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CONSTANT TERM OF SMOOTH H_ψ-SPHERICAL FUNCTIONS ON A REDUCTIVE p-ADIC GROUP

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Abstract. Let ψ be a smooth character of a closed subgroup, H, of a reductive p-adic group G. If P is a parabolic subgroup of G such that PH is open in G, we define the constant term of every smooth function on G which transforms by ψ under the right action of G. The example of mixed models is given: it includes symmetric spaces and Whittaker models. In this case a notion of cuspidal function is defined and studied. It leads to finiteness theorems.

1. Introduction

The theory of the constant term first appears in the theory of automorphic forms as an important tool in the spectral decomposition of $L^2(G/\Gamma)$, for Γ an arithmetic subgroup of a reductive group G defined over $\mathbb{Q}$. The terminology comes from the case of GL_2, where the constant term is related to the constant term of Fourier series on $U/U \cap \Gamma$, where U is some unipotent subgroup of G.

It was later modified and used by Harish-Chandra in his treatment of the Plancherel formula for the real and p-adic reductive groups (cf. [12], [19]).

Later, in the case of p-adic groups, Casselman (cf. [8]) has found interpretation of the work of Harish-Chandra in terms of Jacquet modules. More recently Bernstein (cf. [1], see also [6]) vastly generalized this approach. In this article, we use Bernstein's theory to extend the results to functions which transform by a smooth character under certain subgroups of G. This includes the case of symmetric spaces, Whittaker models and mixed models (see below). The construction for Whittaker models is related to the work of Casselman and Shalika [9].

An important aspect of our work is that the constant term is defined for all smooth functions.

Let us be more precise.

Let F be a non-Archimedean local field. Let G be the group of F-points of a connected reductive group, $\underline{G}$, defined over F. Let H be a closed subgroup of G and ψ be a smooth character of H. An H_ψ-fixed linear form on a smooth G-module is a linear form which transforms by ψ under H. We assume that:

(1.1) (P, P^-) is a pair of opposite parabolic subgroups of G such that P^-H is open.

We have in mind the following examples of this situation.

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Example 1 (Symmetric spaces)**.** The group H is the fixed point set in G of a rational involution σ defined over F of the group $\underline{G}$, P is a σ-parabolic subgroup of G, i.e. P and $\sigma(P)$ are opposite, and F is of characteristic different from 2. For the purpose of induction, we do not limit ourselves to ψ trivial.

Notice that G itself appears as a symmetric space for $G \times G$, where $\sigma(x, y) = (y, x)$. This will be referred as "the group case".

Example 2 (Whittaker models)**.** The group H is the unipotent radical, U_0, of a minimal parabolic subgroup P_0 of G, P is a parabolic subgroup of G which contains P_0, and ψ is a nondegenerate smooth character of U_0 (see Definition 5.1).

Example 3 (Mixed models)**.** Let Q be a parabolic subgroup of G, with Levi subgroup L and unipotent radical U_Q. Let H' be the fixed point set of a rational involution, σ', of L. If σ' is not trivial, we assume that F is of characteristic different from 2. We take $H = H'U_Q$ and ψ is nondegenerate (see Definition 5.1). We assume that P^-Q is open and that $P^- \cap L$ is a σ'-parabolic subgroup of L.

We denote by δ_P the modulus function of P. Let V be a smooth G-module, V_P its normalized Jacquet module along P. Assuming (1.1) and denoting by M the common Levi subgroup of P and P^-, our first result (cf. Theorem 3.4) is the definition of a natural linear map

$$j_{P^-} : V^{*H_\psi} \mapsto V_P^{*(M\cap H)_{\psi_P}},$$

from the space of H_ψ-fixed linear forms on V to the space of $(M \cap H)_{\psi_P}$-fixed linear forms on V_P, where ψ_P is equal to the product of the restrictions to $M \cap H$ of ψ and $\delta_P^{1/2}$.

In Example 1, this definition has already been given by Kato and Takano [15] and Lagier [16], when V is admissible. This definition is related to the behavior on the maximal split torus, A, of the center of M, of the generalized coefficients $c_{\xi,v}$, $v \in V, \xi \in V^{*H_\psi}$. The latter are defined by $c_{\xi,v}(g) = \langle \xi, \pi(g^{-1})v\rangle, g \in G$. For the group itself viewed as a symmetric space of $G \times G$, as was mentioned above this construction is due to Casselman (cf. [8]) for admissible representations and to Bernstein (cf. [1], see also [6]) for general smooth representations.

Using results of Bernstein on smooth representations (the generalized Jacquet Lemma and the second adjointness theorem; see section 2), we generalize this map j_{P^-} to arbitrary smooth representations and to the more general situation described above.

This allows us to define the constant term along P, f_P, of any smooth function, f, on G transforming under ψ on the right. In Example 1, when f is a generalized coefficient $c_{\xi,v}$ of an admissible representation, this has already been defined in [16], Proposition 2, where it is shown that the restriction of f_P to A is determined by the restriction of f to A. It seems that it is different for smooth modules.

In Example 1 (resp., Example 2) an H_ψ-fixed linear form, ξ, on a smooth G-module is said to be H_ψ-cuspidal if and only if $j_{P^-}(\xi)$ is zero for all pairs (P, P^-) where P is a proper σ-parabolic subgroup of G and $P^- = \sigma(P)$ (resp., P contains P_0 and P^- contains a given opposite parabolic subgroup of P_0). There is a similar definition for Example 3. We denote by $V_{cusp}^{*H_\psi}$ the space of such linear forms.

Let $X(G)$ be the group of unramified characters of G which has the structure of a complex torus. We will denote by $X(G)_H$ the group of elements of $X(G)$ which are trivial on H. An $X(G)_H$-component of equivalence classes of irreducible

representations of G is a set $\{\pi \otimes \chi \mid \chi \in X(G)_H\}$ of irreducible representations of G, where π is an irreducible representation of G.

Let K be a compact open subgroup of G. Our second result asserts that for Example 1 (Theorem 4.4), there exists a finite number of $X(G)_H$-components of irreducible representations, (O_i), such that every smooth irreducible representation of G, (π, V), which has a nonzero vector fixed by K and such that $V_{cusp}^{*H_\psi}$ is nonzero, is equivalent to at least one representation in the O_i. One has also a finiteness result for Examples 2 and 3 (see Theorem 5.6).

The final results (Theorem 4.5 and Theorem 5.7) assert that for any smooth representation (π, V) of finite length, i.e. admissible of finite type, the dimension of V^{*H_ψ} is finite, this for Examples 1, 2 and 3. For Example 2, the result was already known and due to Bushnell and Henniart [7]. For some particular cases of Example 3 (Klyachko models), Offen and Sayag proved in [17] that, for irreducible unitary representations of $GL(n, F)$, the dimension is one or zero.

We hope to use these results, as well as the notion of the constant term of smooth functions, for harmonic analysis. Proposition 3.17 gives some insight into the constant term of wave packets. Notice that our Theorem 3.4 leads, when one views the group as a symmetric space, to a curious operation which associates, to an invariant distribution on G, an invariant distribution on a Levi subgroup (see section 4.6).

2. Preliminaries

2.1. **Notation.** If E is a vector space, E^* will denote its dual. If E is real, $E_{\mathbb{C}}$ will denote its complexification.

If G is a group, $g \in G$ and X is a subset of G, $g.X$ will denote gXg^{-1}. If (π, V) is a representation of G, V^G will denote the space of invariants under G.

We will use conventions as in [19]. Let F be a non-Archimedean local field. One considers various algebraic groups defined over F, and a sentence such as:

(2.1) "let A be a split torus" will mean "let A be the group of F-points of a torus, $\underline{A}$, split and defined over F".

With these conventions, let G be a connected reductive linear algebraic group.

Let A be a split torus of G. Let $X_*(A)$ be the group of one-parameter subgroups of A. This is a free abelian group of finite type. Such a group will be called a lattice. One fixes a uniformizing element ϖ of F. One denotes by $\Lambda(A)$ the image of $X_*(A)$ in A by the morphism of groups $\underline{\lambda} \mapsto \underline{\lambda}(\varpi)$. By this morphism $\Lambda(A)$ is isomorphic to $X_*(A)$.

If (P, P^-) are two opposite parabolic subgroups of G, we will denote by M their common Levi subgroup and by A_M or A the maximal split torus of its center. We denote by U (resp., U^-) the unipotent radical of P (resp., P^-). We define the set A^- (resp., A^{--}) of P to be the antidominant (resp., strictly antidominant) elements in A. More precisely, if $\Sigma(P, A)$ is the set of roots of A in the Lie algebra of P, and $\Delta(P, A)$ is the set of simple roots, one has:

$$A^- \text{ (resp., } A^{--}) = \{a \in A \mid |\alpha(a)|_F \leq 1 \text{ (resp., } < 1),\ \alpha \in \Delta(P, A)\}.$$

We define similarly A^+ and A^{++} by reversing the inequalities. One defines also for $\varepsilon > 0$:

$$A^-(\varepsilon) = \{a \in A \mid |\alpha(a)|_F \leq \varepsilon,\ \alpha \in \Delta(P, A)\}.$$

Let M_0 be a minimal Levi subgroup of G, and let A_0 be the maximal split torus of the center of M_0. We choose a minimal parabolic subgroup, P_0, of G with Levi subgroup M_0. One chooses a maximal compact subgroup of G, K_0, which is the stabilizator of a special point of the apartment associated to A_0 of the Bruhat-Tits building of G. If P is a parabolic subgroup of G which contains A_0, we denote by P^- the opposite parabolic subgroup of G to P which contains A_0 and by M the intersection of P and P^-.

For the following result, see [8], Prop. 1.4.4:

(2.2) There exists a decreasing sequence of compact open subgroups of G, K_n, $n \in \mathbb{N}$ such that for all $n \in \mathbb{N}^*$, $K = K_n$ is normal in K_0 and for every parabolic subgroup, P, which contains P_0, one has:
1) $K = K_{U^-} K_M K_U$, where $K_{U^-} = K \cap U^-$, $K_M = K \cap M$, $K_U = K \cap U$.
2) For all $a \in A^-$, $aK_U a^{-1} \subset K_U$, $a^{-1} K_{U^-} a \subset K_{U^-}$.
3) The sequence K_n forms a neighborhood basis of the identity in G.

One says that K has an Iwahori factorization with respect to (P, P^-) if 1) and 2) are satisfied. In that case, one has:

(2.3) For all $a \in A^{--}$, one has:
$$\bigcup_{n \in \mathbb{N}} a^{-n} K_U a^n = U.$$

2.2. **Second adjointness theorem.** If P is a parabolic subgroup of G, one denotes by δ_P its modulus function. We keep the notation of the preceding paragraph. We choose an A_0 that contains A and a P_0 that is included in P. Let (π, V) be a smooth representation of G and let P be a parabolic subgroup of G, with unipotent radical U and Levi subgroup M. One denotes by (π_P, V_P) the tensor product of the quotient of V by the M-submodule generated by the $\pi(u)v - v$, $u \in U, v \in V$ with the representation of M on $\mathbb{C}$ given by $\delta_P^{-1/2}$. We call it the normalized Jacquet module of V along P. We denote by j_P the natural projection map from V to V_P.

The following result is due to J. Bernstein (cf. [1]). This is a generalization of a result of W. Casselman [8] and a consequence of his stabilization theorem (cf. [1]; see [6], Theorem 1 for a published version) and of the description of the canonical duality between V_P and $\check{V}_{P^-}$ (cf. [6], section 5 and [8]). Here (π, V) is a smooth representation of G. Let j_P (resp., j_{P^-}) denote the canonical projection of V (resp., of the smooth dual $\check{V}$ of V) onto V_P (resp., $\check{V}_{P^-}$).

(2.4) Let (π, V) be a smooth representation of G. Then there exists a unique nondegenerate M-invariant bilinear form $\langle ., . \rangle_P$ on $\check{V}_{P^-} \times V_P$, such that for all compact open subgroups K and for all $a \in A^{--}$ we have:
$$\delta_P^{1/2}(a^n) \langle j_{P^-}(\check{v}), \pi_P(a^n) j_P(v) \rangle_P = \langle \check{v}, \pi(a^n) v \rangle, \ \ n \geq n_K(a),$$
where $n_K(a) \in \mathbb{N}$ depends only on K and a but not on $V, v \in V^K, \check{v} \in \check{V}^K$.

We define:

$$\Theta_P := \Delta(P_0 \cap M, A_0)$$

and for $\varepsilon > 0$:

$$A_0^-(P, <\varepsilon) := \{a \in A_0^- \mid |\alpha(a)|_F < \varepsilon, \alpha \in \Delta(P_0, A_0) \setminus \Theta_P\}. \tag{2.5}$$

Lemma 2.1. *Assume that P contains P_0. Let K be a compact open subgroup of G. Then there exists $\varepsilon_K < 1$ such that for every smooth G-module (π, V), one has:*

$$\delta_P^{1/2}(a)\langle j_{P^-}(\check{v}), \pi_P(a) j_P(v)\rangle_P = \langle \check{v}, \pi(a)v\rangle,$$

for $a \in A_0^-(P, <\varepsilon_K), v \in V^K, \check{v} \in \check{V}^K$.

Proof. It is clearly enough to prove the statement when $K = K_n$, $n \in \mathbb{N}^*$. We assume this in the sequel. One chooses a_0 in A^{--} and sets $b = a_0^{n_K(a_0)}$. Let

$$\varepsilon_K := \inf\{|\alpha(b)|_F \mid \alpha \text{ root of } A \text{ in } P\}.$$

We have:

$$A_0^-(P, <\varepsilon_K) \subset bA_0^-. \tag{2.6}$$

Let a be an element of A_0^-. We will use properties of some distributions on G. First one denotes by e_K the normalized Haar measure on K, that we view as a compactly supported distribution on G. We will use similar notation for other compact subgroups of G, not necessarily open. If $g \in G$, we will denote also by g the Dirac measure at g. We will denote the convolution of two compactly supported distributions e, e' on G simply by ee'. Then we define

$$h(a) := e_K a e_K. \tag{2.7}$$

As K has an Iwahori factorization and $a, b \in A_0^-$, it is is well known that one has:

$$h(ba) = h(b)h(a). \tag{2.8}$$

Denoting by $K_U^a = a^{-1}K_U a$, we have also:

$$h(a) = a e_{K_U^a} e_K. \tag{2.9}$$

Let $v \in V^K$, $\check{v} \in \check{V}^K$. One has:

$$\langle \check{v}, \pi(ba)v\rangle = \langle \check{v}, \pi(h(ba))v\rangle.$$

From (2.8), one has:

$$\langle \check{v}, \pi(ba)v\rangle = \langle \check{v}, \pi(h(b))\pi(h(a))v\rangle.$$

As $h(b)e_K = h(b)$ and $v, \check{v}$ are K-invariant, one has:

$$\langle \check{v}, \pi(ba)v\rangle = \langle \check{v}, \pi(b)\pi(h(a))v\rangle.$$

From (2.4) and the definition of b, one deduces:

$$\langle \check{v}, \pi(ba)v\rangle = \delta_P^{1/2}(b)\langle j_{P^-}(\check{v}), \pi_P(b) j_P(\pi(h(a))v)\rangle_P.$$

But from (2.9), from the fact that K_U^a is a compact subgroup of U and from the intertwining properties of j_P, one deduces:

$$j_P(\pi(h(a)v)) = \delta_P^{1/2}(a)\pi_P(a) j_P(v).$$

So we get:

$$\langle \check{v}, \pi(ab)v\rangle = \delta_P^{1/2}(ba)\langle j_{P^-}(\check{v}), \pi_P(ba) j_P(v)\rangle.$$

Taking into account (2.6), this proves the lemma. □

3. H_ψ-FIXED LINEAR FORMS

3.1. **H_ψ-fixed linear forms and Jacquet modules.** Let H be a closed subgroup of G, and let ψ be a smooth complex character of G, which means that the kernel of ψ is open in G. Define:

$$H_\psi := \{(h, \psi(h)) \in H \times \mathbb{C}^* \mid h \in H\}.$$

If (π, V) is a complex representation of H, one has a representation (π_ψ, V) of H_ψ, given by

$$\pi_\psi((h, \psi(h))) = \psi(h)\pi(h), h \in H. \tag{3.1}$$

Let (P, P^-) be a pair of opposite parabolic subgroups of G, with common Levi subgroup M, and assume that P^-H is open. We claim that:

(3.2) The map $P^- \times H \to P^-H$, $(p, h) \mapsto ph$, is open.

In fact, looking at P^-H as an orbit of $P^- \times H$ in G for a suitable action, the proof of Lemma 12 (iii) in [3] shows that P^-H is homeomorphic with the quotient of $P^- \times H$ by the stabilizer of the neutral element. But the projection of $P^- \times H$ on this quotient is open. This proves our claim.

Examples of this situation were given in the introduction.

The proof of [16], Lemma 2 easily adapts to our situation to give:

Lemma 3.1. *Let P be as above. If K is a compact open subgroup of G, there exists a compact open subgroup of G, $K' \subset K$, such that for all smooth modules V, $\xi \in V^{*H_\psi}$:*

$$\langle \xi, av\rangle = \langle e_{K'}\xi, av\rangle, \ \ v \in V^K, \ \ a \in A_M^-, \tag{3.3}$$

where $e_{K'}\xi$ is the element of $\check{V}$ defined by

$$\langle e_{K'}\xi, v\rangle = \langle \xi, \pi(e_{K'})v\rangle, \ \ v \in V.$$

If K has an Iwahori factorization with respect to (P, P^-), one can take for K' any compact open subgroup of G contained in the open set $K \cap (K_{U^-}K_M Ker\ \psi)$ (see (3.2)), or equivalently contained in $K_{U^-}K_M(K \cap Ker\ \psi)$.

Remark 3.2. For Example 1, Kato and Takano [15, Lemma 4.6] proved the existence of arbitrarily small σ-stable, compact open subgroups, K, with an Iwahori factorization such that

$$K = K_{U^-}K_M(K \cap H).$$

In that case one can take $K' = K$.

For Example 2, and $K = K_n$ as in (2.2) and small enough so that $K_{U_0} \subset \mathrm{Ker}\ \psi$, one sees also that one can take $K' = K$.

Definition 3.3. For any compact open subgroup of G having an Iwahori factorization K for (P, P^-), we fix K' as in the previous lemma and we define $n'_K(a) := n_{K'}(a)$ for $a \in A^{--}$, where $n_K(a)$ has been defined in (2.4). With the notation of Lemma 3.1, we also set $\varepsilon'_K := \varepsilon_{K'}$. If K is any compact open subgroup of G, we choose a compact open subgroup $\underline{K}$ of G, contained in K, having an Iwahori factorization for (P, P^-), and we set $n'_K(a) = n'_{\underline{K}}(\mathrm{a})$, $\varepsilon'_K = \varepsilon'_{\underline{K}}$.

From the definition of the action of H_ψ on any smooth G-module (π, V) (cf. (3.1)), one sees that:

(3.4) One has $\xi \in V^{*H_\psi}$ if and only if $\xi \in V^*$ and $\xi(\pi(h)v) = \psi(h^{-1})\xi(v)$, for all $v \in V, h \in H$.

Theorem 3.4. *Let (P, P^-) be a a pair of opposite parabolic subgroups of G such that PH is open in G. Let A be the maximal split torus M of the center of their common Levi subgroup. For every smooth module V, $\xi \in V^{*H_\psi}$, there exists a unique $j_{P^-}(\xi) \in V_P^{*(M\cap H)_{\psi_P}}$ such that for all $a \in A^{--}$:*

$$\langle \xi, \pi(a^n)v\rangle = \delta_P^{1/2}(a^n)\langle j_{P^-}(\xi), \pi_P(a^n)j_P(v)\rangle,\ \ v \in V^K,\ \ n \geq n'_K(a) \tag{3.5}$$

and

$$\langle \xi, \pi(a)v\rangle = \delta_P^{1/2}(a)\langle j_{P^-}(\xi), \pi_P(a)j_P(v)\rangle,\ \ v \in V^K,\ \ a \in A^-(\varepsilon'_K). \tag{3.6}$$

Proof. First, we prove that there is at most one linear form on V_P satisfying (3.5) and (3.6) for all compact open subgroups of G. If there exist two linear forms on V_P with these properties, let us denote by η their difference. Then one has, in particular, for all compact open subgroups, K, of G, with $n = n'_K(a)$:

$$\langle \eta, \pi_P((a)^n)j_P(v)\rangle = 0,\ \ v \in V^K.$$

One chooses K with an Iwahori factorization for (P, P^-). Then, from Bernstein's generalized Jacquet Lemma (cf. [1], 5.4), one has

$$j_P(V^K) = V_P^{K_M}, \tag{3.7}$$

and $\pi_P(a^n)$ acts bijectively on $V_P^{K_M}$. Hence η vanishes on $V_P^{K_M}$, for all such K. Hence η is equal to zero, which proves the unicity statement of the theorem. Notice that the unicity statement has been proved without using the $(M\cap H)_{\psi_P}$-invariance of the linear forms. The proof above shows:

(3.8) Let $a \in A^{--}$ and $n \in \mathbb{N}^*$. There is at most one linear form η on $V_P^{K_M}$ such that

$$\langle \xi, \pi(a^n)v\rangle = \delta_P^{1/2}(a)\langle \eta, \pi_P(a^n)j_P(v)\rangle, v \in V^K.$$

Now we turn to the existence. One chooses a compact open subgroup K with an Iwahori factorization with respect to (P, P^-), and one defines a linear form on $V_P^{K_M}$, η_K by:

$$\langle \eta_K, x\rangle := \langle j_{P^-}(e_{K'}\xi), x\rangle_P, x \in V_P^{K_M}.$$

Then, using Lemma 3.1 and equation (3.7), one sees that:

$$\langle \xi, \pi(a^n)v\rangle = \delta_P^{1/2}(a^n)\langle \eta_K, \pi_P(a^n)j_P(v)\rangle, v \in V^K, a \in A^-, n \geq n_{K'}(a) \tag{3.9}$$

and

$$\langle \xi, av\rangle = \delta_P^{1/2}(a)\langle \eta_K, \pi_P(a)j_P(v)\rangle, v \in V^K, a \in A^-(\varepsilon'_K). \tag{3.10}$$

Now, let K_1 be another compact open subgroup with an Iwahori factorization with respect to (P, P^-) and contained in K. Using (3.8), the fact that $V_P^{(K_1)_M}$ contains $V_P^{K_M}$, and the properties above of η_K and η_{K_1}, one gets:

$$\eta_{K_1|V_P^{K_M}} = \eta_K.$$

This proves the existence of a linear form η on $j_P(V)$ such that for any compact open subgroup K with an Iwahori factorization with respect to (P, P^-), one has:

$$\eta_{|V_P^{K_M}} = \eta_K.$$

Then from (3.9) and (3.10) the linear form η satisfies analogous of (3.5) and (3.6) for any compact open subgroup K with an Iwahori factorization with respect to (P, P^-). But the definition of $n'_K(a)$ and ε'_K implies that it is true also for any open compact subgroup of G. It remains to prove the $(M \cap H)_{\psi_P}$-invariance of η. But it is clear that if $m \in M \cap H$, the linear form $\psi_P(m^{-1})\eta \circ \pi_P(m) \in V_P^*$ satisfies analogous of (3.5) and (3.6), maybe for a different choice of the n'_K, as $\pi(m)V^K = V^{mKm^{-1}}$, this for all compact open subgroups. From the proof of the unicity (see Equation (3.8)), one concludes that this linear form is equal to η. So η has the required properties. □

Remark 3.5. (i) It was shown in the proof of Theorem 3.4, that property (3.5), for a compact open subgroup K of G with an Iwahori factorization, and for a single a, characterizes the restriction of $j_{P^-}(\xi)$ to $V_P^{K_M}$. By choosing an element in $A^-(\varepsilon_K)$, one deduces similarly that (3.6) for such a K characterizes also the restriction of $j_{P^-}(\xi)$ to $V_P^{K_M}$.

(ii) The proof of unicity (see Equation (3.8)) shows that if a linear form on V_P enjoys the same properties as $j_{P^-}(\xi)$ but for a different choice of the n or ε, it is equal to $j_{P^-}(\xi)$. This remark was already used in order to show the $(M \cap H)_{\psi_P}$-invariance of η.

(iii) The theorem generalizes results of Kato and Takano (cf. [15]) on the one hand and of Lagier (cf. [16]) on the other hand, for admissible representations, for Example 1. Our proof is close to the proof of Lagier. The proof of Kato and Takano takes into account Remark 3.2 and uses standard sections of $V_P^{K_M}$. This point of view might work here also for Examples 1 and 2.

From the unicity statement in the theorem, one deduces:

Proposition 3.6. *The map $j_{P^-} : V^{*H_\psi} \mapsto V^{*(M\cap H)_{\psi_P}}$ is linear.*

Proposition 3.7. *Assume that P contains P_0. Let K be a compact open subgroup of G. For every smooth G-module (π, V), one has:*

$$\delta_P^{1/2}(a)\langle j_{P^-}(\xi), \pi_P(a)j_P(v)\rangle_P = \langle \xi, \pi(a)v\rangle,\ a \in A_0^-(P, < \varepsilon'_K).$$

Proof. The proof is similar to the proof of Lemma 2.1. □

Lemma 3.8. *Let V, W be smooth G-modules and let $\phi : V \to W$ be a morphism of G-modules. If $\eta \in W^{*H_\psi}$, let $\xi = \eta \circ \phi \in V^{*H_\psi}$. Let $j_P(\phi)$ be the morphism of M-modules $j_P(\phi) : V_P \to W_P$ deduced from ϕ. Then one has:*

$$j_{P^-}(\xi) = j_{P^-}(\eta) \circ j_P(\phi).$$

Proof. It is a simple exercise to prove that $j_{P^-}(\eta) \circ j_P(\phi) \in V^{*(M\cap H)_{\psi_P}}$ has the characterization properties (3.5) and (3.6) of $j_{P^-}(\xi)$, by using the corresponding properties for $j_{P^-}(\eta)$. The lemma follows from the unicity statement in Theorem 3.4. □

Proposition 3.9. *(i) We keep the preceding notation. Let V be a smooth G-module and $\xi \in V^{*H_\psi}$. Let (Q, Q^-) be another pair of opposite parabolic subgroups of G*

with $Q \subset P$, $Q^- \subset P^-$ and such that Q^-H is open. Denote by $R = Q \cap M$, $R^- = Q^- \cap M$. We assume that $R^-(M \cap H)$ is open.

Then (R, R^-) is a pair of opposite parabolic subgroups of M. Taking into account the equality $V_Q = (V_P)_R$, one has:

$$j_{Q^-}(\xi) = j_{R^-}(j_{P^-}(\xi)).$$

(ii) If $h \in H$, $(h.P^-)H$ is open, where $h.P^- = hP^-h^{-1}$. Denote by $T_h : V_P \mapsto V_{h.P}$ the map given by $j_P(v) \to j_{h.P}(\pi(h)v)$. One has:

$$\langle j_{P^-}(\xi), j_P(v)\rangle = \psi(h)\langle j_{h.P^-}(\xi), j_{h.P}(\pi(h)v)\rangle, v \in V, h \in H.$$

Proof. The proof of the proposition is essentially contained in [15] (Proposition 5.9 and Lemma 6.6) and [16] (Theorem 3 and proof of Proposition 3). One can choose a maximal split torus A_0 of G, contained in the common Levi subgroup, L, of Q and Q^-. Moreover let P_0 be a minimal parabolic subgroup of G contained in Q and containing A_0. For $\varepsilon > 0$ one has:

$$A_0^-(Q, < \varepsilon) = A_0^-(P, < \varepsilon) \cap A_0^-(R, < \varepsilon).$$

Let K be a compact open subgroup of G. Then, it follows easily from a double application of Proposition 3.7, that for ε small enough, one has:

$$\langle \xi, \pi(a)v\rangle = \langle j_{R^-}(j_{P^-}(\xi)), \pi_Q(a)j_Q(v)\rangle, \ \ a \in A_0^-(Q, < \varepsilon).$$

Let A_Q be the maximal split torus of the center of L. The uniqueness statement in Theorem 3.4 (see also Remark 3.5) and the inclusion $A_Q^-(\varepsilon) \subset A_0^-(Q, < \varepsilon)$ imply (i).

(ii) The proof is also essentially in [15, Lemma 6.6] and [16, proof of Proposition 3]. It is an easy consequence of the unicity statement in Theorem 3.4 and Remark 3.5. □

3.2. **Families of H_ψ-fixed linear forms.**

Definition 3.10. Let X be the set of points of an algebraic variety defined over $\mathbb{C}$ (resp., complex, resp., real manifold). Let V be a complex vector space. Let Λ be a lattice. We say that a family of characters of Λ, $\chi = (\chi_x)_{x\in X}$, is regular (resp., holomorphic, resp., C^∞) if for all $\lambda \in \Lambda$, the map $x \mapsto \chi_x(\lambda)$ is regular (resp., holomorphic, resp., C^∞) on X.

A family of smooth admissible representations of G in V, $(\pi_x, V_x)_{x\in X}$, is said to be regular (resp., holomorphic, resp., C^∞) if:

(i) The action of some compact open subgroup of G on V under π_x does not depend on $x \in X$.

(ii) For each $v \in V$ and $g \in G$, the map $x \mapsto \pi_x(g)v$, which has its values in a finite dimensional vector space of V by the admissibility and (i), is a regular function on X (resp., holomorphic, resp., C^∞).

(iii) For each parabolic subgroup P of G with Levi subgroup M, there exists a finite set of regular (resp., holomorphic, resp., C^∞) families of characters of $\Lambda(A_M)$, $\mathcal{X} = ({}^1\chi, \ldots, {}^p\chi)$ depending on $x \in X$, such that:

(3.11) For all $x \in X$, the $\Lambda(A_M)$-module $(V_x)_P$ is of type $({}^1\chi_x, \ldots, {}^p\chi_x)$ (see Definition 6.2 in the Appendix).

A family of linear forms on V, $(\xi_x)_{x\in X}$, is said to be regular (resp., holomorphic, resp., C^∞) if for all $v \in V$, the map $x \mapsto \xi_x(v)$ is regular (resp., holomorphic, resp., C^∞).

The example we have in mind is the family of parabolically induced representations from a representation of a Levi subgroup of G, twisted by unramified characters of this Levi subgroup. Property (iii) follows from the Geometric Lemma (see e.g. [10], Lemma 7 (i)).

Proposition 3.11. *Let $(\pi_x, V)_{x\in X}$ (resp., $(\xi_x)_{x\in X}$) be a regular (resp., holomorphic, resp., C^∞) family of smooth admissible representations of G (resp., of H_ψ-fixed linear forms under π_x). Then: (i) For all $v \in V$, the map $x \mapsto \langle j_{P^-}(\xi_x), j_P(v)\rangle$ is regular (resp., holomorphic, resp., C^∞) on X.*

(ii) Let $F(X, V)$ be the space of all maps from X to V which take their values in a finite dimensional space and which are regular (resp., holomorphic, resp., C^∞). Let π be the representation of G on $F(X, V)$ given by:

$$(\pi(g)\phi)(x) = \pi_x(g)\phi(x),\ \phi \in F(X, V),\ x \in X.$$

π is a smooth representation of G. Let Y be a compact subset of X, and let μ be a measure on Y. Let ξ be the H_ψ-fixed linear form on $F(X, V)$ given by:

$$\xi(\phi) = \int_Y \xi_y(\phi(y))d\mu(y),\ \phi \in F(X, V).$$

Then

$$\langle j_{P^-}(\xi), j_P(\phi)\rangle = \int_Y \langle j_{P^-}(\xi_y), j_P(\phi(y))\rangle d\mu(y). \tag{3.12}$$

Proof. (i) The fact that $F(X, V)$ is a smooth G-module follows from Definition 3.10 (i). Let $a \in A^{--}$ and $v \in V$. We will study the sequence:

$$u_x(n) := \delta_P^{1/2}(a^n)\langle j_{P^-}(\xi_x), (\pi_x)_P(a^n)v\rangle.$$

Using (3.11) for $a' = a^{-1}$ one sees that

$$\langle j_{P^-}(\xi_x), ((\pi_x)_P(a') -^1 \chi_x(a'))\dots((\pi_x)_P(a') -^p \chi_x(a'))\pi_P(a^n)v\rangle = 0.$$

By expanding the product, one gets easily:

$$u_x(n-p) + \theta_1(x)u_x(n-p+1) + \dots + \theta_p(x)u_x(n) = 0,\ n \in \mathbb{N}, \tag{3.13}$$

where, for all k, $x \mapsto \theta_k(x)$ is a regular (resp., holomorphic, resp., C^∞) function on X.

From Theorem 3.4, for n large enough, independent of $x \in X$, $u_x(n)$ is equal to $\langle \xi_x, \pi_x(a^n)v\rangle$; hence it is regular (resp., holomorphic, resp., C^∞) in $x \in X$. The recursion relation (3.13) shows that the maps $x \mapsto u_x(n)$ enjoy the same property for all $n \in \mathbb{N}$, and in particular for $n = 0$. This proves (i).

(ii) Using the properties of $j_{P^-}(\xi_x)$, $x \in X$, given by Theorem 3.4, and part (i), one sees easily that the right-hand side of (3.12) defines an $(M \cap H)_{\psi_P}$-fixed linear form on $F(X, V)_P$ which has the characterization properties of $j_{P^-}(\xi)$. (ii) follows. □

3.3. **Constant term of smooth functions.** Let $C^\infty(G/H, \psi)$ be the space of smooth functions on G, i.e. left invariant by at least a compact open subgroup of G and which transform under the right regular representation of H by ψ. The left regular representation L of G endows this space with a structure of a smooth G-module. We denote by δ_e or δ the Dirac measure in e. This is an H_ψ-fixed linear form on $C^\infty(G/H, \psi)$. Let (P, P^-) be as in (1.1).

Definition 3.12. We recall that P^-H is assumed to be open. For $f \in C^\infty(G/H,\psi)$ we denote by f_P the element of $C^\infty(M/M\cap H,\psi_P)$ defined by

$$f_P(mM\cap H) = \langle j_{P^-}(\delta_{eH}), \delta_P^{1/2}(m) j_P(L_{m^{-1}}f)\rangle, m \in M.$$

We call it the constant term of f along P.

One sees easily that:

(3.14) The map $f \mapsto f_P$ is a morphism of P-modules from $C^\infty(G/H,\psi)$ endowed with the left regular representation of P to $C^\infty(M/M\cap H,\psi_P)$, on which M acts by the tensor product of the left regular representation of M with $\delta_P^{1/2}$ and the unipotent radical, U, of P acts trivially.

If (π, V) is a smooth G-module and $\xi \in V^{*H_\psi}$, $v \in V$ we denote by $c_{\xi,v} \in C^\infty(G/H,\psi)$ the function defined by

$$c_{\xi,v}(g) = \langle \xi, \pi(g^{-1})v\rangle,\ g \in G, \tag{3.15}$$

so that the map c_ξ from V to $C^\infty(G/H,\psi)$, $v \mapsto c_{\xi,v}$ is a morphism of G-modules. We call $c_{\xi,v}$ a generalized coefficient of (π, V).

Proposition 3.13. *With the previous notation one has:*

$$(c_{\xi,v})_P = c_{j_{P^-}(\xi), j_P(v)},\ v \in V^{*H_\psi},\ v \in V.$$

Proof. When $V = C^\infty(G/H,\psi)$ and ξ is the Dirac measure δ_{eH}, the formula is precisely the definition of the constant term. More generally, one has $\xi = \delta_e \circ c_\xi$. Also, one has $L_m c_{\xi,v} = c_{\xi,\pi(m)v}$. The proposition follows from Lemma 3.8 and from Definition 3.12. □

Proposition 3.14. *(i) Let V be a smooth G-submodule of $C^\infty(G/H,\psi)$. The map $f \to f_P$ is the unique morphism of P-modules from V to $C^\infty(M/M\cap H,\psi_P)$, endowed with the tensor product of the left action of M with $\delta_P^{1/2}$ and the trivial action of U, such that:*

For all compact open subgroups, K, of G, and $f \in V^K$,

$$f(a^{-1}) = \delta_P^{1/2}(a) f_P(a^{-1}), a \in A^-(\varepsilon'_K). \tag{3.16}$$

(ii) If A_0 is a maximal split torus contained in $P \cap P^-$, and P_0 is a minimal parabolic subgroup which contains A_0 and is contained in P, one has

$$f(a^{-1}) = \delta_P^{1/2}(a) f_P(a^{-1}), a \in A_0^-(P, < \varepsilon'_K).$$

Proof. From Definition 3.12, Theorem 3.4 and (3.14), the map $f \to f_P$ has the required properties except unicity. Now suppose T is such a map. As T is a morphism of P modules, the linear form on V, $f \to T(f)(e)$ goes through the quotient to a linear form η on V_P. It is fixed under $(M\cap H)_{\psi_P}$. Let ξ be the H_ψ-fixed linear form on V given by the restriction of the Dirac measure δ_e to V. Then using (3.16) and Remark 2, one sees easily that $j_{P^-}(\xi) = \eta$. When one applies Proposition 3.13 to our choice of V and ξ, and takes into account that $c_{\xi,f} = f$ for $f \in V$, one gets:

$$(T(f))(e) = f_P(e).$$

From the M-equivariance of T and from (3.14), one deduces:

$$(T(f))(m) = f_P(m), f \in V, m \in M.$$

This proves unicity. □

Remark 3.15. This definition generalizes the definition given by N. Lagier for generalized coefficients of admissible representations ([16], Proposition 2) in Example 1.

Proposition 3.16. *With the notation of Proposition* 3.9*, one has:*

(i)

$$f_Q = (f_P)_R,\ f \in C^\infty(G/H, \psi).$$

(ii)

$$f_{h.P}(hmh^{-1}) = \psi(h^{-1})(L_{h^{-1}}f)_P(m),\ h \in H, m \in M, f \in C^\infty(G/H, \psi).$$

Proof. The proof is an immediate consequence of Definition 3.12 and Proposition 3.9. □

3.4. **Constant term of wave packets.** The constant term of smooth compactly supported functions modulo H seems mysterious (see sections 4.5 and 4.6 below). On the other hand, at least in the group case, these functions appear as wave packets of coefficients of admissible representations (cf. [19]). Still in the group case, the constant term of the coefficients of the representations occurring in the wave packets is well understood (cf. loc. cit.) So it seems to us that the following proposition, which expresses the constant term of a wave packet as a wave packet of constant terms, might be useful to answer the question (4.15) below.

Proposition 3.17. *We use the notation of Proposition* 3.11 *(iii). Let* $\phi \in F(X, V)$, $f = c_{\xi,\phi}$, $f_x = c_{\xi_x,\phi(x)}, x \in X$. *Then, for* $g \in G$ *(resp.,* $m \in M$*), the map* $x \mapsto f_x(g)$ *(resp.,* $x \mapsto (f_x)_P(m)$*) is regular (resp., holomorphic, resp.,* C^∞*) on* X *and*

$$f(g) = \int_Y f_y(g)d\mu(y),\ f_P(m) = \int_Y (f_y)_P(m)d\mu(y).$$

Proof. The assertions on f are immediate. The assertions on f_P follow from Propostion 3.11. □

4. Reductive symmetric spaces

4.1. If J is an algebraic group, one denotes by $Rat(J)$ the group of its rational characters defined over F. Let A_G be the largest split torus in the center of G. Let us define: $\mathfrak{a}_G = Hom_{\mathbb{Z}}(Rat(G), \mathbb{R})$. The restriction of rational characters from G to A_G induces an isomorphism:

$$Rat(G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq Rat(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{4.1}$$

One has the canonical map $H_G : G \to \mathfrak{a}_G$, which is defined by

$$e^{\langle H_G(x),\chi\rangle} = |\chi(x)|_F,\ x \in G, \chi \in Rat(G), \tag{4.2}$$

where $|.|_F$ is the normalized absolute value of F. The kernel of H_G, which is denoted by G^1, is the intersection of the kernels of the characters of G, $|\chi|_F$, $\chi \in Rat(G)$. One defines $X(G) = Hom(G/G^1, \mathbb{C}^*)$, which is the group of unramified characters of G. One will use similar notation for Levi subgroups of G.

One denotes by $\mathfrak{a}_{G,F}$, resp. $\tilde{\mathfrak{a}}_{G,\mathbb{F}}$, the image of G, resp. A_G, by H_G. Then G/G^1 is isomorphic to the lattice $\mathfrak{a}_{G,F}$.

There is a surjective map:

$$(\mathfrak{a}_G^*)_{\mathbb{C}} \to X(G) \to 1, \tag{4.3}$$

which associates to $\chi \otimes s$ the character $g \mapsto |\chi(g)|_F^s$ (cf. [19], I.1.(1)). The kernel is a lattice and it defines a structure of a complex algebraic variety on $X(G)$ of dimension $\dim_{\mathbb{R}} \mathfrak{a}_G$.

From the canonical isomorphism (4.1), one deduces

$$A_G^1 = A_G \cap G^1. \tag{4.4}$$

Moreover A_G^1 is the largest compact subgroup of A_G.

Notice that, with the notation of section 2.1, $\Lambda(A_G) \cap A_G^1$ is trivial. In fact if $\underline{\lambda} \in X_*(A_G)$ and $\underline{\lambda}(\varpi)$ is an element of A^1, then $\underline{\lambda} \in X^*(G)$ is trivial: one reduces to see this, by using products, to a similar statement for a one dimensional torus, where it is clear. Hence from (4.4):

(4.5) The map $\Lambda(A_G) \to G/G^1$ is injective.

Let (P, P^-) be a pair of opposite parabolic subgroups of G and let M be their common Levi subgroup. Denote by $Z(G)$ the center of G. Then one has

$$\Lambda(A_M) \cap Z(G) = \Lambda(A_G). \tag{4.6}$$

In fact, only the inclusion of the left-hand side in the right-hand side is nontrivial. Let $a \in \Lambda(A_M) \cap Z(G)$ and $\underline{\lambda}$ be a one-parameter subgroup of A_M, defined over F, with $\underline{\lambda}(\varpi) = a$. Let α be a root of A_M in the Lie algebra of G. Then $\alpha \circ \underline{\lambda}$ is a rational character of $GL(1, F)$ whose image of ω is trivial; hence it is trivial. It implies that the image of $\underline{\lambda}$ commutes with P and with P^-. By the density of P^-P for the ordinary topology one sees that this image commutes with G. But G is Zariski dense in $\underline{G}$ ([4], Corollary 18.3). Hence, as $\underline{\lambda}$ is defined over F, it is a one-parameter subgroup in the center of G whose image is an F-split torus. Hence it is an F-split torus contained in A_G. It follows that $\underline{\lambda}(\omega)$ is an element of $\Lambda(A_G)$.

4.2. **σ-parabolic subgroups.** Recall that G is the group of F-points of a connected reductive group, $\underline{G}$, defined over F. Let σ be a rational involution of $\underline{G}$, defined over F. Let H be the group of F-points of an open F-subgroup of the fixed point set of σ. We will also denote by σ the restriction of σ to G.

A parabolic subgroup P of G, is called a σ-parabolic subgroup if P and $\sigma(P)$ are opposite parabolic subgroups. Then $M := P \cap \sigma(P)$ is the σ-stable Levi subgroup of P. For this part, we assume:

(4.7) The characteristic of F is different from 2.

Then one deduces from [14], Proposition 13.4:

(4.8) P^-H is open in G and $(P, \sigma(P))$ satisfies (1.1), where $P^- = \sigma(P)$. Moreover δ_P restricted to H is trivial.

We recall also that (cf. loc. cit. Corollary 6.16):

(4.9) There are only a finite number of H-conjugacy classes of σ-parabolic subgroups of G.

4.3. **H-cuspidal linear forms and functions.**

Definition 4.1. Let ψ be a smooth character of H. An H_ψ-fixed linear form, ξ, on a smooth G-module V is said to be H_ψ-cuspidal if for any proper σ-parabolic subgroup P of G, one has:

$$j_{P^-}(\xi) = 0.$$

An element f of $C^\infty(G/H,\psi)$ is said to be H_ψ-cuspidal if for any proper σ-parabolic subgroup P of G, one has:

$$(L_g f)_P = 0, g \in G.$$

We denote by $C^\infty(G/H,\psi)_{cusp}$ the vector space of H-cuspidal smooth functions on G/H.

Notice that it would be more correct to say (σ,ψ)-cuspidal instead of H_ψ-cuspidal as the definition does not depend only on H but on σ. Nevertheless, as in applications σ is fixed, we have decided to choose H_ψ-cuspidal.

In view of Propostions 3.9 and 3.16, one sees that one can limit P to a set of representatives of the H-conjugacy classes of maximal proper σ-parabolic subgroups of G.

From the definition, it is clear that:

(4.10) An H_ψ-fixed linear form, ξ, on a smooth G-module V is H_ψ-cuspidal if and only if every generalized coefficient, $c_{\xi,v}$, $v \in V$, is H_ψ-cuspidal.

4.4. **Finiteness theorems.** A split torus is said to be σ-split if all its elements are anti-invariant by σ. Let $A_\emptyset$ be a maximal σ-split torus of G, let $M_\emptyset$ be the centralizer of $A_\emptyset$ in G and let $P_\emptyset$ be a minimal σ-parabolic subgroup of G whose σ-stable Levi subgroup is equal to $M_\emptyset$ (see [14] for the existence). We denote also by $A_{\emptyset,G}$ the maximal σ-split torus in A_G. We denote by $\Lambda^+(A_\emptyset)$ the set of $P_\emptyset$ dominant elements in $\Lambda(A_\emptyset)$. Let $\Sigma(G,A_\emptyset)$ be the set of roots of $A_\emptyset$ in the Lie algebra of G. It is a root system (cf. [14]). We denote by $\Delta(G,A_\emptyset)$ the set of simple roots of the set of roots of $A_\emptyset$ in the Lie algebra of $P_\emptyset$. If Θ is a subset of $\Delta(G,A_\emptyset)$ we denote by $\langle\Theta\rangle_\emptyset$ the subroot system of $\Sigma(G,A_\emptyset)$ generated by Θ. Let P_Θ be the parabolic subgroup of G which contains $A_\emptyset$ and such that the set of roots of $A_\emptyset$ in its Lie algebra is equal to the union of the set of roots of $A_\emptyset$ in the Lie algebra of $P_\emptyset$ and $\langle\Theta\rangle_\emptyset$. It is a σ-parabolic subgroup of G.

Let $C > 0$. We denote by $\Lambda^+(A_\emptyset)$ (resp., $\Lambda^+(A_\emptyset,C)$) the set of elements a in $\Lambda(A_\emptyset)$ such that $|\alpha(a)|_F \geq 1$ (resp., $C \geq |\alpha(a)|_F \geq 1$) for all $\alpha \in \Delta(G,A_\emptyset)$.

Lemma 4.2. *Let $\Lambda(A_\emptyset)_G$ be the intersection of $\Lambda(A_\emptyset)$ with the center of G. It is a subgroup of $\Lambda(A_G)$ and $\Lambda^+(A_\emptyset,C)$ is a union of finitely many orbits of $\Lambda(A_\emptyset)_G$.*

Proof. As $\Lambda(A_\emptyset)$ is a subgroup of $\Lambda(A_{M_\emptyset})$, one deduces the first assertion from (4.6). To prove the second assertion, let us consider the morphism of abelian groups $\phi : \Lambda(A_\emptyset) \to \mathbb{R}^{\Sigma(G,A_\emptyset)}$ given by $a \mapsto (\mathrm{Log}\,|\alpha(a)|_F)$. The image of $\Lambda^+(A_\emptyset,C)$ is clearly finite. Hence it is enough to prove that the kernel of ϕ is equal to $\Lambda(A_\emptyset)_G$. Let a be an element of its kernel. The image, ϖ', of the uniformizing element ϖ by a one-parameter subgroup of $GL(1,F)$ satisfies $|\varpi'|_F = 1$ if and only if this one-parameter subgroup is constant. It follows easily that $\alpha(a)$ is equal to one for all roots of $A_\emptyset$. Hence a commutes to $P_\emptyset$ and to $\sigma(P_\emptyset)$. Hence, by the density of $P_\emptyset\sigma(P_\emptyset)$, it commutes with G. Our assertion on the kernel of ϕ follows. □

Lemma 4.3. *Let K be a compact open subgroup of G. Then there exists a finite set $F_{\emptyset,K} \in \Lambda(A_\emptyset)^+$ such that the restriction of every element of $C^\infty(G/H,\psi)^K_{cusp}$ to $\Lambda^+(A_\emptyset)$ is zero outside $F_{\emptyset,K}\Lambda(A_\emptyset)_G$.*

Proof. Let A_0 be a maximal split torus contained in $M_\emptyset$. Hence it contains $A_\emptyset$. Let P_0 be a minimal parabolic subgroup of G contained in $P_\emptyset$ and containing A_0. Let

$\varepsilon > 0$ and $C = \varepsilon^{-1}$. Using the notation of (2.5), we denote by

$$\Lambda^+(P, > C) := \{a^{-1} \mid a \in \Lambda(A_\emptyset) \cap A_0^-(P, < \varepsilon)\}.$$

From Proposition 3.14(ii), one sees that there exists $C > 0$ such that for all $P = P_\Theta$:

$$f_{|\Lambda^+(P,>C)} = \delta_P^{1/2}(a)(f_P)_{|\Lambda^+(P,>C)}, f \in C^\infty(G/H, \psi)^K.$$

If moreover $f \in C^\infty(G/H, \psi)^K_{cusp}$, one has $f_P = 0$ for all $P = P_\Theta$, where Θ is equal to $\Delta(G, A_\emptyset) \setminus \{\alpha\}$ for some $\alpha \in \Delta(G, A_\emptyset)$. But the complementary in $\Lambda^+(A_\emptyset)$ of the union of $\Lambda^+(P, > C)$ for all such P is equal to $\Lambda^+(A_\emptyset, C)$. Then the lemma follows from the previous lemma. □

Let $(A_i)_{i\in I}$ be a set of representatives of the H-conjugacy classes of maximal σ-split tori of G. This set is finite (cf. [14], 6.10 and 6.16). One assumes that this set contains $A_\emptyset$. The tori A_i are conjugated under G (cf. [13], Proposition 1.16). One chooses, for all $i \in I$, an element x_i of G, such that $x_i.A_\emptyset = A_i$. One takes $x_\emptyset$ equal to the neutral element e of G.

One fixes a minimal σ-parabolic subgroup $P_\emptyset$ of G, which contains $A_\emptyset$. Let $\overline{W(A_\emptyset)}$ be a set of representatives in G of the quotient, $W(A_\emptyset)$, of the normalizer of $A_\emptyset$ in G by its centralizer $M_\emptyset$ in G. One chooses a subset $\overline{\mathcal{W}}^G$ of the set $\{x_i w \mid w \in \overline{W(A_\emptyset)}\}$, which is a set of representatives of the open $(H, P_\emptyset)$-double cosets of G (cf. [13, Theorem 3.2]). It is a finite set. Moreover, if $y \in \overline{\mathcal{W}}^G$, $y.A_\emptyset$ is a maximal σ-split torus of G and $y.P_\emptyset$ is a σ-split parabolic subgroup of G.

The Cartan decomposition (cf. [2], [11]) ensures the existence of a compact subset Ω of G, such that:

$$G = \bigcup_{y\in\overline{\mathcal{W}}^G} \Omega\Lambda^+(A_\emptyset)y^{-1}H. \tag{4.11}$$

Let $X(G)_\sigma$ be the neutral component of the set $\{\chi \in X(G) \mid \chi \circ \sigma = \chi^{-1}\}$. Notice that elements of $X(G)_\sigma$ are trivial on H.

Theorem 4.4. *(i) Let K be a compact open subgroup of G. Let $\mathcal{X}$ be a finite family of characters of $\Lambda(A_G)$. The space $C^\infty(G/H, \psi)^K_{cusp}(\mathcal{X})$ of elements of $C^\infty(G/H, \psi)^K_{cusp}$ which are of type $\mathcal{X}$ under the left regular action of $\Lambda(A_G)$ (see Definition 6.2) is finite dimensional.*

(ii) The elements of $C^\infty(G/H, \psi)^K_{cusp}$ have their support in a compact set modulo $\Lambda(A_\emptyset)_G$ depending only on K.

(iii) Let (π, V) be a smooth G-module of finite length. Let $V^{*H_\psi}_{cusp}$ be the space of H_ψ-fixed H_ψ-cuspidal linear forms. This space is finite dimensional.

(iv) There exists a finite number of irreducible representations of G, (π_i), such that any irreducible representation (π, V) of G, having a nonzero vector fixed by K and with $V^{*H_\psi}_{cusp}$ nonreduced to zero, is equivalent to a representation $\pi_i \otimes \chi$ for an element χ of $X(G)_\sigma$.

Proof. (i) From the compacity of Ω, there exist $g_1, \ldots, g_n \in \Omega$ such that Ω is contained in the union of the sets $Kg_1, \ldots, Kg_n$. Define $A_y := y.A_\emptyset$, $\Lambda(A_y)^+ = y.\Lambda(A_\emptyset)^+$ for $y \in \mathcal{W}^G$. Notice that $\Lambda(A_y)$ is equal to $y.\Lambda(A_\emptyset)$ and the intersection $\Lambda(A_y)$ with the center of G is equal to $\Lambda(A_\emptyset)_G$. Then, by the Cartan decomposition (4.11), one sees that the function f is determined by the restriction of $L_{yg_i^{-1}}f$ to $\Lambda(A_y)^+$, when y varies in $\mathcal{W}^G$ and $i = 1, \ldots, n$. But if $g \in G$, $L_g f$ is invariant

under $g.K$. Taking into account that $\Lambda(A_\emptyset)_G \subset \Lambda(A_G)$ (cf. Lemma 4.2), Lemma 4.3, applied to $y.A_\emptyset$, but changing K, together with the properties of functions of type $\mathcal{X}$ (see (6.3)), shows that there is a finite subset of G/H such that $f \in C^\infty(G/H,\psi)^K_{cusp}(\mathcal{X})$ is zero if and only if it is zero on this finite set. The restriction map to this set is injective. This implies (i).

(ii) is proved in the same way.

(iii) From the fact that V is of finite length, one sees that there exists a finite family of characters of $\Lambda(A_G)$ such that every element of V is of type $\mathcal{X}$. Every G-module map between V and another G-module has its image in the space of vectors of type $\mathcal{X}$. From Definitions 4.1 and 6.2, one sees easily that the space $C^\infty(G/H,\psi)_{cusp}(\mathcal{X})$, of elements of $C^\infty(G/H,\psi)_{cusp}$ of type $\mathcal{X}$ under the left regular action of $\Lambda(A_G)$ is a smooth submodule of $C^\infty(G/H,\psi)$. Let us consider the linear map:

$$\psi_V : Hom_G(V, C^\infty(G/H,\psi)_{cusp}(\mathcal{X})) \to V^{*H_\psi}_{cusp},$$

given by

$$T \mapsto \delta_{eH} \circ T.$$

It is easily seen to be well defined, i.e. that its image is actually in $V^{*H_\psi}_{cusp}$ (see Proposition 3.13). Also, one checks easily that $\xi \mapsto c_\xi$ is the inverse of this linear map; in particular, ψ_V is bijective. As V is of finite length, it is generated by a finite set. One can choose a compact open subgroup K of G, which fixes each of these generators. An element T of $Hom_G(V, C^\infty(G/H,\psi)_{cusp})$ is given by the image of these generators, that have to lie in $C^\infty(G/H,\psi)^K_{cusp}(\mathcal{X})$. As this space is finite dimensional by (i), this implies that $Hom_G(V, C^\infty(G/H,\psi)_{cusp}(\mathcal{X}))$ is finite dimensional. As ψ_V is bijective, this implies (iii).

(iv) One will apply Lemma 6.1 of the Appendix to $\Lambda = \Lambda(A_G)$, $\Lambda' = G/G^1$ (see (4.5)). We use the notation of this lemma. With this notation, $X(G)$ (resp. $X(G)_\sigma$) identifies to $X(\Lambda')$ (resp. $X(\Lambda')_\sigma$). Let O be equal to $\{\chi \in X(\Lambda) \mid \chi_{|\Lambda^+} = \psi_{|\Lambda^+}\}$. If χ_0 is an element of O, O is equal to $\chi_0 X(\Lambda)^-$. Then Lemma 6.1 shows that, up to twisting by some $\chi \in X(G)_\sigma$, one can assume that the restriction of the central character of π to $\Lambda(A_G)$ varies in a finite set.

It remains to prove that there is only a finite set of equivalence classes of irreducible representations with the required property and with a given restriction of the central character to $\Lambda(A_G)$. If V is a smooth G-module, one defines a linear map $\phi_V : V \otimes V^{*H_\psi}_{cusp} \to C^\infty_c(G/H,\psi)_{cusp}$ given by $v \otimes \xi \mapsto c_{\xi,v}$. From (4.10), one sees that it is well defined. If V is irreducible, this map is easily seen to be injective. But if (V_j) are such inequivalent representations, the images of the maps ϕ_{V_j} above are in direct sum. From (i), applied to $\mathcal{X}$ reduced to an element, one deduces (iv). □

Theorem 4.5. *Let (π, V) be a smooth module of finite length. Then the dimension of V^{*H_ψ} is finite.*

Proof. We prove the theorem by induction on the dimension of the algebraic group G. If this dimension is zero, the theorem is clear. Assume now that this dimension is greater than zero. When G has no proper σ-parabolic subgroup, one sees from the definition that $V^{*H_\psi} = V^{*H_\psi}_{cusp}$. Hence it is finite dimensional by point (iii) of the previous theorem. When G has at least a proper σ-parabolic subgroup, let $\mathcal{P}$ be a set of representatives of H-conjugacy classes of proper σ-parabolic subgroups of G. This set is nonempty, by our hypothesis, and finite (cf. (4.9)). If $P \in \mathcal{P}$, let us

denote by M its σ-stable Levi subgroup. Then $M/M\cap H$ is a reductive symmetric space. Define a linear map $j_{\mathcal{P}}$ from V^{*H_ψ} to $\bigoplus_{P\in\mathcal{P}} V_P^{*(M\cap H)_{\psi_P}}$, where M denotes the σ-stable Levi subgroup of P, by

$$j_{\mathcal{P}}(\xi) = (j_{P^-}(\xi))_{P\in\mathcal{P}},\ \ \xi\in V^{*H}.$$

From Proposition 3.9 (ii), one sees that the kernel $j_{\mathcal{P}}$ is precisely $V_{cusp}^{*H_\psi}$; hence it is finite dimensional by Theorem 4.4 (iii). For $P\in\mathcal{P}$, V_P is a smooth module of finite length. Then by using the induction hypothesis and the finiteness of $\mathcal{P}$, one sees that the dimension of $\bigoplus_{P\in\mathcal{P}} V_P^{*(M\cap H)_{\psi_P}}$ is finite. The theorem follows immediately. □

4.5. **Some questions.** *We take ψ to be trivial in this part. It would be interesting to determine the constant term of compactly supported functions, in particular for those supported in P^-H.*

Let P be a σ-parabolic subgroup of G, M its σ-stable Levi subgroup, and U its unipotent radical. Let $V := C_c^\infty(G/H)$ be the space of smooth compactly supported functions on G/H. It is also the space of locally constant functions with compact support. Let $X = C_c^\infty(G/H)_{PH}$ be the subpace of $C_c^\infty(G/H)$ with support contained in PH. It is a smooth P-module. If $f\in X$, we define a function f^P on $M\cap H$ by

$$f^P(mM\cap H) = \delta_P^{1/2}(m)\int_U f(muM\cap H)du.$$

Lemma 4.6. *At least if F is of characteristic zero, one has:*

$$\text{If } f\in C_c^\infty(G/H)_{PH}, \text{ then } f^P\in C_c^\infty(M/M\cap H).$$

Moreover the map $f\mapsto f^P$ goes through the quotient to X_P and defines an isomorphism of M-modules between X_P with $C_c^\infty(M/M\cap H)$. From the exactness of the Jacquet functor, this gives a natural imbedding of M-modules of $C_c^\infty(M/M\cap H)$ into V_P, which allows us to identify $C_c^\infty(M/M\cap H)$ as an M-submodule of V_P.

Proof. Along the lines of the proof of Lemma 4 of [5], one can see that:

Assume F is of characteristic zero. Let Ω be a subset of PH which is compact modulo H. Then

$$\Omega\subset\Omega_U\Omega_M H, \tag{4.12}$$

where Ω_M is a compact subset of M modulo $M\cap H$ and Ω_U is a compact subset of the unipotent radical, U, of P.

Also, let us prove that:

(4.13) Let $u\in U$, $m\in M$, $h\in H$, such that $umh\in M$. Then one has $u=1$ and $h\in M$.

In fact, one has $umh = m'$ with $m'\in M$; hence $h = u'm^{-1}m'$, where $u'\in U$. Using that $\sigma(h)=h$ and $\sigma(P)\cap P = M$, one sees that $h\in M$ and $u'=1$, which proves (4.13). Together with (4.12), it shows that if $f\in C_c^\infty(G/H)$ has its support in Ω, f^P has its support in Ω_M. □

Now, let ξ be an H-invariant distribution on G/H. The restriction of $j_{P^-}(\xi)$ to $C_c^\infty(M/M\cap H)$ defines an $M\cap H$-invariant distribution, ξ_P, on $M/M\cap H$.

(4.14) The map $\xi \to \xi_P$ is a linear map from the space of H-invariant distributions on G/H to the space of $H\cap M$-invariant distributions on $M/H\cap M$.

It would be interesting to elucidate this curious operation on distributions.

4.6. **The group case.** Assume $G = G_1 \times G_1$, and $\sigma(x,y) = (y,x)$. The group H is the diagonal of G. Then G/H is identified with G_1 by the map $(x,y)H \mapsto xy^{-1}$, and $C_c^\infty(G/H)$ is identified with $C_c^\infty(G_1)$, i.e. to the space of compactly supported locally finite functions on G_1. Then a σ-parabolic subgroup P of G is simply a parabolic subgroup of G of the form $P_1 \times P_1^-$, where P_1^- is a parabolic subgroup opposite to P_1. We will denote by M_1 the common Levi subgroup of P_1 and P_1^-. In this case the constant term provides a P-module map

(4.15) $f \mapsto f_P$ from $C^\infty(G_1)$ to $C^\infty(M_1)$.

For coefficients of admissible representations, the constant term defined here coincides with the ordinary constant term.

It would be interesting to determine this map, at least for compactly supported functions. One might use the Harish-Chandra Plancherel formula [19] *and Proposition* 3.17.

(4.16) The map $\xi \mapsto \xi_P$ of section 4.5 is a linear map from the space of distributions on G_1, invariant by conjugacy under G_1 to the space of distributions on M_1, invariant by conjugacy under M_1.

It would be interesting to know if this operation is trivial and, if not, how it behaves on the Dirac measure at the neutral element, on orbital integrals and on characters.

5. Whittaker models and mixed models

Let Q be a parabolic subgroup of G with Levi subgroup L and unipotent radical U_Q. Let A_0 be a maximal split torus of G contained in L and let P_0 be a minimal parabolic subgroup of G contained in Q and which contains A_0. Let U_0 be the unipotent radical of A_0. Then the unipotent radical U_0 of P_0^- contains U_Q. Let Δ (resp., Δ_L) be the set of simple roots of A_0 in the Lie algebra of U_0 (resp., $L\cap U_0$)). If $\alpha \in \Delta$, let U_α be the corresponding subgroup of the unipotent radical of P_0. One has:

$$U_\alpha \subset U_Q, \alpha \in \Delta \setminus \Delta_L.$$

.

Definition 5.1. 1) A smooth character ψ of U_Q is said to be (A_0, P_0)-nondegenerate if the restriction to every U_α, $\alpha \in \Delta \setminus \Delta_L$, is nontrivial.

2) With the notation of Example 2 (Whittaker models) in the Introduction, a smooth character of U_0 is said to be nondegenerate if it is (A_0, P_0)-nondegenerate for some maximal split torus, A_0, of G contained in P_0. The notion is independant of A_0.

3) With the notation of Example 3 in the Introduction, let $P'_\emptyset$ be a minimal σ'-parabolic subgroup of L, which contains a maximal σ'-split torus $A_\emptyset$. Let $M'_\emptyset$ be

the centralizer in L of $A_\emptyset$. Let A_0 be a maximal split torus of $M'_\emptyset$ which contains $A_\emptyset$. Let P_0 be a minimal parabolic subgroup of G containing A_0, such that:

$$P_0 \subset Q, P_0 \cap L \subset P'_\emptyset.$$

A smooth character of H, in Example 3, is said to be nondegenerate if its restriction to U_Q is (A_0, P_0)-nondegenerate for a pair (A_0, P_0) as above.

In Examples 2 and 3, we fix a nondegenerate smooth character of H, and we choose (A_0, P_0) as above. In Example 2, one takes $Q := P_0$ and L equal to the centralizer M_0 of A_0.

Lemma 5.2. *Let K be a compact open subgroup of G. There exists $\varepsilon > 0$ such that for all $\alpha \in \Delta \setminus \Delta_L$ and for all $a \in A_0$, the inequality $|\alpha(a)|_F < \varepsilon$ implies that the restriction of ψ to $U_\alpha \cap a^{-1}Ka$ is nontrivial.*

Proof. One has a filtration of U_α (see [18]), $(U_{\alpha,n})_{n\in\mathbb{Z}}$, by compact open subgroups such that:

(5.1)
1) $U_{\alpha,n} \subset U_{\alpha,n+1}, n \in \mathbb{Z}$, $\bigcup_{n\in\mathbb{Z}} U_{\alpha,n} = U_\alpha$.
2) $(U_{\alpha,n})_{n\in\mathbb{Z}}$ is a basis of neighborhoods of e in U_α.
3) There exists $C > 0$ such that if $a \in A_o$ satisfies $|\alpha(a)|_F > C$ one has $U_{\alpha,n+1} \subset aU_{\alpha,n}a^{-1}$, $n \in \mathbb{Z}$.

There exists $n, n' \in \mathbb{N}$ such that $n < n'$, K contains $U_{\alpha,n}$ and there exists $x \in U_{\alpha,n'}$ and $x \notin \operatorname{Ker} \psi$. From (5.1) 3), one sees that the lemma holds with $\varepsilon = C^{n-n'}$. □

Lemma 5.3. *Let K and ε be as in the previous lemma. If $f \in C^\infty(G/H, \psi)$ is K-invariant, one has:*

(5.2) $f(a) = 0$ *if $a \in A_0$ satisfies $|\alpha(a)|_F < \varepsilon$ for some $\alpha \in \Delta \setminus \Delta_L$.*

Proof. Assume $a \in A_0$ satisfies $|\alpha(a)|_F < \varepsilon$ for some $\alpha \in \Delta \setminus \Delta_L$. Then, from the previous lemma, let us choose $x \in U_\alpha$ with $x = a^{-1}ka$ for some $k \in K$ and with $\psi(x)$ different from one. Then $ax = ka$ implies

$$\psi(x)f(a) = f(a).$$

The lemma follows. □

For Example 2, we define $A_\emptyset := A_0$, $\Delta' := \emptyset$. In that case, one also has $\Delta_L = \emptyset$. For Example 3, let Δ' be the set of simple roots of $A_\emptyset$ in the Lie algebra of $P'_\emptyset$.

Let $\varepsilon, C > 0$ and let $\Lambda^+(A_\emptyset, C, \varepsilon)$ be equal to

$$\{a \in \Lambda(A_\emptyset) \mid 1 \le |a^\alpha|_F \le C \text{ if } \alpha \in \Delta' \text{ and } \varepsilon \le |a^\alpha|_F \le C \text{ if } \alpha \in \Delta \setminus \Delta_L\}.$$

Lemma 5.4. *Let $\Lambda(A_\emptyset)_G$ be the intersection of $\Lambda(A_\emptyset)$ with the center of G. It is a subgroup of $\Lambda(A_G)$ and $\Lambda^+(A_\emptyset, C, \varepsilon)$ is a finite union of orbits of $\Lambda(A_\emptyset)_G$.*

Proof. Let Δ'' be the union of Δ' and $\Delta \setminus \Delta_L$. One considers the morphism $\Lambda(A_\emptyset) \mapsto \mathbb{R}^{\Delta''}$, given by $a \to (\operatorname{Log} |\alpha(a)|_F)$. As in the proof of Lemma 4.2, one sees that an element a of the kernel is such that for all $\alpha \in \Delta''$, $\alpha(a) = 1$. But, as $a \in A_\emptyset$, this implies easily that $\alpha(a) = 1, \alpha \in \Delta$. Thus, as in the proof of Lemma 4.2, a is an element of the center of G. The rest of the lemma is proved like Lemma 4.2. □

For Example 3, let $\mathcal{W}^L$ be as in section 4.4, (see (4.11)) for the pair (L, σ'). Denote by $\Lambda_L^+(A_\emptyset)$ the set of $P'_\emptyset$-dominant elements of $\Lambda(A_\emptyset)$. One deduces from (4.11) and from the equality $G = KLU_Q$, where K is a well-chosen maximal compact subgroup of G, that there exists a compact subset Ω of G, such that

$$G = \bigcup_{y \in \overline{\mathcal{W}}^L} \Omega \Lambda_L^+(A_\emptyset) y^{-1} H. \tag{5.3}$$

For Example 2, there exists a compact subset Ω such that

$$G = \Omega \Lambda(A_0) U_0. \tag{5.4}$$

We introduce some more notation.

Example 3: If $\Theta \subset \Delta$, let $P_{0,\Theta}$ be the parabolic subgroup of G containing P_0 and which has the centalizer of $\bigcap_{\alpha \in \Theta} \mathrm{Ker}\ \alpha$ as Levi subgroup. Let $P'_0 = P_0 \cap L$. This is a minimal parabolic subgroup of L. Similarly we define $P'_{0,\Theta} \subset L$, for $\Theta \subset \Delta_L$. One has

$$P_{0,\Theta} \cap L = P'_{0,\Theta \cap \Delta_L}.$$

Now, if $\Theta' \subset \Delta'$, one defines in a similar way a σ'-parabolic subgroup of L, which contains $P'_\emptyset$, $P'_{\Theta'}$. Now we define a finite set of parabolic subgroups of G.

In Example 3, if $\alpha \in \Delta'$, let $P^\alpha = P_{0,\Theta}$ be such that $\Delta - \Delta_L \subset \Theta$ and such that $P^\alpha \cap L = P'_{\Delta' \setminus \{\alpha\}}$.

If $\alpha \in \Delta \setminus \Delta_L$, one defines $P^\alpha = P_{0,\Delta \setminus \{\alpha\}}$.

We define $\Delta'' = (\Delta \setminus \Delta_L) \cup \Delta'$. On checks easily that, for $C > 1$:

$$\Lambda(A_\emptyset) \cap A_0^+(P^\alpha, > C) = \{a \in \Lambda(A_\emptyset) \mid |a^\alpha| > C\},\ \alpha \in \Delta''. \tag{5.5}$$

If R is a parabolic subgroup of G which contains A_0, R^- will denote its opposite parabolic subgroup which contains A_0. If $\alpha \in \Delta''$, it is easy to see that the intersection of $(P^\alpha)^-$ with L is a σ'-parabolic subgroup of L and that $(P^\alpha)^- H$ is open. As P_0 contains U_Q, one sees that if P is a parabolic subgroup of G which contains P_0, one has $P \cap H = U_Q(H' \cap P)$. Taking into account the second part of (4.8), one sees that

$$\delta_P^{1/2} \text{ is trivial on } M \cap H, \text{ for } P = y.P^\alpha,\ \alpha \in \Delta'',\ y \in \mathcal{W}^L, \text{ where } M = P \cap P^-. \tag{5.6}$$

In Example 2, if α is in Δ, we set $P^\alpha = P_{0,\Delta \setminus \{\alpha\}}$.

We give here an ad hoc definition of cuspidality. We will not discuss here its naturality. The important fact is that it is strong enough to lead to the next theorem and that it allows us to prove Theorem 5.7 below.

Definition 5.5. For Example 2 (resp. Example 3), an H_ψ-fixed linear form, ξ, on a smooth G-module (resp., an element f of $C^\infty(G/H, \psi)$) is said to be H_ψ-cuspidal if and only if $j_{P^-}(\xi)$ (resp., for all $g \in G$, $(L_g f)_P$) is zero for all pairs (P, P^-) of opposite parabolic subgroups of G of the form P^α, $\alpha \in \Delta$ (resp. $(y.(P^\alpha)^-, y.P^\alpha), y \in \mathcal{W}^L, \alpha \in \Delta''$). We denote by $V_{cusp}^{*H_\psi}$ (resp., $C^\infty(G/H, \psi)_{cusp}$) the space of such linear forms (resp., functions).

Theorem 5.6. *(i) Let K be a compact open subgroup of G. Fix a finite family $\mathcal{X}$ of characters of $\Lambda(A_G)$. The space $C^\infty(G/H, \psi)_{cusp}^K(\mathcal{X})$ of elements of $C^\infty(G/H, \psi)_{cusp}^K$ which are of type $\mathcal{X}$ under the left regular action of $\Lambda(A_G)$ (see Definition 6.2) is finite dimensional.*

(ii) The elements of $C^\infty(G/H,\psi)^K_{cusp}$ have their support in a compact set modulo $\Lambda(A_G)$ depending only on K.

(iii) Let (π, V) be a smooth G-module of finite length. The space $V^{*H_\psi}_{cusp}$ is finite dimensional.

(iv) Let κ be a complex character of the center of G. There exists a finite number of irreducible representations (π_i, V_i) of G, with κ as central character, such that any irreducible representation (π, V) of G, having this central character, a nonzero vector fixed by K and with $V^{*H_\psi}_{cusp}$ nonzero is equivalent to one of the representations (π_i, V_i).

Proof. In view of Lemmas 5.3, 5.4 and (5.5), the proof is analogous to the proof of Theorem 4.4. □

Let $\alpha \in \Delta''$, $P = P^\alpha$, and $M = P \cap P^-$. Then one sees easily that $(M, Q \cap M, M \cap H, \psi_P)$ satisfies the same properties as (G, Q, H, ψ). It is here that it is useful to allow, in Example 3, that ψ restricted to H' is any smooth character of H'. Hence one proves the following theorem as in Theorem 4.5.

Theorem 5.7. *Let (π, V) be a smooth module of finite length. Then the dimension of V^{*H_ψ} is finite.*

6. Appendix

Lemma 6.1. *Let $\Lambda \subset \Lambda'$ be two lattices, i.e. finitely generated abelian free groups. Let σ be an involution of Λ' which preserves Λ. We define two lattices:*

$$\Lambda^+ = \{x \in \Lambda \mid \sigma(x) = x\}, \qquad \Lambda^- = \{x \in \Lambda \mid \sigma(x) = -x\}.$$

Let $X(\Lambda) = Hom(\Lambda, \mathbb{C}^*)$ be the group of complex characters of Λ. It is a complex torus. Let $X(\Lambda)^-$ be the set of elements of $X(\Lambda)$ which are trivial on Λ^+. We define similar notation for Λ' by replacing Λ by Λ'. Let $X(\Lambda')^{-\sigma}$ be the group of σ-anti-invariant elements of $X(\Lambda')$. It is a real abelian Lie group. Let $X(\Lambda')_\sigma$ be its neutral component.

Then one has:

(i) The restriction of an element of $X(\Lambda')_\sigma$ to Λ belongs to $X(\Lambda)^-$.

(ii) One defines an action of $X(\Lambda')_\sigma$ on $X(\Lambda)^-$ by multiplication by the restriction to Λ of elements of $X(\Lambda')_\sigma$. This action has a finite number of orbits.

Proof. (i) Let χ' be an element of $X(\Lambda')_\sigma$ and $x \in \Lambda^+$. Then, from the anti-invariance of χ', one has $\chi'(x) = \chi'(x)^{-1}$; hence it is equal to 1 or -1. By the connectivity of $X(\Lambda')_\sigma$, one sees that it is equal to 1. This proves (i).

(ii) Let $L = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $L' = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$. Then σ induces a linear involution of L and L'. One has

$$L = L^+ \oplus L^-, \tag{6.1}$$

where $L^+ = \Lambda^+ \otimes_{\mathbb{Z}} \mathbb{R}$ and $L^- = \Lambda^- \otimes_{\mathbb{Z}} \mathbb{R}$. If $\nu \in L^*_{\mathbb{C}}$ (resp., $\nu' \in L'^*_{\mathbb{C}}$), it determines a complex character χ_ν of Λ (resp., $\chi'_{\nu'}$ of Λ') defined by:

$$\chi_\nu(x) = e^{\nu(x)}, x \in \Lambda, \qquad \chi'_{\nu'}(x') = e^{\nu'(x')}, x' \in \Lambda'.$$

Let $\chi \in X(\Lambda)^-$. The restriction of χ to $\Lambda^+ \oplus \Lambda^-$ being trivial on Λ^+, one sees from (6.1), that there exists $\nu \in L^*_{\mathbb{C}}$, trivial on L^+, hence anti-invariant by σ, with:

(6.2) The restrictions of χ_ν and χ to $\Lambda^+ \oplus \Lambda^-$ are equal.

The anti-invariance by σ of ν allows us to extend it in a σ-anti-invariant linear form on L', ν'. One sees easily that $\chi'_{\nu'}$ is an element of $X(\Lambda')_\sigma$. This implies that, in an orbit of the action of $X(\Lambda')_\sigma$ on $X(\Lambda)^-$, there always exists an element of $X(\Lambda)^-$ trivial on $\Lambda^+ \oplus \Lambda^-$. But the equality:

$$2x = (x + \sigma(x)) + (x - \sigma(x)), \quad x \in \Lambda$$

shows that $2\Lambda \subset \Lambda^+ \oplus \Lambda^-$. Hence $\Lambda^+ \oplus \Lambda^-$ is of finite index in Λ and there is only a finite number of characters of Λ trivial on $\Lambda^+ \oplus \Lambda^-$. The lemma follows. $\square$

Definition 6.2. Let Λ be a lattice and let $\mathcal{X} = (\chi_1, \dots, \chi_n)$ be a finite family of characters of Λ. Let (π, V) be a representation of Λ. A vector v in V is said to be of type $\mathcal{X}$ if

$$(\pi(a) - \chi_1(a)) \cdots (\pi(a) - \chi_n(a))v = 0, \quad a \in \Lambda.$$

The representation itself is said to be of type $\mathcal{X}$ if any vector in V is of type $\mathcal{X}$.

A function f on Λ is said to be of type $\mathcal{X}$ if it is of type $\mathcal{X}$ for the left regular representation of Λ. We denote by $F(\Lambda)_{\mathcal{X}}$ the vector space of functions of type $\mathcal{X}$.

A function of type $\mathcal{X}$ is Λ-finite under left translations (see e.g [10, Lemma 14]). Reciprocally a Λ-finite function is of type $\mathcal{X}$ for some $\mathcal{X}$. From loc. cit., one has:

(6.3) If $\mathcal{X}$ is given, there exists a finite set, for which the restriction of elements of $F(\Lambda)_{\mathcal{X}}$ is injective.

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