valued frame is right-similar to a Parseval frame (see Definition 4.1 below), we can focus on Parseval frames. For ease of reference we present in the following corollary the main result of Theorem 3.1 formulated directly for Parseval frames.

Corollary 3.3. *Given a Parseval operator-valued frame $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$, then*

$$\{\{L_j^* V\theta_A\}_{j\in\mathbb{J}} \mid V \in B(\ell^2(\mathbb{J}) \otimes H_o), V^*V = P_A\}$$

is the collection of all Parseval operator-valued frames in $\mathcal{F}$. The correspondence is one-to-one: if $V \in B(\ell^2(\mathbb{J}) \otimes H_o), V^*V = P_A$, and $\{B_j := L_j^* V\theta_A\}_{j\in\mathbb{J}} \in \mathcal{F}$, then $V = \theta_B\theta_A^*$.

Proof. We need only to show that when $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ is Parseval and $M \in \mathcal{M}_A$, then the operator-valued frame $\Phi_A^{-1}(M) := \{B_j : L_j^* M\theta_A\}_{j\in\mathbb{J}}$ is Parseval if and only if M is a partial isometry. This is clear since $\theta_B\theta_B^* = M\theta_A\theta_A^* M^* = MP_A M^* = MM^*$ and as remarked after equation (8), $\{B_j\}_{j\in\mathbb{J}}$ is Parseval if and only if $\theta_B\theta_B^*$ is a projection. Finally, $V = M = \Phi_A(\{B_j\}_{j\in\mathbb{J}}) = \theta_B\theta_A^*$ since $S_A^{-1} = I$. □

4. Similarity and composition of frames

For operator-valued frames with ranges in H_o where $\dim H_o > 1$ there are two natural distinct notions of similarity, which are called 'right' and 'left'.

Definition 4.1. Let $\{A_j\}_{j\in\mathbb{J}}$,$\{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$. We say that:

(i) $\{B_j\}_{j\in\mathbb{J}}$ is right-similar (resp., right unitarily equivalent) to $\{A_j\}_{j\in\mathbb{J}}$ if there is an invertible operator (resp., unitary operator) $T \in B(H)$ such that $B_j = A_jT$ for all $j \in \mathbb{J}$.

(ii) $\{B_j\}_{j\in\mathbb{J}}$ is left-similar (resp., left unitarily equivalent) to $\{A_j\}_{j\in\mathbb{J}}$ if there is an invertible operator (resp., unitary operator) $R \in B(H_o)$ such that $B_j = RA_j$ for all $j \in \mathbb{J}$.

When $\dim H_o = 1$, the left similarity is trivial (a multiplication by a nonzero scalar) and the right similarity is just the usual similarity of the vector frames (corresponding to the operators, i.e., functionals, by the Riesz Representation Theorem).

We leave to the reader the following simple results about right similarity.

Lemma 4.2. *Let $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ have frame bounds a and b, i.e., $aI \le S_A \le bI$, let $T \in B(H)$ be an invertible operator, and let $\{B_j := A_jT\}_{j\in\mathbb{J}}$. Then*

(i) $\{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ and $\frac{a}{\|T^{-1}\|^2}I \le S_B \le b\|T\|^2 I$. In particular, if T is unitary, then $\{B_j\}_{j\in\mathbb{J}}$ has the same bounds as $\{A_j\}_{j\in\mathbb{J}}$. Assuming that $\{A_j\}_{j\in\mathbb{J}}$ is Parseval, then $\{B_j\}_{j\in\mathbb{J}}$ is Parseval if and only if T is unitary.

(ii) $\theta_B = \theta_A T$ and $S_B = T^*S_AT$.

(iii) $\Phi_A(\{B_j\}_{j\in\mathbb{J}}) = \theta_A T S_A^{-1}\theta_A^*$.

Now we characterize right-similar frames.

Proposition 4.3. *Let $\{A_j\}_{j\in\mathbb{J}}, \{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ and let $M := \Phi_A(\{B_j\}_{j\in\mathbb{J}})$. Then the following conditions are equivalent:*

(i) $B_j = A_jT$ for all $j \in \mathbb{J}$ for some invertible operator $T \in B(H)$, i.e., $\{A_j\}_{j\in\mathbb{J}}$ and $\{B_j\}_{j\in\mathbb{J}}$ are right-similar.

(ii) $\theta_B = \theta_A T$ for some invertible $T \in B(H)$.

(iii) $M = P_AMP_A$ is invertible in $B(P_A\ell^2(\mathbb{J}) \otimes H_o)$.

(iv) $P_B = P_A$.

If the above conditions are satisfied, the invertible operator T in (i) and (ii) is uniquely determined and $T = S_A^{-1}\theta_A^*\theta_B$.

In the case that $\{A_j\}_{j\in\mathbb{J}}$ is Parseval, then $\{B_j\}_{j\in\mathbb{J}}$ is Parseval if and only if the operator T in (i), or equivalently in (ii), is unitary.

Proof. (i) $\Longleftrightarrow$ (ii) One implication is given by Lemma 4.2 and the other is immediate.

(ii) $\Longrightarrow$ (iii) We have $\theta_B = \theta_A T = (\theta_A T S_A^{-1}\theta_A^*)\theta_A$. Let $N := \theta_A T S_A^{-1}\theta_A^*$. Then

$$\begin{aligned} N^*N &= (\theta_A T S_A^{-1}\theta_A^*)^*(\theta_A T S_A^{-1}\theta_A^*) = \theta_A S_A^{-1}T^*S_AT S_A^{-1}\theta_A^* \\ &\ge a\|T\|^2\theta_A S_A^{-2}\theta_A^* \ge \frac{a\|T\|^2}{b}\theta_A S_A^{-1}\theta_A^* = \frac{a\|T\|^2}{b}P_A, \end{aligned}$$

and $NP_A = (\theta_A T S_A^{-1}\theta_A^*)P_A = N$. Thus $N \in \mathcal{M}_A$ and by the injectivity of the map Φ_A in Theorem 3.1, $M = \theta_A T S_A^{-1}\theta_A^*$. We now see that also $P_AM = M$, i.e.,

$M = P_A M P_A$. Furthermore,

$$M(\theta_A T^{-1} S_A^{-1} \theta_A^*) = \theta_A T T^{-1} S_A^{-1} \theta_A^* = \theta_A S_A^{-1} \theta_A^* = P_A,$$

and similarly, $(\theta_A T^{-1} S_A^{-1} \theta_A^*) M = P_A$. Thus M is invertible in $B(P_A \ell^2(\mathbb{J}) \otimes H_o)$.

(iii) $\Longrightarrow$ (iv) By Theorem 3.1, $P_B = [M]$, the range projection of M. By hypothesis, M is invertible in $B(P_A \ell^2(\mathbb{J}) \otimes H_o)$; hence $[M] = P_A$.

(iv) $\Longrightarrow$ (ii) Since $\theta_B = P_B \theta_B = P_A \theta_B = \theta_A (S_A^{-1} \theta_A^* \theta_B)$, it suffices to show that $S_A^{-1} \theta_A^* \theta_B$ has an inverse, namely $S_B^{-1} \theta_B^* \theta_A$. Indeed,

$$S_A^{-1} \theta_A^* \theta_B S_B^{-1} \theta_B^* \theta_A = S_A^{-1} \theta_A^* P_B \theta_A = S_A^{-1} \theta_A^* P_A \theta_A = S_A^{-1} \theta_A^* \theta_A = I.$$

Similarly,

$$S_B^{-1} \theta_B^* \theta_A S_A^{-1} \theta_A^* \theta_B = S_B^{-1} \theta_B^* P_A \theta_B = S_B^{-1} \theta_B^* P_B \theta_B = S_B^{-1} \theta_B^* \theta_B = I.$$

The uniqueness is then easily established. □

Remark 4.4. (i) For every operator-valued frame $\{A_j\}_{j\in\mathbb{J}}$, $\{B_j := A_j S_A^{-1/2}\}_{j\in\mathbb{J}}$ is a Parseval operator-valued frame. Thus every operator-valued frame is right-similar to a Parseval operator-valued frame. Every operator-valued frame right unitarily equivalent to $\{B_j\}_{j\in\mathbb{J}}$ is also Parseval and right-similar to $\{A_j\}_{j\in\mathbb{J}}$.

(ii) The equivalence of (i) and (iv) is the higher multiplicity analog of [15, Proposition 2.6 and Corollary 2.8].

Now we consider left similarity.

Lemma 4.5. *Let $\{A_j\}_{j\in\mathbb{J}}$ be an operator-valued frame having frame bounds a and b and let $R \in B(H_o)$ be an invertible operator. Then*

(i) *$\{B_j := RA_j\}_{j\in\mathbb{J}}$ is an operator-valued frame and $\frac{a}{\|R^{-1}\|^2} I \le S_B \le b\|R\|^2 I$. In particular, if R is unitary, then $\{B_j\}_{j\in\mathbb{J}}$ has the same frame bounds as $\{A_j\}_{j\in\mathbb{J}}$.*

(ii) *$\theta_B = (I \otimes R)\theta_A$, $S_B = \theta_A^*(I \otimes R^* R)\theta_A$, $P_B = [(I \otimes R)\theta_A]$ is the range projection of $(I \otimes R)\theta_A$, and $\Phi_A(\{B_j\}_{j\in\mathbb{J}}) = (I \otimes R)P_A$.*

(iii) *Assume that $\{A_j\}_{j\in\mathbb{J}}$ is Parseval. Then $\{B_j\}_{j\in\mathbb{J}}$ is Parseval if and only if $P_A(I \otimes R^* R)P_A = P_A$ if and only if $P_B = (I \otimes R)P_A(I \otimes R^*)$. In particular, this holds if R is an isometry.*

(iv) *P_B is unitarily equivalent to P_A.*

Proof. (i) Obvious.

(ii) For every $R \in B(H_o)$, $h \in H_o$, and $j \in \mathbb{J}$, from the definition of L_j we have that $L_j R h = e_j \otimes Rh = (I \otimes R)L_j h$, i.e., $L_j R = (I \otimes R)L_j$. Then

$$\theta_B = \sum_{j\in\mathbb{J}} L_j R A_j = \sum_{j\in\mathbb{J}} (I \otimes R) L_j A_j = (I \otimes R)\theta_A.$$

Consequently, $S_B = \theta_B^* \theta_B = \theta_A^*(I \otimes R^* R)\theta_A$. Clearly, $M := (I \otimes R)P_A \in \mathcal{M}_A$ and since $\theta_B = (I \otimes R)\theta_A = M\theta_A$, by the injectivity of Φ_A in Theorem 3.1, it follows that $\Phi_A(\{B_j\}_{j\in\mathbb{J}}) = M$. This can also be verified directly from

$$\Phi_A(\{B_j\}_{j\in\mathbb{J}}) = \theta_B S_A^{-1} \theta_A^* = (I \otimes R)\theta_A S_A^{-1} \theta_A^* = (I \otimes R)P_A.$$

(iii) Immediate from (ii).

(iv) Denote by $N(X)$ the null projection of the operator X. Then

$$P_B^\perp = [(I \otimes R)(\theta_A)]^\perp = [(I \otimes R)(\theta_A \theta_A^*)^{1/2}]^\perp = N\left((\theta_A \theta_A^*)^{1/2}(I \otimes R^*)\right).$$

Now $x \in N\left((\theta_A\theta_A^*)^{1/2}(I\otimes R^*)\right)$ if and only if $x \in (I\otimes R^*)^{-1}N\left((\theta_A\theta_A^*)^{1/2}\right)$; thus

$$P_B^\perp = (I\otimes R^{-1})^* P_A^\perp = [(I\otimes R^{-1})^* P_A^\perp] \sim [P_A^\perp (I\otimes R^{-1})] = P_A^\perp,$$

where we use the well-known fact $[X] \sim [X^*]$. Since $P_B \sim P_A$ (e.g., see Theorem 3.1), it follows that P_B and P_A are unitarily equivalent. □

In Proposition 4.3 we have seen that the invertible operator implementing the right-similarity of two operator-valued frames is uniquely determined and that it must be a unitary operator when both frames are Parseval. The following example shows that neither of these conclusions holds in the case of left-similarities.

Example 4.6. Let $H := \ell^2(\mathbb{J})$, $H_o := \mathbb{C}^2$, Q_1, Q_2 be two orthogonal (rank-one) projections in $B(H_o)$, let $P := I\otimes Q_1$, and let $\{A_j := L_j^*\, P|_{P\ell^2(\mathbb{J})\otimes H_o}\}_{j\in\mathbb{J}}$. By Example 2.2, $\{A_j\}_{j\in\mathbb{J}}$ is a Parseval operator-valued frame. Moreover, for every $\lambda \neq 0$, $R := Q_1 + \lambda Q_2$ is an invertible operator and

$$\begin{aligned} RA_j &= RL_j^*\, P|_{P\ell^2(\mathbb{J})\otimes H_o} = L_j^*(I\otimes R)\, P|_{P\ell^2(\mathbb{J})\otimes H_o} \\ &= L_j^*(I\otimes (Q_1+\lambda Q_2))\,(I\otimes Q_1)|_{P\ell^2(\mathbb{J})\otimes H_o} = L_j^*\, P|_{P\ell^2(\mathbb{J})\otimes H_o} = A_j. \end{aligned}$$

If $A_j\}_{j\in\mathbb{J}}$ and $\{B_j = RA_j\}_{j\in\mathbb{J}}$ are two left-similar Parseval operator-valued frames, then $P_B = (I\otimes R)P_A(I\otimes R^*)$ by Lemma 4.5 and furthermore there is a unitary operator U for which $P_B = UP_AU^*$. The following example shows that it can be impossible to choose $U \in I\otimes B(H_o)$.

Example 4.7. Let $H := \ell^2(\mathbb{J})$, $H_o := \mathbb{C}^2$, P_1, P_2 be two orthogonal projections in $B(H)$, $Q_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, Q_3 := \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, and $R := \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}$. Define $P := P_1\otimes Q_1 + P_2\otimes Q_2$, $\{A_j := L_j^*\, P|_{P\ell^2(\mathbb{J})\otimes H_o}\}_{j\in\mathbb{J}}$, and $\{B_j := RA_j\}_{j\in\mathbb{J}}$. Then by Example 2.2, $\{A_j\}$ is a Parseval operator-valued frame and $P_A = P$. Since

$$P(I\otimes R^*R)P = P_1\otimes (Q_1R^*RQ_1) + P_2\otimes (Q_2R^*RQ_2) = P_1\otimes Q_+P_2\otimes Q_2 = P,$$

by Lemma 4.5 (i) and (iii), $\{B_j\}_{j\in\mathbb{J}}$ is also a Parseval operator-valued frame and

$$\begin{aligned} P_B &= (1\otimes R)\theta_A\theta_A^*(1\otimes R^*) = (1\otimes R)P(1\otimes R^*) \\ &= P_1\otimes RQ_1R^* + P_2\otimes RQ_2R^* = P_1\otimes Q_1 + P_2\otimes Q_3. \end{aligned}$$

This implies that if $P_B = UPU^*$ for some unitary U, then $U \notin I\otimes B(H_o)$. Indeed if $U = 1\otimes W$ for some unitary $W \in B(H_o)$, then by the above computation,

$$P_B = P_1\otimes WQ_1W^* + P_2\otimes WQ_2W^* = P_1\otimes Q_1 + P_2\otimes Q_3,$$

which is impossible since $WQ_1W^*WQ_2W^* = 0$ while $Q_1Q_3 \neq 0$.

Operator-valued frames can be both right and left-similar. The following proposition determines when this occurs.

Proposition 4.8. *Let $\{A_j\}_{j\in\mathbb{J}} \in \mathcal{F}$, $R \in B(H_o)$ be an invertible operator, and let $\{B_j := RA_j\}_{j\in\mathbb{J}}$. Then the following conditions are equivalent.*

(i) *$\{B_j\}_{j\in\mathbb{J}}$ and $\{A_j\}_{j\in\mathbb{J}}$ are right-similar.*
(ii) *$(I\otimes R)P_A = P_A(I\otimes R)P_A$ is invertible in $B(P_A\ell^2(\mathbb{J})\otimes H_o)$.*
(iii) *$P_A^\perp(I\otimes R)P_A = 0$ and $P_A^\perp(I\otimes R^{-1})P_A = 0$.*
(iv)

$$RA_i = A_iS_A^{-1}\sum_{j\in\mathbb{J}} A_j^*RA_j \quad \textit{for every } i\in\mathbb{J} \tag{14}$$

and

$$R^{-1}A_i = A_i S_A^{-1} \sum_{j\in\mathbb{J}} A_j^* R^{-1} A_j \quad \textit{for every } i \in \mathbb{J}. \tag{15}$$

(v)

$$RA_i = A_i S_A^{-1} \sum_{j\in\mathbb{J}} A_j^* R A_j \quad \textit{for every } i \in \mathbb{J}$$

and

$$\left(\sum_{j\in\mathbb{J}} A_j^* R A_j\right)^{-1} = S_A^{-1}(\sum_{j\in\mathbb{J}} A_j^* R^{-1} A_j) S_A^{-1}. \tag{16}$$

If R is unitary, then these conditions are also equivalent to

(vi) $(I \otimes R)P_A = P_A(I \otimes R)$.

(vii) *R commutes with $A_j S_A^{-1} A_i^*$ for all $i, j \in \mathbb{J}$.*

Proof. (i) $\Longleftrightarrow$ (ii) Let $M := \Phi_A(\{B_j\}_{j\in\mathbb{J}})$. Then by Lemma 4.5, $M = (I \otimes R)P_A$. By Proposition 4.3, we see that $\{B_j\}_{j\in\mathbb{J}}$ is right-similar to $\{A_j\}_{j\in\mathbb{J}}$ if and only if $MP_A = P_A M P_A$ is invertible in $B(\ell^2(\mathbb{J}) \otimes H_o)$.

(ii) $\Longleftrightarrow$ (iii) Set

$$I \otimes R := \begin{pmatrix} P_A(I\otimes R)P_A & P_A(I\otimes R)P_A^\perp \\ 0 & P_A^\perp(I\otimes R)P_A^\perp \end{pmatrix}$$

and

$$(I \otimes R)^{-1} := \begin{pmatrix} P_A(I\otimes R^{-1})P_A & P_A(I\otimes R^{-1}))P_A^\perp \\ P_A^\perp(I\otimes R^{-1}))P_A & P_A^\perp(I\otimes R^{-1}))P_A^\perp \end{pmatrix}.$$

Then it is immediate to verify that $P_A^\perp(I \otimes R^{-1}))P_A = 0$ if and only if

$$(P_A(I\otimes R)P_A)(P_A(I\otimes R^{-1})P_A) = P_A,$$

and if and only if $P_A(I \otimes R)P_A$ is invertible.

(iii) $\Longleftrightarrow$ (iv) Recall that $(I\otimes R)L_j = L_j R$ for all $j \in \mathbb{J}$ and that $P_A = \theta_A S_A^{-1}\theta_A^*$. Therefore

$$(I\otimes R)P_A = (I\otimes R)\theta_A S_A^{-1}\theta_A^* = (I\otimes R)\sum_{j\in\mathbb{J}} L_j A_j S_A^{-1}\theta_A^* = \sum_{j\in\mathbb{J}} L_j R A_j S_A^{-1}\theta_A^*$$

and

$$\begin{aligned} P_A(I\otimes R)P_A &= \theta_A S_A^{-1}\theta_A^*(I\otimes R)\theta_A S_A^{-1}\theta_A^* \\ &= (\sum_{j\in\mathbb{J}} L_j A_j) S_A^{-1}(\sum_{i,j\in\mathbb{J}} A_i^* L_i^*(I\otimes R)L_j A_j) S_A^{-1}\theta_A^* \\ &= (\sum_{j\in\mathbb{J}} L_j A_j) S_A^{-1}(\sum_{i,j\in\mathbb{J}} A_i^* R L_i^* L_j A_j) S_A^{-1}\theta_A^* \\ &= (\sum_{j\in\mathbb{J}} L_j A_j) S_A^{-1}(\sum_{j\in\mathbb{J}} A_j^* R A_j) S_A^{-1}\theta_A^*. \end{aligned}$$

Hence $P_A^\perp(I\otimes R^{-1})P_A = 0$ if and only if

$$\sum_{j\in\mathbb{J}} L_j R A_j S_A^{-1}\theta_A^* = (\sum_{j\in\mathbb{J}} L_j A_j) S_A^{-1}(\sum_{j\in\mathbb{J}} A_j^* R A_j) S_A^{-1}\theta_A^*. \tag{17}$$

By multiplying (17) on the left by L_i^* and on the right by θ_A, we obtain (14). Conversely, by multiplying (14) on the left by L_i and on the right by $S_A^{-1}\theta_A^*$ and

summing over $i \in \mathbb{J}$ we obtain back (17). Thus $P_A^{\perp}(I \otimes R^{-1})P_A = 0$ is equivalent to (14). By the same argument, $P_A^{\perp}(I \otimes R^{-1})P_A = 0$ is equivalent to (15).

(iv) $\Longleftrightarrow$ (v) Assume that (14) and (15) hold. Then

$$A_i^* A_i = A_i^* R^{-1} A_i S_A^{-1} (\sum_{j \in \mathbb{J}} A_j^* R A_j) \quad \text{for all } i,$$

and hence by summing over $i \in \mathbb{J}$ and then multiplying on the left and on the right by $S_A^{-1/2}$ we obtain

$$S_A^{-1/2} (\sum_{i \in \mathbb{J}} A_i^* R^{-1} A_i) S_A^{-1} (\sum_{j \in \mathbb{J}} A_j^* R A_j) S_A^{-1/2} = I.$$

Similarly,

$$S_A^{-1/2} (\sum_{i \in \mathbb{J}} A_i^* R A_i) S_A^{-1} (\sum_{j \in \mathbb{J}} A_j^* R^{-1} A_j) S_A^{-1/2} = I.$$

Thus

$$\left(S_A^{-1/2} (\sum_{j \in \mathbb{J}} A_j^* R A_j) S_A^{-1/2} \right)^{-1} = S_A^{-1/2} (\sum_{j \in \mathbb{J}} A_j^* R^{-1} A_j) S_A^{-1/2},$$

and hence (16) holds. Conversely, if (14) and (16) hold, then by multiplying (14) on the right by the right-hand side of (16) we easily obtain (15).

Assume now that R is unitary.

(iii) $\Longleftrightarrow$ (vi) Obvious.

(vi) $\Longrightarrow$ (vii) As seen above in the course of the proof of (iii) $\Longleftrightarrow$ (iv),

$$(I \otimes R)P_A = \sum_{i,j \in \mathbb{J}} L_j R A_j S_A^{-1} A_i^* L_i^*$$

and

$$P_A (I \otimes R) = \sum_{i,j \in \mathbb{J}} L_j A_j S_A^{-1} A_i^* R L_i^*.$$

Thus

$$\sum_{i,j \in \mathbb{J}} L_j R A_j S_A^{-1} A_i^* L_i^* = \sum_{i,j \in \mathbb{J}} L_j A_j S_A^{-1} A_i^* R L_i^*.$$

By multiplying this identity on the left by L_j^* and on the right by L_i and recalling (4) we obtain (vii).

(vii) $\Longrightarrow$ (iv) For every $i \in \mathbb{J}$,

$$A_i S_A^{-1} \sum_{j \in \mathbb{J}} A_j^* R A_j = R A_i S_A^{-1} \sum_{j \in \mathbb{J}} A_j^* A_j = R A_i,$$

i.e., (14) holds. Since $R^{-1} = R^*$ also commutes with all $A_j S_A^{-1} A_i^*$, the same argument shows that (15) too holds. □

Composition of frames. Let $A = \{A_j\}_{j \in \mathbb{J}}$ be an operator-valued frame in $B(H, H_o)$ and $B = \{B_m\}_{m \in \mathbb{M}}$ be an operator-valued frame in $B(H_o, H_1)$. Then it is easy to check that $\{C_{(m,j)} := B_m A_j\}_{m \in \mathbb{M}, j \in \mathbb{J}}$ is an operator-valued frame in $B(H, H_1)$. The operator-valued frame $\{C_{(m,j)}\}_{(m,j) \in \mathbb{M} \times \mathbb{J}}$ is called the composition of the frames $A = \{A_j\}_{j \in \mathbb{J}}$ and $B = \{B_m\}_{m \in \mathbb{M}}$.

Proposition 4.9. *Let $\{C_{(m,j)} := B_m A_j\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$ be the composition of the operator-valued frames $\{A_j\}_{j\in\mathbb{J}}$ and $B = \{B_m\}_{m\in\mathbb{M}}$. Then*

(i) *$\theta_C = (I_{\mathbb{J}} \otimes \theta_B)\theta_A$ and $S_C = \theta_A^*(I_{\mathbb{J}} \otimes S_B)\theta_A$, where $I_{\mathbb{J}}$ is the identity on $l^2(\mathbb{J})$. In particular, if $\{A_j\}_{j\in\mathbb{J}}$ and $\{B_m\}_{m\in\mathbb{M}}$ are Parseval, then $\{C_{(m,j)}\}_{m\in\mathbb{M},j\in\mathbb{J}}$ is Parseval.*

(ii) *If P_A commutes with $(I_{\mathbb{J}}\otimes S_B)$, then $P_C = (I_{\mathbb{J}}\otimes\theta_B)P_A(I_{\mathbb{J}}\otimes S_B^{-1})P_A(I_{\mathbb{J}}\otimes\theta_B^*)$.*

Proof. (i) Let $\{e_j\}_{j\in\mathbb{J}}$ and $\{f_m\}_{m\in\mathbb{M}}$ denote the standard orthonormal bases of $\ell^2(\mathbb{J})$ and $\ell^2(\mathbb{M})$, respectively. Then for any $x \in H$,

$$\begin{aligned}(I_{\mathbb{J}} \otimes \theta_B)\theta_A(x) &= (I_{\mathbb{J}} \otimes \theta_B)\sum_{j\in\mathbb{J}}(e_j \otimes A_j x) = \sum_{j\in\mathbb{J}}(e_j \otimes \theta_B(A_j x))\\ &= \sum_{m\in\mathbb{M},j\in\mathbb{J}}(e_j \otimes f_m \otimes B_m A_j x) = \theta_C(x).\end{aligned}$$

The formula for S_C now follows directly, as well as the Parseval case.

(ii) If P_A commutes with $(I_{\mathbb{J}} \otimes S_B)$, then

$$P_A(I_{\mathbb{J}} \otimes S_B^{-1})P_A = (P_A(I_{\mathbb{J}} \otimes S_B)P_A)^{-1};$$

hence $S_C^{-1} = S_A^{-1}\theta_A^*(I_{\mathbb{J}} \otimes S_B^{-1})\theta_A S_A^{-1}$, and thus

$$\begin{aligned}P_C &= \theta_C S_C^{-1}\theta_C^* = (I_{\mathbb{J}} \otimes \theta_B)\theta_A S_A^{-1}\theta_A^*(I_{\mathbb{J}} \otimes S_B^{-1})\theta_A S_A^{-1}\theta_A^*(I_{\mathbb{J}} \otimes \theta_B)\\ &= (I_{\mathbb{J}} \otimes \theta_B)P_A(I_{\mathbb{J}} \otimes S_B^{-1})P_A(I_{\mathbb{J}} \otimes \theta_B).\end{aligned}$$

□

Remark 4.10. (i) If the composition of an operator-valued frame $\{B_m\}_{m\in\mathbb{I}}$ with the frame $\{A_j\}_{j\in\mathbb{J}}$ is the same as the composition with the frame $\{A'_j\}_{j\in\mathbb{J}}$, then $\{A_j\}_{j\in\mathbb{J}} = \{A'_j\}_{j\in\mathbb{J}}$. Indeed, if $B_m A_j = B_m A'_j$ for all $m \in \mathbb{M}$ and $j \in \mathbb{J}$, then

$$\sum_{m\in M} B_m^* B_m A_j = S_B A_j = S_B A'_j,$$

and hence $A_j = A'_j$, since S_B is invertible.

(ii) Assume that $\bigcup_{j\in\mathbb{J}} A_j H_o$ is dense in H_o. Then the composition of $\{A_j\}_{j\in\mathbb{J}}$ with $\{B_m\}_{m\in\mathbb{M}}$ equals the composition of $\{A_j\}_{j\in\mathbb{J}}$ with $\{B'_m\}_{m\in\mathbb{M}}$ if and only if $\{B_m\}_{i\in\mathbb{I}} = \{B'_m\}_{m\in\mathbb{M}}$.

Given an operator-valued frame $\{A_j\}_{j\in\mathbb{J}}$ with $A_j \in B(H, H_o)$ and given an arbitrary vector frame $\{B_m\}_{m\in\mathbb{M}}$ with $B_m \in B(H_o, \mathbb{C})$, we can view the vector frame $\{C_{(m,j)} := B_m A_j\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$ to be a "decomposition" of the original operator-valued frame.

This decomposition is of course not unique. For instance, in the discussion after Definition 2.1, we chose $\{B_m\}_{m\in\mathbb{M}}$ to be an orthonormal basis $\{e_m\}_{m\in\mathbb{M}}$ of H_o and then $C_{(m,j)} := B_m A_j$ corresponds to the vector frame $\{A_j^* e_m\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$. If $\{B'_m\}_{m\in\mathbb{M}}$ is (right) similar to $\{B_m\}_{m\in\mathbb{M}}$, i.e., $B'_m = B_m R$ for some invertible operator $R \in B(H_o)$ and all $m \in \mathbb{M}$ (i.e., if B'_m is a Riesz frame (see Remark 2.6), then $C'_{(m,j)} := B_m R A_j$ corresponds to the vector frame $\{A_j^* R^* e_m\}_{m\in\mathbb{M},j\in\mathbb{J}}$ and provides a vector frame decomposition both of the operator-valued frame $\{A_j\}_{j\in\mathbb{J}}$ and also of the left-similar operator-valued frame $\{RA_j\}_{j\in\mathbb{J}}$. $\{A_j^* R^* e_m\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$ is similar to $\{A_j^* e_m\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$ only if R satisfies the conditions of Proposition P: left right. However, if $T \in B(H)$ is invertible, $\{TA_j^* e_m\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$ is by definition

similar to $\{A_j^* e_m\}_{(m,j)\in\mathbb{M}\times\mathbb{J}}$ and provides a vector decomposition of the operator-valued frame $\{A_j T^*\}_{j\in\mathbb{J}}$.

5. Dual frames, complementary frames, and disjointness

Dual frames. A vector frame $\{b_j\}_{j\in\mathbb{J}}$ on a Hilbert space H is said to be the dual of another vector frame $\{a_j\}_{j\in\mathbb{J}}$ on H if

$$x = \sum_{j\in\mathbb{J}} (x, a_j) b_j \quad \text{for all } x \in H.$$

It is well known and easy to see that this condition can be reformulated in terms of the analysis operators θ_b and θ_a of the two frames as

$$\theta_b^* \theta_a = I.$$

In particular, $\{S_a^{-1} a_j\}_{j\in\mathbb{J}}$ has analysis operator $\theta_a S_a^{-1}$ and is called the *canonical dual* of the frame $\{a_j\}_{j\in\mathbb{J}}$ (other duals are called *alternate duals*). When $\{a_j\}_{j\in\mathbb{J}}$ is Parseval, the identity

$$x = \theta_a^* \theta_a x = \sum_{j\in\mathbb{J}} (x, a_j) a_j \quad \text{for all } x \in H$$

is called the *reconstruction formula* (see [15, Sections 1.2, 1.3]).

These notions extend naturally to operator-valued frames:

Definition 5.1. Let $\{A_j\}_{j\in\mathbb{J}}, \{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$. Then $\{B_j\}_{j\in\mathbb{J}}$ is called a dual of $\{A_j\}_{j\in\mathbb{J}}$ if $\theta_B^* \theta_A = I$. The operator-valued frame $\{A_j S_A^{-1}\}_{j\in\mathbb{J}}$ is called the *canonical dual frame* of $\{A_j\}_{j\in\mathbb{J}}$.

Remark 5.2. (i) Notice that $\theta_B^* \theta_A = I$ if and only if $\theta_A^* \theta_B = I$; i.e., $\{B_j\}_{j\in\mathbb{J}}$ is a dual of $\{A_j\}_{j\in\mathbb{J}}$ if and only if $\{A_j\}_{j\in\mathbb{J}}$ is a dual of $\{B_j\}_{j\in\mathbb{J}}$ (cf. [15, Proposition 1.13]).

(ii) If we consider two operator-valued frames $\{A_j\}_{j\in\mathbb{J}}$ with $A_j \in B(H_1, H_o)$ and $\{B_j\}_{j\in\mathbb{J}}$ with $B_j \in B(H_2, H_o)$ with a given unitary $U : H_1 \mapsto H_2$, and if we want to keep track of the different Hilbert spaces, then we would define duality between them (relative to the choice of the unitary U) by asking that $\theta_B^* \theta_A = U$, or, equivalently, that $\theta_A^* \theta_B = U^*$.

In general, duals are far from unique. Theorem 3.1 provides a natural way to parametrize the collection of all the dual frames of a given operator-valued frame. Recall that for $\{A_j\}_{j\in\mathbb{J}}, \{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$, there is a unique $M \in B(\ell(\mathbb{J}) \otimes H_o)$ such that $\theta_B = M\theta_A$, $M = MP_A$ and M^*M is invertible in $B(P_A \ell^2(\mathbb{J}) \otimes H_o)$; namely, $M := \Phi_A(\{B_j\}_{j\in\mathbb{J}})$. Then in the two-by-two matrix relative to the decomposition $P_A + P_A^\perp = I \otimes I_o$, $M = \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}$ and $X^*X + Y^*Y$ is invertible.

Proposition 5.3. *Let $\{A_j\}_{j\in\mathbb{J}}, \{B_j\}_{j\in\mathbb{J}} \in \mathcal{F}$ and let $M := \Phi_A(\{B_j\}_{j\in\mathbb{J}})$. Then the following conditions are equivalent.*

(i) *$\{B_j\}_{j\in\mathbb{J}}$ is a dual of $\{A_j\}_{j\in\mathbb{J}}$.*

(ii) *$\sum_{j\in\mathbb{J}} B_j^* A_j = I$, where the convergence is in the strong operator topology.*

(iii) *$P_A M P_A = \theta_A^* S_A^{-2} \theta_A$.*

In particular, $\{B_j\}_{j\in\mathbb{J}}$ is the canonical dual of $\{A_j\}_{j\in\mathbb{J}}$ if and only if $M = P_A M P_A$.

Proof. (i) $\Longleftrightarrow$ (ii). Obvious by (11).

(i) $\Longleftrightarrow$ (iii) By Theorem 3.1, there is a unique operator $M \in B(\ell(\mathbb{J}) \otimes H_o)$ with $M = MP_A$ and $M^*MP_A\,|_{P_A\ell^2(\mathbb{J})\otimes H_o}$ invertible, for which $B_j = L_j^* M\theta_A$ for all $j \in \mathbb{J}$, or, equivalently, for which $\theta_B = M\theta_A$. Then $I = \theta_A^*\theta_B = \theta_A^* M\theta_A$ if and only if

$$P_A M P_A = \theta_A^* S_A^{-1}\theta_A^* M\theta_A S_A^{-1}\theta_A^* = \theta_A^* S_A^{-2}\theta_A.$$

If for some $M = MP_A$, $P_A M P_A = \theta_A^* S_A^{-2}\theta_A$, which is invertible, then also $M^*MP_A\,|_{P_A\ell^2(\mathbb{J})\otimes H_o}$ is invertible.

By definition, $\{B_j\}_{j\in\mathbb{J}}$ is the canonical dual of $\{A_j\}_{j\in\mathbb{J}}$ if and only if

$$\theta_B = \theta_A S_A^{-1} = \big(\theta_A S_A^{-2}\theta_A^*\big)\theta_A,$$

i.e., if and only if $M = \theta_A S_A^{-2}\theta_A^*$. □

By the above proposition, the only operator-valued frames that have a unique dual frame are those with range projection $P_A = I \otimes I_o$. By Remark 2.6, these are the Riesz operator-valued frames (cf. [15, Corollary 2.26]).

Given an operator-valued frame $\{A_j\}_{j\in\mathbb{J}}$, set

$$M := \Phi_A(\{B_j\}_{j\in\mathbb{J}}) = \begin{pmatrix} X & 0 \\ Y & 0 \end{pmatrix}.$$

Proposition 5.3 (iii) states that dual frames are characterized by $X = \theta_A S_A^{-2}\theta_A^*$ (by $X = P_A$ if $\{A_j\}_{j\in\mathbb{J}}$ is Parseval).

Frames that are right-similar to $\{A_j\}_{j\in\mathbb{J}}$ are characterized by $Y = 0$ (see Proposition 4.3). Thus the only frame that is both right-similar and dual to $\{A_j\}_{j\in\mathbb{J}}$ corresponds to $X = \theta_A S_A^{-2}\theta_A^*$ and $Y = 0$, and hence it is the canonical dual of $\{A_j\}_{j\in\mathbb{J}}$. Equivalently, if two right-similar frames are both the dual of a given frame, then they are equal. To summarize:

Proposition 5.4. (i) *(cf.* [15, Proposition 1.14]*) If two operator-valued frames are right-similar and are dual of the same operator-valued frame , then they are equal.*

(ii) *If two Parseval operator-valued frames are one the dual of the other, then they are equal.*

In general there are infinitely many dual frames of a given operator-valued frame that are left-similar to it.

Example 5.5. Let $H := \ell^2(\mathbb{J})$, $H_o := \mathbb{C}^2$, $Q_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $R := \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$. Define $P := I \otimes Q_1$ and define the operator-valued frames $\{A_j := L_j^*\, P|_{P\ell(\mathbb{J})\otimes H_o}\}_{j\in\mathbb{J}}$, and $\{B_j := RA_j\}_{j\in\mathbb{J}}$. Then by Example 2.2, $\{A_j\}$ is a Parseval operator-valued frame and $P_A = P$. Now $M = (I \otimes R)P$, $PMP = P$ and hence $\{B_j := RA_j\}_{j\in\mathbb{J}}$ is a dual of $\{A_j\}_{j\in\mathbb{J}}$ for every λ. $P^\perp MP = I \otimes \begin{pmatrix} 0 & 0 \\ \lambda & 1 \end{pmatrix}$ however; hence every λ defines a different operator-valued frame $\{B_j\}_{j\in\mathbb{J}}$.

Disjoint frames and complementary frames. The treatment given in [15, Chapter 2] for the vector case generalizes without difficulties to the higher multiplicity case. For the reader's convenience we present the definitions and one of the key arguments.

Definition 5.6. Let $\{A_j\}_{j\in\mathbb{J}}$ and $\{B_j\}_{j\in\mathbb{J}}$ be two operator-valued frames with $A_j \in B(H_A, H_o)$ and $B_j \in B(H_B, H_o)$ for all $j \in \mathbb{J}$. Then the two frames are called:

(i) disjoint if $\{A_j \oplus B_j\}_{j\in\mathbb{J}}$ is an operator-valued frame;

(ii) strongly disjoint (also called orthogonal) if there are two invertible operators $T_A \in B(H_A)$ and $T_B \in B(H_B)$ such that $\{A_jT_A \oplus B_jT_B\}_{j\in\mathbb{J}}$ is a Parseval operator-valued frame on $H_A \oplus H_B$;

(iii) strongly complementary if there are two invertible operators $T_A \in B(H_A)$ and $T_B \in B(H_B)$ such that $\{A_jT_A \oplus B_jT_B\}_{j\in\mathbb{J}}$ is an orthonormal operator-valued frame on $H_A \oplus H_B$.

Proposition 5.7. *Let $\{A_j\}_{j\in\mathbb{J}}$ and $\{B_j\}_{j\in\mathbb{J}}$ be two operator-valued frames as in the Definition* 5.6. *Then the two frames are:*

(i) *disjoint if and only if $P_AH_{\mathbb{J}} \cap P_BH_{\mathbb{J}} = \{0\}$ and $P_AH_{\mathbb{J}} + P_BH_{\mathbb{J}}$ is closed;*

(ii) *strongly disjoint if and only if $P_AP_B = 0$ if and only if $\theta_A^*\theta_B = 0$ if and only if $\theta_B^*\theta_A = 0$;*

(iii) *strongly complementary if and only if $P_A + P_B = I \otimes I_o$.*

Proof. (i) $\{A_j \oplus B_j\}_{j\in\mathbb{J}}$ is an operator-valued frame if and only if $\{A_jT_A \oplus B_jT_B\}_{j\in\mathbb{J}}$ is also an operator-valued frame for any choice of invertible operators T_A and T_B. Since right-similarities do not change the frame projections (see Proposition 4.3), we can assume that both frames are Parseval. Since $(A_j \oplus B_j)(x \oplus y) = A_jx + B_jy$ for all $x \in H_A,\ y \in H_B$, a simple computation shows that $(A_j \oplus B_j)^*z = A_j^*z \oplus B_j^*z$ for all $z \in H_o$; hence

$$\begin{aligned}\big((A_j \oplus B_j)^*(A_j \oplus B_j)(x \oplus y), (x \oplus y)\big) &= (A_jx + B_jy, A_jx + B_jy)\\ &= (A_jx, A_jx) + (B_jy, B_jy) + (A_jx, B_jy) + (B_jy, A_jx)\\ &= (A_j^*A_jx, x) + (B_j^*B_jy, y) + (B_j^*A_jx, y) + (A_j^*B_jy, x).\end{aligned}$$

Now sum over $\mathbb{J}$ using the fact (see (11)) that all the series converge in the strong operator topology, e.g., $\sum_{j\in\mathbb{J}} A_j^*B_j = \theta_A^*\theta_B$ and similarly for the other series. Thus

$$\begin{aligned}\sum_{j\in\mathbb{J}}&\big((A_j \oplus B_j)^*(A_j \oplus B_j)(x \oplus y), (x \oplus y)\big)\\ &= (\theta_A^*\theta_Ax, x) + (\theta_B^*\theta_By, y) + (\theta_B^*\theta_Ax, y) + (\theta_A^*\theta_By, x)\\ &= \big(\theta_Ax + \theta_By, \theta_Ax + \theta_By\big).\end{aligned}$$

As in the proof of [15, Theorem 2.9], the projections P_A and P_B satisfy the condition in (i) if and only if

$$a\|\theta_Ax + \theta_By\| \le \sqrt{\|\theta_Ax\|^2 + \|\theta_By\|^2} = \sqrt{\|x\|^2 + \|y\|^2} = \|x \oplus y\| \le b\|\theta_Ax + \theta_By\|$$

for some $a, b > 0$. By definition, this is precisely the condition that guarantees that $\{A_j \oplus B_j\}_{j\in\mathbb{J}}$ is an operator-valued frame.

(ii) It is clear that the three conditions $P_AP_B = 0$, $\theta_A^*\theta_B = 0$, and $\theta_B^*\theta_A = 0$ are all equivalent and that they also imply the condition in (i). If these conditions hold, by the proof of (i), $\big(S_{A\oplus B}(x \oplus y), (x \oplus y)\big) = (S_Ax, x) + (S_By, y)$ for all $x \in H_A,\ y \in H_B$. In particular, $S_{A\oplus B} = I_A \otimes I_B$ if (and only if) $S_A = I_A$ and $S_B = I_B$; i.e., it is sufficient to choose right-similarities that make the two operator-valued frames Parseval to obtain that their direct sum is also Parseval. Conversely, assume without loss of generality that the direct sum of the frames is already

Parseval and hence both frames are Parseval. Then, again by the computation in the first part of the proof,

$$\|\theta_A x\|^2 + \|\theta_A y\|^2 = \|x\|^2 + \|y\|^2 = \big(S_{A\oplus B}(x \oplus y), (x \oplus y)\big) = \|\theta_A x + \theta_B y\|^2.$$

This clearly implies that the ranges of θ_A and θ_B are orthogonal.

(iii) From the proof of (ii), it is easy to see that if $P_A P_B = 0$, then $P_{A+B} = P_A + P_B$. The rest of the proof is then obvious. □

Corollary 5.8. *Let $\{A_j\}_{j\in\mathbb{J}}$ be an operator-valued frame on a Hilbert space H, with range in H_o.*

(i) *Up to right unitary equivalence, the collection of strong complements of $\{A_j\}_{j\in\mathbb{J}}$ is uniquely parametrized by*

$$\big\{\{L_j^* T\}_{j\in\mathbb{J}} \mid T = P_A^\perp T P_A^\perp \geq 0,\, T \text{ invertible in } B(P_A^\perp \ell^2(\mathbb{J}) \otimes H_o)\big\}.$$

(ii) *Up to right unitary equivalence, the collection of operator-valued frames strongly disjoint from $\{A_j\}_{j\in\mathbb{J}}$ is uniquely parametrized by*

$$\big\{\{L_j^* T\}_{j\in\mathbb{J}} \mid P \leq P_A^\perp, T = PTP \geq 0,\, T \text{ invertible in } B(P\ell^2(\mathbb{J}) \otimes H_o)\big\}.$$

6. Unitary systems and groups

General unitary systems and local commutants. In the terminology of [9] and [15], a unitary system $\mathcal{U}$ on a Hilbert space H is simply a collection of unitary operators that includes the identity. Following the customary terminology for (vector) frames, we introduce the analogous notion for operator-valued frames:

Definition 6.1. An operator $A \in B(H, H_o)$ is called an operator frame generator for a unitary system $\mathcal{U}$ if $\{AU^*\}_{U\in\mathcal{U}}$ is an operator-valued frame. If $\{AU^*\}_{U\in\mathcal{U}}$ is Parseval, A is said to be a Parseval operator frame generator.

If $\dim H_o = 1$, i.e., A corresponds to a vector ψ, then AU^* corresponds to the vector $U\psi$. Recall from [15, Proposition 3.1] that given a wavelet generator for a unitary system $\mathcal{U}$, i.e., a vector ψ such that $\{U\psi\}_{U\in\mathcal{U}}$ is an orthonormal basis, then a vector ϕ is a Parseval frame generator from $\mathcal{U}$ if and only if $\phi = V\psi$ for a co-isometry V such that $(VU - UV)\psi = 0$ for every $U \in \mathcal{U}$. The *local commutant* at ψ is defined in [9] as the collection

$$C_\psi(\mathcal{U}) := \{T \in B(H) \mid (TU - UT)\psi = 0 \quad \text{for all} \quad U \in \mathcal{U}\}.$$

The multi-dimensional analog of an orthonormal basis is a collection of partial isometries with mutually orthogonal equivalent domains spanning the Hilbert space, or equivalently, an operator-valued frame with frame projection $\theta_A\theta_A^* = I$.

Proposition 6.2. *Suppose that $A \in B(H, H_o)$ is a frame generator for a unitary system $\mathcal{U}$ for which $\theta_A\theta_A^* = I$. Then an operator $B \in B(H, H_o)$ is a Parseval frame generator for $\mathcal{U}$ if and only if $B = AV^*$ for some co-isometry V such that $(VU - UV)A^* = 0$ for every $U \in \mathcal{U}$. If $B = AV^*$ for such a co-isometry, then $V := \theta_B^*\theta_A$.*

Proof. Assume B is a Parseval frame generator for $\mathcal{U}$ and let $V := \theta_B^*\theta_A$. Then by the hypotheses, $\theta_B^*\theta_A\theta_A^*\theta_B = I$; i.e., V is a co-isometry and

$$B = BI = L_I\theta_B = L_I\theta_A\theta_A^*\theta_B = AV^*.$$

Since $AU^* = L_U^*\theta_A$ and $BU^* = L_U^*\theta_B A$ for all $U \in \mathcal{U}$, we have

$$AU^*V^* = L_U^*\theta_A\theta_A^*\theta_B = L_U^*\theta_B = BU^* = AV^*U^*.$$

Moreover, if $AV^* = AW^*$ and $A(WU - UW)^* = 0$ for every $U \in \mathcal{U}$, then also $AU^*V^* = AU^*W^*$ and hence $\theta_A V^* = \theta_A W^*$, whence $V = W$. Conversely, if V is a co-isometry and $A(VU - UV)^* = 0$ for every $U \in \mathcal{U}$, then

$$\sum_{U\in\mathcal{U}} UVA^*AV^*U^* = V\left(\sum_{U\in\mathcal{U}} UA^*AU^*\right)V^* = VV^* = I,$$

whence AV^* is a Parseval frame generator for $\mathcal{U}$. □

Discrete group representations. Let G be a discrete group, not necessarily countable and let λ be the left regular representation of G (resp., let ρ be the right regular representation of G). Denote by $\mathcal{L}(G) \subset B(\ell^2(G))$, (resp., $\mathcal{R}(G)$) the von Neumann algebra generated by the unitaries $\{\lambda_g\}_{g\in G}$ (resp., $\{\rho_g\}_{g\in G}$). It is well known that both $\mathcal{L}(G)' = \mathcal{R}(G)$ and $\mathcal{R}(G)' = \mathcal{L}(G)$ are finite von Neumann algebras that share a faithful trace vector χ_e, where $\{\chi_g\}_{g\in G}$ is the standard basis of $\ell^2(G)$.

Let H_o be a Hilbert space and I_o be the identity of $B(H_o)$. Then we call $\lambda \otimes id : G \ni g \longrightarrow \lambda_g \otimes I_o$ the left regular representation of G with multiplicity H_o. Set $H_G = \ell^2(G) \otimes H_o$.

Given a unitary representation (G, π, H), denote by $\pi(G)''$ the von Neumann algebra generated by $\{\pi_g\}_{g\in G}$. Operator frame generators, if any, for the unitary system $\{\pi_g\}_{g\in G}$, are called generators for the representation. Explicitly:

Definition 6.3. Let (G, π, H) be a unitary representation of the discrete group G on the Hilbert space H. Then an operator $A \in B(H, H_o)$ is called a frame generator (resp. a Parseval frame generator) with range in H_o for the representation if $\{A_g := A\pi_{g^{-1}}\}_{g\in G}$ is an operator-valued frame (resp. a Parseval operator-valued frame).

Before characterizing those representations that have an operator frame generator and then parametrizing its generators, we need the following preliminary lemma.

Lemma 6.4. *Let A and B be two generators with range in H_o for a unitary representation (G, π, H). Then*

(i) $\theta_A\pi_g = (\lambda_g \otimes I_o)\theta_A$ *for all* $g \in G$.

(ii) $\theta_A^*\theta_B$ *is in the commutant* $\pi(G)'$ *of* $\pi(G)''$. *In particular,* $S_A \in \pi(G)'$ *and* $AS_A^{-1/2}$ *is a Parseval frame generator.*

(iii) $\theta_A T\theta_B^* \in \mathcal{R}(G) \otimes B(H_o)$ *for any* $T \in \pi(G)'$. *In particular, we have that* $P_A \in \mathcal{R}(G) \otimes B(H_o)$.

(iv) $P_A \sim P_B$, *where the equivalence is in* $\mathcal{R}(G) \otimes B(H_o)$, *i.e., it is implemented by a partial isometry belonging to* $\mathcal{R}(G) \otimes B(H_o)$.

Proof. (i) For all g, $q \in G$ and $h \in H_o$, one has

$$L_{gq}h = \chi_{gq} \otimes h = \lambda_g\,\chi_q \otimes h = (\lambda_g \otimes I_o)\,(\chi_q \otimes h) = (\lambda_g \otimes I_o)\,L_q h.$$

Thus

$$\begin{aligned}\theta_A\pi_g &= \sum_{p\in G} L_p A\pi_{p^{-1}}\pi_g = \sum_{p\in G} L_p A\pi_{p^{-1}g} = \sum_{q\in G} L_{gq} A\pi_{q^{-1}}\\ &= \sum_{q\in G}(\lambda_g\otimes I_o)L_q A\pi_{q^{-1}} = (\lambda_g\otimes I_o)\theta_A.\end{aligned}$$

(ii) For all $g\in G$ one can apply (i) twice and obtain

$$\theta_A^*\theta_B\pi_g = \theta_A^*(\lambda_g\otimes I_o)\theta_B = \pi_g\theta_A^*\theta_B.$$

Thus, $\theta_A^*\theta_B\in\pi(G)'$. In particular, setting $B=A$ we have $S_A=\theta_A^*\theta_A\in\pi(G)'$. Then $AS_A^{-1/2}\pi_{g^{-1}} = A\pi_{g^{-1}}S_A^{-1/2}$ for all $g\in G$, and hence $AS_A^{-1/2}$ is a Parseval frame generator.

(iii) For all $g\in G$ and $T\in\pi(G)'$, applying (i) twice, one obtains

$$\theta_A T\theta_B^*(\lambda_g\otimes I_o) = \theta_A T\pi_g\theta_B^* = \theta_A\pi_g T\theta_B^* = (\lambda_g\otimes I_o)\theta_A T\theta_B^*.$$

Therefore,

$$\theta_A T\theta_B^*\in(\mathcal{L}(G)\otimes I_o)' = \mathcal{L}(G)'\otimes(I_o)' = \mathcal{R}(G)\otimes B(H_o).$$

Setting $A=B$ and $T=S_A^{-1}$, we see that $P_A=\theta_A S_A^{-1}\theta_A^*\in\mathcal{R}(G)\otimes B(H_o)$.

(iv) By passing if necessary to $AS_A^{-1/2}$ (resp., $BS_B^{-1/2}$) which by (ii) is a Parseval frame generator and by Proposition 4.3 has frame projection P_A (resp., P_B), we can assume, without loss of generality, that both θ_A and θ_B are isometries. Then the partial isometry $V=\theta_B\theta_A^*\in\mathcal{R}(G)\otimes B(H_o)$ implements the equivalence, i.e., $V^*V=P_A$ and $VV^*=P_B$. □

Given a countable group, in [15, Theorems 3.8, 3.11, Proposition 6.2], Han and Larson have identified its representations that have a multi-frame (vector) generator with the subrepresentations of its left regular representation with finite multiplicity. Lemma 6.4 can be used to easily reobtain their result with an increase in generality.

Theorem 6.5. *A unitary representation (G,π,H) of a discrete group is unitarily equivalent to a subrepresentation of $\lambda\otimes id$ with multiplicity H_o if and only if (G,π,H) has an operator frame generator with range in H_o.*

Proof. Assume that (G,π,H) is unitarily equivalent to, and hence can be identified with, a subrepresentation of $\lambda\otimes id$, the restriction $\lambda\otimes id\,|_{PH_G}$ for some projection $P\in(\mathcal{L}(G)\otimes I_o)'=\mathcal{R}(G)\otimes B(H_o)$. Let $H:=PH_G$, $A:=L_e^*P|_H$, and

$$A_g = L_e^*P|_H\,(\lambda_g\otimes I_o)\;P|_H = L_e^*(\lambda_g\otimes I_o)\;P|_H \quad \text{for all } g\in G.$$

Then

$$\begin{aligned}S_A &= \sum_{g\in G} A_g^*A_g = \sum_{g\in G} P(\lambda_g\otimes I_o)L_eL_e^*(\lambda_{g^{-1}}\otimes I_o)\;P|_H\\ &= P(\sum_{g\in G} L_gL_g^*)\;P|_H = I|_H\,.\end{aligned}$$

This shows that A is a (Parseval) frame generator for (G,π,H).

Conversely, assume that $A\in B(H,H_o)$ is a frame generator for (G,π,H). Then $\theta_A S_A^{-1/2}$ is an isometry onto the subspace P_AH_G, $S_A^{-1/2}$ commutes with π, and P_A

commutes with the left regular representation with multiplicity H_o by Lemma 6.4 (ii) and (iii). Then by Lemma 6.4 (i),

$$\theta_A S_A^{-1/2}\pi_g = \theta_A \pi_g S_A^{-1/2} = (\lambda_g \otimes I_o)\theta_A S_A^{-1/2} = (\lambda_g \otimes I_o)\ P_A|_{P_A H_G}\ \theta_A S_A^{-1/2}$$

for all $g \in G$; i.e., π is unitarily equivalent to $\lambda \otimes id\ P_A|_{P_A H_G}$. □

From the above proof it is easy to obtain the following:

Remark 6.6. If $A \in B(H, H_o)$ is a frame generator for (G, π, H), then the equivalence of (G, π, H) and $(G, \lambda \otimes id\ |_{P_A H_G}, P_A H_G)$ is implemented by the isometry $\theta_A S_A^{-1/2}$. An isometry V implements this equivalence if and only if $V = \theta_A S_A^{-1/2} U$ for some unitary operator $U \in \pi(G)'$.

It is well known and easy to see that two subrepresentations of the left regular representation with multiplicity H_o, $(\lambda \otimes id)\ P|_{PH_G}$ and $(\lambda \otimes id)\ Q|_{QH_G}$, are equivalent if and only if $P \sim Q$ in $\mathcal{R}(G) \otimes B(H_o)$. In other words, the equivalence classes of subrepresentations of the left regular representation with fixed multiplicity H_o are identified with the collection of equivalence classes of projections of the von Neumann algebra $\mathcal{R}(G) \otimes B(H_o)$.

Theorem 6.5 permits us to characterize those operator-valued frames labeled by a discrete group G that have a frame generator. For simplicity's sake, because of Remark 4.4 we need to consider only Parseval operator-valued frames.

Proposition 6.7. *Let G be a discrete group and let $\{A_g\}_{g\in G}$ be a Parseval operator-valued frame in $B(H, H_o)$. Then there is a unitary representation π of G on H for which $A_g = A_e \pi_{g^{-1}}$ for all $g \in G$ if and only if $A_{gp}A_{gq}^* = A_p^* A_q$ for all $p, q, g \in G$.*

Proof. Assume $A_g = A_e \pi_{g^{-1}}$ for some unitary representation π and for all $g \in G$. Then

$$A_{gp}A_{gq}^* = A_e \pi_{(gp)^{-1}} (\pi_{(gp)^{-1}})^* A_e^* = A_e \pi_{p^{-1}} \pi_{g^{-1}} \pi_g (\pi_q^{-1})^* A_e^* = A_p^* A_q$$

for all $p, q, g \in G$. Assume now that $A_{gp}A_{gq}^* = A_p^* A_q$. Then for every $g \in G$, by the proof of Lemma 6.4 (i),

$$(\lambda_g \otimes I_o)\theta_A \theta_A^* (\lambda_g \otimes I_o)^* = \sum_{p,q\in G} (\lambda_g \otimes I_o) L_p A_p A_q^* L_q^* (\lambda_g \otimes I_o)^* \tag{18}$$

$$= \sum_{p,q\in G} L_{gp} A_p A_q^* L_{gq} = \sum_{r,s\in G} L_r A_{g^{-1}r} A_{g^{-1}s}^* L_s \tag{19}$$

$$= \sum_{r,s\in G} L_r A_r A_s^* L_s = \theta_A \theta_A^*. \tag{20}$$

This proves that the projection $P_A = \theta_A \theta_A^* \in \mathcal{R}(G) \otimes B(H_o)$. But then the operator-valued weight $\{A_g\}_{g\in G}$ can be identified with the compression to P_A of the left regular representation $\lambda_g \otimes I_o$ which has an operator frame generator. Explicitly, again by the proof of Lemma 6.4 (i),

$$A_g = L_g^* \theta_A = L_e^* P_A (\lambda_{g^{-1}} \otimes I_o) P_A \theta_A = A_e \theta_A^* (\lambda_{g^{-1}} \otimes I_o) P_A \theta_A.$$

Since $(\lambda_g \otimes I_o)P_A$ is a unitary representation of G on the Hilbert space P_A, then $\pi_g := \theta_A^* (\lambda_g \otimes I_o) P_A \theta_A$ is a unitary representation of G on the Hilbert space H. Thus $A_g := A_e \pi_{g^{-1}}$; i.e., A_e is a frame generator for (G, π, H). □

Remark 6.8. (i) Proposition 6.7 is a generalization of the following known result for group-indexed frames: When $\dim H_o = 1$, i.e., in the case of a Parseval vector frame $\{x_g\}_{g\in G}$, the necessary and sufficient condition for that frame to have a generator for some unitary representation of G (necessarily unitarily equivalent to a subrepresentation of the left regular representation) is that

$$\langle x_{gp}, x_{gq}\rangle = \langle x_p, x_q\rangle \quad \text{for all} \quad p, q, g \in G. \tag{21}$$

This result can be deduced from the material in Chapter 3 of [15]; however, it was not stated explicitly in that paper. Condition (21) is clearly equivalent to the condition that the range of the analysis operator is invariant under the left regular representation of the group on the analysis space, and so the frame can be obtained from the standard orthonormal basis for this representation by simply projecting, thereby obtaining the required subrepresentation of G. We note that Nga Nguyen has written an exposition of this in a forthcoming article stemming from her thesis research, along with some extensions to frames satisfying this condition which are not necessarily Parseval, where the situation is more complicated. We also note that some special cases, notably where G is a cyclic group on a finite dimensional Hilbert space, have been independently proven and used by others.

(ii) More is true in a case where the group is abelian (see also Remark 7.5). If G is abelian, then Cor. 3.14 or Theorem 6.3 of [15] states that all group frames for a unitary representation of G on the same Hilbert space are unitarily equivalent. So in the abelian case two frames both satisfying the condition (21) are necessarily unitarily equivalent.

(iii) We do not know if there is a similar necessary and sufficient condition for frames indexed by a unitary system, or at least by some structured unitary system, such as a Gabor system.

7. Parametrization of operator frame generators

Theorem 3.1 shows how to parametrize all operator-valued frames with a given multiplicity in terms of a single operator-valued frame . This general result can be applied to parametrize all operator frame generators for a unitary representation of a discrete group in terms of a single operator frame generator.

Theorem 7.1. *Let $A \in B(H, H_o)$ be a frame generator for the unitary representation (G, π, H).*

(i) *If $B(H_G) \ni M = MP_A$ and $M^*M|_{P_AH_G}$ is invertible in $B(P_AH_G)$, then $L_e^*M\theta_A$ is a frame generator for (G, π, H) if and only if $M \in \mathcal{R}(G) \otimes B(H_o)$.*

(ii) *The collection $\mathcal{F}_G$ of all the operator frame generators for (G, π, H) with the same multiplicity H_o is uniquely parametrized as*

$$\mathcal{F}_G = \{L_e^*M\theta_A \mid M \in \mathcal{R}(G) \otimes B(H_o), M = MP_A,\, M^*M|_{P_AH_G} \text{ is invertible}\}.$$

(iii) *If A is a Parseval generator, the collection of all the Parseval operator frame generators for (G, π, H) with multiplicity H_o is uniquely parametrized as*

$$\{L_e^*V\theta_A \mid V \in \mathcal{R}(G) \otimes B(H_o), V^*V = P_A\}. \tag{22}$$

If $B = L_e^*V\theta_A$ with $V \in \mathcal{R}(G) \otimes B(H_o)$ and $V^*V = P_A$, then $V = \theta_B\theta_A^*$.

(iv) *If A is a Parseval generator and $P \sim P_A$ in $\mathcal{R}(G) \otimes B(H_o)$, then $P = P_B$ for $B = L_e^*V\theta_A$ for some $V \in \mathcal{R}(G) \otimes B(H_o)$ with $V^*V = P_A$ and $VV^* = P$.*

Proof. (i) If $M = MP_A$ is an operator in $\mathcal{R}(G) \otimes B(H_o)$ and $M^*M|_{P_AH_G}$ is invertible, by Theorem 3.1 (ii), $\{L_g^* M\theta_A\}_{g\in G}$ is an operator-valued frame. But then for all $g \in G$,

$$L_g^* M\theta_A = (L_e^*\lambda_{g^{-1}} \otimes I_o)M\theta_A = L_e^* M\lambda_{g^{-1}} \otimes I_o\theta_A = L_e^* M\theta_A\pi_{g^{-1}},$$

by Lemma 6.4 (i); i.e., $L_e^* M\theta_A$ is the generator of $\{L_g^* M\theta_A\}_{g\in G}$. Conversely, assume that $L_e^* M\theta_A$ is an operator frame generator for (G, π, H); i.e., assume that $\{L_e^* M\theta_A\pi_{g^{-1}}\}_{g\in G}$ is a frame. For all $g \in G$ we have

$$L_e^* M\theta_A\pi_{g^{-1}} = (L_e^*\lambda_{g^{-1}} \otimes I_o)(\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o)\theta_A = L_g^*(\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o)\theta_A.$$

Set $N_g := (\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o)P_A$. Then obviously $N_g = N_gP_A$ and

$$\begin{aligned} N_g^*N_g &= P_A(\lambda_g \otimes I_o)M^*(\lambda_{g^{-1}} \otimes I_o)(\lambda_g \otimes I_o)M(\lambda_{g^{-1}} \otimes I_o)P_A \\ &= (\lambda_g \otimes I_o)P_AM^*MP_A(\lambda_{g^{-1}} \otimes I_o). \end{aligned}$$

Since by hypothesis $P_AM^*MP_A$ is invertible in $B(P_AH_G)$ and since $\lambda_g \otimes I_o$ commutes with P_A, $N_g^*N_g$ is also invertible in $B(P_AH_G)$. But then, by the uniqueness part of Theorem 3.1, $N_g = N_e = M$; i.e., M commutes with $\lambda_g \otimes I_o$ for all $g \in G$ and hence $M \in \mathcal{R}(G) \otimes B(H_o)$.

(ii) If $B \in B(H, H_o)$ is an operator frame generator for (G, π, H), then by Theorem 3.1, $B\pi_{g^{-1}} = L_g^* M\theta_A$ for some unique $M = MP_A$ for which $M^*M|_{P_AH_G}$ is invertible. In particular, $B = L_e^* M\theta_A$ and hence $M \in \mathcal{R}(G) \otimes B(H_o)$ by the above proof.

(iii) and (iv) The rest of the proof follows by the same arguments and Corollary 3.3. □

Special cases of operator frame generators arise from right or left similarities. First we need the following lemma.

Lemma 7.2. *Let $A \in B(H, H_o)$ be an operator frame generator for (G, π, H).*

(i) *Let $R \in B(H_o)$ be invertible. Then RA is an operator frame generator for (G, π, H).*

(ii) *Let $T \in B(H)$ be invertible. Then $\{A\pi_{g^{-1}}T\}_{g\in G}$ has a generator (necessarily AT) if and only if $T \in \pi(G)'$.*

(iii) *Let $T \in B(H)$ be invertible. If $T \in \pi(G)''$, then AT is an operator frame generator for (G, π, H) and $AT = L_e^*(Y \otimes I_o)\theta_A$ for some invertible operator $Y \in \mathcal{R}(G)$. If T is unitary, then Y can be chosen to be unitary.*

Proof. (i) Obvious.

(ii) The sufficiency of the condition is clear. For the necessity, assume that $\{A\pi_{g^{-1}}T\}_{g\in G}$, has a generator B, i.e., $A\pi_{g^{-1}}T = B\pi_{g^{-1}}$ for all $g \in G$. Then $\theta_AT = \theta_B$ by Lemma 4.2 and hence $T = S_A^{-1}\theta_A^*\theta_B \in \pi(G)'$ by Lemma 6.4 (ii).

(iii) By Lemma 6.4 (ii), T commutes with $S_A^{-1/2}$; hence

$$AT = L_e^*\theta_AT = L_e^*\theta_ATS_A^{-1}\theta_A^*\theta_A = L_e^*(\theta_AS_A^{-1/2}TS_A^{-1/2}\theta_A^*)\theta_A.$$

By Lemma 6.4 (i) and (ii),

$$\theta_AS_A^{-1/2}\pi_gS_A^{-1/2}\theta_A^* = \theta_A\pi_gS_A^{-1}\theta_A^* = (\lambda_g \otimes I_o)\theta_AS_A^{-1}\theta_A^* = (\lambda_g \otimes I_o)P_A.$$

Since $\theta_A S_A^{-1/2}$ is a unitary operator in $B(H, P_A H_G)$ and since the unitary group $\{\pi_g \mid g \in G\}$ (resp., $(\lambda_g \otimes I_o)\ P_A|_{P_A H_G}$) generates the von Neumann algebra $\pi(G)''$ (resp., $P_A(\mathcal{L}(G) \otimes I_o)\ P_A|_{P_A H_G}$), the map

$$\pi(G)'' \ni X \mapsto \theta_A S_A^{-1/2} X S_A^{-1/2} \theta_A^* \in P_A(\mathcal{L}(G) \otimes I_o)\ P_A|_{P_A H_G}$$

is a (spatial) isomorphism of von Neumann algebras. Let $Q \otimes I_o$ be the central support of P_A, so $Q \in \mathcal{R}(G) \cap \mathcal{L}(G)$. It is well known [18, Proposition 5.5.5] that the map

$$(\mathcal{L}(G) \otimes I_o)\ (Q \otimes I_o)|_{(Q \otimes I_o) H_G} \ni X \mapsto X\ P_A|_{P_A H_G} \in P_A(\mathcal{L}(G) \otimes I_o)\ P_A|_{P_A H_G}$$

is also an isomorphism. Thus $\theta_A S_A^{-1/2} T S_A^{-1/2} \theta_A^* = (Z \otimes I_o) P_A$ for some operator $Z \in \mathcal{L}(G)$ for which $(Z \otimes I_o)(Q \otimes I_o)|_{(Q \otimes I_o) H_G}$ is invertible and then we have $AT = L_e^*(Z \otimes I_o) P_A \theta_A$. By passing if necessary to $ZQ + Q^\perp \in \mathcal{L}(G)$, we can assume without loss of generality that Z is invertible. If T is unitary, we can similarly assume that Z too is unitary.

Recall that the involution J, defined by $J(x\chi_e) := x^*\chi_e$ for all $x \in \mathcal{L}(G)$ and then extended to $\ell^2(\mathbb{J})$, establishes the conjugate linear isomorphism of $\mathcal{L}(G)$ and $\mathcal{R}(G)$, $\mathcal{L}(G) \ni x \mapsto JxJ \in \mathcal{R}(G)$. For all $h \in H_o$,

$$(Z^* \otimes I_o) L_e h = Z^* \chi_e \otimes h = JZJ\chi_e \otimes h = (JZJ \otimes I_o) L_e h;$$

hence

$$AT = L_e^*(Z \otimes I_o)\theta_A = L_e^*((JZJ)^* \otimes I_o)\theta_A = L_e^*(JZ^*J \otimes I_o)\theta_A.$$

Let $Y := JZ^*J$ and $M := (Y \otimes I_o) P_A$. Then $Y \in \mathcal{R}(G)$; hence $M \in \mathcal{R}(G) \otimes B(H_o)$ and $M = MP_A$. Furthermore, Y is invertible (unitary if T and hence Z are unitary); hence $M^*M = P_A(YY^* \otimes I_o)P_A$ is invertible in $B(P_A H_G)$. Then by Theorem 7.1, $AT = L_e^*(Y \otimes I_o)\theta_A = L_e^* M \theta_A$ is an operator frame generator for (G, π, H). □

A reformulation of statement (ii) is that if two operator-valued frames with generators A and B are right-similar, then the (unique) similarity operator must belong to $\pi(G)'$. Using this fact, the characterization of right-similarity for general operator-valued frames carries through easily for operator-valued frames with a generator as follows.

Proposition 7.3. *Let A and B be frame generators with the values in the same space H_o for a unitary representation (G, π, H). Then the following conditions are equivalent:*

(i) *$B = AT$ for some invertible operator $T \in \pi(G)'$;*

(ii) *$B\pi_{g^{-1}} = A\pi_{g^{-1}}T$ for all $g \in G$ for some invertible operator $T \in B(H)$;*

(iii) *$\theta_B = \theta_A T$ for some invertible $T \in B(H)$;*

(iv) *$P_A \theta_B S_A^{-1} \theta_A^*$ is invertible in $B(P_A H_G)$;*

(iv′) *$B = L_e^* M \theta_A$ for some $M \in \mathcal{R}(G) \otimes B(H_o)$ with $M = MP_A$ and such that $P_A M P_A$ is invertible in $B(P_A H_G)$;*

(v) *$P_B = P_A$.*

Corollary 7.4. *Let $A \in B(H, H_o)$ be an operator frame generator for (G, π, H). Then all the operator frame generators for (G, π, H) are left-similar to A if and only if $P_A \in (\mathcal{L}(G) \cap (\mathcal{R}(G)) \otimes I_o$.*

Proof. P_A belongs to the center $(\mathcal{L}(G) \cap \mathcal{R}(G)) \otimes I_o$ of $\mathcal{R}(G) \otimes B(H_o)$ if and only if there are no projections $\mathcal{R}(G) \otimes B(H_o)$ that are different but Murray-von Neumann equivalent to it. By Lemma 6.4 and Proposition 7.3 this is equivalent to the condition that any operator frame generator for (G, π, H) is left-similar to A. □

Remark 7.5. (i) Corollary 7.4 provides a higher multiplicity analog of Proposition 3.13 in [15].

(ii) If the group G is abelian, then so is $\mathcal{L}(G) = \mathcal{R}(G)$ and hence $\mathcal{R}(G) \otimes I_o$ is the center of $\mathcal{R}(G) \otimes B(H_o)$. Thus in particular if $\dim H_o = 1$, $\mathcal{R}(G) \otimes B(H_o)$ is abelian and all operator frame generators for (G, π, H) are left-similar. This generalizes Cor. 3.14 (and Theorem 6.3) of [15], which states that, for vector group frames, if G is abelian, then all group frames for a unitary representation of G on the same Hilbert space are unitarily equivalent. (See also Remark 6.8 (i) in the present article.)

To simplify notation, we formulate the next result directly for Parseval operator frame generators.

Proposition 7.6. *Let A, $B \in B(H, H_o)$ be Parseval frame generators for (G, π, H).*

(i) *$B = AU$ for some unitary $U \in \pi(G)'$ if and only if $B = L_e^* W \theta_A$ for some unitary $W \in \mathcal{R}(G) \otimes B(H_o)$ with $WP_A = P_A W$, again if and only if $P_B = P_A$.*

(ii) *Let $U \in B(H_o)$ be unitary. Then $B = UA$ if and only if $B = L_e^*(I \otimes U)\theta_A$. If $B = UA$, then $P_B = (I \otimes U)P_A(I \otimes U^*)$.*

(iii) *$B = AU$ for some unitary $U \in \pi(G)''$ if and only if $B = L_e^*(W \otimes I_o)\theta_A$ for some unitary $W \in \mathcal{R}(G)$. If $B = L_e^*(W \otimes I_o)\theta_A$, then $P_B = (W \otimes I_o)P_A(W^* \otimes I_o)$.*

Proof. (i) By Proposition 4.3, $P_B = P_A$ if and only if the operator-valued frames $\{B\pi_{g^{-1}}\}_{g \in G}$ and $\{A\pi_{g^{-1}}\}_{g \in G}$ are right unitarily equivalent. By Lemma 7.2, this unitary equivalence holds if and only if $B = AU$ for some unitary $U \in \pi(G)'$. Also, by Proposition 4.3, $P_B = P_A$ if and only if $\theta_B \theta_A^*$ is a unitary in $B(P_A H_G)$ and hence it is the compression to $P_A H_G$ of a unitary $W \in \mathcal{R}(G) \otimes B(H_o)$ that commutes with P_A.

(ii) This is immediate from Lemma 4.5.

(iii) Assume that $B = AU$ for a unitary $U \in \pi(G)''$. In the proof of Lemma 7.2 (iii) we can choose Z to be unitary and hence $W := JZ^*J \in \mathcal{R}(G)$ is also unitary. Then $\theta_B = (W \otimes I_o)\,\theta_A$; hence $P_B = (W \otimes I_o)P_A(W^* \otimes I_o)$. On the other hand, if $B = L_e^*(W \otimes I_o)\theta_A$ for some unitary $W \in \mathcal{R}(G)$, then by the same argument as in the proof of Lemma 7.2 (iii) we see that

$$B = L_e^*(JW^*J \otimes I_o)\,\theta_A = L_e^*\theta_A(\theta_A^*(JW^*J \otimes I_o)\theta_A),$$

where $JW^*J \in \mathcal{L}(G)$ and hence $\theta_A^*(JW^*J \otimes I_o)\theta_A$ is a unitary in $\pi(G)''$. □

Remark 7.7. For vector frames ($\dim H_o = 1$), Han and Larson [15, Theorem 6.17] have shown that given a Parseval frame generator η for (G, π, H), the collection of all the (vector) Parseval frame generators for (G, π, H) is parametrized by the

unitary group of $\pi(G)''$; namely, it coincides with $\{U\eta \mid U \in \pi(G)'', U \text{ is unitary}\}$. This result is also a consequence of Theorem 7.1, since the partial isometry V intertwining P_A and P_B can be extended to a unitary W because the von Neumann algebra $\mathcal{R}(G)$ is finite. However, Proposition 7.6 shows why this result does not hold when $\dim H_o > 1$.

8. Homotopy of operator frame generators

The objective of this section is to prove the following theorem:

Theorem 8.1. *Let (G, π, H) be a unitary representation of a discrete group G and assume that the collection $\mathcal{F}_G$ of all the operator frame generators with range in H_o for (G, π, H) is nonempty.*

(i) *If $\dim H_o < \infty$, then $\mathcal{F}_G$ is path-connected in the norm topology.*

(ii) *If $\dim H_o = \infty$, then $\mathcal{F}_G$ is path-connected in the norm topology if and only if the von Neumann algebra $\mathcal{R}(G)$ generated by the right regular representation of G is diffuse (i.e. has no nonzero minimal projections).*

As we will point out in the proof of the theorem, it is easy to reduce the problem to showing that any two Parseval operator frame generators are homotopic. The latter property is obviously true in the case when $\dim H_o = 1$, because then (by [15, Theorem 6.17], see also Remark 7.7 and Proposition 7.6), the collection of Parseval (vector) frame generators for (G, π, H) is parametrized by the unitary group of the von Neumann algebra $\pi(G)''$, which is well known to be path-connected in the norm topology. In the general case, however, by Theorem 7.1 (iii) all Parseval operator frame generators for (G, π, H) are parametrized by the partial isometries of the algebra $\mathcal{R}(G) \otimes B(H_o)$ that have the same initial projection, the frame projection of a fixed Parseval operator frame generator. When $\dim H_o = \infty$, this class of partial isometries is not path-connected in the norm topology. It is, however, path-connected in the strong operator topology when $\mathcal{L}(G)$ has no nonzero minimal projections, and this is sufficient for the path-connectedness in the norm topology of the operator frame generators. In order to do that we need to introduce some notation and preliminary results.

Let V, W be partial isometries in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection, i.e., $V^*V = W^*W$, and hence with range projections, VV^*, WW^* Murray-von Neumann equivalent ($VV^* \sim WW^*$). We say that

$$V \approx W$$

if there is a norm continuous path of partial isometries

$$\{V(t) \in \mathcal{R}(G) \otimes B(H_o) \mid t \in [0,1]\}$$

such that $V^*(t)V(t) = V^*V$ for all $t \in [0,1]$, $V(0) = V$, and $V(1) = W$. Clearly, $\approx$ is an equivalence relation for the class of partial isometries that have the same initial projection. Adaptating the proof of [23, Theorem 3.1] with a slight change yields the following equivalent characterization. For the reader's convenience we sketch the proof.

Lemma 8.2. *Let V, W be partial isometries in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection. Then $V \approx W$ if and only if VV^* and WW^* are unitarily equivalent in $\mathcal{R}(G) \otimes B(H_o)$.*

Proof. If VV^* and WW^* are unitarily equivalent, then $(WW^*)^\perp \sim (VV^*)^\perp$; i.e., there is a partial isometry $Z \in \mathcal{R}(G) \otimes B(H_o)$ with $ZZ^* = (WW^*)^\perp$ and with $Z^*Z = (VV^*)^\perp$. Then a simple computation shows that the operator

$$U := WV^* + Z \in \mathcal{R}(G) \otimes B(H_o)$$

is unitary and that $W = UV$. Since U is homotopic in the norm topology to the identity, choose a norm continuous path of unitaries $U(t) \in \mathcal{R}(G) \otimes B(H_o)$ with $U(0) = I$ and $U(1) = U$. Then $V(t) := U(t)V$ is the required norm continuous path of partial isometries with the same initial projection that joins $V(0) = V$ and $V(1) = W$. This establishes that $V \approx W$. Conversely, if $V \approx W$ and $V(t)$ is a norm continuous path of partial isometries with the same initial projection that joins $V(0) = V$ with $V(1) = W$, then $P(t) := V(t)V(t)^*$ is a norm continuous path of projections joining $P(0) = VV^*$ with $P(1) = WW^*$. It is well known that homotopy of projections implies unitary equivalence. □

If $\dim H_o = \infty$, there are partial isometries $V, W \in \mathcal{R}(G) \otimes B(H_o)$ with the same initial projection but with range projections that are not unitarily equivalent, e.g., $V = I \otimes I_o$ and $W = I \otimes Z$ with $Z \in B(H_o)$ a nonunitary isometry. By Lemma 8.2, V and W cannot be joined by a norm continuous path of partial isometries all with the same initial projection. The norm continuity of such a path of partial isometries, however, is only sufficient but is not always necessary for the existence of a norm continuous path of Parseval frame generators joining L_e^*V with L_e^*W. The existence of the latter is, in view of Theorem 7.1 (iii), equivalent to the existence of a path of partial isometries $V(t)$ joining V and W for which $L_e^*V(t)$ is norm continuous. It is convenient to denote the existence of such a path by using the following notation:

Let V, W be partial isometries in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection. We say that

$$V \underset{e}{\sim} W \tag{23}$$

if there is a path of partial isometries $\{V(t) \in \mathcal{R}(G) \otimes B(H_o) \mid t \in [0,1]\}$ such that $L_e^*V(t)$ is norm continuous, $V^*(t)V(t) = V^*V$ for all $t \in [0,1]$, $V(0) = V$, and $V(1) = W$.

Clearly, $\underset{e}{\sim}$ is also an equivalence relation for partial isometries that have the same initial projection and $V \approx W$ implies $V \underset{e}{\sim} W$.

A key ingredient in the proof of Theorem 8.1 is that a finite trace in a von Neumann algebra is strongly continuous (actually, σ-weakly, but we do not need this here).

Denote by $\tau(X) = (X\chi_e, \chi_e)$ for $X \in \mathcal{R}(G)$ the normalized trace on $\mathcal{R}(G)$. Denote by $E := \mathcal{R}(G) \otimes B(H_o) \mapsto B(H_o)$ the corresponding *slice map*, namely, the bounded linear extension of the map $E(X \otimes Y) = \tau(X)Y$ for all $X \in \mathcal{R}(G)$ and all $Y \in B(H_o)$. It is easy to see that the map E is positive, that is, $E(Z) \geq 0$ when $Z \geq 0$, or, equivalently, $E(Z_1) \leq E(Z_2)$ when $Z_1 \leq Z_2$. Also, the map E is normal, that is, $E(Z_\gamma) \uparrow E(Z)$ when $Z_\gamma \uparrow Z$, or equivalently, E is σ-weakly continuous.

The bridge between trace and norm is given by the following lemma.

Lemma 8.3. *$E(Z) = L_e^* Z L_e$ for all $Z \in \mathcal{R}(G) \otimes B(H_o)$.*

Proof. It is enough to verify that the two maps agree on elementary tensors. Indeed, for all $h \in H_o$ and all $X \in \mathcal{R}(G)$, $Y \in B(H_o)$ we have

$$\begin{aligned} L_e^*(X \otimes I_o)L_e h &= L_e^* X\chi_e \otimes h = L_e^*(\sum_{g\in G}(X\chi_e, \chi_g)\chi_g) \otimes h \\ &= L_e^* \sum_{g\in G}(\chi_g \otimes (X\chi_e, \chi_g)h) = (X\chi_e, \chi_e)h = \tau(X)h. \end{aligned}$$

Thus $L_e^*(X \otimes I_o)L_e = \tau(X)I_o$ and hence

$$L_e^*(X \otimes Y)L_e = L_e^*(I \otimes Y)(X \otimes I_o Y)L_e = Y L_e^*(X \otimes I_o Y)L_e = \tau(X)Y = E(X \otimes Y).$$

□

Notice that the trace τ on $\mathcal{R}(G)$ is always finite, while the trace $\tau \otimes tr$ on $\mathcal{R}(G) \otimes B(H_o)$ is finite only if $\dim H_o < \infty$. Thus, given two partial isometries with the same initial projection, we want to construct a strongly continuous path of partial isometries that connects them, where the strong convergence occurs in the first component of the tensor product. This will be achieved via the following key lemma.

Lemma 8.4. *Assume that $\dim H_o = \infty$ and that $\mathcal{R}(G)$ has no minimal projections. Let R be a projection in $\mathcal{R}(G)$ with $R \sim R^\perp$, let $\{Q_n\}_{1\le n\le N}$ with $N \le \infty$ be a collection of infinite projections in $B(H_o)$, and let $I = \sum_{n=1}^N F_n$ be a decomposition of the identity into mutually orthogonal central projections $F_n \in \mathcal{R}(G) \cap \mathcal{L}(G)$. Then there is a path of partial isometries $\{W(t) \mid t \in [0,1]\}$ in $\mathcal{R}(G) \otimes B(H_o)$ such that*

(i) *$L_e^* W(t)$ is norm continuous,*
(ii) *$W(t)^* W(t) = \sum_{n=1}^N F_n \otimes Q_n$ for all $t \in [0,1]$,*
(iii) *$W(t)W(t)^* \le \sum_{n=1}^N F_n \otimes Q_n$ for all $t \in [0,1]$,*
(iv) *$W(0) = \sum_{n=1}^N F_n \otimes Q_n$, and*
(v) *$W(1)W(1)^* = \sum_{n=1}^N RF_n \otimes Q_n$.*

Proof. The reduced von Neumann algebra $R\mathcal{R}(G)R := R\mathcal{R}(G)\, R|_{R\ell^2(G)}$ has no minimal projections; thus it contains a strongly continuous increasing net of projections $\{R(t) \mid t \in [0,1]\}$ with $R(0) = 0$, $R(1) = R$. For instance, $R(t)$ can be obtained from the spectral resolution of a positive generator of a maximal abelian subalgebra of $R\mathcal{R}(G)R$. Since $R^\perp \sim R$, there is a unitary $U \in \mathcal{R}(G)$ such that $R^\perp = URU^*$. Since Q_n is an infinite projection in $B(H_o)$, there exist partial isometries $S_{1,n}, S_{2,n} \in B(H_o)$ such that

$$S_{1,n}^* S_{1,n} = S_{2,n}^* S_{2,n} = Q_n \quad \text{and} \quad S_{1,n}S_{1,n}^* + S_{2,n}S_{2,n}^* = Q_n.$$

Notice that $S_{1,n}$, $S_{2,n}$ are the generators of the Cuntz algebra $\mathcal{O}_2$ represented on $Q_n H_o$. Define for $t \in [0,1]$,

$$W(t) := \sum_{n=1}^N \Big((R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t)F_n \otimes S_{1,n} + R(t)U^* F_n \otimes S_{2,n} \Big).$$

By definition, $W(0) = \sum_{n=1}^{N} F_n \otimes Q_n$. Since $R(t) \perp (R(t) + UR(t)U^*)^\perp$ for all $t \in [0,1]$, and $S_{2,n}^* S_{1,n} = S_{1,n}^* S_{2,n} = 0$ for all n, it follows that

$$\begin{aligned}
&W(t)^* W(t) \\
&= \sum_{n=1}^{N} \Big((R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + (R(t) + UR(t)U^*)^\perp R(t) F_n \otimes Q_n S_{1,n} \\
&+ (R(t) + UR(t)U^*)^\perp R(t) U^* F_n \otimes Q_n S_{2,n} + R(t)(R(t) + UR(t)U^*)^\perp F_n \otimes S_{1,n}^* Q_n \\
&+ R(t) F_n \otimes S_{1,n}^* S_{1,n} + R(t) U^* F_n \otimes S_{1,n}^* S_{2,n} \\
&+ UR(t)(R(t) + UR(t)U^*)^\perp F_n \otimes S_{2,n}^* Q_n \\
&+ UR(t) F_n \otimes S_{2,n}^* S_{1,n} + UR(t) U^* F_n \otimes S_{2,n}^* S_{2,n} \Big) \\
&= \sum_{n=1}^{N} \Big((R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t) F_n \otimes S_{1,n}^* S_{1,n} \\
&+ UR(t) U^* F_n \otimes S_{2,n}^* S_{2,n} \Big) \\
&= \sum_{n=1}^{N} \Big((R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t) F_n \otimes Q_n + UR(t) U^* F_n \otimes Q_n \Big) \\
&= \sum_{n=1}^{N} F_n \otimes Q_n.
\end{aligned}$$

Using the fact that $UR(t)U^* \perp (R(t) + UR(t)U^*)^\perp$ and $UR(t)U^* \perp R(t)$ for all t, a similar computation yields

$$\begin{aligned}
&W(t) W(t)^* \\
&= \sum_{n=1}^{N} \Big((R(t) + UR(t)U^*)^\perp F_n \otimes Q_n + R(t) F_n \otimes S_{1,n} S_{1,n}^* + R(t) F_n \otimes S_{2,n} S_{2,n}^* \Big) \\
&= \sum_{n=1}^{N} \Big(((R(t) + UR(t)U^*)^\perp + R(t)) F_n \otimes Q_n \Big) \\
&\leq \sum_{n=1}^{N} F_n \otimes Q_n.
\end{aligned}$$

In particular,

$$W(1)W(1)^* = \sum_{n=1}^{N} ((R + R^\perp)^\perp + R) F_n \otimes Q_n = \sum_{n=1}^{N} R F_n \otimes Q_n.$$

Thus $\{W(t) \mid t \in [0,1]\}$ is a path of partial isometries of $\mathcal{R}(G) \otimes B(H_o)$ that satisfies conditions (ii), (iii), (iv), and (v). We now show that the condition (i) is also satisfied. Let $0 \leq t < t' \leq 1$ and $\Delta R := R(t') - R(t)$. Then

$$W(t') - W(t) = \sum_{n=1}^{N} \Big(-(\Delta R + U \Delta R U^*) F_n \otimes Q_n + \Delta R F_n \otimes S_{1,n} + \Delta R U^* F_n \otimes S_{2,n} \Big).$$

By using the facts that $\Delta R \perp U\Delta RU^*$ and $Q_n S_{i,n} = S_{i,n} Q_n = S_{i,n}$ for $i = 1, 2$ and all n, we obtain

$$\begin{aligned}
&(W(t') - W(t))(W(t') - W(t))^* \\
&= \sum_{n=1}^{N} \Big((\Delta R + U\Delta RU^*)^2 F_n \otimes Q_n - (\Delta R + U\Delta RU^*)\Delta RF_n \otimes S_{1,n}^* \\
&- (\Delta R + U\Delta RU^*)U\Delta RF_n \otimes S_{2,n}^* - \Delta R(\Delta R + U\Delta RU^*)F_n \otimes S_{1,n} \\
&+ (\Delta R)^2 F_n \otimes S_{1,n}S_{1,n}^* + \Delta RU\Delta RF_n \otimes S_{1,n}S_{2,n}^* \\
&- \Delta RU^*(\Delta R + U\Delta RU^*)F_n \otimes S_{2,n} + \Delta RU^*\Delta RF_n \otimes S_{2,n}S_{1,n}^* \\
&+ (\Delta R)^2 F_n \otimes S_{2,n}S_{2,n}^* \Big) \\
&= \sum_{n=1}^{N} \Big((\Delta R + U\Delta RU^*)F_n \otimes Q_n - \Delta RF_n \otimes S_{1,n}^* - U\Delta RF_n \otimes S_{2,n}^* \\
&- \Delta RF_n \otimes S_{1,n} + \Delta RF_n \otimes S_{1,n}S_{1,n}^* - \Delta RU^*F_n \otimes S_{2,n} + \Delta RF_n \otimes S_{2,n}S_{2,n}^* \Big) \\
&= \sum_{n=1}^{N} \Big((2\Delta R + U\Delta RU^*)F_n \otimes Q_n - \Delta RF_n \otimes (S_{1,n}^* + S_{1,n}) \\
&- U\Delta RF_n \otimes S_{2,n}^* - \Delta RU^*F_n \otimes S_{2,n} \Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&E\big((W(t') - W(t))(W(t') - W(t))^*\big) \\
&= \sum_{n=1}^{N} \Big(\tau(\Delta RF_n)(3Q_n - (S_{1,n} + S_{1,n}^*)) - \tau(\Delta RU^*F_n)S_{2,n} - \tau(U\Delta RF_n)S_{2,n}^* \Big) \\
&\leq \sum_{n=1}^{N} 7\tau(\Delta RF_n)I_o = 7\tau(\Delta R)I_o.
\end{aligned}$$

Hence

$$\|L_e^* W(t') - L_e^* W(t)\|^2 = \|E\big((W(t') - W(t))(W(t') - W(t))^*\big)\| \leq 7\tau(\Delta R).$$

Since $\mathcal{R}(G)$ is finite, $\tau(R(t))$ is continuous and hence $L_e^* W(t)$ is norm continuous, which concludes the proof. □

Now we can proceed to prove Theorem 8.1.

Proof. It is well known that in any von Neumann algebra (or, more in general, unital C^*-algebra), positive invertible operators are homotopic to the identity. But then, the operator frame generator A for (G, π, H) is homotopic to the Parseval operator frame generator $AS_A^{-1/2}$ by Lemma 6.4 (ii). Thus, to prove the path-connectedness in the norm topology of $\mathcal{F}_G$ it is enough to prove that the collection of Parseval operator frame generators for (G, π, H) is path-connected in the norm topology. By (22), this collection is parametrized by

$$\{L_e^* V\theta_A \mid V \in \mathcal{R}(G) \otimes B(H_o), V^*V = P_A\},$$

and since θ_A is an isometry, we only need to prove that $V \underset{e}{\sim} W$ for any two partial isometries V and W in $\mathcal{R}(G) \otimes B(H_o)$ with the same initial projection P_A, i.e., $V^*V = W^*W = P_A$.

(i) The algebra $\mathcal{R}(G) \otimes B(H_o)$ is finite because both $\mathcal{R}(G)$ and $B(H_o)$ are finite; hence the equivalence of the projections VV^* and WW^* implies their unitary equivalence. But then $V \approx W$ by Lemma 8.2 , and hence $V \underset{e}{\sim} W$. This proves that $\mathcal{F}_G$ is norm connected.

(ii) We prove first that the condition is necessary. Assume that $\mathcal{R}(G)$ has a nonzero minimal projection Q. Then Q belongs to a finite type I subfactor of $\mathcal{R}(G)$. Indeed if $c(Q)$ is the central cover of Q, which is the smallest projection in the center $\mathcal{L}(G) \cap \mathcal{R}(G)$ of $\mathcal{R}(G)$ that majorizes Q, then $c(Q)$ is minimal in $\mathcal{L}(G) \cap \mathcal{R}(G)$. But then the reduced von Neumann algebra $\mathcal{R}(G)c(Q) := c(Q)\mathcal{R}(G)\, c(Q)|_{c(Q)\ell^2(G)}$ is a factor, it is finite because so is $\mathcal{R}(G)$, and it is of type I because it contains the minimal projection Q. Let $\{E_{i,j}\}_{1 \le i,j \le n}$ be a set of matrix units for $\mathcal{R}(G)c(Q)$, i.e., $E_{i,j}^* = E_{j,i}$, $E_{i,k}E_{h,j} = \delta_{h,k}E_{i,j}$ for all $1 \le i,j,h,k \le n$, $\sum_{i=1}^n E_{i,i} = c(Q)$, and $\mathcal{R}(G)c(Q) = \{\sum_{i,j=1}^n c_{i,j}E_{i,j} \mid c_{i,j} \in \mathbb{C}\}$. Then every element Z in the factor $\mathcal{R}(G)c(Q) \otimes B(H_o)$ has the unique matricial form

$$Z = \sum_{i,j=1}^n E_{i,j} \otimes Z_{i,j} \quad \text{for } Z_{i,j} \in B(H_o). \tag{24}$$

Therefore, it is easy to see from Lemma 8.3 that

$$L_e^* Z L_e = \frac{1}{n} \sum_{i=1}^n Z_{i,i}. \tag{25}$$

Since $\mathcal{R}(G)c(Q) \otimes B(H_o)$ is an infinite type I factor, there is a proper isometry $V \in \mathcal{R}(G)c(Q) \otimes B(H_o)$, i.e., $V^*V = c(Q) \otimes I_o$ but $VV^* \neq c(Q) \otimes I_o$. To prove that $\mathcal{F}_G$ is not path-connected in the norm topology, it will be enough to show that the two Parseval operator-valued frame generators $L_e^* c(Q) \otimes I_o$ and $L_e^* V$ cannot be connected by any norm continuous path of arbitrary operator-valued frame generators.

Assume otherwise. Then by Theorem 7.1 (ii), there is a path of operators $M(t) \in \mathcal{R}(G) \otimes B(H_o)$ with $M(t)c(Q) \otimes I_o = M(t)$, $M(t)^*M(t)c(Q) \otimes I_o$ is invertible in $\mathcal{R}(G)c(Q) \otimes B(H_o)$, $M(0) = c(Q)$, $M(1) = V$, and $L_e^* M(t)$ is norm continuous. By (24), $M(t) = \sum_{i,j=1}^n E_{i,j} \otimes M_{i,j}(t)$ for a (unique) collection $M_{i,j}(t) \in B(H_o)$. Then for all $s,t \in [0,1]$, by (25),

$$\begin{aligned}
&(L_e^* M(s) - L_e^* M(t))(L_e^* M(s) - L_e^* M(t))^* \\
&\qquad = L_e^* \Big(\sum_{i,j=1}^n E_{i,j} \otimes \sum_{k=1}^n (M(s)_{i,k} - M(t)_{i,k})((M(s)_{k,j} - M(t)_{k,j})^* \Big) L_e \\
&\qquad = \frac{1}{n} \sum_{i,k=1}^n (M(s)_{i,k} - M(t)_{i,k})((M(s)_{k,i} - M(t)_{k,i})^*.
\end{aligned}$$

Thus the norm continuity of $L_e^* M(t)$ is equivalent to the norm continuity of each $M_{i,j}(t) \in B(H_o)$ for $1 \le i,j \le n$, and the latter is equivalent to the norm continuity of $M(t)$. But then, $M(t)^*M(t)$ is norm continuous and by the norm continuity of the inverse (e.g., see [14, Problem 100]), $(M(t)^*M(t))^{-1}$ is also norm continuous in $\mathcal{R}(G)c(Q) \otimes B(H_o)$. As a consequence, $P(t) := M(t)(M(t)^*M(t))^{-1}M(t)*$ is a

norm continuous path, and it is immediate to see (cf. Theorem 3.1) that the $P(t)$ are projections. But this is impossible since $P(0) = c(Q)$ and $P(1) = VV^*$ are not unitarily equivalent and hence not homotopic.

We prove now that if $\mathcal{R}(G)$ has no nonzero minimal projections, then $V \underset{e}{\sim} W$. By the standard type decomposition of von Neumann algebras (for these and other von Neumann algebra properties, see [18]), there is a (unique) central projection $F^{(1)} \in \mathcal{L}(G) \cap \mathcal{R}(G)$ for which $(\mathcal{L}(G) \cap \mathcal{R}(G))F^{(1)}$ is diffuse, i.e., has no atoms and hence is isomorphic to $L^\infty(\mathbb{R})$, and $(\mathcal{L}(G) \cap \mathcal{R}(G))(F^{(1)})^\perp$ is atomic and hence $\mathcal{R}(G)(F^{(1)})^\perp$ is a direct sum of type II_1 factors. Since it is immediate to verify that $V \underset{e}{\sim} W$ if and only if both

$$VF^{(1)} \otimes I_o \underset{e}{\sim} WF^{(1)} \otimes I_o \quad \text{and} \quad V(F^{(1)})^\perp \otimes I_o \underset{e}{\sim} W(F^{(1)})^\perp \otimes I_o,$$

we can consider separately the cases where the center of $\mathcal{R}(G)$ is diffuse and where it is atomic.

Consider first the case where $\mathcal{R}(G)$ has diffuse center. Then there is a strongly continuous increasing net of central projections $F(t) \in \mathcal{L}(G) \cap \mathcal{R}(G)$ such that $F(0) = 0$ and $F(1) = I$. Let

$$V(t) := V(F(t)^\perp \otimes I_o) + W(F(t) \otimes I_o).$$

Since $F(t) \otimes I_o$ is in the center of $\mathcal{R}(G) \otimes B(H_o)$, we see that

$$V(t)^*V(t) = V^*V(F(t)^\perp \otimes I_o) + W^*W(F(t) \otimes I_o) = V^*V = P_A$$

for all $t \in [0,1]$ and $V(0) = V, V(1) = W$. Moreover, for all $s < t \in [0,1]$,

$$V(t)-V(s) = V(F(s)-F(t))\otimes I_o+W(F(t)-F(s))\otimes I_o = (W-V)(F(t)-F(s))\otimes I_o;$$

hence

$$\begin{aligned}(V(t) - V(s))(V(t) - V(s))^* \\ &= (F(t) - F(s)) \otimes I_o\big((W - V)(W - V)^*\big)(F(t) - F(s)) \otimes I_o \\ &\le ||W - V||^2(F(t) - F(s)) \otimes I_o \\ &\le 4(F(t) - F(s)) \otimes I_o.\end{aligned}$$

But then,

$$\begin{aligned}||L_e^*V(t) - L_e^*V(s)||^2 &= ||E((V(t) - V(s))(V(t) - V(s))^*)|| \\ &\le 4||E((F(t) - F(s)) \otimes I_o|| \\ &= 4\tau((F(t) - F(s)).\end{aligned}$$

By the strong (actually σ-weak) continuity of τ, $L_e^*V(t)$ is norm continuous, and hence $V \underset{e}{\sim} W$. Following the terminology introduced in [1] for wavelet generators for the unitary system, we say that the path constructed in the case where $\mathcal{R}(G)$ has diffuse center is *a direct path*.

Consider now the key case where $\mathcal{R}(G)$ has no nonzero minimal projections and the center of $\mathcal{R}(G)$ is atomic. Then the identity $I \in \mathcal{R}(G)$ can be decomposed (uniquely) into a sum $I = \sum_{n=1}^N F_n$ of $N \le \infty$ mutually orthogonal projections F_n minimal in the center $\mathcal{R}(G) \cap \mathcal{L}(G)$. Notice that since $\mathcal{R}(G)$ has a finite faithful trace

τ, the decomposition is at most countable. The minimality of the projections F_n implies that each reduced von Neumann algebra $\mathcal{R}(G)F_n := \mathcal{R}(G)F_n\mathcal{R}(G)\ F_n\mid_{F_n\ell^2(G)}$ is a factor, and being finite and with no minimal projections, it is of type II_1. Let

$$F^{(2)} := \sum\{F_n \mid V^*VF_n \otimes I_o \not\sim F_n \otimes I_o\},$$
$$F^{(3)} := \sum\{F_n \mid (VV^*)^\perp F_n \otimes I_o \sim F_n \otimes I_o, F_n \leq (F^{(2)})^\perp\},$$
$$F^{(4)} := \sum\{F_n \mid (VV^*)^\perp F_n \otimes I_o \not\sim F_n \otimes I_o, F_n \leq (F^{(2)})^\perp\}.$$

Thus $F^{(2)} + F^{(3)} + F^{(4)} = I$. Reasoning as above, we can consider separately the cases where $F^{(2)} = I$ and $F^{(2)} = 0$.

Assume first that $F^{(2)} = I$, i.e.,

$$W^*WF_n \otimes I_o = V^*VF_n \otimes I_o \not\sim F_n \otimes I_o \quad \text{for all } n.$$

Then

$$WW^*F_n \otimes I_o \not\sim F_n \otimes I_o \quad \text{and} \quad VV^*F_n \otimes I_o \not\sim F_n \otimes I_o \quad \text{for all } n. \tag{26}$$

Since the factor $\mathcal{R}(G)\otimes B(H_o)$ is infinite, (26) implies that $(VV^*)^\perp F_n\otimes I_o \sim F_n\otimes I_o$ for every n, and hence $(VV^*)^\perp \sim I \otimes I_o$ and similarly, $(WW^*)^\perp \sim I \otimes I_o$. But then VV^* and WW^* are unitarily equivalent, hence $V \approx W$ by Lemma 8.2, and thus $V \underset{e}{\sim} W$.

When $F^{(2)} = 0$, i.e., $F^{(3)} + F^{(4)} = I$, we have $W^*W = V^*V \sim I \otimes I_o$. Since $\mathcal{R}(G)$ is a direct sum of type II_1 factors and in every II_1 factor there are projections equivalent to their orthogonal complement, we can fix a projection $R \in \mathcal{R}(G)$ with $R \sim R^\perp$. As in each of the infinite factors we have $RF_n\otimes I_o \sim R^\perp F_n\otimes I_o \sim F_n\otimes I_o$, it follows that $R\otimes I_o \sim R^\perp\otimes I_o \sim I\otimes I_o$. Fix a partial isometry $V_o \in \mathcal{R}(G)\otimes B(H_o)$ with $V_o^*V_o = V^*V = W^*W$ and $V_o^*V_o = R \otimes I_o$. We claim that $V \underset{e}{\sim} V_o$. The same argument will prove that $W \underset{e}{\sim} V_o$ and hence that $V \underset{e}{\sim} W$, which will conclude the proof.

Assume next that $F^{(3)} = I$. Then

$$VV^* \sim V^*V \sim I \otimes I_o \sim R \otimes I_o \quad \text{and} \quad (VV^*)^\perp \sim I \otimes I_o \sim R^\perp \otimes I_o = (R \otimes I_o)^\perp.$$

Thus VV^* and $V_o^*V_o$ are unitarily equivalent; hence $V \approx V_o$, and hence $V \underset{e}{\sim} V_o$.

Finally consider the case where $F^{(4)} = I$, namely where $V^*V \sim I \otimes I_o$ but $(VV^*)^\perp F_n\otimes I_o \not\sim F_n\otimes I_o$ for every n, which is the crux of the proof. If for a certain n the projection $(VV^*)^\perp F_n \otimes I_o$ is finite and hence $(\tau\otimes tr)((VV^*)^\perp F_n \otimes I_o) < \infty$, let $Q_n^\perp \in B(H_o)$ be a finite projection with

$$tr(Q_n^\perp) > \frac{(\tau \otimes tr)((VV^*)^\perp F_n \otimes I_o)}{\tau(F_n)}.$$

Then $Q_n \sim I_o$. Since $\mathcal{R}(G)F_n$ is a type II_1 factor, it contains a projection R_n with trace

$$\tau(R_n) = \frac{(\tau \otimes tr)((VV^*)^\perp F_n \otimes I_o)}{\tau(F_n)tr(Q_n^\perp)}.$$

Equivalently,

$$(\tau \otimes tr)(R_n \otimes Q_n^{\perp}) = (\tau \otimes tr)((VV^*)^{\perp} F_n \otimes I_o)$$

and hence, again because $\mathcal{R}(G)F_n$ is a factor, $R_n \otimes Q_n^{\perp} \sim (VV^*)^{\perp} F_n \otimes I_o$. If for a certain n the projection $(VV^*)^{\perp} F_n \otimes I_o$ is infinite (but still $(VV^*)^{\perp} F_n \otimes I_o \not\sim F_n \otimes I_o$ by the definition of $F^{(4)}$), there is a projection $Q_n^{\perp} \in B(H_o)$ with $Q_n \sim I_o$ and for which $F_n \otimes Q_n^{\perp} \sim (VV^*)^{\perp} F_n \otimes I_o$. In this case, set $R_n := F_n$, so for all n,

$$(VV^*)^{\perp} F_n \otimes I_o \sim R_n \otimes Q_n^{\perp}$$

with $R_n = R_n F_n$. Moreover,

$$(R_n \otimes Q_n^{\perp})^{\perp} F_n \otimes I_o = F_n \otimes Q_n + R_n^{\perp} F_n \otimes Q_n^{\perp} \sim F_n \otimes I_o \sim VV^* F_n$$

because $Q_n \sim I_o$ and $Q_n^{\perp} \not\sim I_o$. Thus $VV^* F_n \otimes I_o$ and $F_n \otimes Q_n + R_n^{\perp} F_n \otimes Q_n^{\perp}$ are unitarily equivalent for all n and hence VV^* and $\sum_{n=1}^N F_n \otimes Q_n + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}$ are unitarily equivalent in $\mathcal{R}(G) \otimes B(H_o)$. Choose a unitary $U \in \mathcal{R}(G) \otimes B(H_o)$ such that $UVV^*U^* = \sum_{n=1}^N F_n \otimes Q_n + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}$. Then $V \approx UV$.

Now apply Lemma 8.4 to the fixed projection $R \sim R^{\perp}$, and the sequences of central projections F_n and infinite projection Q_n that we have constructed for $1 \leq n \leq N$. Thus we obtain a path of partial isometries $\{W(t) \mid t \in [0,1]\}$ in $\mathcal{R}(G) \otimes B(H_o)$, where $L_e^* W(t)$ is norm continuous, $W(t)^* W(t) = \sum_{n=1}^N F_n \otimes Q_n$ and $W(t)W(t)^* \leq \sum_{n=1}^N F_n \otimes Q_n$ for all $t \in [0,1]$, $W(0) = \sum_{n=1}^N F_n \otimes Q_n$, and $W(1)W(1)^* = \sum_{n=1}^N RF_n \otimes Q_n$. Then let

$$V(t) := \Big(W(t) + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}\Big) UV \quad \text{for} \quad t \in [0,1].$$

Since the initial projections and the range projections of all the partial isometries $W(t)$ are orthogonal to $\sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}$, $W(t) + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}$ are partial isometries for all $t \in [0,1]$. Therefore $V(t)$ are also partial isometries belonging to $\mathcal{R}(G) \otimes B(H_o)$, and all have initial projection $P_A = (UV)^* UV$. Since $L_e^* W(t)$ is norm continuous, so is $L_e^* V(t)$. Thus by definition, $UV = V(0) \underset{e}{\sim} V(1)$. Moreover, the range projection of $V(1)$ is unitarily equivalent to the range projection $R \otimes I_o$ of V_o. Indeed,

$$\begin{aligned} V(1)V(1)^* &= \Big(W(1) + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}\Big) UVV^*U^* \Big(W(1)^* + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp}\Big) \\ &= W(1)W(1)^* + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp} \\ &= \sum_{n=1}^N RF_n \otimes Q_n + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp} \\ &\sim R \otimes I_o \end{aligned}$$

and

$$(V(1)V(1)^*)^{\perp} = \sum_{n=1}^N R^{\perp} F_n \otimes Q_n + \sum_{n=1}^N R_n^{\perp} F_n \otimes Q_n^{\perp} \sim R^{\perp} \otimes I_o = (R \otimes I_o)^{\perp}.$$

But then, $V(1) \approx V_o$. Since we have already established that $V \approx UV \underset{e}{\sim} V(1)$, we conclude in this case too that $V \underset{e}{\sim} V_o$, thus completing the proof. $\square$

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