A NEW APPROACH TO CLASSIFICATION OF INTEGRAL QUADRATIC FORMS OVER DYADIC LOCAL FIELDS

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ABSTRACT. In 1963, O’Meara solved the classification problem for lattices over dyadic local fields in terms of Jordan decompositions. In this paper we translate his result in terms of good BONGs. BONGs (bases of norm generators) were introduced in 2003 as a new way of describing lattices over dyadic local fields. This result and the notions we introduce here are a first step towards a solution of the more difficult problem of representations of lattices over dyadic fields.

1. INTRODUCTION

Since the main result of this paper is given in terms of BONGs, which were introduced in [1], we now give a reminder of some of the definitions and results in that paper which we will use here.

Throughout this paper \( F \) is a dyadic local field, \( \mathcal{O} \) the ring of integers, \( p \) the prime ideal, \( \mathcal{O}^\times := \mathcal{O} \setminus p \) the group of units, \( e := \text{ord}_2 \) and \( \pi \) is a fixed prime element. For \( a \in \hat{F} \) we denote its quadratic defect by \( \delta(a) \) and let \( \Delta = 1 - 4\rho \) be a fixed unit with \( \delta(\Delta) = 4\mathcal{O} \).

We denote by \( d : \hat{F}/\hat{F}^2 \longrightarrow \mathbb{N} \cup \{\infty\} \) the order of the “relative quadratic defect” \( d(a) = \text{ord}_2^{-1}\delta(a) \). If \( a = \pi^k \varepsilon \), with \( \varepsilon \in \mathcal{O}^\times \), then \( d(a) = 0 \) if \( R \) is odd and \( d(a) = d(\varepsilon) = \text{ord}_2 \delta(\varepsilon) \) if \( R \) is even. Thus \( d(\hat{F}) = \{0, 1, 3, \ldots, 2e - 1, 2e, \infty\} \). This function satisfies the domination principle \( d(ab) \geq \min\{d(a), d(b)\} \).

If \( \alpha \) is a positive integer then \( (1 + p^\alpha)\hat{F}^2 = \{a \in \hat{F} \mid d(a) \geq \alpha\} \) and \( (1 + p^\alpha)\mathcal{O}^\times \times \mathcal{O} = \{a \in \hat{F} \mid d(a) \geq \alpha\} \). For convenience we set \( (1 + p^\alpha)\hat{F}^2 := \{a \in \hat{F} \mid d(a) \geq \alpha\} \) and \( (1 + p^\alpha)\mathcal{O}^\times := \{a \in \mathcal{O}^\times \mid d(a) \geq \alpha\} \) for any \( \alpha \in \mathbb{R} \cup \{\infty\} \). Thus \( (1 + p^\alpha)\hat{F}^2 = \hat{F}^2 \) for \( \alpha > 2e \) and \( (1 + p^\alpha)\hat{F}^2 = \hat{F} \) for \( \alpha \leq 0 \). If \( d \) is the smallest element in \( d(\hat{F}) \) s.t. \( \alpha \leq d \) then \( (1 + p^\alpha)\hat{F}^2 = (1 + p^d)\hat{F}^2 \).

We denote by \( (\cdot, \cdot)_p : \hat{F}/\hat{F}^2 \times \hat{F}/\hat{F}^2 \longrightarrow \{\pm 1\} \) the Hilbert symbol, which is a non-degenerate bilinear symmetric form.

If \( a \in \hat{F} \), we denote by \( N(a) \) the norm group \( N(F(\sqrt{\pi})/F) = \{b \in \hat{F} \mid (a, b)_p = 1\} \). If \( b \in \hat{F} \) and \( d(a) + d(b) > 2e \) then \( (a, b)_p = 1 \). However if \( \alpha \notin \hat{F}^2 \) then there is \( b \in \hat{F} \) with \( d(b) = 2e - d(a) \) s.t. \( (a, b)_p = -1 \). (For \( d(a) \) odd this is just [3] Lemma 3]. If \( d(a) = 2e \) and \( b \in \hat{F} \) is arbitrary with \( d(b) = 0 \) then \( a \in \Delta \hat{F}^2 \) and \( \text{ord}_2 b \) is odd.

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An element \(x\) of a lattice \(L\) is called a norm generator of \(L\) if \(nL = Q(x)O\). A sequence \(x_1, \ldots, x_n\) of vectors in \(FL\) is called a basis of norm generators (BONG) for \(L\) if \(x_1\) is a norm generator for \(L\) and \(x_2, \ldots, x_n\) is a BONG for \(pr_{x_1}L\). A BONG uniquely determines a lattice, so if \(x_1, \ldots, x_n\) is a BONG for \(L\), we will write \(L = \langle x_1, \ldots, x_n \rangle\). If moreover \(Q(x_i) = a_i\) we say that \(L \cong \langle a_1, \ldots, a_n \rangle\) relative to the BONG \(x_1, \ldots, x_n\). If \(L \cong \langle a_1, \ldots, a_n \rangle\) then \(\det L = a_1 \cdots a_n\).

If \(x_1, \ldots, x_n\) are mutually orthogonal vectors with \(Q(x_i) = a_i\), \(L = Ox_1 \perp \cdots \perp Ox_n\) and \(V = Fx_1 \perp \cdots \perp Fx_n\) then we say that \(L \cong \langle a_1, \ldots, a_n \rangle\) and \(V \cong [a_1, \ldots, a_n]\) relative to the basis \(x_1, \ldots, x_n\).

If \(L\) is binary with \(nL = aO\), we denote by \(a(L) := \det L a^{-2}\) and by \(R(L) := \text{ord }\text{vol }L - 2\text{ord }nL = \text{ord }a(L)\). \(a(L) \in \hat{F}/\hat{O}^{\times 2}\) is an invariant of \(L\) and it determines the class of \(L\) up to scaling. If \(L \cong a, \beta \gg\) then \(a(L) = \frac{\beta}{a}\).

We denote by \(A = A_F \subset \hat{F}/\hat{O}^{\times 2}\) the set of all possible values of \(a(L)\), where \(L\) is an arbitrary binary lattice. We have \(A = \{a \in \frac{1}{2}O \mid a \neq 0, \exists (-a) \subseteq O\}\). If \(a = R\) and \(d(-a) = d\), then \(a \in \frac{1}{4}O\) means \(R \geq -2e\), while \(d(-a) \subseteq O\) means \(R + d = \text{ord }(-a) \geq 0\).

If \(a(L) = a = \pi^R\) with \(d(-a) = d\) then:

- \(L\) is nonmodular, proper modular or improper modular iff \(R > 0\), \(R = 0\), resp. \(R < 0\).
- \(R\) is odd then \(R > 0\).

The inequality \(R + 2e > 0\) becomes equality iff \(a \in -\frac{1}{4}O^{\times 2}\) or \(a \in -\frac{1}{2}O^{\times 2}\). We have \(a(L) = -\frac{1}{4}\), resp. \(a(L) = -\frac{1}{2}\), when \(L \cong \pi^r A(0,0)\), resp. \(\pi^r A(2, 2\rho)\), for some integer \(r\).

The inequality \(R + d \geq 0\) becomes equality iff \(a \in -\frac{1}{4}O^{\times 2}\).

A special type of BONG is the so-called “good BONG”. If \(L \cong \langle a_1, \ldots, a_n \rangle\) relative to some BONG \(x_1, \ldots, x_n\) and \(ord a_i = R_i\) we say that the BONG \(x_1, \ldots, x_n\) is good if \(R_i \leq R_{i+2}\) for any \(1 \leq i \leq n - 2\).

**Remark.** The condition \(R_i \leq R_{i+2}\) for \(1 \leq i \leq n - 2\) is equivalent to the condition that the sequence \((R_i + R_{i+1})\) is increasing.

A set \(x_1, \ldots, x_n\) of orthogonal vectors with \(Q(x_i) = a_i\) and \(ord a_i = R_i\) is a good BONG for some lattice iff \(R_i \leq R_{i+2}\) for all \(1 \leq i \leq n - 2\) and \(a_{i+1}/a_i \in A\) for all \(1 \leq i \leq n - 1\). The condition \(a_{i+1}/a_i \in A\) is equivalent to \(R_{i+1} - R_i + 2e \geq 0\) and \(R_{i+1} - R_i + d(-a_i a_{i+1}) \geq 0\). As consequences of \(a_{i+1}/a_i \in A\), if \(R_{i+1} - R_i\) is odd then it is positive, if \(R_{i+1} - R_i = -2e\) then \(a_{i+1}/a_i \in -\frac{1}{4}O^{\times 2}\) or \(-\frac{1}{2}O^{\times 2}\) and if \(R_{i+1} - R_i + d(-a_i a_{i+1}) = 0\) then \(a_{i+1}/a_i \in -\frac{1}{4}O^{\times 2}\).

The good BONGs enjoy some properties similar to those of orthogonal bases. If \(L \cong \langle a_1, \ldots, a_n \rangle\) relative to some good BONG \(x_1, \ldots, x_n\) and \(ord a_i = R_i\) then \(L^2 \cong \langle a_1^{-1}, \ldots, a_n^{-1} \rangle\) relative to the good BONG \(x_1^2, \ldots, x_1^2\), where \(x_1^2 = Q(x)^{-1}x_i\). Also if for some \(1 \leq i \leq j \leq n\) we have \(\langle x_{i}, \ldots, x_{j} \rangle \cong \langle b_1, \ldots, b_j \rangle\) relative to some other good BONG \(y_1, \ldots, y_j\) then \(L \cong \langle a_1, \ldots, a_{i-1}, b_1, \ldots, b_j, a_{i+1}, \ldots, a_n \rangle\) relative to the good BONG \(x_1, \ldots, x_{i-1}, y_1, \ldots, y_j, x_{i+1}, \ldots, x_n\). There are some differences though from the orthogonal bases. E.g. the relation \(L = \langle x_1, \ldots, x_i \rangle \perp \langle x_{i+1}, \ldots, x_n \rangle\) holds iff \(R_i \leq R_{i+1}\).

The orders \(R_i = ord a_i\) are independent of the choice of the good BONGs and they are in 1-1 correspondence with the invariants \(t, \dim L_k, s_k := sL_k\) and \(nL_k^a\),
where $L = L_1 \perp \ldots \perp L_i$ is a Jordan splitting. More precisely, if $s_k = p^{u_k}$, $nL^{s_k} = p^{v_k}$ and $n_k = \sum_{1 \leq k} \dim L_k$ then the sequence $R_{n_{i-1}+1}, \ldots, R_{n_k}$ is $r_k, \ldots, r_k$ if $L_k$ is proper (i.e. if $r_k = u_k$), and it is $u_k, 2r_k - u_k, \ldots, u_k, 2r_k - u_k$ otherwise; see [1] Lemma 4.7.

The good BONGs are closely connected with the maximal norm splittings. A splitting $L = L_1 \perp \ldots \perp L_i$ is called a maximal norm splitting if $sL_1 \supseteq \ldots \supseteq sL_i$ and $\dim L_i \leq 2$, $L_i$ is modular and $nL_i = nL^{\perp L_i}$ for all $1 \leq i \leq t$. Condition $nL_i = nL^{\perp L_i}$ is equivalent to $nL_1 \supseteq \ldots \supseteq nL_i$ and $nL_i^t \subseteq \ldots \subseteq nL_i^t$. If we put together the BONGs of the components $L_1, \ldots, L_i$ of a maximal norm splitting we get a good BONG for $L$. Conversely any good BONG of a lattice can be obtained by putting together some BONGs of the components of some maximal norm splitting. Moreover, the splitting can be chosen such that all binary components are improper modular. An explicit algorithm for finding a maximal norm splitting and, hence, a good BONG of a lattice is provided in [2, Section 7].

2. THE INVARIANTS $\alpha_i$

Let $L$ be a lattice over the dyadic field $F$. Let $L \cong \langle a_1, \ldots, a_n \rangle$ relative to a good BONG and let $R_i := \text{ord} a_i$. Also let $L = L_1 \perp \ldots \perp L_i$ be a Jordan decomposition. We keep the notation of [4], $s_k := sL_k$, $g_k := gL^{s_k}$, $w_k := wL^{s_k}$ but, in order to avoid confusion, we write $a_k$ for O’Meara’s $a_k$. Also we denote $r_k = \text{ord} s_k$, $u_k = \text{ord} a_k = \text{ord} nL^{s_k}$. Associated to our splitting we have the Jordan chain $L_{(1)} \subset \ldots \subset L_{(i)}$ and the inverse Jordan chain $L_{(i)}^* \supset \ldots \supset L_{(1)}^*$, where $L_{(k)} := L_1 \perp \ldots \perp L_k$ and $L_{(k)}^* := L_k \perp \ldots \perp L_i$.

Since the $R_i$'s are invariants of $L$ we will write $R_i = R_i(L)$.

**Definition 1.** For any $1 \leq i \leq n - 1$ we define $\alpha_i = \alpha_i(L)$ by:

$$\alpha_i := \min \{(R_{i+1} - R_i)/2 + e \cup \{R_{i+1} - R_j + d(-a_ja_{j+1}) | 1 \leq j \leq t\}$$

$$\cup \{R_{j+1} - R_i + d(-a_ja_{j+1}) | i \leq j < n\}).$$

Apparently $\alpha_i(L)$ defined this way depends on the choice of the good BONG. We will show later that, in fact, it depends only on $L$. For the time being we will mean $\alpha_i(L)$ with respect to a given good BONG. We now give some properties of the $\alpha_i$'s.

**Lemma 2.1.** If $k \leq i < l$ then, in the set defining $\alpha_i$, we can replace $(R_{i+1} - R_i)/2 + e$ and all the terms corresponding to indices $k \leq j < l$ by $\alpha_{i-k+1}(\langle a_k, \ldots, a_l \rangle)$. In particular, $\alpha_i \leq \alpha_{i-k+1}(\langle a_k, \ldots, a_l \rangle)$.

**Proof.** By definition $\alpha_{i-k+1}(\langle a_k, \ldots, a_l \rangle) = \min \{(R_{i+1} - R_i)/2 + e \cup \{R_{i+1} - R_j + d(-a_ja_{j+1}) | k \leq j \leq l\} \cup \{R_{j+1} - R_i + d(-a_ja_{j+1}) | i \leq j < l\})$, hence the conclusion.

**Lemma 2.2.** The sequence $(R_i + \alpha_i)$ is increasing and the sequence $(-R_{i+1} + \alpha_i)$ is decreasing.

**Proof.** Let $1 \leq i \leq h \leq n - 1$. We have $R_i + R_{i+1} \leq R_h + R_{h+1}$. From Definition 1 we get $R_i + \alpha_i = \min \{(R_{i+1} + R_i)/2 + e \cup \{R_{i+1} - R_j + d(-a_ja_{j+1}) | 1 \leq j \leq i\} \cup \{R_{j+1} + d(-a_ja_{j+1}) | i \leq j < n\})$ and $-R_{i+1} + \alpha_i = \min \{(-R_{i+1} + R_i)/2 + e \cup \{-R_j + d(-a_ja_{j+1}) | 1 \leq j \leq i\} \cup \{-R_{j+1} - R_i + d(-a_ja_{j+1}) | i \leq j < n\})$, and similarly for $R_h + \alpha_h$ and $-R_{h+1} + \alpha_h$. In order to prove that $R_i + \alpha_i \leq R_h + \alpha_h$
we show that the elements in the set that has $R_i + \alpha_i$ as its minimum are less than or equal to the corresponding elements for $R_h + \alpha_h$. The same holds for $-R_{i+1} + \alpha_i \geq -R_{h+1} + \alpha_h$.

The proof is straightforward and uses the fact that $R_i + R_{i+1}$ is an increasing sequence. For terms involving $d(-a_ja_{j+1})$ we consider the cases $j \leq i$, $i \leq j \leq h$ and $h \leq j$ and use the inequalities among $R_i + R_{i+1}$, $R_j + R_{j+1}$ and $R_h + R_{h+1}$ that occur in each case.

**Corollary 2.3.** Suppose that $1 \leq i \leq j \leq n - 1$ and $R_i + R_{i+1} = R_j + R_{j+1}$. Then:

(i) $R_i + \alpha_i = \ldots = R_j + \alpha_j$ and $-R_{i+1} + \alpha_i = \ldots = -R_{j+1} + \alpha_j$.

(ii) $R_k = R_i$ for any $k, l \in [i, j + 1]$ of the same parity and $\alpha_k = \alpha_l$ for any $k, l \in [i, j]$ of the same parity.

(iii) If $\alpha_k = (R_{k+1} - R_k)/2 + e$ for some $i \leq k \leq j$ then $\alpha_k = (R_{k+1} - R_k)/2 + e$ for all $i \leq k \leq j$.

In the particular case when $j = i + 1$ we get the following statement:

If $1 \leq i \leq n - 2$ and $R_i = R_{i+2}$ then $R_i + \alpha_i = R_{i+1} + \alpha_{i+1}$, $-R_{i+2} + \alpha_i = -R_{i+1} + \alpha_{i+1}$ and $\alpha_i = (R_{i+1} - R_i)/2 + e$ is equivalent to $\alpha_{i+1} = (R_{i+2} - R_{i+1})/2 + e$.

**Proof.** For (i) we note that $R_i + R_{i+1} = (R_i + \alpha_i) - (-R_{i+1} + \alpha_i)$ and $R_j + R_{j+1} = (R_j + \alpha_j) - (-R_{j+1} + \alpha_j)$ and use Lemma 2.2. By using the fact that $R_k + R_{k+1}$ is an increasing sequence we get $R_i + R_{i+1} = R_{i+1} + R_{i+2} = \ldots = R_j + R_{j+1}$, which is equivalent to the first part of (ii). For the second part of (ii) use (i). Finally (iii) follows from $R_i + \alpha_i = \ldots = R_j + \alpha_j$, $R_i + R_{i+1} = \ldots = R_j + R_{j+1}$ and the fact that $\alpha_k = (R_{k+1} - R_k)/2 + e$ is equivalent to $R_k + \alpha_k = (R_k + R_{k+1})/2 + e$.

**Lemma 2.4.** Suppose that $1 \leq i < n$ and $1 \leq k \leq h < l \leq n$.

Then:

(i) If $h \leq i$ then all terms in the definition of $\alpha_i$ corresponding to indices $k \leq j \leq h$ can be replaced by $R_{i+1} - R_{h+1} + \alpha_{h-k+1}(< a_k, \ldots, a_l >)$. In particular, all terms with $1 \leq j \leq h$ can be replaced by $R_{i+1} - R_{h+1} + \alpha_h$.

(ii) If $i \leq h$ then all terms in the definition of $\alpha_i$ corresponding to indices $h \leq j < l$ can be replaced by $R_h - R_i + \alpha_{h-k+1}(< a_k, \ldots, a_l >)$. In particular, all terms with $h \leq j < n$ can be replaced by $R_h - R_i + \alpha_h$.

**Proof.** By Lemma 2.1 we have $\alpha_{h-k+1}(< a_k, \ldots, a_l >) \geq \alpha_h$.

(i) By Lemma 2.2 we have $\alpha_i \leq R_{i+1} - R_{h+1} + \alpha_h \leq R_{i+1} - R_{h+1} + \alpha_{h-k+1}(< a_k, \ldots, a_l >)$. If $k \leq j \leq h$ then $\alpha_{h-k+1}(< a_k, \ldots, a_l >) \leq R_{h+1} - R_j + d(\alpha_{j+1})$. Therefore if we add $R_{i+1} - R_{h+1} + \alpha_{h-k+1}(< a_k, \ldots, a_l >)$ to the set that defines $\alpha_i$ and remove any one of $R_{i+1} - R_j + d(-a_{j+1})$ with $k \leq j \leq h$ then $\alpha_i$ does not change.

(ii) By Lemma 2.2 we have $\alpha_i \leq R_h - R_i + \alpha_h \leq R_h - R_i + \alpha_{h-k+1}(< a_k, \ldots, a_l >)$. If $h \leq l$ then $\alpha_{h-k+1}(< a_k, \ldots, a_l >) \leq R_{j+1} - R_h + d(-a_{j+1})$ so $R_h - R_i + \alpha_{h-k+1}(< a_k, \ldots, a_l >) \leq R_{j+1} - R_i + d(-a_{j+1})$. Thus if we add $R_h - R_i + \alpha_{h-k+1}(< a_k, \ldots, a_l >)$ to the set that defines $\alpha_i$ and remove any one of $R_{i+1} - R_j + d(-a_{j+1})$ with $h \leq j < l$ then $\alpha_i$ does not change.

If we take $k = 1$ and $l = n$ then $\alpha_{h-k+1}(< a_k, \ldots, a_l >)$ becomes $\alpha_h( < a_1, \ldots, a_n >) = \alpha_h(L) = \alpha_h$ so we get the second claims of (i) and (ii).

**Corollary 2.5.** For any $1 \leq i \leq n - 1$ we have:

(i) $\alpha_i = \min \{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_{i+1}), R_{i+1} - R_i + \alpha_{i-1}, R_{i+1} - R_i + \alpha_{i+1}\}$.
(ii) \( \alpha_i = \min \{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + \alpha_{i-1}(\langle a_1, \ldots, a_i \rangle), R_{i+1} - R_i + \alpha_i(\langle a_{i+1}, \ldots, a_n \rangle)\}. \)

(The terms that do not make sense, i.e. \( R_{i+1} - R_i + \alpha_{i-1} \) and \( R_{i+1} - R_i + \alpha_i(\langle a_1, \ldots, a_i \rangle) \) when \( i = 1 \), or \( R_{i+1} - R_i + \alpha_{i+1} \) and \( R_{i+1} - R_i + \alpha_{i}(\langle a_{i+1}, \ldots, a_n \rangle) \) when \( i = n - 1 \), are ignored.)

Proof. (i) By Lemma 2.4 (i), resp. (ii), in the set defining \( \alpha_i \), \( R_{i+1} - R_i + \alpha_{i-1} \) can replace all the terms \( R_{i+1} - R_j + d(-a_ja_{j+1}) \) with \( 1 \leq j \leq i - 1 \), while \( R_{i+1} - R_i + \alpha_{i+1} \) replaces all \( R_{j+1} - R_i + d(-a_ja_{j+1}) \) with \( i + 1 \leq j < n \). Therefore \( \alpha_i = \min \{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + \alpha_{i-1}, R_{i+1} - R_i + \alpha_{i+1}\} \).

(ii) Same as (i) but this time the terms corresponding to \( 1 \leq j \leq i - 1 \) are replaced by \( R_{i+1} - R_i + \alpha_{i-1}(\langle a_1, \ldots, a_i \rangle) \) and those corresponding to \( i + 1 \leq j < n \) by \( R_{i+1} - R_i + \alpha_{i}(\langle a_{i+1}, \ldots, a_n \rangle) \).

Remark 2.6. We have \( L^2 \cong \langle a_1^2, \ldots, a_n^2 \rangle \) with \( a_i^2 = a_{ni+1-i}^{-1} \) and \( R_i^2 := \text{ord} a_i^2 = -R_{ni+1-i} \). One can easily see that \( \alpha_i^2 := \alpha_i(L^2) = \alpha_{n-i} \). Also, the \( \alpha_i 's \) are invariant to scaling.

Lemma 2.7. If \( 1 \leq i \leq n - 1 \) then:

(i) \( \alpha_i \geq 0 \) with equality iff \( R_{i+1} - R_i = -2e \).

(ii) If \( R_{i+1} - R_i \geq 2e \) then \( \alpha_i = (R_{i+1} - R_i)/2 + e \).

(iii) If \( R_{i+1} - R_i \leq 2e \) then \( \alpha_i \geq R_{i+1} - R_i \) with equality iff \( R_{i+1} - R_i = 2e \) or it is odd.

(iv) \( \alpha_i \) is an odd integer unless \( \alpha_i = (R_{i+1} - R_i)/2 + e \).

Proof. We use induction on \( n \). For \( n = 1 \) our lemma is vacuous.

For the induction step let \( 1 \leq i \leq n - 1 \) and let \( L^\prime = \langle a_1, \ldots, a_i \rangle \) and \( L^n = \langle a_1, \ldots, a_n \rangle \). By Corollary 2.5(ii) we have \( \alpha_i = \min \{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + \alpha, R_{i+1} - R_i + \beta\} \), where \( \alpha = \alpha_{i-1}(L') \) and \( \beta = \alpha_{i}(L^n) \). (We ignore \( \alpha \) and \( \beta \) whenever they are not defined.) By the induction hypothesis \( \alpha, \beta \) satisfy (i)-(iv) of the lemma.

We have \( (R_{i+1} - R_i)/2 + e \geq 0 \) with equality iff \( R_{i+1} - R_i = -2e \) and \( R_{i+1} - R_i + d(-a_ia_{i+1}) \geq 0 \) with equality iff \( a_{i+1}/a_i \in O^2 \), which implies \( R_{i+1} - R_i = -2e \). If \( R_{i+2} - R_{i+1} > 2e \) then \( \beta = (R_{i+2} - R_{i+1})/2 + e \geq 2e \) so \( R_{i+1} - R_i + \beta > R_{i+1} - R_i + 2e \geq 0 \). Similarly with \( R_{i+1} - R_i + \alpha \) if \( R_{i+1} - R_i > 2e \). If \( R_{i+2} - R_{i+1} \leq 2e \) then, by the induction hypothesis, \( \beta \geq R_{i+2} - R_{i+1} \) with equality iff \( R_{i+2} - R_{i+1} \) is odd or it is \( 2e \). Thus \( R_{i+1} - R_i + \beta \geq R_{i+2} - R_i \geq 0 \) with equality iff \( R_{i+2} = R_{i+1} \) and \( R_{i+2} - R_{i+1} \) is odd or \( 2e \). Suppose this happens. If \( R_{i+2} - R_{i+1} = 2e \) then \( R_{i+1} - R_i = R_{i+1} - R_{i+2} = -2e \). If \( R_{i+2} - R_{i+1} \) is odd then so is \( R_{i+1} - R_i = R_{i+1} - R_{i+2} \) so both must be positive. But this is impossible. Similar results hold for \( R_{i+1} - R_i + \alpha \) when \( R_i - R_{i-1} \leq 2e \). Thus we have (i).

If \( R_{i+1} - R_i \geq 2e \) then \( \alpha_i, \beta \geq 0 \) so \( R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + \alpha, R_{i+1} - R_i + \beta \geq R_{i+1} - R_i \geq (R_{i+1} - R_i)/2 + e \). Hence \( \alpha_i = (R_{i+1} - R_i)/2 + e \) and we have (ii).

We now prove (iii). If \( R_{i+1} - R_i = 2e \) then (ii) implies that \( \alpha_i = (R_{i+1} - R_i)/2 + e = 2e \) so we are done. If \( R_{i+1} - R_i < 2e \) is odd then \( d(-a_ia_{i+1}) = 0 \) and \( \alpha, \beta \geq 0 \) so \( \alpha_i = \min \{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i\} = R_{i+1} - R_i \). Finally if \( R_{i+1} - R_i < 2e \) is even then \( ord a_{i+1} = R_i + R_{i+1} \) is even so \( d(-a_ia_{i+1}) > 0 \). Also \( R_i - R_{i-1}, R_{i+2} - R_{i+1} \geq R_i - R_{i+1} > -2e \) \( (R_{i-1} \leq R_{i+1} \) and \( R_i \leq R_{i+2} \) so by (i)
\[\alpha, \beta > 0.\] We have \(R_{i+1} - R_i + d(-a_i a_{i+1}) > R_{i+1} - R_i + \alpha + R_{i+1} - R_i + \beta > R_{i+1} - R_i.\]

Since also \((R_{i+1} - R_i)/2 + e > R_{i+1} - R_i\) (we have \(R_{i+1} - R_i < 2e\)) we get \(\alpha_i > (R_{i+1} - R_i)/2 + e > R_{i+1} - R_i.\)

We now prove (iv). If \(R_{i+1} - R_i \geq 2e\) then (ii) implies \(\alpha_i = (R_{i+1} - R_i)/2 + e\) so (iv) is vacuous. If \(R_{i+1} - R_i < 2e\) is odd then (iii) implies \(\alpha_i = R_{i+1} - R_i\) so \(\alpha_i\) is odd. If \(R_{i+1} - R_i < 2e\) is even then again \(\alpha_i = (R_{i+1} - R_i)/2 + e\).\]

\begin{proof}
(ii) \(\alpha_i = (R_{i+1} - R_i)/2 + e = 0\) or \(\alpha_i = (R_{i+1} - R_i)/2 + e \geq (R_{i+1} - R_i)/2 + e > \alpha_i.\)

We have \(R_{i+1} - R_i + 2e \geq 0\). Contradiction. If \(\alpha_i = R_{i+1} - R_i + \alpha\) then \(\alpha_i\) is odd unless \(\alpha_i\) is not odd, which would imply \(\alpha = (R_i - R_{i-1})/2 + e\). So \(\alpha_i = R_{i+1} - R_i + (R_{i-1} - R_i)/2 + e \geq (R_{i+1} - R_i)/2 + e > \alpha_i.\) (We have \(R_{i+1} \geq R_{i-1}\).) Contradiction. Similar results hold if \(\alpha_i = R_{i+1} - R_i + \beta\) since \(R_{i+1} - R_i + (R_{i+2} - R_{i+1})/2 + e \geq (R_{i+1} - R_i)/2 + e > \alpha_i.\) (We have \(R_{i+2} \geq R_{i+1}\).)
\end{proof}

\begin{corollary}
(i) \(\alpha_i \in \mathbb{Z}\) except when \(R_{i+1} - R_i\) is odd and \(e > 2e\).
(ii) \(\alpha_i \leq 2e, = 2e\) or \(> 2e\) if \(R_{i+1} - R_i\) is \(< 2e, = 2e\) or \(> 2e\) accordingly.
(iii) \(\alpha_i \in ([0,2e]\cap \mathbb{Z}) \cup ((2e,\infty) \cap \frac{1}{2}\mathbb{Z})\).
\end{corollary}

\begin{proof}
(i) If \(R_{i+1} - R_i > 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e\). If \(R_{i+1} - R_i\) is even then \(\alpha_i \in \mathbb{Z}\), while if it is odd then \(\alpha_i \in \frac{1}{2}\mathbb{Z}\). Suppose now that \(R_{i+1} - R_i \leq 2e\). If \(R_{i+1} - R_i \) is odd then \(\alpha_i = R_{i+1} - R_i \in \mathbb{Z}\). If \(R_{i+1} - R_i\) is even then either \(\alpha_i\) is an odd integer or \(\alpha_i = (R_{i+1} - R_i)/2 + e \in \mathbb{Z}\).

(ii) If \(R_{i+1} - R_i < 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e < 2e\). If \(R_{i+1} - R_i = 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e = 2e\). If \(R_{i+1} - R_i > 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e > 2e\).

(iii) We have \(\alpha_i \geq 0\). If \(\alpha_i \leq 2e\) then \(R_{i+1} - R_i \leq 2e\) so \(\alpha_i \in \mathbb{Z}\). If \(\alpha_i > 2e\) then \(R_{i+1} - R_i > 2e\) so \(\alpha_i = (R_{i+1} - R_i)/2 + e \in (2e,\infty) \cap \frac{1}{2}\mathbb{Z}\).
\end{proof}

\begin{corollary}
In each of the following cases, \(\alpha_i\) depends only on \(R_{i+1} - R_i\):
(i) \(\alpha_i = (R_{i+1} - R_i)/2 + e\)
(ii) \(\alpha_i \in \mathbb{Z}\) except when \(R_{i+1} - R_i\) is odd and \(e > 2e\).
(iii) \(\alpha_i \in ([0,2e]\cap \mathbb{Z}) \cup ((2e,\infty) \cap \frac{1}{2}\mathbb{Z})\).
\end{corollary}

\begin{proof}
(i) If \(R_{i+1} - R_i \geq 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e\) by Lemma 2.7(ii). If \(R_{i+1} - R_i = 2e\) then \(\alpha_i = 0 = (R_{i+1} - R_i)/2 + e\). If \(R_{i+1} - R_i > 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e \geq 1\) so \(\alpha_i = 1 = (R_{i+1} - R_i)/2 + e\).

(ii) We use Lemma 2.7(ii) and (iii). If \(R_{i+1} - R_i \geq 2e\) then \(\alpha_i = (R_{i+1} - R_i)/2 + e < R_{i+1} - R_i\). If \(R_{i+1} - R_i < 2e\) then \(\alpha_i = R_{i+1} - R_i < (R_{i+1} - R_i)/2 + e\). In both cases \(\alpha_i = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i\}\).
\end{proof}

\begin{lemma}
Let \(a\) be a norm generator of a lattice \(L\) and let \(w \geq 2s L\) be a fractional ideal. Then \(w = w L\) iff \(g L = a O^2 + w\) and we have either \(w = 2s L\) or \(\text{ord } a + \text{ord } w = 0\).
\end{lemma}

\begin{proof}
For the necessity, see [1, 93A]. For the sufficiency it is enough to prove that, given another fractional ideal \(w'\) satisfying the hypothesis of the lemma, we have \(w = w'\). Suppose that \(w \neq w'\). We may assume that \(w \supset w'\). Since \(w \supset w' \supset 2s L\) we must have that \(\text{ord } a + \text{ord } w = 0\). Let \(w = b O\). Then \(a + b \in a O^2 + w = g L = a O^2 + w'\). So \(a + b = a o^2 + b'\) for some \(a \in O\) and \(b' \in w' \subseteq w\). It follows that \(1 + b/a = a^2 + b'/a\), which implies that \(\text{ord } (1 + b/a) \subseteq b'/a \subseteq a^{-1}w\).
\end{proof}
On the other hand \( \operatorname{ord} b/a = \operatorname{ord} a^{-1}w \) is odd and, since \( bO = w \subseteq gL \subseteq aO \) and \( bO = w \supseteq 2sL \supseteq 4aO \), we have \( 4O \subset b/aO \subset O \). By [I; 63:5] we get 
\[ d(1 + b/a) = b/aO = a^{-1}w, \] 
a contradiction. 
\[ \square \]

**Lemma 2.11.** Let \( J_1, \ldots, J_n \) be lattices in the same quadratic space and let \( J = \sum J_k \). If \( a_k \) and \( a \) are norm generators for \( J_k \) and \( a \) and \( J \), respectively, then:
\[ gJ = \sum gJ_k + 2sJ \quad \text{and} \quad wJ = \sum wJ_k + \sum a^{-1}d(aa_k) + 2sJ. \]

**Proof.** We have \( gJ_k \subseteq gJ \) and \( 2sJ \subseteq gJ \) so \( gJ \supseteq \sum gJ_k + 2sJ \). For the reverse inclusion note that \( Q(J) \subseteq \sum Q(J_k) + 2sJ \). Thus \( gJ = Q(J) + 2sJ \subseteq \sum (Q(J_k) + 2sJ_k) + 2sJ = \sum gJ_k + 2sJ \).

We have \( aO^2 \subseteq gO \) and \( 2aO = 2nJ \subseteq gJ \) so \( gJ = aO^2 + 2aO + gJ = aO^2 + 2aO + \sum gJ_k + 2sJ = aO^2 + \sum a_kO^2 + \sum wJ_k + 2sJ \). But \( aO^2 + \sum a_kO^2 + 2aO = g((a, a_1, \ldots, a_n)) \). (We have \( s((a, a_1, \ldots, a_n)) = n((a, a_1, \ldots, a_n)) = aO \).) But \( w((a, a_1, \ldots, a_n)) = \sum \operatorname{ord} a_k(a_k) \). So \( gJ = g((a, a_1, \ldots, a_n)) + \sum wJ_k + 2sJ = aO^2 + \sum a^{-1}d(aa_k) + 2aO + \sum wJ_k + 2sJ = aO^2 + \sum a^{-1}d(aa_k) + 2sJ \). (Recall, \( aO \subseteq 2sJ \).) 

Let \( w = \sum wJ_k + \sum a^{-1}d(aa_k) + 2sJ \). We have \( gJ = aO^2 + 2sJ \) and \( 2sJ \subseteq w \). By Lemma 2.10 in order to prove that \( w = wJ \) we still need to prove that \( w = 2sJ \) or \( \operatorname{ord} a + \operatorname{ord} w \) is odd. If \( w \neq 2sJ \), i.e. \( w \supseteq 2sJ \), then \( w = a^{-1}d(aa_k) \) or \( w = wJ_k \) for some \( k \). Suppose that \( w = wJ_k \). We cannot have \( wJ_k = 2sJ_k \subseteq 2sJ \). So \( \operatorname{ord} a_k + \operatorname{ord} wJ_k \) is odd, which implies that \( \operatorname{ord} a + \operatorname{ord} wJ_k \) is odd unless \( \operatorname{ord} a_k \) is odd. But this would imply that \( a_kO = a^{-1}d(aa_k) \subseteq w = wJ_k \), so \( wJ_k = a_kO \), which contradicts the fact that \( \operatorname{ord} a_k + \operatorname{ord} wJ_k \) is odd. Finally if \( w = a^{-1}d(aa_k) \) then \( \operatorname{ord} a + \operatorname{ord} w = \operatorname{ord} d(aa_k) \) is odd unless \( a_k \in \Delta F^2 \). (If \( \alpha \in F \) has odd order then \( \operatorname{ord} \alpha = \alpha O \) has odd order. If \( \operatorname{ord} \alpha \) is even then \( \operatorname{ord} \alpha \neq \alpha O \) (mod 2) is even if \( \operatorname{ord} \alpha = 2e \), i.e. \( \alpha \in \Delta F^2 \).) But this implies that \( \operatorname{ord} a_k = 4a_kO \), i.e. \( w = a^{-1}d(aa_k) = 4a_kO \subseteq 2sJ \), a contradiction. 

\[ \square \]

**Lemma 2.12.** Suppose that \( nL_k = nL^2_k \), \( nL_{k+1} = nL^{2k+1} \) and \( a_k \) and \( a_{k+1} \) are norm generators for \( L_k \) and \( L_{k+1} \), respectively. If \( u_k + u_{k+1} \) is even, then
\[ f_k = s_k^{-2}d(aa_k a_{k+1}) + a_k s_k^{-2}wL^*_k + a_k wL^*_k + 2p(u_k + u_{k+1})/2 - r_k. \]

**Proof.** We have \( L^{2k} = s_k L^*_k \) and \( L^{2k+1} = s_k L^*_k \) and \( L^{2k+1} = s_k L^*_k \). Now \( L_{k+1} \subseteq L^{2k+1} \subseteq L^{2k+1} \) and \( L_k \subseteq L^*_k \). Thus \( a_k \) and \( a_{k+1} \) is a norm generator for \( L^*_k \) and \( L^*_k \). Let \( \pi \equiv 0 \pmod{2} \). Also \( \pi \equiv 0 \pmod{2} \) \( a_k \) is a norm generator for \( L^*_k \). 

By Lemma 2.11 we get \( w = a_k^{-1}d(aa_k a_{k+1}) + wL^*_k + wL^*_k + 2s_k = a_k^{-1}d(aa_k a_{k+1}) + s_k wL^*_k + wL^*_k + 2s_k = a_k^{-1}d(aa_k a_{k+1}) + s_k wL^*_k + wL^*_k + 2s_k = a_k^{-1}d(aa_k a_{k+1}) + s_k wL^*_k + wL^*_k + 2s_k. \)

By [I; 93:26] we have \( s_k L^*_k = d(aa_k a_{k+1}) + a_k wL^*_k + a_k wL^*_k + 2p(u_k + u_{k+1})/2 + r_k = d(aa_k a_{k+1}) + a_k^{-1}d(aa_k a_{k+1}) + a_k wL^*_k + a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k + 2a_k wL^*_k. \)

But \( a_k \) and \( a_{k+1} \) is a norm generator for \( L^*_k \). 

By [I; 93:25] \( u_k \leq u_{k+1} \), so \( u_k \leq u_{k+1} \) and \( u_k \leq 2u_k \geq u_{k+1} - 2u_{k+1} \). Thus \( a_k^{-1}d(aa_k a_{k+1}) + a_k a_k^{-1}d(aa_k a_{k+1}) + a_k a_k^{-1}d(aa_k a_{k+1}) + a_k a_k^{-1}d(aa_k a_{k+1}) + a_k a_k^{-1}d(aa_k a_{k+1}) + a_k a_k^{-1}d(aa_k a_{k+1}) \). Also \( a_k^2 wL^*_k \) \( \subseteq a_k wL^*_k \) for \( a_k wL^*_k \) and \( a_k wL^*_k \). 

\[ \square \]
Also \( \text{ord } a_k + s_k = u_k + r_k \geq (u_k + u_{k+1})/2 + r_k \) (we have \( u_{k+1} \leq u_k \)) and
\( \text{ord } a_k + s_k = u_k + r_k \geq (u_k + u_{k+1})/2 + r_k \) (we have \( u_k - 2r_k \geq u_{k+1} - 2r_{k+1} \)).
Hence \( 2a_k + s_k \leq 2p^{(u_k + u_{k+1})/2 + r_k} \).

By removing all unnecessary terms (which are included in others) we get
\( s_k^2 = d(a_k a_{k+1}) + a_k + s_k^2 u_k L(k) + a_k u_k L^*(k+1) + 2p^{(u_k + u_{k+1})/2 + r_k} \).
When we divide by \( s_k^2 \) we get the desired result.

Suppose \( L \equiv \langle a_1, \ldots, a_n \rangle \) relative to the good BONG \( x_1, \ldots, x_n \). Let \( L = L^1 \perp \ldots \perp L^m \) be a maximal norm splitting with all the binary components improper such that \( x_1, \ldots, x_n \) is obtained by putting together the BONGs of
\( L^1, \ldots, L^m \). We choose the Jordan decomposition \( L = L_1 \perp \ldots \perp L_k \) with components obtained by putting together the \( L^i \)'s of the same scale (see also the proof of [H, Lemma 4.7]). So the \( L^i \)'s with \( s L^i = s_k \) make a maximal norm splitting for \( L_k \), those with \( s L^j \supseteq s_k \) a maximal norm splitting for \( L_i \) and those with \( s L^j \subseteq s_k \) a maximal norm splitting for \( L^*(k+1) \). By putting together the BONGs of the components of these maximal norm splittings we get good BONGs for \( L_k \) and \( L^*(k+1) \).
It follows that \( L_k = \langle x_{n_k-1+1}, \ldots, x_{n_k} \rangle \), \( L_i = \langle x_1, \ldots, x_{n_i} \rangle \) and \( L^*(k+1) = \langle x_{n_{k+1}}, \ldots, x_{n_{k+1}} \rangle \).
Also \( nL_k = nL_k^* \).
For any \( L^i \) with \( s L^i = s_k \) we have \( L^i \subseteq L_k \subseteq L^2 \) and \( nL^i = nL_{k^2}^* = nL_{k^2}^* \).

**Lemma 2.13.** (i) For any \( n_k - 1 + 1 \leq i \leq n_k \) we have \( R_i = u_k \) if \( i \equiv n_k - 1 + 1 \) (mod 2) and \( R_i = 2r_k - u_k \) if \( i \equiv n_k - 1 \) (mod 2).

(ii) For any \( n_k - 1 + 1 \leq i \leq n_k \) we have \( R_i = u_k \) if \( i \equiv n_k + 1 \) (mod 2) and \( R_i = 2r_k - u_k \) if \( i \equiv n_k \) (mod 2).

(iii) \( \pm a_{n_k-1+1} \) and \( \pm \pi^{2u_k-2r_k} a_{n_k} \) are norm generators for \( L_k \) and for \( L^2 \).

**Proof.** If \( L_k \) is improper then \( \dim L_k \) is even so \( n_k - 1 \equiv n_k \) (mod 2). Also the sequence \( R_{n_k-1+1}, \ldots, R_{n_k} \) is \( u_k, 2r_k - u_k, \ldots, u_k, 2r_k - u_k \) so we get both (i) and (ii).
If \( L_k \) is proper then \( u_k = r_k \) and the sequence \( R_{n_k-1+1}, \ldots, R_{n_k} \) is \( r_k, \ldots, r_k \).

But \( u_k = r_k \), so \( r_k = u_k = 2r_k - u_k \) and again we get both (i) and (ii).
(iii) We have \( L_k = \langle a_{n_k-1+1}, \ldots, a_{n_k} \rangle \) so \( a_{n_k-1+1} \) is a norm generator for \( L_k \).
We have \( L_k^1 = \langle a_1, \ldots, a_{n_k-1} \rangle \) so \( a_{n_k-1}^{-1} \) is a norm generator for \( L_k^1 = p^{-r_k} \).
Therefore \( \pi^{2r_k} a_{n_k}^{-1} \) is a norm generator for \( L_k \).

But \( \text{ord } a_{n_k} = 2r_k - u_k \), so \( \pi^{2u_k-2r_k} a_{n_k}^{-1} \) differs from \( a_{n_k}^{-1} \) by the square of a unit.
Since \( \pi^{2u_k-2r_k} a_{n_k}^{-1} \) is a norm generator for \( L_k \) so is \( \pi^{2u_k-2r_k} a_{n_k}^{-1} \).
Since \( gL_k \) is an additive group, \( -a_{n_k-1} \) and \( -\pi^{2u_k-2r_k} a_{n_k} \) will also be norm generators for \( L_k \).
We have \( L_k \subseteq L^2 \) and \( nL_k = nL^2_k \) so \( \pm a_{n_k-1+1} \) and \( \pm \pi^{2u_k-2r_k} a_{n_k} \) are norm generators for \( L^2_k \) as well.

We now want to find relations between the \( \alpha_i \)'s and the O'Meara invariants \( \omega_k \) and \( f_k \). In particular, this will prove that the \( \alpha_i \)'s are invariants of the lattice \( L \), i.e. they do not depend on the choice of the BONG of \( L \).

**Lemma 2.14.** \( \text{ord } \omega L = \min \{ R_1 + \alpha_1, R_1 + \alpha_1 + e \} \).
If \( n = 1 \) we ignore \( R_1 + \alpha_1 \).
If moreover \( L_1 \) is not unary then \( \omega L = R_1 + \alpha_1 \).

**Proof.** Note that if \( L_1 \) is not unary, in particular if \( L_1 \) is binary, then \( R_1 = u_1 \geq 2r_1 - u_1 = R_2 \) so \( \alpha_1 \leq (R_2 - R_1)/2 + e \leq e \). Hence \( \min \{ R_1 + \alpha_1, R_1 + e \} = R_1 + \alpha_1 \) and so the two statements of the lemma are equivalent.
We use induction on \( m \), the number of components in the maximal norm splitting we fixed for \( L \). Suppose first that \( m = 1 \). If \( L = L^1 \) is unary then \( \omega L = 2sL = 2p^{R_1} \).
so \(\text{ord } mL^1 = R_1 + e\), as claimed. If \(L = L^1\) is binary and so improper modular then we may assume that it is unimodular since the statement is invariant upon scaling. Hence \(R_1 + R_2 = 0\) and \(R_1 = \text{ord } nL > \text{ord } sL = 0\). Now \(a_1 \in Q(L)\) is a norm generator. Thus by [4, 93:10] there is \(b \in mL\) such that \(L \cong A(a_1, b)\). Also if \(mL \supset 2sL = 2\mathcal{O}\) then \(mL = b\mathcal{O}\). Suppose first that \(mL = 2\mathcal{O}\). Then \(b \in 2\mathcal{O}\) so \(\text{ord } b \geq e\). Thus \(d(-a_1 a_2) = d(-\det L) = d(1 - a_1 b) \geq \text{ord } a_1 b \geq R_1 + e\) so \(R_2 - R_1 + d(-a_1 a_2) = 2R_2 + d(-a_1 a_2) \geq - R_1 + e\). On the other hand \((R_2 - R_1)/2 + e = - R_1 + e\) so \(\alpha_1 = \text{min}\{ (R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1 a_2)\} = - R_1 + e\). Thus \(\text{ord } mL^1 = e = R_1 + \alpha_1\). If \(mL \supset 2sL = 2\mathcal{O}\) then \(mL = b\mathcal{O}\) and \(\alpha_1 + \text{ord } b = \text{odd}\). Also \(\text{ord } a_1 = \text{ord } mL \leq \text{ord } 2sL = e\) and \(\text{ord } b = \text{ord } mL < \text{ord } 2sL = e\). It follows that \(\text{ord } a_1 b < 2e\) and it is odd. Hence \(d(-a_1 a_2) = d(1 - a_1 b) = \text{ord } a_1 b = R_1 + \text{ord } b\) so \(R_2 - R_1 + d(-a_1 a_2) = - R_1 + \text{ord } b\). Also \((R_2 - R_1)/2 + e = - R_1 + e > - R_1 + \text{ord } b\). It follows that \(\alpha_1 = - R_1 + \text{ord } b = - R_1 + \text{ord } mL\). So \(\text{ord } mL = R_1 + \alpha_1\).

We now prove the induction step. We have \(L = L^1 \perp L'\), where \(L' = L^2 \perp \ldots \perp L^m\). Now let \(a\) and \(a'\) be norm generators for \(L^1, L'\). We have \(mL^1 = mL\) since \(a\) is also a norm generator for \(L\). By Lemma 2.11 we have \(mL = mL^1 + mL^1 + a^{-1}b(aa')\).

\((a^{-1}b(aa')) = 0\) and \(2sL = 2sL \subseteq mL^1\) can be ignored.) Since \(a^{-1}b(aa') = \text{ord } a' + d(aa')\) it follows that \(\text{ord } mL = \text{min}\{ \text{ord } mL^1, \text{ord } mL^1, \text{ord } a' + d(aa')\}\).

If \(L^1\) is unary then \(R_1 \leq R_2, L^1 \cong < a_1 \succ \) and \(L' \cong < a_2, \ldots, a_n \succ\). We take \(a = a_1\) and \(a' = - a_2\). We have \(\text{ord } a' = R_2, mL^1 = R_1 + e\) and \(\text{ord } mL^1 = \text{min}\{ R_2 + \alpha_1(L'), R_2 + e\}\). It follows that \(\text{ord } mL = \text{min}\{ R_1 + e, R_2 + \alpha_1(L'), R_2 + e, R_2 + d(-a_1 a_2)\}\). Since \(R_2 + e \geq R_1 + e\), it can be removed. By Corollary 2.5 (ii) \(\alpha_1 = \text{min}\{ (R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1 a_2)\}\). Thus \(\text{min}\{ R_1 + \alpha_1, R_1 + e\} = \text{min}\{ (R_1 + R_2)/2 + e, R_2 + d(-a_1 a_2)\}\). But \(R_2 \geq R_1\), so \((R_1 + R_2)/2 + e \geq R_1 + e\). Thus \(\text{min}\{ R_1 + \alpha_1, R_1 + e\} = \text{min}\{ R_2 + d(-a_1 a_2), R_2 + \alpha_1(L'), R_1 + e\}\).

If \(L^1\) is binary then \(R_1 \geq R_2\). \(L^1 \cong < a_1, a_2 \succ \) and \(L' \cong < a_3, \ldots, a_n \succ\). We prove that \(\text{ord } mL = R_3 + \alpha_1\). We take \(a = \pi^{2a_1 - 2a_2} a_2\) and \(a' = - a_3\). (See Lemma 2.13(iii).) We have \(mL^1 = R_1 + \alpha_1(L^1), mL' = \text{min}\{ R_3 + \alpha_1(L'), R_3 + e\}\) and \(\text{ord } a' + d(aa') = R_3 + d(-a_2 a_3)\). Thus \(\text{ord } mL = \text{min}\{ R_1 + \alpha_1(L^1), R_3 + \alpha_1(L'), R_3 + d(-a_2 a_3), R_3 + e\}\). But \(e \geq (R_2 - R_1)/2 + e \geq \alpha_1(L^1)\), so \(R_3 + e \geq R_3 + \alpha_1(L^1)\) and so \(R_3 + e\) can be removed. On the other hand \(\alpha_1 = \text{min}\{ \alpha_1(L), R_1 - R_3 + d(-a_2 a_3), R_3 + R_1 + \alpha_1(L')\}\). (We have \(\alpha_1(L^1) = \text{min}\{ (R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1 a_2)\}\) and, by Lemma 2.4(ii), \(R_1 - R_1 + \alpha_1(L') = R_3 - R_1 + \alpha_1(< a_3, \ldots, a_n \succ\).)

We can replace all \(R_3 - R_1 + d(-a_3 a_3)\) with \(j \geq 3\). So \(R_1 + \alpha_1 = \text{min}\{ R_1 + \alpha_1(L^1), R_3 + d(-a_2 a_3), R_3 + \alpha_1(L')\}\) and \(\text{ord } mL\).

Lemma 2.15. If \(L_k\) is unary then \(w_k = s_k(f_{k-1} + f_k + 2\mathcal{O})\). (The term \(f_{k-1}\) is ignored if \(k = 1\) and \(f_k\) is ignored if \(k = t\).)

Proof. Since \(L_k\) is unary we have \(s_k = a_k \mathcal{O}\) and \(u_k = r_k\). Also \(mL_k = 2s_k\).

We have \(L^{2k} = \big( \bigcup_{j < k} s_kL^1_{L_j} \big) \perp L_k \perp \big( \bigcup_{j > k} L_j \big)\). The first orthogonal sum is included in \(s_k^{-1} s_kL^{2k-1}\), while the last one is included in \(L^{2k+1}\). Hence \(L^{2k} \subseteq L_k + s_k^{-1} s_kL^{2k-1} + L^{2k+1}\). The reverse inclusion follows from [4, 93:24] so \(L^{2k} = L_k + s_k^{-1} s_kL^{2k-1} + L^{2k+1}\). Now \(a_k\) is a norm generator for both \(L^{2k}\) and \(L_k, \pi^{-2(k-r-1)} a_k^{-1} a_{k+1}\) for \(s_k^{-1} s_kL^{2k-1}\) and \(a_{k+1}\) for \(L^{2k+1}\). By Lemma 2.11 we have \(w_k = a_k^{-1} d(\pi^{-2(k-r-1)} a_k a_{k+1}) + w_{L_k} + w(s_k^{-1} s_kL^{2k-1}) + wL^{2k+1}\).
2s_k = s_k^{-1}s_k\partial(a_k a_{k+1}) + s_k^{-1}s_k\partial(a_k a_{k+1}) + s_k^{-1}s_k\partial w_{k-1} + w_{k+1} + 2s_k. \ (We \ ignore \ a_k\partial(a_k a_{k+1}) = 0.)

If u_k + u_{k+1} is even then by (193:26) we have $s_k^2f_k = \partial(a_k a_{k+1}) + a_k w_{k+1} + a_k w_k + 2p(u_k + u_{k+1})/2 + r_k$ 2. This formula also holds in the case when $u_k + u_{k+1}$ is odd if we drop the last term. Indeed, in this case $s_k^2f_k = a_k a_{k+1}O$ but ord $a_k a_{k+1}$ is odd so $\partial(a_k a_{k+1}) = a_k a_{k+1}O$ and we also have $a_k w_{k+1}, a_k w_k \subseteq a_k a_{k+1}O$. It follows that $s_k^2f_k = s_k^2\partial(a_k a_{k+1}) + s_k^{-1}a_k w_{k-1} + s_k^{-1}a_k w_k + 2p(u_k + u_{k+1})/2 - r_k - s_k^2\partial(a_k a_{k+1}) + s_k^2 a_k w_{k+1} + s_k^{-1}a_k w_k + 2p(u_k + u_{k+1})/2 - r_k + 2O$.

If $u_k + u_{k+1}$ is odd we ignore $2p(u_k + u_{k+1})/2 - r_k$. But $r_k = u_k$, so $(u_k + u_{k+1})/2 - r_k = (u_k - 2r_k + (u_k - 2r_k) - u_k)/2 \geq 0$ and $(u_k + u_{k+1})/2 - r_k = (u_k - u_{k+1})/2 \geq 0$. Hence $2p(u_k + u_{k+1})/2 - r_k = 2p(u_k + u_{k+1})/2 - r_k \subseteq 2O$ so these terms can be ignored. Thus $s_k^2f_k = s_k^2\partial(a_k a_{k+1}) + s_k^{-1}a_k w_{k-1} + s_k^{-1}a_k w_k + s_k^{-1}a_k w_{k+1} + s_k^{-1}a_k w_k + 2s_k = w_k + s_k^{-1}a_k a_{k+1} w_k + s_k^{-1}a_k w_k = w_k.$ (We have $s_k^2\partial(a_k a_{k+1}) = s_k^2\partial(a_k a_{k+1}) \subseteq O$ and $s_k^{-1}a_k a_{k+1} = a_k a_{k+1}O \subseteq O$.)

Lemma 2.16. Let $1 \leq i \leq n - 1$. Then:

(i) If $n_k - i < u_k$ for some $1 \leq k \leq t$, then $R_i + \alpha_i = \text{ord} w_k$ and $-R_i + 1 + \alpha_i = \text{ord} w_{k-1}^2$.

(ii) Suppose that $i = n_k$ for some $1 \leq k \leq t - 1$. If $R_i + 1 - R_i$ is even or \leq 2e then $\alpha_i = \text{ord} f_k$; otherwise $\alpha_i = (R_i + 1 - R_i)/2 + e$, ord $f_k = R_i + 1 - R_i = 2\alpha_i - 2e$ and both $\alpha_i$ and ord $f_k > 2e$.

Proof. (i) Note that $R_i + R_i + 1 = u_k + 2r_k - u_k = 2r_k$. Thus if $R_i + \alpha_i = \text{ord} w_k$ then $-R_i + 1 + \alpha_i = \text{ord} w_k - 2r_k = \text{ord} s_k^2w_k = \text{ord} w_{k-1}^2$ so it is enough to prove the first part of the statement. Also $R_{n_k + 1} + R_{n_k + 1} + 2 = 2r_k + u_k$, $L_{(k-1)} \equiv a_{1}, \ldots, a_{i-1} >$ and $L_{(k)} \equiv a_{1}, \ldots, a_{n} >$. Note that $R_i \geq R_{i+1}$. If $a$ and $b$ are norm generators for $L_{(k)}$ and $s_k L_{(k-1)}$, respectively, then $nL_{(k)} = p_{fe} = p_{u_k} = nL_{(k)}$. Therefore $a$ is also a norm generator for $L_{(k)}$. By Lemma 2.11 we have $w_k = wL_{(k-1)} = w(s_k L_{(k-1)}) + w L_{(k)} + a^{-1}d(ab) + 2s_k$, which implies that ord $w_k = \min\{\text{ord} w(s_k L_{(k-1)}), \text{ord} w L_{(k)}, \text{ord} b + \text{ord}(ab), r_k + e\}$. Now $L_k$ is not unary so ord $w_{(k)} = R_k + \alpha_1(L_{(k-1)})$ by Lemma 2.14. Also $L_{(k-1)} \equiv a_{1}, \ldots, a_{i-1} >$ and ord $a_{i-1} = -R_i + 1$ so by Lemma 2.14 we have ord $L_{(k-1)} = \min\{-R_i + 1 + \alpha_1(L_{(k-1)}), -R_i + 1 + e\} = \min\{-R_i + 1 + \alpha_1(L_{(k-1)}), -R_i + 1 + e\}$.

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$r_k + e = (R_t + R_{t+1})/2 + e$. Thus $\text{ord } w_k = \min \{R_t + \alpha_1(L^*_k(k)), R_t + R_{t+1} - R_{t-1} + d(-a_1-a_2), R_t + R_{t+1} - R_{t-1} + \alpha_2(L^*_{k-1}), R_t + R_{t+1} - R_{t-1} + e, (R_t + R_{t+1} + e)/2\}$. But $R_{t-1} \leq R_{t+1} \leq R_t$, so $R_t + R_{t+1} - R_{t-1} + e \geq (R_t + R_{t+1} + e)/2 + e = R_t + (R_{t+1} - R_t)/2 + e \geq R_t + \alpha_1(L^*_k(k))$, so the last two terms can be removed. By Lemma 2.4(i), $R_{t+1} - R_{t} + \alpha_2(L^*_{k-1}) = R_{t+1} - R_{t-1} + \alpha_2(a_1, \ldots, a_{t-1})$ replaces all the terms in the definition of $\alpha_t$ with $1 \leq j \leq t-2$, while by Lemma 2.1, $\alpha_1(L^*_k(k)) = \alpha_1(a_1, \ldots, a_t)$ replaces $(R_{t+1} - R_t)/2 + e$ and the terms with $i \leq j < t$. Hence $\alpha_t = \min \{R_{t+1} - R_{t} + \alpha_2(L^*_{k-1}), R_{t+1} - R_{t-1} + d(-a_1-a_2), \alpha_1(L^*_k(k))\}$.

It follows that $R_t + \alpha_t = \min \{R_t + R_{t+1} - R_{t-1} + \alpha_2(L^*_{k-1}), R_t + R_{t+1} - R_{t-1} + d(-a_1-a_2), R_t + \alpha_1(L^*_k(k))\} = \text{ord } w_k$.

(ii) Since $i = n_k$, we have $R_t = 2r_k - u_k$, $R_{t+1} = u_k + 1$, $L(k) = \langle a_1, \ldots, a_t \rangle$ and $L^*_{k+1} = \langle a_1, \ldots, a_{t+1} \rangle$. We have $R_{t+1} - R_t = u_k + u_{k+1} - 2r_k$, so $R_{t+1} - R_t$ is even if $u_k + u_{k+1}$ is even. If $u_k + u_{k+1}$ and $R_{t+1} - R_t$ are odd then $f_k = s_k^{-2}a_ku_k$, so $\text{ord } f_k = u_k + u_{k+1} - 2r_k = R_{t+1} - R_t$. If $R_{t+1} - R_t < 2e$ then $\alpha_t = R_{t+1} - R_t = \text{ord } f_k$, while if $R_{t+1} - R_t > 2e$ then $\alpha_t = (R_{t+1} - R_t)/2 + e$ (See Lemma 2.7(ii) and (iii)).

Suppose now that $R_{t+1} - R_t$ is even. By Lemma 2.12 we have $f_k = s_k^{-2}u_k\alpha_k + a_k u_k^2 s_k^2 w_k L^*_{k+1} + \alpha_k u_k + 2p(u_k + u_{k+1})/2 - r_k$. We take $\alpha_k = \pi u_k^{-2r_k}a_k$ and $\alpha_{k+1} = -u_k$. (See Remark 2.6(iii)). Thus $\text{ord } d(a_k u_k + 1) = \text{ord } s_k^{-2}u_k\alpha_k + a_k u_k^2 s_k^2 w_k L^*_{k+1} + \alpha_k u_k + 2p(u_k + u_{k+1})/2 - r_k$ and so $s_k^{-2}u_k\alpha_k + a_k u_k^2 s_k^2 w_k L^*_{k+1} + \alpha_k u_k + 2p(u_k + u_{k+1})/2 - r_k = -2r_k + u_k + u_{k+1} + d(-a_1-a_{k+1}) = R_{t+1} - R_t$. Thus $\text{ord } f_k = \min \{R_{t+1} - R_t + \alpha_2(L^*_{k+1}), R_{t+1} - R_t + e\}$. Since $R_{t+1} - R_t = R_{t+1} - R_t + e$, we get $\alpha_t = -u_k$, $\alpha_{t+1} = -u_{k+1}$, $\alpha_{t+2} = -u_{k+2}$, and $\alpha_{t+3} = \alpha_{t+4} = \ldots = -u_{\alpha_t}$. Since $\alpha_1(L^*_k(k)) = \alpha_{t+1}(L^*_k(k))$ (see Remark 2.6) and $\alpha_{k+1} = u_{k+1} = R_{t+1}$ we have $\alpha_k = \min \{R_{t+1} - R_t + \alpha_2(L^*_{k+1}), R_{t+1} - R_t + e\}$. Finally $2p(u_k + u_{k+1})/2 - r_k = (u_k + u_{k+1})/2 - r_k + e = ((R_{t+1} - R_t)/2 + e, R_{t+1} - R_t + d(-a_1-a_{k+1}), R_{t+1} - R_t + \alpha_2(L^*_{k+1}), R_{t+1} - R_t + \alpha_1(L^*_{k+1}), R_{t+1} - R_t + e)$. Thus $\text{ord } f_k = \min \{R_{t+1} - R_t + \alpha_2(L^*_{k+1}), R_{t+1} - R_t + e\}$. But $R_{t+1} - R_t - u_k = u_{k+1} = R_{t+1}$ so $R_{t+1} - R_t + e \geq (R_{t+1} - R_t)/2 + e$, so it can be ignored. So $\text{ord } f_k = \min \{R_{t+1} - R_t + e, R_{t+1} - R_t + d(-a_1-a_{k+1}), R_{t+1} - R_t + \alpha_2(L^*_{k+1}), R_{t+1} - R_t + \alpha_1(L^*_{k+1})\}$, which, by Corollary 2.5(i), is equal to $\alpha_t$. (Recall, $L(k) = \langle a_1, \ldots, a_t \rangle$ and $L^*_{k+1} = \langle a_1, \ldots, a_{t+1} \rangle$.)

**Corollary 2.17.** (i) If $L_k$ is not unary and $i = n_{k-1} + 1$ or $n_k - 1$ then $\alpha_i^{-1} w_k = \alpha_i$.

(ii) If $L_k$ is unary and $i = n_k$ then $\alpha_i^{-1} w_k = \min \{\alpha_i, \alpha_i, e\}$. (We ignore $\alpha_i$ if $i = 1$, and $\alpha_i$ if $i = n_k$.)

**Proof.** (i) In both cases when $i = n_{k-1} + 1$ or $n_k - 1$ we have $R_t = u_k = \text{ord } a_k$. Hence $\text{ord } w_k = \alpha_t + e = \text{ord } a_k + \alpha_i = \text{ord } a_i$. So $\alpha_i^{-1} w_k = \alpha_i$.

(ii) We have $s_k = \alpha_k O$ and, by Lemma 2.15, $\text{ord } s_k = \text{ord } f_k + 2d$ so $\alpha_i^{-1} w_k = f_k - f_k + e = f_k = f_k - f_k + 2d$. Thus $\text{ord } a_i^{-1} w_k = \min \{\text{ord } f_k, \text{ord } f_k, e\}$ and we have to prove that it is equal to $\min \{\alpha_i, \alpha_i, e\}$. Now $i = n_k - 1 = n_k - 1$ so, by Lemma 2.16(ii), we have the following cases: $\alpha_i = \text{ord } f_k$ or $\alpha_i = \text{ord } f_k > 2e$. But if $\alpha_i = \text{ord } f_k$ or $\alpha_i = \text{ord } f_k > 2e$ then they cannot be ordered in $\min \{\alpha_i, \alpha_i, e\}$ and $\min \{\text{ord } f_k, \text{ord } f_k, e\}$, respectively. So $\alpha_i = \text{ord } f_k$ or $\alpha_i = \text{ord } f_k > 2e$ so they cannot be ordered. Thus $\min \{\alpha_i, \alpha_i, e\} = \min \{\text{ord } f_k, \text{ord } f_k, e\}$. □
3. Main theorem

In this section we state and prove the main result of this paper, the classification of integral lattices over dyadic local fields in terms of good BONGs. It is well known that this problem was first solved by O’Meara in [1] Theorem 93:28. Since our proof uses O’Meara’s result we first state Theorem 93:28.

Throughout this section $L,K$ are two lattices with $L \cong \langle a_1, \ldots, a_n \rangle$ and $K \cong \langle b_1, \ldots, b_n \rangle$ relative to good BONGs. In terms of Jordan decompositions we write $L = L_1 \perp \ldots \perp L_t$ and $K = K_1 \perp \ldots \perp K_{t'}$. Let $s_k = sL_k, s'_k = sK_k, g_k = gL^s_k, g'_k = gK'^s_k, w_k = wL^s_k, w'_k = wK'^s_k, f_k = f_k(L)$ and $f'_k = f_k(K)$. Let $a_k$ and $b_k$ be norm generators for $L^s_k$ and $K'^s_k$, respectively. We say that $L$ and $K$ are of the same fundamental type if

$$t = t', \quad \dim L_k = \dim K_k, \quad s_k = s'_k, \quad g_k = g'_k$$

for $1 \leq k \leq t$. These conditions are equivalent to

$$t = t', \quad \dim L_k = \dim K_k, \quad s_k = s'_k, \quad w_k = w'_k, \quad a_k \cong b_k \quad (\text{mod } w_k)$$

for $1 \leq k \leq t$. We now state O’Meara’s Theorem 93:28.

**Theorem 93:28.** Let $L,K$ be lattices with the same fundamental type such that $FL \cong FK$. Let $L(1) \subset \cdots \subset L(t)$ and $K(1) \subset \cdots \subset K(t')$ be Jordan chains for $L$ and $K$. Then $L \cong K$ if and only if the following conditions hold for $1 \leq i \leq t - 1$:

(i) $\det L(k)/\det K(k) \cong 1$ (mod $f_k$);

(ii) $FL(k) \to FK(k) \perp [a_{k+1}]$ when $f_k \subset 4a_{k+1}w_{k+1}^{-1}$;

(iii) $FL(k) \to FK(k) \perp [a_k]$ when $f_k \subset 4a_kw_k^{-1}$.

We now state our main result.

**Theorem 3.1.** Let $L,K$ be two lattices with $FL \cong FK$ and let $L \cong \langle a_1, \ldots, a_n \rangle$ and $K \cong \langle b_1, \ldots, b_n \rangle$ relative to good BONGs. Let $R_i = R_i(L) = \ord a_i, S_i = R_i(K) = \ord b_i, x_i = x_i(L)$ and $b_i = x_i(K)$. Then $L \cong K$ iff:

(i) $R_i = S_i$ for $1 \leq i \leq n$;

(ii) $x_i = b_i$ for $1 \leq i \leq n - 1$;

(iii) $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$ for $1 \leq i \leq n - 1$;

(iv) $[b_1, \ldots, b_{i-1}] \to [a_1, \ldots, a_i]$ for any $1 < i < n$ s.t. $\alpha_{i-1} + \alpha_i > 2e$.

**Proof.** Condition 3.1(i) is equivalent to $t = t', \dim L_k = \dim K_k, s_k = s'_k$ and $nL^s_k = nK'^s_k$, i.e. $a_kO = b_kO$. (See [1] Lemma 4.7.) Suppose this happens. Denote as before $n_k = \dim L(k) = \dim K(k), p^{rk} = s_k$ and $p^{uk} = nL^s_k = a_kO$.

As in the previous section, we choose a Jordan splitting of $L$ such that $L_k \cong \langle a_{n_k-1+1}, \ldots, a_{n_k} \rangle$. Hence for any $1 \leq k \leq n$, $a_k$ can be either $\pm a_{n_k-1+1}$ or $\pm \pi^{2u_k-2v_n}a_{n_k}$. We choose a Jordan splitting for $K$ with the same property.

Assuming that 3.1(i) holds, Lemma 2.16 and Corollary 2.17(ii) imply that 3.1(ii) is equivalent to $\forall 1 \leq k \leq t$ and $f_k = f_k'$ for $1 \leq k \leq t - 1$.

From here the proof of Theorem 3.1 consists of two steps:

1. Assuming that 3.1(i) and (ii) hold, we prove that condition 3.1(iii) is equivalent to $a_k \cong b_k \quad (\text{mod } w_k)$ for any $1 \leq k \leq t$ and condition 93:28(i).

2. Assuming that 3.1(i)-(iii) hold, we prove that condition 3.1(iv) is equivalent to conditions 93:28(ii) and (iii). \[
\square
\]
Lemma 3.2. Suppose that $L, K$ satisfy conditions 3.1(i) and 3.1(ii). If $R_{i-1} = R_{i+1}$ for some $1 < i < n$ then:

(i) If 3.1(iii) holds at $i - 2$ or $i - 2 = 0$ then 3.1(iii) holds at $i$.

(ii) If 3.1(iii) holds at $i + 1$ or $i + 1 = n$ then 3.1(iii) holds at $i - 1$.

Proof. (i) We have $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \min\{d(a_1 \cdots a_i - 2b_1 \cdots b_i - 2), d(-a_i a_i), d(-b_i - b_i)\}$. (If $i - 2 = 0$ we ignore $d(a_1 \cdots a_i - 2b_1 \cdots b_i - 2)$.) But $d(a_1 \cdots a_i - 2b_1 \cdots b_i) \geq \alpha_{i-2} \geq R_{i-1} - R_{i+1} + \alpha_i = \alpha_i$. (We have $R_{i-1} + \alpha_i \geq -R_{i+1} + \alpha_i$.) Also $d(-a_i a_i) = R_{i+1} - R_{i-1} + d(-a_i a_i) \geq \alpha_i$. Similarly $d(-b_i - b_i) \geq \alpha_i$. Hence $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$.

(ii) is similar. This time $R_{i+1} + \alpha_i \geq R_{i-1} + \alpha_i$ so $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i \geq R_{i-1} - R_{i+1} + \alpha_i = \alpha_i$. (If $i + 1 = n$ then $d(a_1 \cdots a_n b_1 \cdots b_n) = \infty > \alpha_{i-2}$.) Also $d(-a_i a_i) = R_{i+1} - R_{i-1} + d(-a_i a_i) \geq \alpha_i$ and similarly $d(-b_i b_i) \geq \alpha_i$.

Lemma 3.3. Assuming that 3.1(i) and (ii) hold, condition 3.1(iii) is equivalent to $a_k \equiv b_k \pmod{w_k}$ for any $1 \leq k \leq t$ and condition 93:28(i).

Proof. We have $L_k \equiv <a_1, \ldots, a_n>$ and $K_k \equiv b_1, \ldots, b_n>$. Hence $d(L_k) = a_1 \cdots a_n$ and $d(K_k) = b_1 \cdots b_n$. Since the two determinants have the same order, $R_i = \cdots + R_n$, the condition $d(L_k)/d(K_k) \equiv 1 (\text{mod } f_k)$ is equivalent to $d(a_1 \cdots a_n b_1 \cdots b_n) \geq \text{ord } f_k$. Let $i = n_k$. We claim that $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \text{ord } f_k$ is equivalent to $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$. By Lemma 2.16(ii) we have either $\alpha_i = \text{ord } f_k$ or $\alpha_i, \text{ord } f_k > 2e$. In the first case our claim is obvious and in the second both $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \text{ord } f_k$ and $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$ are equivalent to $a_1 \cdots a_i b_1 \cdots b_i \in F^2$.

Thus condition 3.1(iii) at indices $i = n_k$ with $1 \leq k \leq t - 1$ is equivalent to 93:28(i). Assume these equivalent conditions hold. We want to prove that condition $a_k \equiv b_k \pmod{w_k}$ at indices $1 \leq k \leq t$ such that $L_k$ is not unary is equivalent to condition 3.1(iii) at $i = n_k - 1 + 1$, while if $L_k$ is unary then it holds unconditionally.

Note that $a_k \equiv b_k \pmod{w_k}$ is equivalent to $b_k/a_k \equiv 1 (\text{mod } a_k^{-1}w_k)$, i.e. to $d(a_k b_k) = d(b_k/a_k) \geq \text{ord } a_k^{-1}w_k$. We will take $a_k = a_{n_k+1} = a_i$ and $b_k = b_{n_k+1} = b_i$. So our condition is equivalent to $d(a_i b_i) \geq \text{ord } a_i^{-1}w_i$, where $i = n_k - 1 + 1$.

If $L_k$ is unary then $d(a_i b_i) = \text{min}\{\alpha_i, \alpha_i, \alpha_i\}$ by Corollary 2.17(ii), where $i = n_k - 1 + 1 = n_k$. Since $i - 1 = n_k - 1$ and $i = n_k$, condition 3.1(iii) is satisfied for both. Thus $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i \geq \text{ord } a_k^{-1}w_k$ and $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i \geq \text{ord } a_i^{-1}w_i$. (If $k = 1$ so $i = n_0 + 1 = 1$ we ignore $a_1$ and we have $d(a_1 b_1) \geq \alpha_1 \geq \text{ord } a_1^{-1}w_1$.) If $k = t$ so $i = n_t$ we ignore $a_i$ and, since $a_1 \cdots a_n = \det FM = \det FN = b_1 \cdots b_n$ in $F/F^2$, we get $d(a_n b_n) = d(a_1 \cdots a_{n_k-1} b_1 \cdots b_{n_k-1}) \geq \alpha_{n_k-1} \geq \text{ord } a_k^{-1}w_k$.) Thus condition $a_k \equiv b_k \pmod{w_k}$ is superfluous when $L_k$ is unary.

Suppose now that $L_k$ is not unary and let $i = n_k - 1 + 1$. By Corollary 2.17(i) we have $d(a_i b_i) = \text{ord } a_i^{-1}w_i = \alpha_i$. We will prove that $d(a_i b_i) \geq \text{ord } a_k^{-1}w_k = \alpha_i$ is equivalent to the condition 3.1(iii) at $i$ i.e. to $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$. If $k = 1$ so $i = n_0 + 1 = 1$ this is obvious. If $k > 1$ so $i > 1$ note that $-R_i + \alpha_i \geq -R_i + \alpha_i$ and $R_i = u_k \geq 2R_k - u_k = R_{i+1}$ so $\alpha_{i-1} \geq \alpha_i$. We have $i = n_k - 1$ so $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_{i-1} \geq \alpha_i$ and so $d(a_i b_i) \geq \alpha_i$ is equivalent to $d(a_1 \cdots a_i b_1 \cdots b_i) \geq \alpha_i$ by the domination principle.
To complete the proof we show that 3.1(iii) is true if it is true for \(i = n_k\), where \(1 \leq k \leq t - 1\), and for \(i = n_{k-1} + 1\), where \(1 \leq k \leq t\) and \(L_k\) is not unary. To do this we use Lemma 3.2.

Let \(1 \leq k \leq t\). For any \(n_{k-1} + 1 < i < n_k\) we have \(R_{i-1} = R_{i+1}\) (they are both \(u_k\) or \(2r_k - u_k\)) so by Lemma 3.2(i) if 3.1(iii) holds for \(i - 2\) or \(i - 2 = 0\) it will also hold for \(i\). Thus, since 3.1(iii) is true for \(n_{k-1}\) (or \(n_{k-1} = 0\) if \(k = 1\)), it will also be true by induction for any \(n_{k-1} + 2 \leq i < n_k\) with \(i \equiv n_{k-1} + 1\) (mod 2). Similarly since 3.1(iii) is true at \(n_{k-1} + 1\), it will also be true by induction for any \(n_{k-1} + 1 \leq i < n_k\) with \(i \equiv n_{k-1} + 1\) (mod 2). Hence 3.1(iii) holds for any \(n_{k-1} < i < n_k\). Since 3.1(iii) also holds for any \(i = n_k\) with \(1 \leq k \leq t - 1\) it will hold for all \(1 \leq i \leq n - 1\).

\[\square\]

**Lemma 3.4.** If \(1 < i < n\) and \(R_{i-1} = R_{i+1}\) then \(\alpha_{i-1} + \alpha_i \leq 2e\).

**Proof.** We have \(\alpha_{i-1} + \alpha_i \leq (R_i - R_{i+1})/2 + e = (R_{i+1} - R_{i-1})/2 + e = (R_{i+1} - R_{i-1})/2 + 2e\) so if \(R_{i-1} = R_{i+1}\) then \(\alpha_{i-1} + \alpha_i \leq 2e\).

\[\square\]

**Lemma 3.5.** Let \(V, W\) be two quadratic spaces over \(F\). We have:

(i) If \(\dim V - \dim W = 1\) and \(H\) is a hyperbolic plane then \(W \rightarrow V \iff V \rightarrow W \perp H\).

(ii) If \(\dim V = \dim W\) and \(a \in \hat{F}\) then \(W \rightarrow V \perp [a] \iff V \rightarrow W \perp [a\det V \det W]\).

(iii) If \(\dim V = \dim W\), \(a, b \in \hat{F}\) and \((ab, \det V \det W)_p = 1\) (in particular, if \(d(ab) + d(\det V \det W) > 2e\) then \(W \rightarrow V \perp [a] \iff V \rightarrow W \perp [b]\).

**Proof.** This is a direct consequence of [4, 63:21]. For (iii) we also use the fact that if \(xy = zt\) then \([x, y] \cong [z, t]\) iff \(z \rightarrow [x, y]\), which in turn is equivalent to \((x, y)\).

\[\square\]

**Lemma 3.6.** Suppose that \(L, K\) satisfy the conditions 3.1(i)-(iii) (or, equivalently, they have the same fundamental type and they satisfy the condition 93:28(ii)). Then:

(i) If \(f_k \in 4a_kw_k^{-1}\) and both \(a_k\) and \(b_k\) are norm generators for \(L^k\) then \(FL(k) \rightarrow FK(k) \perp [a_k]\) is equivalent to \(FL(k) \rightarrow FK(k) \perp [b_k]\), and also to \(FK(k) \rightarrow FL(k) \perp [b_k]\).

(ii) If \(f_k \in 4a_{k+1}w_{k+1}^{-1}\) and both \(a_{k+1}\) and \(b_{k+1}\) are norm generators for \(L^{k+1}\) then \(FL(k) \rightarrow FK(k) \perp [a_{k+1}]\) is equivalent to \(FL(k) \rightarrow FK(k) \perp [b_{k+1}]\) and also to \(FK(k) \rightarrow FL(k) \perp [b_{k+1}]\).

**Proof.** (i) \(f_k \in 4a_kw_k^{-1}\) is equivalent to \(ord f_k + ord a_k^{-1}w_k > 2e\). We have \(a_k^{-1}w_k \geq 2a_k^{-1}s_k \geq 2O\), so \(ord a_k^{-1}w_k \leq e < ord f_k\). Since \(a_k, b_k\) are both norm generators for \(L^k\), we have \(d(a_k, b_k) \geq ord a_k^{-1}w_k\). Since also \(d(det L(k) \det K(k)) \geq ord f_k > ord a_k^{-1}w_k\), we also have \(d(a_k, b_k) \det L(k) \det K(k) \geq ord a_k^{-1}w_k\). Since \(d(det L(k) \det K(k)) \geq d(a_k, b_k) \geq ord f_k + ord a_k^{-1}w_k > 2e\) we get by Lemma 3.5(iii) that \(FL(k) \rightarrow FK(k) \perp [a_k] \iff FL(k) \rightarrow FK(k) \perp [b_k]\). Similarly, since \(d(det L(k) \det K(k)) \geq d(a_k, b_k) \det L(k) \det K(k) \geq ord f_k + ord a_k^{-1}w_k > 2e\), we have \(FL(k) \rightarrow FK(k) \perp [a_k]\) iff \(FL(k) \rightarrow FK(k) \perp [b_k] \det L(k) \det K(k)\) which, by Lemma 3.5(ii), is equivalent to \(FK(k) \rightarrow FL(k) \perp [b_k]\).

(ii) The proof is the same as (i) but with \(a_k, b_k, w_k\) replaced by \(a_{k+1}, b_{k+1}, w_{k+1}\).
Lemma 3.7. Suppose that $L, K$ satisfy the conditions 3.1(i)-(iii). If $1 \leq k \leq t-1$ then:

(i) If $f_k \subset 4a_kw_k^{-1}$ then $FL_{(k)} \rightarrow FK_{(k)} \uparrow [a_k]$ iff $[b_1, \ldots, b_{i-1}] \rightarrow [a_1, \ldots, a_i]$, with $i = n_k$.

(ii) If $f_k \subset 4a_{k+1}w_{k+1}^{-1}$ then $FL_{(k)} \rightarrow FK_{(k)} \uparrow [a_{k+1}]$ iff $[b_1, \ldots, b_{i-1}] \rightarrow [a_1, \ldots, a_i]$, with $i = n_k + 1$.

Proof. (i) We take $b_k = -\pi^{2\nu_k-2r_k}b_k$ as a norm generator for $K^{2k}$, so for $L^{2k}$. (See Lemma 2.13(iii).) By Lemma 3.6(i) $FL_{(k)} \rightarrow FK_{(k)} \uparrow [a_k]$ iff $FL_{(k)} \rightarrow FK_{(k)} \uparrow [b_k]$, i.e. iff $[a_1, \ldots, a_i] \rightarrow [b_1, \ldots, b_i] \uparrow [-b_i] \equiv [b_1, \ldots, b_{i-1}] \uparrow H$. By Lemma 3.5(i) this is equivalent to $[b_1, \ldots, b_{i-1}] \rightarrow [a_1, \ldots, a_i]$.

(ii) We take $b_{k+1} = a_i$ as a norm generator for $L^{2k+1}$. By Lemma 3.6(ii) $FL_{(k)} \rightarrow FK_{(k)} \uparrow [a_{k+1}]$ iff $FK_{(k)} \rightarrow FL_{(k)} \uparrow [b_{k+1}]$, i.e. iff $[b_1, \ldots, b_{i-1}] \rightarrow [a_1, \ldots, a_{i-1}] \uparrow [a_i] \equiv [a_1, \ldots, a_i]$.

Lemma 3.8. (i) If $i = n_k > n_{k-1} + 1$ then $\alpha_{i-1} + \alpha_i > 2$ if $f_k \subset a_kw_k^{-1}$.

(ii) If $i = n_k + 1 < n_{k+1}$ then $\alpha_{i-1} + \alpha_i > 2$ if $f_k \subset a_{k+1}w_{k+1}^{-1}$.

(iii) If $i = n_k = n_{k-1} + 1$ then $\alpha_{i-1} + \alpha_i > 2$ if $f_k \subset a_kw_k^{-1}$ or $f_{k-1} \subset a_kw_k^{-1}$.

(In (ii) we ignore the condition $f_k \subset a_kw_k^{-1}$ if $i = 1$.)

Proof. (i) Condition $f_k \subset 4a_kw_k^{-1}$ is equivalent to $\alpha_k^1w_k + \alpha_k + 2$. By Corollary 2.17(i) we have $\alpha_k^1w_k = \alpha_{k-1}$. By Lemma 2.16(ii) we have either $\alpha_i = \alpha_k$ or $\alpha_k$, $\alpha_k > 2$.

(ii) We have $f_k \subset 4a_{k+1}w_{k+1}^{-1}$ if $\alpha_{k+1}^1w_{k+1} > 2$. By Corollary 2.17(i) $\alpha_{k+1}^1w_{k+1} = \alpha_i$ and by Lemma 2.16(ii) $\alpha_d, \alpha_{d-1}$ are either equal or they are both $> 2e$. Thus $\alpha_{k+1}^1w_{k+1} > 2e$ if $\alpha_{k+1} > 2e$.

(iii) $f_{k-1} \subset 4a_kw_k^{-1}$ and $f_k \subset 4a_kw_k^{-1}$ are equivalent to $\alpha_{k-1} + \alpha_k > 2$, resp. $\alpha_k > 2$. By Corollary 2.17(ii) we have $\alpha_k^1w_k = \min\{\alpha_{k-1}, \alpha_k, e\} > 0$. By Lemma 2.16(ii) we have that $\alpha_k = \alpha_{k-1}$ or $\alpha_{k-1}, \alpha_{k-1} > 2e$ and $\alpha_k = \alpha_i < 2e$. Therefore $\alpha_{k-1} < 4a_kw_k^{-1}$ and $f_k \subset 4a_kw_k^{-1}$ are equivalent to $\alpha_{k-1} + \alpha_k > 2e$, resp. $\alpha_k > 2e$. Obviously either of them implies $\alpha_{k-1} + \alpha_k > 2e$.

Lemma 3.9. Assuming that 3.1(i)-(iii) hold, condition 3.1(iv) is equivalent to 9.3.28(ii) and (iii).

Proof. Take $1 < i < n$. If $n_k - 1 < i < n_k$ for some $1 \leq k \leq t$ then $R_{i-1} = R_{i+1}$, by Lemma 2.13, so, by Lemma 3.4, $\alpha_{i-1} + \alpha_i \leq 2e$, which makes 3.1(iv) vacuous at $i$. Therefore we can restrict ourselves to $i = n_k$ or $n_k + 1$ for some $1 \leq k \leq t - 1$. We have three cases:

1. if $n_k$ and $\dim L_k > 1$, i.e. $i = n_k > n_{k-1} + 1$. By Lemma 3.8(i) $f_k \subset 4a_kw_k^{-1}$ is equivalent to $\alpha_{i-1} + \alpha_i > 2e$. On the other hand if $f_k \subset 4a_kw_k^{-1}$ then
we note that if $w$ comes $k$ will prove that the condition 3.1(iv) at index $[i_{n+1}]$ is equivalent, by Lemma 3.7(i), to $0 < k < t$ which is either $R$ or $a_k$. Thus 93:28(ii) is superfluous at $k = 1$. Next we note that if $k = 1$ then 3.1(iv) is vacuous at $i = n_0 + 1 = 1$. On the other hand 93:28(ii) is vacuous at index $k - 1 = 0$. Also if $f_1 \subset 4a_1w_k^{-1}$ then $FL(R) \to FK(R)$ is equivalent to $[a_1, \ldots, b_n] = (a_1, \ldots, a_n)$ (we have $i = n_0 + 1 = n = n_0$). But this follows from $[a_1, \ldots, a_n] \ni [b_1, \ldots, b_n]$. Thus 93:28(ii) is superfluous at index $k - 1 = t - 1$. Now suppose that $1 < k < t$. By Lemma 3.8(iii) we have $\alpha_{i-1} + \alpha_{i} > 2e$ if $f_{k-1} \subset 4a_kw_k^{-1}$ or $f_k \subset 4a_kw_k^{-1}$. To complete the proof we note that if $f_{k-1} \subset 4a_kw_k^{-1}$ then $FL(R) \to FK(R)$ is equivalent to $[a_1, \ldots, b_n] = (a_1, \ldots, a_n)$ by Lemma 3.7(ii) (we have $i = n_0 + 1$) and if $f_k \subset 4a_kw_k^{-1}$ then $FL(R) \to FK(R)$ is equivalent to $[a_1, \ldots, b_n] = (a_1, \ldots, a_n)$ by Lemma 3.7(i) (we have $i = n_0$). $\square$

4. THE 2-ADIC CASE

In this section we will assume that $F$ is 2-adic, i.e. that $e = 1$.

In [4] O’Meara gives a solution to the classification problem in the 2-adic case which only involves the Jordan invariants $t$, $\dim L_k$, $s_k$ and $n_k := n_L$. The invariants $s_k$ and $n_k$ are no longer necessary since they can be written as $s_k = n_k$ and $n_k = 2s_k$. A similar phenomenon occurs when we use good BONGs instead of Jordan decompositions. This time the invariants $\alpha_i$ are no longer necessary.

Lemma 4.1. If $e = 1$ then $\alpha_i = 1$ if $R_i + 1 - R_i = 1$ and $\alpha_i = (R_i + 1 - R_i)/2 + 1$ otherwise.

Proof. We have $R_i + 1 - R_i \geq -2e = -2$ and if $R_i + 1 - R_i$ is negative then it is even. Thus $R_i + 1 - R_i$ is either $-2$ or it is $0$. If $R_i + 1 - R_i = -2e = -2$ or $R_i + 1 - R_i = 2 - 2e = 0$ or $R_i + 1 - R_i \geq 2e = 2$ then $\alpha_i = (R_i + 1 - R_i)/2 + e = (R_i + 1 - R_i)/2 + 1$ by Corollary 2.9(i). If $R_i + 1 - R_i = 1$, which is odd and $< 2e$, we have $\alpha_i = R_i + 1 - R_i = 1$ by Lemma 2.7(iii). $\square$

Since the $\alpha_i$’s are uniquely defined by the $R_i$’s, condition (ii) of the main theorem is superfluous since it follows from (i). Also, ord $a_1 \cdots a_i = ord b_1 \cdots b_i$ so ord $a_1 \cdots a_i b_1 \cdots b_i$ is even. So if $R_i + 1 - R_i \leq 1$ we have $d(a_1 \cdots a_i b_1 \cdots b_i) \geq 1 \geq \alpha_i$. So condition (iii) is superfluous if $R_i + 1 - R_i \leq 1$. If $R_i + 1 - R_i = 2$ then $\alpha_2 = 2$, while if $R_i + 1 - R_i > 2$ then $\alpha_i > 2$. Thus in these cases (iii) becomes $a_1 \cdots a_i b_1 \cdots b_i \in \hat{F}^2 \cup \Delta \hat{F}^2$ if $R_i + 1 - R_i = 2$ and $a_1 \cdots a_i b_1 \cdots b_i \in \hat{F}^2$ if $R_i + 1 - R_i > 2$. Finally, it is easy to see that the condition $\alpha_{i-1} + \alpha_i > 2$ from
Hence if \( \alpha \) is satisfied iff \( \text{Lemma 5.1.} \)

Theorem 4.2. Suppose that \( F \) is 2-adic, \( L \equiv < a_1, \ldots, a_n \gg \) and \( K \equiv < b_1, \ldots, b_n \gg \) relative to good BONGs, \( R_i = R_i(L) = \text{ord } a_i, S_i = R_i(K) = \text{ord } b_i \) and \( FL \equiv FK \).

Then \( L \equiv K \) if and only if the following conditions hold:

(i) \( R_i = S_i \) for any \( 1 \leq i \leq n \).

(ii) For any \( 1 \leq i \leq n-1 \) we have \( a_1 \cdots a_ib_1 \cdots b_i \in \hat{F}^2 \cup \hat{\Delta}_{\hat{F}}^2 \) if \( R_{i+1} - R_i = 2 \), and \( a_1 \cdots a_ib_1 \cdots b_i \in \hat{F}^2 \) if \( R_{i+1} - R_i > 2 \).

(iii) \( [b_1, \ldots, b_{i-1}] \rightarrow [a_1, \ldots, a_i] \) for any \( 1 < i < n \) s.t. \( R_{i-1} < R_{i+1} \) and \( (R_i - R_{i-1}, R_{i+1} - R_i) \neq (1, 1) \).

5. Remarks

1. The binary case

If \( L \equiv < \alpha, \beta \gg \) and \( \eta \in \mathbb{O}^\times \) then \([1, 3.12]\) states that \( L \equiv < \eta\alpha, \eta\beta \gg \) iff \( \eta \in g(a(L)) = g\left(\frac{2}{a}\right) \).

The function \( g : \mathcal{A} \rightarrow Sgp(O^\times/O^\times\times) \) was introduced in \([1, \text{Definition 6}]\). Here \( SgpH \) is the set of all subgroups of a group \( H \). We recall the definition of \( g \).

Definition. If \( a = \pi^R \varepsilon \in \mathcal{A} \) and \( d(-a) = d \) then:

I. If \( R > 2e \) then \( g(a) = O^\times\times \).

II. If \( R \leq 2e \) then:

\[
g(a) = \begin{cases} (1 + p^{R/2+e})O^\times\times & \text{if } d > e - R/2, \\ (1 + p^{R+d})O^\times\times \cap N(-a) & \text{if } d \leq e - R/2. \end{cases}
\]

The following lemma gives a more compact formula for \( g(a) \).

Lemma 5.1. If \( a \in \mathcal{A} \) and ord \( a = R \) and \( d(-a) = d \) then \( g(a) = (1 + p^{\alpha(a)})O^\times\times \cap N(-a) \), where \( \alpha(a) = \min\{R/2 + e, R + d\} \).

Proof. By \([1, 3.16]\) we have \( g(a) \subseteq N(-a) \). If \( \eta \in O^\times \) then \( \eta \in g(a) \) iff \( \eta \in N(-a) \) and (I) If \( R > 2e \) then \( d(\eta) \geq O^\times\times \); (II) If \( R \leq 2e \) then \( d(\eta) \geq R + d \) if \( d \leq e - R/2 \), and \( d(\eta) \geq R/2 + e \) if \( d > e - R/2 \). (See \([1, \text{Definition 6}]\).)

We have to prove that the conditions from (I) and (II) are equivalent to \( d(\eta) \geq \alpha(a) \).

If \( R > 2e \) then \( R + d + R/2 + e > 2e \) so \( \alpha(a) > 2e \). Thus \( d(\eta) \geq \alpha(a) \) is equivalent to \( \eta \in O^\times\times \). If \( R \leq 2e \) then \( d \leq e - R/2 \) is equivalent to \( R + d \leq R/2 + e \). Hence if \( d \leq e - R/2 \) then \( \alpha(a) = R + d \) and if \( d > e - R/2 \) then \( \alpha(a) = R/2 + e \). □

If \( n = 2 \) then from \([1, 3.12]\) we have \( < a_1, a_2 > \equiv < \eta a_1, \eta a_2 > \) iff \( \eta \in g(a_2/a_1) \).

By Lemma 5.1 this is equivalent to \( \eta \in N(-a_1a_2) \) and \( d(\eta) \geq \alpha(a_2/a_1) \). The first condition is equivalent to the isometry of quadratic spaces \( [a_1, a_2] \cong [\eta a_1, \eta a_2] \), while the second means \( d(\eta) \geq \alpha(a_2/a_1) \) if \( \min\{R_2 - R_1, 2, R_2 - R_1 + d(-a_1a_2)\} = \alpha_1(< a_1, a_2 >) \), which is consistent with condition (iii) of the main theorem.

Remark 5.2. Since \( \alpha(a_2/a_1) = \alpha_1(< a_1, a_2 >) \) we have by Lemma 5.1 \( g(a_2/a_1) = (1 + p^{\alpha_1(< a_1, a_2 >)})O^\times\times \cap N(-a_1a_2) \). Equivalently, \( g(a(L)) = (1 + p^{\alpha_1(L)})O^\times\times \cap N(-\det FL) \).

1 In \([1, \text{Definition 6}]\) there are some mistakes which we corrected here.
2. The formula for $\alpha_i$

We will now show the heuristical method by which the invariants $\alpha_i$ were found. We want to know, given that $L \cong \langle a_1, \ldots, a_n \rangle$ relative to a good BONG and $1 \leq i \leq n - 1$, how much the product $a_1 \cdots a_i$ can be altered by a change of good BONGs. That is, if $L \cong \langle b_1, \ldots, b_n \rangle$ relative to another good BONG we want to know how big the quadratic defect of $(b_1 \cdots b_i)/(a_1 \cdots a_i)$ can be. So we are looking for a lower bound $\alpha_i = \alpha_i(L)$ for $(d(a_1 \cdots a_i))$.

For any $\eta \in g(a_{i+1}/a_i)$ we have $\langle a_i, a_{i+1} \rangle \cong \langle \eta a_i, \eta a_{i+1} \rangle \cong \langle \eta a_i, \eta a_{i+1} \rangle$ so, by [1] Lemma 4.9(ii), $L \cong \langle a_1, \ldots, a_{i-1}, \eta a_i, \eta a_{i+1}, a_{i+2}, \ldots, a_n \rangle$. By this change of BONGs, $a_1 \cdots a_i$ was changed by the factor $\eta$. We have $\eta \in g(a_{i+1}/a_i)$ which, by Lemma 5.1, implies $d(\eta) \geq \alpha(a_{i+1}/a_i) = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1})\}$. (See Lemma 5.1.) This lower bound can be further decreased if we decrease $d(-a_ia_{i+1})$. This can be done by changing the good BONGs of $\langle a_1, \ldots, a_{i-1}, \eta \rangle$ and $\langle a_1, \ldots, a_{i-1}, \eta \rangle$. If $\langle a_1, \ldots, a_{i+1} \rangle \cong \langle a_1, \ldots, a_{i-1} \rangle$, then $d(-a_ia_{i+1})$ is replaced by $d(-a_ia_{i+1})$. But $d(a_{i+1}/a_{i+1}) \geq a_1(a_{i+1}, a_n \rangle).$ Also, by reason of determinant, $\langle a_1, \ldots, a_{i-1}, \eta \rangle = \min(\langle a_1, \ldots, a_{i-1}, \eta \rangle)$.

The new lower bound for $\eta$ is $\min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d(-a_ia_{i+1}), R_{i+1} - R_i + a_{i+1}(\langle a_1, \ldots, a_{i-1}, \eta \rangle)$, $\langle a_1, \ldots, a_{i+1} \rangle, R_{i+1} - R_i + a_{i+1}(\langle a_1, \ldots, a_{i-1}, \eta \rangle)$ from Corollary 2.5(ii).

In the case $i = 1$ and $n \geq 3$ the formula becomes $\alpha_1 = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1a_2), R_1 - R_1 + a_1(\langle a_2, \ldots, a_n \rangle)\}$. In the case $i = n - 1$ and $n \geq 3$ we have $\alpha_{n-1} = \min\{(R_n - R_{n-1})/2 + e, R_n - R_{n-1} + d(-a_{n-1}a_n), R_n - R_{n-1} + d(-a_{n-2}(\langle a_1, \ldots, a_{n-1} \rangle)\}$. Finally if $i = 1$ and $n = 2$ then $\alpha_1 = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1a_2)\}$. Starting with the case $n = 2$ it is easy to prove by induction that $\alpha_1 = \min\{(R_2 - R_1)/2 + e, R_2 - R_1 + d(-a_1a_2)\}$.

Of course this is only a guess and does not constitute a proof. In fact the relation $d(a_1 \cdots a_i) \geq \alpha_i$ is only proved this way in the particular case when $b_1, \ldots, b_n$ are obtained from $a_1, \ldots, a_n$ through a succession of “binary transformations” of the type $a_1, \ldots, a_n \rightarrow a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ with $1 \leq j \leq n - 1$ and $\eta \in g(a_{j+1}/a_j)$. It is not hard to prove that conditions (i)-(iv) of the main theorem are necessary if $b_1, \ldots, b_n$ are obtained this way. However, for the proof of the necessity in the general case and for the proof of sufficiency the use of O'Meara’s theorem is necessary.

3. In the view of the previous remark there is the natural question that asks whether, given that $L \cong \langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$ relative to good BONGs, there is always a succession of binary transformations as defined above from $a_1, \ldots, a_n$ to $b_1, \ldots, b_n$. The answer to this question is YES but only if we make the
assumption that $F/\mathbb{Q}_2$ is not totally ramified, i.e. that the residual field $\mathcal{O}/p$ has more than 2 elements.

If $|\mathcal{O}/p| = 2$ we have the following counterexample. Let $0 < d < 2e$ be odd and let $R = 2e - 2d$ and $\varepsilon, \eta \in \mathcal{O}^\times$ with $d(\varepsilon) = d$ and $d(\eta) = 2e - d$. It can be proved that $\langle 1, -\pi^R \varepsilon, \varepsilon \eta, -\pi^R \eta \rangle \not\sim \langle \eta, -\pi^R \varepsilon \eta, \varepsilon, -\pi^R \rangle$ but one cannot go from $\langle 1, -\pi^R \varepsilon, \varepsilon \eta, -\pi^R \eta \rangle$ to $\langle \eta, -\pi^R \varepsilon \eta, \varepsilon, -\pi^R \rangle$ through binary transformations.

E.g., if $F = \mathbb{Q}_2$ and we take $d = 1$, so $R = 0$, and $\varepsilon = \eta = -1$ then $\langle 1, 1, 1, 1 \rangle \not\sim 7, 7, 7, 7$. However from $1, 1, 1, 1$ we can go through binary transformations only to $a_1, a_2, a_3, a_4$, where an even number of $a_i$'s belong to $\mathcal{O}^\times$ and the rest to $5\mathcal{O}^\times$. This happens because $g(1) = g(5) = \mathcal{O}^\times \cup 5\mathcal{O}^\times$ so the only binary relations involving 1 and 5 are $\langle 1, 1 \rangle \not\sim \langle 5, 5 \rangle$ and $\langle 1, 5 \rangle \not\sim \langle 5, 1 \rangle$. Similarly from $7, 7, 7, 7$ we can only go to $a_1, a_2, a_3, a_4$, where an even number of $a_i$'s belong to $7\mathcal{O}^\times$ and the rest to $3\mathcal{O}^\times$.

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