

TWISTED FOURIER-MUKAI NUMBER OF A $K3$ SURFACE

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ABSTRACT. We give a counting formula for the twisted Fourier-Mukai partners of a projective $K3$ surface. As an application, we describe all twisted Fourier-Mukai partners of a projective $K3$ surface of Picard number 1.

1. INTRODUCTION

Fourier-Mukai (FM) partners of a projective $K3$ surface S have been studied beginning with Mukai's work [8]. The basic results due to Mukai and Orlov ([10]) are as follows:

- (1) If a 2-dimensional moduli space M of stable sheaves on S is non-empty and compact, then M is a $K3$ surface. When such an M is *fine*, a universal sheaf induces an equivalence $D^b(M) \simeq D^b(S)$.
- (2) Every FM partner of S is isomorphic to a certain 2-dimensional *fine* moduli space of stable sheaves on S .
- (3) A projective $K3$ surface S' is an FM partner of S if and only if their Mukai lattices are Hodge isometric.

Using Mukai-Orlov's theorem (especially (3)), Hosono-Lian-Oguiso-Yau ([5]) derived a counting formula for the FM partners of S .

On the other hand, it has been recognized that studying *twisted* FM partners as well as usual (i.e., untwisted) FM partners of S is important when analyzing the derived category $D^b(S)$ of S . Recall that a twisted FM partner of S is a twisted $K3$ surface (S', α') such that there is an equivalence $D^b(S', \alpha') \simeq D^b(S)$. Căldăraru ([1], [2]), Huybrechts-Stellari ([6], [7]), and Yoshioka ([14]) generalized Mukai and Orlov's results to the twisted situation. When S is untwisted, their results can be stated as follows:

- (1) When a 2-dimensional *coarse* moduli space M of stable sheaves on S is a $K3$ surface, a *twisted* universal sheaf induces an equivalence $D^b(M, \alpha) \simeq D^b(S)$, where α is the obstruction to the existence of a universal sheaf.
- (2) Every twisted FM partner of S is isomorphic to a certain 2-dimensional *coarse* moduli space of stable sheaves on S endowed with a natural twisting.
- (3) A twisted $K3$ surface (S', α') is a twisted FM partner of S if and only if the twisted Mukai lattice of (S', α') and the Mukai lattice of S are Hodge isometric.

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Let $\text{FM}^d(S)$ be the set of the isomorphism classes of the twisted FM partners (S', α') of S with $\text{ord}(\alpha') = d$. The number $\#\text{FM}^d(S)$ is an invariant of the category $D^b(S)$ representing the number of certain geometric origins of $D^b(S)$. The aim of this paper is to present an explicit formula for $\#\text{FM}^d(S)$. It enables us to calculate $\#\text{FM}^d(S)$ from lattice-theoretic information about the Néron-Severi lattice $NS(S)$ and the knowledge of the group $O_{\text{Hodge}}(T(S))$ of the Hodge isometries of the transcendental lattice $T(S)$.

Let $I^d(D_{NS(S)})$ be the set of order $= d$ isotropic elements of the discriminant form $D_{NS(S)}$. From an element $x \in I^d(D_{NS(S)})$ we can construct overlattices M_x, T_x of $NS(S), T(S)$ respectively, and a homomorphism $\alpha_x : T_x \rightarrow \mathbb{Z}/d\mathbb{Z}$. After defining certain subsets $\mathcal{G}_1(M), \mathcal{G}_2(M)$ of the genus $\mathcal{G}(M)$ of a lattice M such that $\mathcal{G}(M) = \mathcal{G}_1(M) \sqcup \mathcal{G}_2(M)$, our formula is stated as follows.

Theorem 1.1 (Theorem 4.2). *For a projective K3 surface S the following formula holds:*

$$\begin{aligned} \#\text{FM}^d(S) = & \sum_x \left\{ \sum_M \# \left(O_{\text{Hodge}}(T_x, \alpha_x) \backslash O(D_M) / O(M) \right) \right. \\ & \left. + \varepsilon(d) \sum_{M'} \# \left(O_{\text{Hodge}}(T_x, \alpha_x) \backslash O(D_{M'}) / O(M') \right) \right\}. \end{aligned}$$

Here x runs over the set $O_{\text{Hodge}}(T(S)) \backslash I^d(D_{NS(S)})$ and the lattices M, M' run over the sets $\mathcal{G}_1(M_x), \mathcal{G}_2(M_x)$ respectively. The natural number $\varepsilon(d)$ is defined by $\varepsilon(d) = 1$ if $d = 1, 2$, and $\varepsilon(d) = 2$ if $d \geq 3$.

When $d = 1$, Theorem 1.1 is Hosono-Lian-Oguiso-Yau’s formula. Even though Theorem 1.1 seems complicated in appearance, it is rather adequate for calculations for the following reasons:

- The genus $\mathcal{G}(M)$ and the homomorphism $O(M) \rightarrow O(D_M)$ have been studied in lattice theory, and much is known (cf. [9]).
- If we can identify the discriminant form $D_{NS(S)}$, it is easy to calculate $I^d(D_{NS(S)})$.
- The group $O_{\text{Hodge}}(T(S))$ and its subgroup $O_{\text{Hodge}}(T_x, \alpha_x)$ are cyclic, and the Euler function of the order of $O_{\text{Hodge}}(T(S))$ must divide $\text{rk}(T(S))$ (cf. [5]). For example, $O_{\text{Hodge}}(T(S)) = \{\pm \text{id}\}$ whenever $\text{rk}(T(S))$ is odd.

In fact, we shall derive more simple formulae for $\#\text{FM}^d(S)$ for several classes of K3 surfaces: Jacobian K3 surfaces, in particular K3 surfaces with $\text{rk}(NS(S)) \geq 13$, K3 surfaces with 2-elementary $NS(S)$, and K3 surfaces with $\text{rk}(NS(S)) = 1$.

As a by-product of the proof of Theorem 1.1, we obtain an upper bound for the twisted FM number of a twisted K3 surface (Proposition 4.3). The sharpness of the estimate is related with a Căldăraru’s problem stated in [1] (Remark 3.7). Note that if a twisted K3 surface (S', α') has an untwisted FM partner, e.g., if $\text{rk}(NS(S')) \geq 12$ ([6]), then Theorem 1.1 gives a counting formula for the twisted FM number of (S', α') .

As an application of Theorem 1.1, we shall describe all twisted FM partners of a K3 surface of Picard number 1 as follows.

Theorem 1.2 (Theorem 5.3). *Let S be a projective K3 surface with $NS(S) = \mathbb{Z}H$, $(H, H) = 2n$. We have $\text{FM}^d(S) \neq \emptyset$ if and only if $d^2 | n$. For a natural number d*

with $d^2|n$, let $\{v_{\sigma,k} \mid (\sigma,k) \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times\}$ be the primitive isotropic vectors in $\widetilde{NS}(S)$ defined by (5.2). Let $M_{\sigma,k}$ be the moduli space of Gieseker stable sheaves on S with Mukai vector $v_{\sigma,k}$, endowed with the obstruction $\alpha_{\sigma,k} \in \text{Br}(M_{\sigma,k})$ to the existence of a universal sheaf.

(1) If $d^2 < n$, then

$$\text{FM}^d(S) = \{ (M_{\sigma,k}, \alpha_{\sigma,k}) \mid (\sigma,k) \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times \}.$$

(2) Assume that $d^2 = n$. Choose a set $\{j\} \subset (\mathbb{Z}/d\mathbb{Z})^\times$ of representatives of $(\mathbb{Z}/d\mathbb{Z})^\times / \{\pm \text{id}\}$. Then

$$\text{FM}^d(S) = \{ (M_{\sigma,k}, \alpha_{\sigma,k}) \mid (\sigma,k) \in \Sigma \times \{j\} \}.$$

Theorem 1.2 is a twisted generalization of a result of [4] and [12].

This paper is organized as follows. In Sect. 2.1, we prepare some lattice theory following [9]. In Sect. 2.2, we recall from [1] and [6] several facts about twisted K3 surfaces. In Sect. 3.1, we describe a quotient set of $\text{FM}^d(S)$ in terms of lattice theory. In Sections 3.2 and 3.3, we describe the fibers of the quotient map in terms of lattice theory. In Sect. 4, we derive Theorem 1.1 and its corollaries. In Sect. 5, we prove Theorem 1.2.

Notation 1.3. By an *even lattice*, we mean a free \mathbb{Z} -module L equipped with a non-degenerate symmetric bilinear form $(,) : L \times L \rightarrow \mathbb{Z}$ satisfying $(x, x) \in 2\mathbb{Z}$ for all $x \in L$. The group of the isometries of L is denoted by $O(L)$. For two lattices L and M , $\text{Emb}(L, M)$ is the set of the primitive embeddings of L into M . An element $l \in L$ is said to be *isotropic* if $(l, l) = 0$. The *hyperbolic plane* U is the even indefinite unimodular lattice $\mathbb{Z}e + \mathbb{Z}f$, $(e, e) = (f, f) = 0$, $(e, f) = 1$.

By a *K3 surface*, we mean a projective K3 surface over \mathbb{C} . The Néron-Severi (resp. transcendental) lattice of a K3 surface S is denoted by $NS(S)$ (resp. $T(S)$). Let $\widetilde{NS}(S) := H^0(S, \mathbb{Z}) \oplus NS(S) \oplus H^4(S, \mathbb{Z})$ and $\widetilde{H}(S, \mathbb{Z}) := \bigoplus_{i=0}^2 H^{2i}(S, \mathbb{Z})$, which are equipped with the Mukai pairing. We denote $\Lambda_{K3} := U^3 \oplus E_8^2$ and $\widetilde{\Lambda}_{K3} := U \oplus \Lambda_{K3} = U^4 \oplus E_8^2$.

2. PRELIMINARIES

2.1. Even lattices. For an even lattice L , we can associate a finite Abelian group $D_L := L^\vee/L$ equipped with a quadratic form $q_L : D_L \rightarrow \mathbb{Q}/2\mathbb{Z}$, $q_L(x + L) = (x, x) + 2\mathbb{Z}$ for $x \in L^\vee$. We call (D_L, q_L) the *discriminant form* of L . There is a natural homomorphism $r_L : O(L) \rightarrow O(D_L, q_L)$, whose kernel is denoted by $O(L)_0$. We often write just (D_L, q) or D_L (resp. r) instead of (D_L, q_L) (resp. r_L). Set

$$I^d(D_L) := \{ x \in D_L \mid q_L(x) = 0 \in \mathbb{Q}/2\mathbb{Z}, \text{ord}(x) = d \}.$$

Proposition 2.1 ([9]). *Let M be a primitive sublattice of an even unimodular lattice L with the orthogonal complement M^\perp . Then*

- (1) *There is an isometry $\lambda : (D_M, q_M) \xrightarrow{\cong} (D_{M^\perp}, -q_{M^\perp})$.*
- (2) *For two isometries $\gamma_M \in O(M)$ and $\gamma_{M^\perp} \in O(M^\perp)$, there is an isometry $\gamma_L \in O(L)$ such that $\gamma_L|_{M \oplus M^\perp} = \gamma_M \oplus \gamma_{M^\perp}$ if and only if $\lambda \circ r_M(\gamma_M) = r_{M^\perp}(\gamma_{M^\perp}) \circ \lambda$.*

Two even lattices L and L' are said to be *isogenous* if $L \otimes \mathbb{Z}_p \simeq L' \otimes \mathbb{Z}_p$ for every prime number p and $\text{sign}(L) = \text{sign}(L')$. By [9], these are equivalent to the conditions that $(D_L, q) \simeq (D_{L'}, q)$ and $\text{sign}(L) = \text{sign}(L')$. The set of the isometry classes of the lattices isogenous to L is denoted by $\mathcal{G}(L)$.

Proposition 2.2 ([9]). *Let $l(D_L)$ be the minimal number of the generators of D_L . If L is indefinite and $\text{rk}(L) \geq l(D_L) + 2$, then $\mathcal{G}(L) = \{L\}$ and the homomorphism r_L is surjective.*

Let L be a lattice of $\text{sign}(L) = (b_+, b_-)$, $b_+ > 0$. We denote by Ω_L the set of the oriented positive-definite b_+ -planes in $L \otimes \mathbb{R}$, which is an open subset of the oriented Grassmannian and has two connected components. A choice of a component of Ω_L , which is equivalent to a choice of an orientation for a positive-definite b_+ -plane in $L \otimes \mathbb{R}$, is sometimes called an *orientation* of L . For example, for a $K3$ surface S the lattice $\widetilde{NS}(S)$ is of $\text{sign}(\widetilde{NS}(S)) = (2, \rho(S))$. We define the orientation of $\widetilde{NS}(S)$ so that $\mathbb{R}(1, 0, -1) \oplus \mathbb{R}(0, H, 0)$ is of positive orientation, where $H \in NS(S)$ is an ample class.

For a subgroup $\Gamma \subset O(L)$, let Γ^+ be the subgroup of Γ consisting of the orientation-preserving isometries in Γ . The group Γ^+ is of index at most 2 in Γ . We define the subsets $\mathcal{G}_1(L), \mathcal{G}_2(L) \subset \mathcal{G}(L)$ by

$$\begin{aligned} \mathcal{G}_1(L) &:= \{L' \in \mathcal{G}(L) \mid O(L')_0^+ \neq O(L')_0\}, \\ \mathcal{G}_2(L) &:= \{L' \in \mathcal{G}(L) \mid O(L')_0^+ = O(L')_0\}. \end{aligned}$$

We have the obvious decomposition $\mathcal{G}(L) = \mathcal{G}_1(L) \sqcup \mathcal{G}_2(L)$. When b_+ is odd, we have $O(L)_0^+ \neq O(L)_0$ if and only if $-\text{id}_{D_L}$ is contained in $r_L(O(L)^+)$.

2.2. Twisted $K3$ surfaces. We recall some basic facts about twisted $K3$ surfaces following [1] and [6].

Let (S, α) be a twisted $K3$ surface. That is, S is a $K3$ surface and α is an element of the Brauer group $\text{Br}(S)$, the group of the torsion elements of $H^2(\mathcal{O}_S^\times)$. Via the exponential sequence

$$0 \rightarrow \text{Pic}(S) \rightarrow H^2(S, \mathbb{Z}) \rightarrow H^2(\mathcal{O}_S) \rightarrow H^2(\mathcal{O}_S^\times) \rightarrow 1,$$

we have a canonical isomorphism

$$(2.1) \quad \text{Br}(S) \simeq H^2(S, \mathbb{Q}) / (NS(S) \otimes \mathbb{Q} + H^2(S, \mathbb{Z})).$$

We denote the identity of $\text{Br}(S)$ by 0, and the inverse of α by $-\alpha$. A (*rational*) B -field lift of α is a class $B \in H^2(S, \mathbb{Q})$ mapping to α in (2.1). Considering the intersection pairings of B -field lifts and $T(S)$, we also have

$$(2.2) \quad \text{Br}(S) \simeq \text{Hom}(T(S), \mathbb{Q}/\mathbb{Z}).$$

By (2.2), we identify α with a surjective homomorphism $\alpha : T(S) \rightarrow \mathbb{Z}/\text{ord}(\alpha)\mathbb{Z}$, whose kernel is denoted by $T(S, \alpha)$.

Definition 2.3. Set

$$\begin{aligned} O_{\text{Hodge}}(T(S), \alpha) &:= \{g \in O_{\text{Hodge}}(T(S)), g^* \alpha = \alpha\}, \\ \Gamma(S, \alpha) &:= r_{\widetilde{NS}(S)}^{-1}(\lambda \circ r_{T(S)}(O_{\text{Hodge}}(T(S), \alpha))), \end{aligned}$$

where $O_{\text{Hodge}}(T(S))$ is the group of the Hodge isometries of $T(S)$, and $\lambda : O(D_{T(S)}) \xrightarrow{\cong} O(D_{\widetilde{NS}(S)})$ is the isomorphism induced from the isometry $\lambda_{\widetilde{H}(S, \mathbb{Z})} : (D_{T(S)}, q) \xrightarrow{\cong} (D_{\widetilde{NS}(S)}, -q)$.

From the inclusions $T(S, \alpha) \subset T(S) \subset T(S, \alpha)^\vee$, we have the isomorphism

$$O_{\text{Hodge}}(T(S), \alpha) \simeq \left\{ g \in O_{\text{Hodge}}(T(S, \alpha)), r(g)(\alpha^{-1}(\bar{1})) = \alpha^{-1}(\bar{1}) \in D_{T(S, \alpha)} \right\},$$

where $\bar{1} \in \mathbb{Z}/\text{ord}(\alpha)\mathbb{Z}$. In the generic case, $O_{\text{Hodge}}(T(S), \alpha) = \{\text{id}\}$ if $\text{ord}(\alpha) \geq 3$ and $O_{\text{Hodge}}(T(S), \alpha) = \{\pm \text{id}\}$ if $\text{ord}(\alpha) \leq 2$.

Let B be a B -field lift of α and let $\omega_S \in H^2(S, \mathbb{C})$ be a period of S . By definition, the twisted Mukai lattice $\tilde{H}(S, B, \mathbb{Z})$ of (S, α) and B is the lattice $\tilde{H}(S, \mathbb{Z})$ endowed with the period $e^B(\omega_S) = (1, B, \frac{1}{2}(B, B)) \wedge (0, \omega_S, 0)$. Let

$$\begin{aligned} \widetilde{NS}(S, B) &:= e^B(\omega_S)^\perp \cap \tilde{H}(S, B, \mathbb{Z}), \\ T(S, B) &:= \widetilde{NS}(S, B)^\perp \cap \tilde{H}(S, B, \mathbb{Z}). \end{aligned}$$

We have a Hodge isometry $e^B : T(S, \alpha) \simeq T(S, B)$, as each class $l \in T(S)$ is of degree 2.

We need the following criterion for derived equivalence, which is just a combination of known results. It is implicitly proved in Section 7 of [6].

Proposition 2.4 ([1], [6], [14], [7]). *Let S be a K3 surface and let (S', α') be a twisted K3 surface. Then there exists an equivalence $D^b(S', \alpha') \simeq D^b(S)$ if and only if there exists a Hodge isometry $T(S) \simeq T(S', \alpha')$.*

Proof. If there is an equivalence $D^b(S', \alpha') \simeq D^b(S)$, the equivalence is of Fourier-Mukai type by the theorem of Canonaco-Stellari ([3]). It follows from Theorem 0.4 of [6] that $\tilde{H}(S', B', \mathbb{Z})$ is Hodge isometric to $\tilde{H}(S, \mathbb{Z})$, where $B' \in H^2(S', \mathbb{Q})$ is a B -field lift of α' . In particular, $T(S', B')$ is Hodge isometric to $T(S)$.

Conversely, assume that there is a Hodge isometry $T(S) \simeq T(S', \alpha')$. We have a Hodge isometry $g : T(S) \simeq T(S', B')$ for a B -field lift B' of α' . By Proposition 2.1, $\widetilde{NS}(S)$ and $\widetilde{NS}(S', B')$ are isogenous. Because $\widetilde{NS}(S)$ admits an embedding of the hyperbolic plane U , it follows from Proposition 2.2 that g extends to a Hodge isometry $\tilde{\Phi} : \tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(S', B', \mathbb{Z})$. Composing $\tilde{\Phi}$ with the isometry $-\text{id}_{H^0(S)+H^4(S)} \oplus \text{id}_{H^2(S)}$ if necessary, we may assume that $\tilde{\Phi}$ is orientation-preserving. Now we have $D^b(S', \alpha') \simeq D^b(S)$ by Theorem 0.1 of [7]. \square

When $\alpha' = 0$, Proposition 2.4 is Mukai-Orlov's theorem ([8], [10]).

Definition 2.5. Let (S, α) be a twisted K3 surface. A twisted K3 surface (S', α') is called a *twisted Fourier-Mukai (FM) partner* of (S, α) if there exists an equivalence $D^b(S', \alpha') \simeq D^b(S, \alpha)$. The set of the isomorphism classes of the twisted FM partners (S', α') of (S, α) with $\text{ord}(\alpha') = d$ is denoted by $\text{FM}^d(S, \alpha)$. It is easily seen that $\text{FM}^d(S, \alpha)$ is empty unless d^2 divides $\det(T(S, \alpha))$.

Definition 2.6. Set

$$\mathcal{FM}^d(S, \alpha) := \text{FM}^d(S, \alpha) / \sim,$$

where two twisted FM partners $(S_1, \alpha_1), (S_2, \alpha_2) \in \text{FM}^d(S, \alpha)$ are equivalent if there exists a Hodge isometry $g : T(S_1) \simeq T(S_2)$ with $g^* \alpha_2 = \alpha_1$. Denote by

$$\pi : \text{FM}^d(S, \alpha) \rightarrow \mathcal{FM}^d(S, \alpha)$$

the quotient map.

We remark that for a twisted K3 surface (S, α) with a B -field lift $B \in H^2(S, \mathbb{Q})$, the period of S is determined by the natural Hodge isometry

$$H^2(S, \mathbb{Z}) \simeq \left((0, 0, 1)^\perp \cap \tilde{H}(S, B, \mathbb{Z}) \right) / \mathbb{Z}(0, 0, 1).$$

In Section 5, we shall calculate twistings in terms of the discriminant group. The basic idea of the following lemma goes back to [8].

Lemma 2.7. *Let (S, α) be a twisted K3 surface with a B-field lift $B \in H^2(S, \mathbb{Q})$. We denote $d := \text{ord}(\alpha)$ and $\lambda := \lambda_{\widetilde{H}(S, B, \mathbb{Z})} : (D_{\widetilde{NS}(S, B)}, q) \simeq (D_{T(S, B)}, -q)$. Then $(0, 0, \frac{-1}{d}) \in \widetilde{NS}(S, B)^\vee$ and we have the Hodge isometry*

$$(2.3) \quad e^B := \exp(B) \wedge : T(S) \xrightarrow{\simeq} \left\langle \lambda\left(\left(0, 0, \frac{-1}{d}\right)\right), T(S, B) \right\rangle.$$

The twisting α is given by

$$(2.4) \quad T(S) \xrightarrow{e^B} e^B(T(S))/T(S, B) \simeq \left\langle \lambda\left(\left(0, 0, \frac{-1}{d}\right)\right) \right\rangle \simeq \mathbb{Z}/d\mathbb{Z}.$$

Proof. Since $\alpha : T(S) \rightarrow \mathbb{Z}/d\mathbb{Z}$ is surjective, we can take a transcendental cycle $l \in T(S)$ such that $(l, B) = \frac{1}{d} + k$ with $k \in \mathbb{Z}$. Setting $l' := e^B(l) = l + (0, 0, \frac{1}{d} + k) \in T(S, B)^\vee$, we have $e^B : T(S) \simeq \langle l', T(S, B) \rangle$. Since $l' - (0, 0, \frac{1}{d}) \in \widetilde{H}(S, B, \mathbb{Z})$, it follows that $((0, 0, \frac{-1}{d}), m) \in \mathbb{Z}$ for all $m \in \widetilde{NS}(S, B)$. By the relation $\lambda((0, 0, \frac{-1}{d})) = l' \in D_{T(S, B)}$, we have (2.3). Then (2.4) follows from the relation $\alpha(l) = \bar{1} \in \mathbb{Z}/d\mathbb{Z}$. \square

More precisely, the divisibility of $(0, 0, 1) \in \widetilde{NS}(S, B)$ is equal to d , but we do not need this fact in the following.

3. LATTICE-THEORETIC DESCRIPTIONS

3.1. Isotropic elements of the discriminant form. Let (S, α) be a twisted K3 surface. We write $T := T(S, \alpha)$, which is an even lattice of $\text{sign}(T) = (2, 20 - \rho(S))$ equipped with a period. For a twisted FM partner $(S_1, \alpha_1) \in \text{FM}^d(S, \alpha)$ there exists a Hodge isometry $g_1 : T(S_1, \alpha_1) \simeq T$ by the Huybrechts-Stellari theorem (Theorem 0.4 of [6]). Then g_1 and α_1 induce an isomorphism $g_1(T(S_1))/T \xrightarrow{\simeq} \mathbb{Z}/d\mathbb{Z}$. The subgroup $g_1(T(S_1))/T \subset D_T$ is an isotropic cyclic group of order $= d$.

Definition 3.1. Define the map

$$\mu : \mathcal{FM}^d(S, \alpha) \rightarrow O_{\text{Hodge}}(T) \backslash I^d(D_T)$$

by $\mu([(S_1, \alpha_1)]) := [g_1(\alpha_1^{-1}(\bar{1}))]$.

Lemma 3.2. *The map μ is well defined and is injective.*

Proof. The ambiguity of the choice of a Hodge isometry $g_1 : T(S_1, \alpha_1) \simeq T$ comes from the action of $O_{\text{Hodge}}(T)$ so that the class $[g_1(\alpha_1^{-1}(\bar{1}))] \in O_{\text{Hodge}}(T) \backslash I^d(D_T)$ is well defined from the partner (S_1, α_1) . Since the map μ is defined in terms of $T(S_1)$ and α_1 , it is clear that μ is well defined as a map from $\mathcal{FM}^d(S, \alpha)$.

We prove the injectivity of μ . For two partners $(S_i, \alpha_i) \in \text{FM}^d(S, \alpha)$, $i = 1, 2$, and Hodge isometries $g_i : T(S_i, \alpha_i) \simeq T$, write $x_i := g_i(\alpha_i^{-1}(\bar{1})) \in I^d(D_T)$. Assume the existence of a Hodge isometry $\varphi \in O_{\text{Hodge}}(T)$ such that $x_2 = r(\varphi)(x_1)$. Since $r(\varphi)(\langle x_1 \rangle) = \langle x_2 \rangle \subset D_T$, φ extends to the Hodge isometry $\tilde{\varphi} : T(S_1) \simeq T(S_2)$. By the equality $r(\varphi)(g_1(\alpha_1^{-1}(\bar{1}))) = g_2(\alpha_2^{-1}(\bar{1}))$ we have $\tilde{\varphi}^* \alpha_2 = \alpha_1$. Hence $[(S_1, \alpha_1)] = [(S_2, \alpha_2)] \in \mathcal{FM}^d(S, \alpha)$. \square

Proposition 3.3. *If the twisted K3 surface (S, α) has an untwisted FM partner, then the map μ is bijective.*

Proof. It suffices to show the surjectivity. Let S' be an untwisted FM partner of (S, α) . Since there exists a Hodge isometry $T(S') \simeq T$, we may assume from the first that (S, α) itself is untwisted. Take an isotropic element $x \in I^d(D_T) = I^d(D_{T(S)})$. Via the isometry $\lambda : (D_{T(S)}, q) \simeq (D_{\widetilde{NS}(S)}, -q)$, we obtain an isotropic element $\lambda(x) \in I^d(D_{\widetilde{NS}(S)})$. Set

$$\begin{aligned} \widetilde{M}_x &:= \langle \lambda(x), \widetilde{NS}(S) \rangle \subset \widetilde{NS}(S)^\vee, \\ T_x &:= \langle x, T(S) \rangle \subset T(S)^\vee, \end{aligned}$$

which are even overlattices of $\widetilde{NS}(S)$, $T(S)$ respectively.

Via the isometry

$$(D_{T_x}, q) \simeq (\langle x \rangle^\perp / \langle x \rangle, q) \simeq (\langle \lambda(x) \rangle^\perp / \langle \lambda(x) \rangle, -q) \simeq (D_{\widetilde{M}_x}, -q),$$

we obtain an embedding $\widetilde{M}_x \oplus T_x \hookrightarrow \widetilde{\Lambda}_{K3}$ with both \widetilde{M}_x and T_x embedded primitively. Since $\widetilde{NS}(S) \subset \widetilde{M}_x$, the lattice \widetilde{M}_x admits an embedding φ of the hyperbolic plane U . Then the lattice $\Lambda_\varphi := \varphi(U)^\perp \cap \widetilde{\Lambda}_{K3}$ is isometric to the $K3$ lattice Λ_{K3} and has the period induced from T_x . Now the surjectivity of the period map ([13], [15]) assures the existences of a $K3$ surface S_φ and a Hodge isometry $\Phi : H^2(S_\varphi, \mathbb{Z}) \xrightarrow{\sim} \Lambda_\varphi$. Pulling back the homomorphism

$$\alpha_x : T_x \rightarrow T_x/T(S) \simeq \langle x \rangle \simeq \mathbb{Z}/d\mathbb{Z}, \quad \alpha_x(x) = \bar{1}$$

by $\Phi|_{T(S_\varphi)}$, we obtain a twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$.

Since there exists a Hodge isometry $T(S_\varphi, \alpha_\varphi) \simeq \text{Ker}(\alpha_x) = T(S)$, it follows from Proposition 2.4 that $(S_\varphi, \alpha_\varphi) \in \text{FM}^d(S)$. By the construction, we have $\mu([(S_\varphi, \alpha_\varphi)]) = [x]$. \square

In general, there is a condition on the image of μ . Let $B \in H^2(S, \mathbb{Q})$ be a B -field lift of $\alpha \in \text{Br}(S)$. There is a natural isometry $\lambda : (D_T, q) \simeq (D_{\widetilde{NS}(S, B)}, -q)$. We define

$$(3.1) \quad J^d(D_T) := \{x \in I^d(D_T), \text{Emb}(U, \langle \lambda(x), \widetilde{NS}(S, B) \rangle) \neq \phi\}.$$

One can verify that the set $J^d(D_T)$ is independent of the choice of a lift B .

Proposition 3.4. *We have the inclusion*

$$\text{Im}(\mu) \subset O_{\text{Hodge}}(T) \setminus J^d(D_T).$$

Proof. For a twisted FM partner $(S_1, \alpha_1) \in \text{FM}^d(S, \alpha)$, write $[x] := \mu([(S_1, \alpha_1)])$. By Proposition 2.2, it suffices to show that $\widetilde{NS}(S_1)$ and $\widetilde{M} := \langle \lambda(x), \widetilde{NS}(S, B) \rangle$ are isogenous. We have the isometry

$$\begin{aligned} (D_{\widetilde{NS}(S_1)}, q) &\simeq (D_{T(S_1)}, -q) \simeq (\langle x \rangle^\perp / \langle x \rangle, -q) \\ &\simeq (\langle \lambda(x) \rangle^\perp / \langle \lambda(x) \rangle, q) \simeq (D_{\widetilde{M}}, q). \end{aligned}$$

\square

We make the following identification tacitly in Section 4. For $x \in I^d(D_T)$ such that $[x] = \mu([(S_1, \alpha_1)])$, set $T_x := \langle x, T \rangle$. For a Hodge isometry $g_1 : T(S_1, \alpha_1) \simeq T$ with $g_1(\alpha_1^{-1}(\bar{1})) = x$, by Proposition 2.2 there exists an isometry

$$\gamma : \widetilde{NS}(S_1) \xrightarrow{\sim} \langle \lambda(x), \widetilde{NS}(S, B) \rangle \quad \text{with} \quad r(\gamma) \circ \lambda_{\widetilde{H}(S_1, \mathbb{Z})} = \bar{\lambda} \circ r(\bar{g}_1).$$

Here the Hodge isometry $\bar{g}_1 : T(S_1) \simeq T_x$ is induced from g_1 , and the isometry $\bar{\lambda} : (D_{T_x}, q) \simeq (D_{\langle \lambda(x), \widetilde{NS}(S, B) \rangle}, -q)$ is induced from λ .

Let $\lambda_{H^2(S)} : D_{T(S)} \rightarrow D_{NS(S)}$ be the natural isomorphism. When S is untwisted, $NS(S_1)$ is isogenous to $\langle \lambda_{H^2(S)}(x), NS(S) \rangle$.

3.2. Embeddings of the hyperbolic plane. Let $(S_1, \alpha_1) \in \text{FM}^d(S, \alpha)$ be a twisted FM partner of (S, α) . Recall from Section 2.2 that we have defined the finite-index subgroup $\Gamma(S_1, \alpha_1) \subset O(\widetilde{NS}(S_1))$. The subgroup $\Gamma(S_1, \alpha_1)^+ \subset \Gamma(S_1, \alpha_1)$ consists of orientation-preserving isometries in $\Gamma(S_1, \alpha_1)$ (see Section 2.1). We define the map

$$\nu : \pi^{-1}\left(\pi((S_1, \alpha_1))\right) \rightarrow \Gamma(S_1, \alpha_1)^+ \backslash \text{Emb}(U, \widetilde{NS}(S_1))$$

as follows. For a twisted FM partner $(S_2, \alpha_2) \in \pi^{-1}\left(\pi((S_1, \alpha_1))\right)$ there exists a Hodge isometry $g : T(S_2) \xrightarrow{\sim} T(S_1)$ with $g^*\alpha_1 = \alpha_2$. By Proposition 2.2, g can be extended to a Hodge isometry $\tilde{\Phi} : \tilde{H}(S_2, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(S_1, \mathbb{Z})$ such that the isometry $\tilde{\Phi}|_{\widetilde{NS}(S_2)} : \widetilde{NS}(S_2) \xrightarrow{\sim} \widetilde{NS}(S_1)$ is orientation-preserving. Then, by considering $\tilde{\Phi} : H^0(S_2, \mathbb{Z}) + H^4(S_2, \mathbb{Z}) \hookrightarrow \widetilde{NS}(S_1)$, we obtain an embedding $\varphi : U \hookrightarrow \widetilde{NS}(S_1)$. Here we identify $H^0(S_2, \mathbb{Z}) + H^4(S_2, \mathbb{Z})$ with U by identifying $(1, 0, 0)$ with e , and $(0, 0, -1)$ with f .

Lemma 3.5. *The assignment $(S_2, \alpha_2) \mapsto [\varphi]$ defines an injective map*

$$\nu : \pi^{-1}\left(\pi((S_1, \alpha_1))\right) \rightarrow \Gamma(S_1, \alpha_1)^+ \backslash \text{Emb}(U, \widetilde{NS}(S_1)).$$

Proof. First we prove that ν is well defined. Let us be given two Hodge isometries $g, g' : T(S_2) \xrightarrow{\sim} T(S_1)$ with $g^*\alpha_1 = (g')^*\alpha_1 = \alpha_2$. Let $\gamma, \gamma' : \widetilde{NS}(S_2) \xrightarrow{\sim} \widetilde{NS}(S_1)$ be orientation-preserving isometries such that $\gamma \oplus g$ and $\gamma' \oplus g'$ extend to Hodge isometries $\tilde{H}(S_2, \mathbb{Z}) \simeq \tilde{H}(S_1, \mathbb{Z})$. Then the isometry $\gamma' \circ \gamma^{-1} \in O(\widetilde{NS}(S_1))$ preserves the orientation and satisfies $r(\gamma' \circ \gamma^{-1}) \circ \lambda = \lambda \circ r(g' \circ g^{-1})$, so that we have $\gamma' \in \Gamma(S_1, \alpha_1)^+ \cdot \gamma$. Hence the map ν is well defined.

We prove the injectivity of ν . For two partners $(S_i, \alpha_i) \in \pi^{-1}\left(\pi((S_1, \alpha_1))\right)$, $i = 2, 3$, set $[\varphi_i] := \nu((S_i, \alpha_i))$ and assume the existence of an isometry $\gamma \in \Gamma(S_1, \alpha_1)^+$ such that $\varphi_2 = \gamma \circ \varphi_3$. We can extend γ to a Hodge isometry $\tilde{\Phi} : \tilde{H}(S_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(S_1, \mathbb{Z})$ with $(\tilde{\Phi}|_{T(S_1)})^*\alpha_1 = \alpha_1$. By restricting $\tilde{\Phi}$ to $\varphi_2(U)^\perp \cap \tilde{H}(S_1, \mathbb{Z})$, we obtain a Hodge isometry $\Phi : H^2(S_2, \mathbb{Z}) \xrightarrow{\sim} H^2(S_3, \mathbb{Z})$ with $(\Phi|_{T(S_2)})^*\alpha_3 = \alpha_2$. Since γ preserves the orientation, Φ maps the positive cone $NS(S_2)^+$ to the positive cone $NS(S_3)^+$. Composing Φ with an element of the Weyl group $W(S_3)$, we may assume that Φ is effective. Then it follows from the Torelli theorem ([11], [15]) that $(S_3, \alpha_3) \simeq (S_2, \alpha_2)$. \square

Proposition 3.6. *If (S, α) has an untwisted FM partner, then the map ν for every twisted FM partner $(S_1, \alpha_1) \in \text{FM}^d(S, \alpha)$ is bijective.*

Proof. We may assume that (S, α) itself is untwisted. To prove the surjectivity, we shall construct a twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$ from an arbitrary embedding $\varphi : U \hookrightarrow \widetilde{NS}(S_1)$. The lattice $\Lambda_\varphi := \varphi(U)^\perp \cap \tilde{H}(S_1, \mathbb{Z})$ is isometric to the $K3$ lattice Λ_{K3} and possesses the period induced from $T(S_1)$. On the other hand, the lattice $M_\varphi := \varphi(U)^\perp \cap \widetilde{NS}(S_1)$ has the orientation induced from φ and the

orientation of $\widetilde{NS}(S_1)$. That is, we can choose a connected component M_φ^+ of the open set $\{v \in M_\varphi \otimes \mathbb{R}, (v, v) > 0\}$ so that for each vector $v \in M_\varphi^+$ the oriented positive-definite two-plane $\mathbb{R}\varphi(e + f) \oplus \mathbb{R}v \subset \widetilde{NS}(S_1) \otimes \mathbb{R}$ is of positive orientation. By the surjectivity of the period map, there exist a $K3$ surface S_φ and a Hodge isometry $\Phi : H^2(S_\varphi, \mathbb{Z}) \xrightarrow{\cong} \Lambda_\varphi$ such that $\Phi(NS(S_\varphi)^+) = M_\varphi^+$. Pulling back α_1 by $\Phi|_{T(S_\varphi)}$, we obtain a twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$.

Since there is a Hodge isometry

$$T(S_\varphi, \alpha_\varphi) \simeq T(S_1, \alpha_1) \simeq T(S),$$

we have an equivalence $D^b(S_\varphi, \alpha_\varphi) \simeq D^b(S)$ by Proposition 2.4. Hence we have $(S_\varphi, \alpha_\varphi) \in \pi^{-1}(\pi((S_1, \alpha_1)))$.

Since the direct sum of the Hodge isometry $\Phi|_{T(S_\varphi)} : T(S_\varphi) \xrightarrow{\cong} T(S_1)$ satisfying $(\Phi|_{T(S_\varphi)})^* \alpha_1 = \alpha_\varphi$ and the orientation-preserving isometry

$$\varphi \oplus (\Phi|_{NS(S_\varphi)}) : \widetilde{NS}(S_\varphi) \xrightarrow{\cong} \varphi(U) \oplus M_\varphi = \widetilde{NS}(S_1)$$

can be extended to the Hodge isometry $\varphi \oplus \Phi$, we have $\nu((S_\varphi, \alpha_\varphi)) = [\varphi]$. □

Remark 3.7. In his thesis [1], Căldăraru proposed the following problem:

Question 3.8 ([1]). Let (S_1, α_1) be a twisted $K3$ surface. For each untwisted FM partner $S_2 \in \text{FM}(S_1)$ and each Hodge isometry $g : T(S_2) \xrightarrow{\cong} T(S_1)$, we obtain a twisted $K3$ surface $(S_2, g^* \alpha_1)$. Is it true that $(S_2, g^* \alpha_1) \in \text{FM}^d(S_1, \alpha_1)$?

From the construction of the twisted $K3$ surface $(S_\varphi, \alpha_\varphi)$ in the proof of Proposition 3.6, it is immediately verified that the map ν in Lemma 3.5 is bijective if and only if the answer to Căldăraru’s question is positive for the twisted $K3$ surface (S_1, α_1) and each S_2, g .

3.3. Union of the two orbits. Given an embedding $\varphi : U \hookrightarrow \widetilde{NS}(S_1)$, we have the decomposition

$$\Gamma(S_1, \alpha_1) = \Gamma(S_1, \alpha_1)^+ \sqcup \Gamma(S_1, \alpha_1)^+ \cdot (-\text{id}_{\varphi(U)} \oplus \text{id}_{\varphi(U)^\perp}).$$

From this we obtain

$$\Gamma(S_1, \alpha_1) \cdot \varphi = \Gamma(S_1, \alpha_1)^+ \cdot \varphi \cup \Gamma(S_1, \alpha_1)^+ \cdot (-\varphi).$$

Lemma 3.9. *Let $(S_\varphi, \alpha_\varphi)$ be the twisted $K3$ surface constructed in the proof of Proposition 3.6. Then $-\varphi \in \Gamma(S_1, \alpha_1)^+ \cdot \varphi$ if and only if $(S_\varphi, \alpha_\varphi) \simeq (S_\varphi, -\alpha_\varphi)$.*

Proof. It suffices to show that $(S_{-\varphi}, \alpha_{-\varphi}) \simeq (S_\varphi, -\alpha_\varphi)$. Since $M_{-\varphi}^+ = -M_\varphi^+$, we have a Hodge isometry $\Phi : H^2(S_{-\varphi}, \mathbb{Z}) \xrightarrow{\cong} H^2(S_\varphi, \mathbb{Z})$ with $(\Phi|_{T(S_{-\varphi})})^* \alpha_\varphi = \alpha_{-\varphi}$ and $\Phi(NS(S_{-\varphi})^+) = -NS(S_\varphi)^+$. Then $-\Phi : H^2(S_{-\varphi}, \mathbb{Z}) \xrightarrow{\cong} H^2(S_\varphi, \mathbb{Z})$ satisfies $(-\Phi|_{T(S_{-\varphi})})^* (-\alpha_\varphi) = \alpha_{-\varphi}$ and $-\Phi(NS(S_{-\varphi})^+) = NS(S_\varphi)^+$. □

When $\text{ord}(\alpha_1) \leq 2$, we have $\alpha_\varphi = -\alpha_\varphi$ so that $\Gamma(S_1, \alpha_1) \cdot \varphi = \Gamma(S_1, \alpha_1)^+ \cdot \varphi$ for every embedding φ . Hence we obtain the identification

$$\Gamma(S_1, \alpha_1)^+ \backslash \text{Emb}(U, \widetilde{NS}(S_1)) \simeq \Gamma(S_1, \alpha_1) \backslash \text{Emb}(U, \widetilde{NS}(S_1)).$$

Proposition 3.10. *Assume that $\text{ord}(\alpha_1) \geq 3$. For an embedding $\varphi : U \hookrightarrow \widetilde{NS}(S_1)$ the following conditions are equivalent:*

- (i) $(S_\varphi, \alpha_\varphi) \simeq (S_\varphi, -\alpha_\varphi)$,
- (ii) S_φ admits an anti-symplectic automorphism,
- (iii) $\varphi(U)^\perp \cap \widetilde{NS}(S_1) \in \mathcal{G}_1(NS(S_1))$, i.e., $O(NS(S_\varphi))_0^+ \neq O(NS(S_\varphi))_0$.

Proof. (ii) \Rightarrow (i) is obvious.

(i) \Rightarrow (ii): Assume the existence of an automorphism $f : (S_\varphi, \alpha_\varphi) \simeq (S_\varphi, -\alpha_\varphi)$. Since $\text{ord}(\alpha_\varphi) \geq 3$, $\text{ord}(f^*|_{T(S_\varphi)})$ must be even, say $2n$. As $O_{\text{Hodge}}(T(S_\varphi))$ is a cyclic group (cf. Appendix B of [5]), then f^n is an anti-symplectic automorphism of S_φ .

(ii) \Rightarrow (iii): For an anti-symplectic automorphism f , we have $-f^*|_{NS(S_\varphi)} \in O(NS(S_\varphi))_0 - O(NS(S_\varphi))_0^+$.

(iii) \Rightarrow (ii): Assume the existence of an isometry $\gamma \in O(NS(S_\varphi))_0 - O(NS(S_\varphi))_0^+$. Then we can extend the isometry $-\gamma \oplus -\text{id}_{T(S_\varphi)}$ to the anti-symplectic Hodge isometry $\Phi : H^2(S_\varphi, \mathbb{Z}) \simeq H^2(S_\varphi, \mathbb{Z})$ which preserves the positive cone. Composing Φ with an element of the Weyl group $W(S_\varphi)$, we may assume that Φ is effective. \square

4. THE COUNTING FORMULA

Proposition 4.1. *For a twisted K3 surface (S_1, α_1) there exists a bijection*

$$\Gamma(S_1, \alpha_1) \backslash \text{Emb}(U, \widetilde{NS}(S_1)) \simeq \bigsqcup_{M \in \mathcal{G}(NS(S_1))} \left(O_{\text{Hodge}}(T(S_1), \alpha_1) \backslash O(D_M) / O(M) \right).$$

Proof. We can decompose the set $\text{Emb}(U, \widetilde{NS}(S_1))$ as

$$\text{Emb}(U, \widetilde{NS}(S_1)) = \bigsqcup_{M \in \mathcal{G}(NS(S_1))} \left\{ \varphi \in \text{Emb}(U, \widetilde{NS}(S_1)), \varphi(U)^\perp \simeq M \right\}.$$

The isometry group $O(\widetilde{NS}(S_1)) = O(M \oplus U)$ acts transitively on each component $\{ \varphi \in \text{Emb}(U, \widetilde{NS}(S_1)), \varphi(U)^\perp \simeq M \}$ with the stabilizer subgroup $O(M) \subset O(M \oplus U)$. From the surjectivity of $r : O(M \oplus U) \rightarrow O(D_{M \oplus U}) \simeq O(D_M)$ (Proposition 2.2) and the inclusion $O(M \oplus U)_0 \subset \Gamma(S_1, \alpha_1)$, we have

$$\begin{aligned} \Gamma(S_1, \alpha_1) \backslash \text{Emb}(U, \widetilde{NS}(S_1)) &\simeq \bigsqcup_{M \in \mathcal{G}(NS(S_1))} \left(\Gamma(S_1, \alpha_1) \backslash O(M \oplus U) / O(M) \right) \\ &\simeq \bigsqcup_{M \in \mathcal{G}(NS(S_1))} \left(r(\Gamma(S_1, \alpha_1)) \backslash O(D_M) / r(O(M)) \right). \end{aligned}$$

\square

Let (S, α) be a twisted K3 surface with a B-field lift $B \in H^2(S, \mathbb{Q})$. For an isotropic element $x \in I^d(D_{T(S, \alpha)})$ we define the lattice T_x by $T_x = \langle x, T(S, \alpha) \rangle$ and define the homomorphism $\alpha_x : T_x \rightarrow \mathbb{Z}/d\mathbb{Z}$ by

$$\alpha_x : T_x \rightarrow T_x / T(S, \alpha) \simeq \langle x \rangle \simeq \mathbb{Z}/d\mathbb{Z}, \quad \alpha_x(x) = \bar{1}.$$

For a pair (x, M) such that $x \in I^d(D_{T(S, \alpha)}) \stackrel{\lambda}{\simeq} I^d(D_{\widetilde{NS}(S, B)})$ and $\langle \lambda(x), \widetilde{NS}(S, B) \rangle \simeq U \oplus M$, we define the natural number $\tau(x, M)$ by

$$\tau(x, M) := \# \left(O_{\text{Hodge}}(T_x, \alpha_x) \backslash O(D_M) / O(M) \right),$$

where the group $O_{Hodge}(T_x, \alpha_x)$ acts on $O(D_M)$ via $r : O_{Hodge}(T_x, \alpha_x) \rightarrow O(D_{T_x})$ and $\bar{\lambda} : O(D_{T_x}) \simeq O(D_{M \oplus U}) \simeq O(D_M)$. For a natural number d we define the natural number $\varepsilon(d)$ by

$$\varepsilon(d) = \begin{cases} 1, & d = 1, 2, \\ 2, & d \geq 3. \end{cases}$$

From Section 3 and Proposition 4.1, we deduce the formula for $\#FM^d(S)$.

Theorem 4.2. *For a K3 surface S the following formula holds:*

$$\#FM^d(S) = \sum_x \left\{ \sum_M \tau(x, M) + \varepsilon(d) \sum_{M'} \tau(x, M') \right\}.$$

Here x runs over the set $O_{Hodge}(T(S)) \setminus I^d(D_{T(S)})$, and the lattices M, M' run over the sets $\mathcal{G}_1(\langle \lambda_{H^2(S)}(x), NS(S) \rangle), \mathcal{G}_2(\langle \lambda_{H^2(S)}(x), NS(S) \rangle)$, respectively.

We also obtain the following inequality.

Proposition 4.3. *For a twisted K3 surface (S, α) the following inequality holds:*

$$\#FM^d(S, \alpha) \leq \sum_x \left\{ \sum_M \tau(x, M) + \varepsilon(d) \sum_{M'} \tau(x, M') \right\}.$$

Here x runs over the set $O_{Hodge}(T(S, \alpha)) \setminus J^d(D_{T(S, \alpha)})$, where $J^d(D_{T(S, \alpha)})$ is the set defined in (3.1). The lattices M, M' run over the sets $\mathcal{G}_1(M_\varphi), \mathcal{G}_2(M_\varphi)$, respectively, where M_φ is a lattice satisfying $\langle \lambda(x), \widetilde{NS}(S, B) \rangle \simeq U \oplus M_\varphi$.

Corollary 4.4. *When $\widetilde{NS}(S, B)$ contains $U \oplus U$, then*

$$\#FM^d(S, \alpha) = \#(O_{Hodge}(T(S, \alpha)) \setminus I^d(D_{T(S, \alpha)})).$$

Proof. By considering $U^\perp \cap \widetilde{H}(S, B, \mathbb{Z})$, one observes the existence of an untwisted FM partner $S' \in FM^1(S, \alpha)$. We have a Hodge isometry $T(S, \alpha) \simeq T(S')$. Since $\widetilde{NS}(S')$ contains $U \oplus U$, then $NS(S')$ contains U . Now we apply Theorem 4.2 to S' . For $x \in I^d(D_{T(S')})$, the lattice $M_x := \langle \lambda_{H^2(S')}(x), NS(S') \rangle$ admits an embedding of U so that $\mathcal{G}(M_x) = \{M_x\}$ and $\tau(x, M_x) = 1$ by Proposition 2.2. Then $\mathcal{G}(M_x) = \mathcal{G}_1(M_x)$ due to the existence of the isometry $-\text{id}_U \oplus \text{id}_{U^\perp}$. \square

By Corollary 1.10.2 of [9], $\widetilde{NS}(S, B)$ admits an embedding of $U \oplus U$ if $\text{rk}(NS(S)) \geq 13$.

Corollary 4.5. *When the lattice $NS(S)$ is 2-elementary, i.e., $D_{NS(S)} \simeq (\mathbb{Z}/2\mathbb{Z})^a$, then $\#FM^d(S) = 0$ unless $d = 1, 2$. If $d = 1, 2$, we have*

$$\#FM^d(S) = \#(O_{Hodge}(T(S)) \setminus I^d(D_{T(S)})).$$

Proof. Because the lattice $M_x := \langle \lambda_{H^2(S)}(x), NS(S) \rangle$ is also 2-elementary, $\mathcal{G}(M_x) = \{M_x\}$ and $\tau(x, M_x) = 1$ by Theorem 3.6.2 and Theorem 3.6.3 of [9]. \square

5. TWISTED FOURIER-MUKAI PARTNERS OF A K3 SURFACE OF PICARD NUMBER 1

Let S be a K3 surface with $NS(S) = \mathbb{Z}H$, $(H, H) = 2n$. In this section we shall describe all twisted FM partners of S (up to isomorphism) as moduli spaces of stable sheaves on S with explicit Mukai vectors, endowed with natural twistings. The result is a natural generalization of a result of [4] and [12]. First of all, we calculate the twisted FM numbers by Theorem 4.2.

Proposition 5.1. *For a natural number d , we have $\#FM^d(S) = 0$ unless $d^2|n$. If $d^2|n$, then*

$$\#FM^d(S) = \begin{cases} \varphi(d) \cdot 2^{\tau(d^{-2}n)-2} & d \geq 3, d^2 = n, \\ \varphi(d) \cdot 2^{\tau(d^{-2}n)-1} & d = 1, 2 \text{ or } d^2 < n. \end{cases}$$

Here φ is the Euler function and $\tau(m)$ is the number of the prime factors of m . We define $\tau(1) = 1$.

Proof. We have $D_{NS(S)} = \langle \frac{H}{2n} \rangle \simeq \mathbb{Z}/2n\mathbb{Z}$ and $O_{Hodge}(T(S)) = \{\pm id\}$ (cf. [5]). Since

$$I^d(D_{NS(S)}) \neq \emptyset \Leftrightarrow \left(\frac{H}{d}, \frac{H}{d}\right) = \frac{2n}{d^2} \in 2\mathbb{Z} \Leftrightarrow d^2|n,$$

it follows that $\#FM^d(S) = 0$ unless $d^2|n$.

Assume that $d^2|n$. Then we have $I^d(D_{NS(S)}) = \{k\frac{H}{d}, k \in (\mathbb{Z}/d\mathbb{Z})^\times\}$ so that

$$\#(O_{Hodge}(T(S)) \setminus I^d(D_{NS(S)})) = \begin{cases} 1 & d = 1, 2, \\ \frac{1}{2}\varphi(d) & d \geq 3. \end{cases}$$

For $d \geq 3$ and $k \in (\mathbb{Z}/d\mathbb{Z})^\times$, we have $\langle \mathbb{Z}H, k\frac{H}{d} \rangle = \mathbb{Z}\frac{H}{d}$ and

$$\mathcal{G}\left(\mathbb{Z}\frac{H}{d}\right) = \left\{ \mathbb{Z}\frac{H}{d} \right\} = \begin{cases} \mathcal{G}_1(\mathbb{Z}\frac{H}{d}) & d^2 = n, \\ \mathcal{G}_2(\mathbb{Z}\frac{H}{d}) & d^2 < n. \end{cases}$$

Because $O_{Hodge}(T_{k\frac{H}{d}}, \alpha_{k\frac{H}{d}}) = \{id\}$, $O(\mathbb{Z}\frac{H}{d}) = \{\pm id\}$ and

$$(5.1) \quad O(D_{\mathbb{Z}\frac{H}{d}}) \simeq \begin{cases} \{id\} & d^2 = n, \\ (\mathbb{Z}/2\mathbb{Z})^{\tau(d^{-2}n)} & d^2 < n, \end{cases}$$

it follows from Theorem 4.2 that

$$\#FM^d(S) = \begin{cases} \frac{1}{2}\varphi(d) \cdot 1 \cdot 2^{\tau(d^{-2}n)-1} & d^2 = n, \\ \frac{1}{2}\varphi(d) \cdot 2 \cdot 2^{\tau(d^{-2}n)-1} & d^2 < n. \end{cases}$$

Let $d \leq 2$. Since $O_{Hodge}(T_{\frac{H}{d}}, \alpha_{\frac{H}{d}}) = O(\mathbb{Z}\frac{H}{d}) = \{\pm id\}$ and the isomorphisms (5.1) still hold, we have $\#FM^d(S) = 2^{\tau(d^{-2}n)-1} = \varphi(d) \cdot 2^{\tau(d^{-2}n)-1}$ by Theorem 4.2. \square

Now we shall construct the moduli spaces. Since the primitive embeddings of the lattice $\mathbb{Z}H$ into the $K3$ lattice Λ_{K3} are unique modulo $O(\Lambda_{K3})$, we fix an isometry $H^2(S, \mathbb{Z}) \simeq \Lambda_{K3} = U^3 \oplus E_8^2$ so that H is written as $H = nu + v$, where $\{u, v\}$ is a standard basis of the first U . Then the transcendental lattice $T(S)$ is written as $T(S) = \mathbb{Z}(nu - v) \oplus U^2 \oplus E_8^2$. Set $G := nu - v \in T(S)$.

For a natural number d with $d^2|n$, let $d^{-2}n = \prod_{i=1}^{\tau(d^{-2}n)} p_i^{e_i}$ be the prime decomposition. When $d^2 = n$, we put $p_1 = e_1 = 1$. Set

$$\Sigma := \left\{ \{i_1, \dots, i_a\} \subset \{1, 2, \dots, \tau(d^{-2}n)\} \mid \{i_1, \dots, i_a\} \neq \emptyset, \prod_{\{i_1, \dots, i_a\}} p_i^{2e_i} \geq \frac{n}{d^2} \right\}.$$

We have $\#\Sigma = 2^{\tau(d^{-2}n)-1}$. For $\sigma = \{i_1, \dots, i_a\} \in \Sigma$, set $r_\sigma := \prod_{i \in \sigma} p_i^{e_i}$ and $s_\sigma := r_\sigma^{-1}d^{-2}n$. Then r_σ is coprime to s_σ . We have $r_\sigma > s_\sigma$ if $d^2 < n$. Let $k \in (\mathbb{Z}/d\mathbb{Z})^\times$. By the arithmetic progression, we can find a natural number $\tilde{k} \in \mathbb{N}$

coprime to $2n$ with $\tilde{k} \equiv k \in \mathbb{Z}/d\mathbb{Z}$. For example, we may take $\tilde{k} = 1$ if $k = \bar{1}$. Fix such a \tilde{k} for each $k \in (\mathbb{Z}/d\mathbb{Z})^\times$. For $(\sigma, k) \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times$, we define

$$(5.2) \quad v_{\sigma,k} := (dr_\sigma, \tilde{k}H, \tilde{k}^2 ds_\sigma) \in \widetilde{NS}(S).$$

Then $v_{\sigma,k}$ is a primitive isotropic vector in $\widetilde{NS}(S)$ with $\text{div}(v_{\sigma,k}) = d$.

Let $M_{\sigma,k} := M_H(v_{\sigma,k})$ be the coarse moduli space of Gieseker stable sheaves on S with Mukai vector $v_{\sigma,k}$. It follows from [8] that $M_{\sigma,k}$ is non-empty and is a $K3$ surface. Let $\alpha_{\sigma,k} \in \text{Br}(M_{\sigma,k})$ be the obstruction to the existence of a universal sheaf ([2]) with a B -field lift $B_{\sigma,k} \in H^2(M_{\sigma,k}, \mathbb{Q})$. By [6], there exists an orientation-preserving Hodge isometry

$$(5.3) \quad \tilde{\Phi} : \tilde{H}(M_{\sigma,k}, B_{\sigma,k}, \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z}) \quad \text{with } (0, 0, 1) \mapsto v_{\sigma,k},$$

which induces a Hodge isometry

$$(5.4) \quad H^2(M_{\sigma,k}, \mathbb{Z}) \simeq \left(v_{\sigma,k}^\perp \cap \tilde{H}(S, \mathbb{Z}) \right) / \mathbb{Z}v_{\sigma,k}.$$

(We remark that the Hodge isometry (5.4) was proved first by [8].)

Using (5.4), we shall calculate the isometry

$$\lambda_{\sigma,k} := \lambda_{H^2(M_{\sigma,k}, \mathbb{Z})} : (D_{NS(M_{\sigma,k})}, q) \simeq (D_{T(M_{\sigma,k})}, -q).$$

Firstly, direct calculations show that

$$\begin{aligned} NS(M_{\sigma,k}) &\simeq \left(v_{\sigma,k}^\perp \cap \widetilde{NS}(S) \right) / \mathbb{Z}v_{\sigma,k} \\ &\simeq \left(v_{\sigma,k}^\perp \cap \left\langle \frac{v_{\sigma,k}}{d}, \widetilde{NS}(S) \right\rangle \right) / \mathbb{Z} \frac{v_{\sigma,k}}{d} \\ &= \left(\mathbb{Z} \frac{v_{\sigma,k}}{d} \oplus \mathbb{Z} \left(0, -\frac{H}{d}, -2s_\sigma \tilde{k} \right) \right) / \mathbb{Z} \frac{v_{\sigma,k}}{d}. \end{aligned}$$

We define

$$H_{\sigma,k} := \left[\left(0, -\frac{H}{d}, -2s_\sigma \tilde{k} \right) \right] \in NS(M_{\sigma,k}).$$

We have $NS(M_{\sigma,k}) = \mathbb{Z}H_{\sigma,k}$ and $\tilde{k}H_{\sigma,k} = [(r_\sigma, 0, -s_\sigma \tilde{k}^2)]$. Since the Hodge isometry (5.3) is orientation-preserving, $H_{\sigma,k}$ is the ample generator of $NS(M_{\sigma,k})$. For the transcendental lattice we have

$$\begin{aligned} T(M_{\sigma,k}) &\simeq \overline{T(S) \oplus \mathbb{Z}v_{\sigma,k}} / \mathbb{Z}v_{\sigma,k} \\ &= \left(\left\langle \mathbb{Z}v_{\sigma,k} \oplus \mathbb{Z}G, \frac{v_{\sigma,k} + \tilde{k}G}{d} \right\rangle \oplus U^2 \oplus E_8^2 \right) / \mathbb{Z}v_{\sigma,k}. \end{aligned}$$

Here \overline{L} means the primitive hull of the lattice L in $\tilde{H}(S, \mathbb{Z})$. Since $\lambda_{\tilde{H}(S, \mathbb{Z})} \left(\frac{v_{\sigma,k}}{d} \right) = \frac{\tilde{k}G}{d} \in D_{T(S)}$, it follows from Lemma 2.7 and the Hodge isometry (5.3) that the twisting $\alpha_{\sigma,k}$ is given by

$$(5.5) \quad \alpha_{\sigma,k} : T(M_{\sigma,k}) \xrightarrow{\tilde{\Phi} \circ e^{B_{\sigma,k}}} \left\langle \frac{G}{d}, T(S) \right\rangle / T(S) \simeq \left\langle \frac{-\tilde{k}G}{d} \right\rangle \simeq \mathbb{Z}/d\mathbb{Z}.$$

Via the Hodge isometry (5.4), we obtain a Hodge isometry

$$(5.6) \quad g_{\sigma,k} := \tilde{\Phi} \circ e^{B_{\sigma,k}} : T(M_{\sigma,k}) \simeq \left\langle \frac{G}{d}, T(S) \right\rangle, \quad g_{\sigma,k} \left(\left[\frac{\tilde{k}G + v_{\sigma,k}}{d} \right] \right) = \frac{\tilde{k}G}{d}.$$

Set $G_{\sigma,k} := g_{\sigma,k}^{-1}(\frac{G}{d})$, which is a primitive vector in $T(M_{\sigma,k})$. From (5.6), for $(\sigma, k), (\sigma', k') \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times$ we have a Hodge isometry

$$(5.7) \quad g_{\sigma,k}^{\sigma',k'} := g_{\sigma',k'}^{-1} \circ g_{\sigma,k} : T(M_{\sigma,k}) \simeq T(M_{\sigma',k'}), \quad g_{\sigma,k}^{\sigma',k'}(G_{\sigma,k}) = G_{\sigma',k'}.$$

Therefore we have an equivalence $D^b(M_{\sigma,k}) \simeq D^b(M_{\sigma',k'})$ by Proposition 2.4. Note that the Hodge isometries from $T(M_{\sigma,k})$ to $T(M_{\sigma',k'})$ are just $\{\pm g_{\sigma,k}^{\sigma',k'}\}$.

Now we calculate $\lambda_{\sigma,k}$ as follows. We have

$$\begin{aligned} D_{NS(M_{\sigma,k})} &\simeq \left\langle \frac{\tilde{k}H_{\sigma,k}}{2r_\sigma s_\sigma} \right\rangle = \left\langle \left(\frac{1}{2s_\sigma}, 0, \frac{-\tilde{k}^2}{2r_\sigma} \right) \right\rangle, \\ D_{T(M_{\sigma,k})} &\simeq \left\langle \frac{\tilde{k}G_{\sigma,k}}{2r_\sigma s_\sigma} \right\rangle = \left\langle \left(\frac{1}{2s_\sigma}, \tilde{k}du, \frac{\tilde{k}^2}{2r_\sigma} \right) \right\rangle. \end{aligned}$$

Firstly assume that $r_\sigma s_\sigma$ is odd. With respect to orthogonal decompositions

$$D_{NS(M_{\sigma,k})} \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{i=1}^{\tau(r_\sigma s_\sigma)} (\mathbb{Z}/p_i^{e_i}\mathbb{Z}), \quad D_{T(M_{\sigma,k})} \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{i=1}^{\tau(r_\sigma s_\sigma)} (\mathbb{Z}/p_i^{e_i}\mathbb{Z}),$$

we write

$$(5.8) \quad \frac{H_{\sigma,k}}{2r_\sigma s_\sigma} = (\bar{1}, x_{\sigma,k}^1, \dots, x_{\sigma,k}^{\tau(r_\sigma s_\sigma)}), \quad \frac{G_{\sigma,k}}{2r_\sigma s_\sigma} = (\bar{1}, y_{\sigma,k}^1, \dots, y_{\sigma,k}^{\tau(r_\sigma s_\sigma)}).$$

For all i , the elements $x_{\sigma,k}^i$ and $y_{\sigma,k}^i$ generate respective $\mathbb{Z}/p_i^{e_i}\mathbb{Z}$. Since we have

$$\begin{aligned} s_\sigma \frac{\tilde{k}H_{\sigma,k}}{2r_\sigma s_\sigma} + s_\sigma \frac{\tilde{k}G_{\sigma,k}}{2r_\sigma s_\sigma} &= [(1, \tilde{k}ds_\sigma u, 0)] \in H^2(M_{\sigma,k}, \mathbb{Z}), \\ -r_\sigma \frac{\tilde{k}H_{\sigma,k}}{2r_\sigma s_\sigma} + r_\sigma \frac{\tilde{k}G_{\sigma,k}}{2r_\sigma s_\sigma} &= [(0, \tilde{k}dr_\sigma u, \tilde{k}^2)] \in H^2(M_{\sigma,k}, \mathbb{Z}), \end{aligned}$$

the isometry $\lambda_{\sigma,k} = \lambda_{H^2(M_{\sigma,k}, \mathbb{Z})}$ is determined by

$$(5.9) \quad \lambda_{\sigma,k}(x_{\sigma,k}^i) = \begin{cases} y_{\sigma,k}^i & i \in \sigma, \\ -y_{\sigma,k}^i & i \notin \sigma. \end{cases}$$

If $r_\sigma s_\sigma$ is even, put $p_1 = 2$ and write

$$\begin{aligned} \frac{H_{\sigma,k}}{2r_\sigma s_\sigma} &= (x_{\sigma,k}^1, \dots, x_{\sigma,k}^{\tau(r_\sigma s_\sigma)}) \in D_{NS(M_{\sigma,k})} \simeq (\mathbb{Z}/2^{e_1+1}\mathbb{Z}) \oplus \bigoplus_{i=2}^{\tau(r_\sigma s_\sigma)} (\mathbb{Z}/p_i^{e_i}\mathbb{Z}), \\ \frac{G_{\sigma,k}}{2r_\sigma s_\sigma} &= (y_{\sigma,k}^1, \dots, y_{\sigma,k}^{\tau(r_\sigma s_\sigma)}) \in D_{T(M_{\sigma,k})} \simeq (\mathbb{Z}/2^{e_1+1}\mathbb{Z}) \oplus \bigoplus_{i=2}^{\tau(r_\sigma s_\sigma)} (\mathbb{Z}/p_i^{e_i}\mathbb{Z}). \end{aligned}$$

Then $\lambda_{\sigma,k}$ is determined by the same equations as (5.9).

Proposition 5.2. *For the twisted K3 surfaces $(M_{\sigma,k}, \alpha_{\sigma,k})$, the following hold.*

- (1) *The underlying K3 surfaces $\{M_{\sigma,k}, (\sigma, k) \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times\}$ are derived equivalent to each other.*
- (2) *We have $M_{\sigma,k} \simeq M_{\sigma',k'}$ if and only if $\sigma = \sigma'$.*
- (3) *We have $(M_{\sigma,k}, \alpha_{\sigma,k}) \simeq (M_{\sigma,1}, k^{-1}\alpha_{\sigma,1})$.*

Proof. We already proved the assertion (1) using Mukai-Orlov's theorem.

Assume the existence of an effective Hodge isometry $\Phi : H^2(M_{\sigma,k}, \mathbb{Z}) \simeq H^2(M_{\sigma',k'}, \mathbb{Z})$ for $(\sigma, k), (\sigma', k') \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times$. We must have $\Phi(H_{\sigma,k}) = H_{\sigma',k'}$

and $\Phi(G_{\sigma,k}) = \pm G_{\sigma',k'}$. Since the identity $\lambda_{\sigma',k'} \circ r(\Phi|_{NS(M_{\sigma,k})}) = r(\Phi|_{T(M_{\sigma,k})}) \circ \lambda_{\sigma,k}$ holds, it follows from (5.9) that $\sigma = \sigma'$ or $(\sigma')^c$. By the definition of Σ we have $\sigma = \sigma'$.

Conversely let $\sigma = \sigma'$. If we define $\gamma_{\sigma,k}^{\sigma,1} : NS(M_{\sigma,k}) \simeq NS(M_{\sigma,1})$ by $\gamma_{\sigma,k}^{\sigma,1}(H_{\sigma,k}) = H_{\sigma,1}$, then $\gamma_{\sigma,k}^{\sigma,1} \oplus g_{\sigma,k}^{\sigma,1}$ extends to an effective Hodge isometry $H^2(M_{\sigma,k}, \mathbb{Z}) \simeq H^2(M_{\sigma,1}, \mathbb{Z})$. Since the twistings $\alpha_{\sigma,k}$ are given by (5.5), we have $(g_{\sigma,k}^{\sigma,1})^* \alpha_{\sigma,1} = k\alpha_{\sigma,k}$, or equivalently, $(g_{\sigma,k}^{\sigma,1})^*(k^{-1}\alpha_{\sigma,1}) = \alpha_{\sigma,k}$. \square

Finally, we obtain the following theorem.

Theorem 5.3. *Let S be a K3 surface with $NS(S) = \mathbb{Z}H$, $(H, H) = 2n$, and let d be a natural number satisfying $d^2|n$.*

(1) *If $d^2 < n$, then*

$$FM^d(S) = \{(M_{\sigma,k}, \alpha_{\sigma,k}) \mid (\sigma, k) \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times\}.$$

(2) *Assume that $d^2 = n$. Choose a set $\{j\} \subset (\mathbb{Z}/d\mathbb{Z})^\times$ of representatives of $(\mathbb{Z}/d\mathbb{Z})^\times / \{\pm \text{id}\}$. Then*

$$FM^d(S) = \{(M_{\sigma,k}, \alpha_{\sigma,k}) \mid (\sigma, k) \in \Sigma \times \{j\}\}.$$

Note that $-\text{id}$ acts trivially on $(\mathbb{Z}/d\mathbb{Z})^\times$ if $d \leq 2$.

Proof. That $(M_{\sigma,k}, \alpha_{\sigma,k}) \in FM^d(S)$ is a direct consequence of Căldăraru's theorem ([2]). By Proposition 5.1, it suffices to show that the twisted K3 surfaces on the right hand sides are not isomorphic to each other.

(1) By Proposition 5.2, the right hand side is identified with the set

$$\{(M_{\sigma,1}, k\alpha_{\sigma,1}), (\sigma, k) \in \Sigma \times (\mathbb{Z}/d\mathbb{Z})^\times\}.$$

We have $(M_{\sigma,1}, k\alpha_{\sigma,1}) \not\cong (M_{\sigma',1}, k'\alpha_{\sigma',1})$ if $\sigma \neq \sigma'$. Since $d^2 < n$, then $\text{Aut}(M_{\sigma,1}) = \{\text{id}\}$ so that $(M_{\sigma,1}, k\alpha_{\sigma,1}) \not\cong (M_{\sigma,1}, k'\alpha_{\sigma,1})$ if $k \neq k'$.

(2) Let $d^2 = n$. The right hand side is identified with the set

$$\{(M_{\sigma,1}, k\alpha_{\sigma,1}), (\sigma, k) \in \Sigma \times \{j^{-1}\}\}.$$

Similarly as in (1), we have $(M_{\sigma,1}, k\alpha_{\sigma,1}) \not\cong (M_{\sigma',1}, k'\alpha_{\sigma',1})$ if $\sigma \neq \sigma'$. Now the K3 surface $M_{\sigma,1}$ admits an anti-symplectic involution $\iota_{\sigma,1}$ and $\text{Aut}(M_{\sigma,1}) = \{\text{id}, \iota_{\sigma,1}\}$. Therefore $(M_{\sigma,1}, k\alpha_{\sigma,1}) \not\cong (M_{\sigma,1}, k'\alpha_{\sigma,1})$ if $k \neq \pm k'$. \square

When $d = 1$ with $\tilde{k} = 1$, the result is proved in [4] and [12].

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