

## UNIVERSAL RELATIONS ON STABLE MAP SPACES IN GENUS ZERO

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ABSTRACT. We introduce a factorization for the map between moduli spaces of stable maps which forgets one marked point. This leads to a study of universal relations in the cohomology of stable map spaces in genus zero.

### INTRODUCTION

The moduli spaces of stable maps  $\overline{M}_{0,m}(X, d)$  provide examples of Deligne–Mumford stacks whose intersection theory is both interesting and relatively accessible, as indicated by the considerable success of Gromov–Witten theory in genus 0. When the target is a point,  $\overline{M}_{0,m}$  is a smooth projective variety and its cohomology ring has been computed by Keel ([Ke]). Recent studies lead to a more comprehensive view of the cohomology and Chow groups of these moduli spaces for other targets: [BO], [GP], [O1], [C1], [C2], [MM1], [MM2].

Let  $d \in H_2(X)$  be a curve class on a smooth projective variety  $X$ . The space  $\overline{M}_{0,0}(X, d)$  parametrizes maps from rational smooth or nodal curves into  $X$  with image class  $d$ , such that any contracted component contains at least 3 nodes. Over  $\overline{M}_{0,0}(X, d)$  there exists a tower of moduli spaces of stable maps with marked points and morphisms

$$f : \overline{M}_{0,m+1}(X, d) \rightarrow \overline{M}_{0,m}(X, d)$$

forgetting one marked point, such that  $\overline{M}_{0,m+1}(X, d)$  is the universal family over  $\overline{M}_{0,m}(X, d)$ .

In this paper we introduce a factorization of the forgetful map  $f$ , which gradually contracts part of the boundary. This allows a detailed study of the cohomology and Chow rings of  $\overline{M}_{0,m+1}(X, d)$  as algebras over the rings of  $\overline{M}_{0,m}(X, d)$ . We find a series of universal relations over families of stable maps (Theorems 3.3 and 3.4).

This paper is the third in a series dedicated to the intersection rings of these moduli spaces. Previously we have found presentations for the Chow rings  $\overline{M}_{0,m}(\mathbb{P}^n, d)$  for all  $m > 0$  ([MM1], [MM2]). The case  $m = 0$  seemed less accessible from our point of view, as in a sense  $\overline{M}_{0,m}(\mathbb{P}^n, d)$  has more structure when  $m > 0$ . For example, when  $d = 1$ ,  $\overline{M}_{0,0}(\mathbb{P}^n, 1) = \text{Grass}(\mathbb{P}^1, \mathbb{P}^n)$  and  $\overline{M}_{0,1}(\mathbb{P}^n, 1)$  is a flag

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variety. This suggested an indirect approach, understanding  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  by studying the extension of algebras  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, d)) \rightarrow H^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ . We view this step as prototypical for studies of Chow quotients by  $SL_2$ -action from an intersection-theoretical point of view. The description of  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  is completed in [MM3].

The main tool in our study of moduli spaces of stable maps is a set of stability conditions for maps to projective varieties, which engender various spaces birational to  $\overline{M}_{g,m}(X, d)$  via morphisms that contract part of the boundary of  $\overline{M}_{g,m}(X, d)$ . We call these the intermediate spaces of  $\overline{M}_{g,m}(X, d)$ . They were first introduced in [MM1] and in a more general form in [MM2]. The fibred products with  $\overline{M}_{g,m}(X, d)$  of the universal families over these intermediate spaces interpolate between  $\overline{M}_{g,m+1}(X, d)$  and  $\overline{M}_{g,m}(X, d)$ . In the genus 0 case, we denote these spaces  $\overline{U}_m^k(X)$ , with  $0 \leq k \leq l_m$ , where  $l_m$  depends on  $m, d, X$ . When  $X = \mathbb{P}^n$  we drop the mention to the target, and we drop the index  $m$  when  $m = 0$ .

The spaces  $\overline{U}_m^k$  and the morphisms between them prove very valuable in understanding the relations between the cohomology rings of  $\overline{M}_{0,m}(\mathbb{P}^n, d)$  when the number of marked points varies. Indeed, the maps  $\overline{U}_m^{k-1} \rightarrow \overline{U}_m^k$  are blow-ups along regularly embedded codimension two substacks, while  $\overline{U}_m^{l_m} \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$  is a projective bundle when  $m > 0$  or  $d$  is odd. In the remaining case,  $\overline{U}_0^{l_0}$ , restricted to an open subspace of  $\overline{M}_{0,0}(\mathbb{P}^n, d)$ , is also a projective bundle, while over the complement it is a  $\mathbb{Z}_2$ -quotient of a stack obtained by gluing two  $\mathbb{P}^1$ -bundles along a common section. Standard methods allow one to express the cohomology of a projective bundle, or that of a blow-up under suitable assumptions, as algebras over the cohomology of the target. In the present case, however, a couple of technical difficulties arise.

Consider first the case of a projective bundle over a smooth variety (or stack)  $\mathbb{P}(E) \rightarrow Y$ . It is well known that the ring  $H^*(\mathbb{P}(E))$  is generated over  $H^*(Y)$  by the first Chern class of the  $\mathcal{O}(1)$ -bundle, the unique relation coming from the Chern polynomial of  $E$ . In the case of  $\overline{U}_m^{l_m} \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$ , however, it is not clear a priori how to express these Chern classes in terms of known classes on the moduli space. Furthermore, for a blow-up  $\tilde{Y} \rightarrow Y$  along a regular embedding  $i : S \hookrightarrow Y$ , by a formula first applied by Keel ([Ke]),  $H^*(\tilde{Y})$  may be expressed as an algebra over  $H^*(Y)$  generated by the class of the exceptional divisor, provided that the pullback  $i^*$  of cohomologies is surjective. However, this condition is not satisfied in the case of  $\overline{U}_m^{k-1} \rightarrow \overline{U}_m^k$ .

Due to the above reasons, a tool introduced in [MM1], [MM2], the extended cohomology ring  $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ , becomes essential in completing the calculations. The construction of  $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$  and its analogues  $B^*(\overline{U}_m^k)$  are motivated by the structure of the moduli spaces at the level of étale covers. In [MM1],  $B^*$  was introduced as the cohomology ring associated to a network of local regular embeddings (for a precise formulation, see also Definition 2.2 in the present paper). In the case of stable map spaces, the local regular embeddings are those determining the well known boundary stratification of the moduli space. We extend this construction in the case of  $\overline{U}_m^k$ . The cohomology rings are presented as invariant subrings of the  $B^*$  rings. One advantage of  $B^*$  over the usual intersection ring lies in the overall simplification of intersection on the boundary. Indeed, we have mentioned

the natural boundary stratification of  $\overline{M}_{0,m}(\mathbb{P}^n, d)$ , which is customarily indexed by stable trees with degree and marking decorations. In contrast to  $H^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ ,  $B^*$  allows the classes of each of these strata to be decomposed as a polynomial of divisor classes  $D_h$  in a natural way. Here  $D_h$  denotes the fiber product of moduli spaces  $\overline{M}_{0,2}(\mathbb{P}^n, k) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d - k)$ .

Let us consider the map  $\overline{U}_0^{l_0} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$ . Working in the rings  $B^*$ , we are able to write explicitly the relations on  $\overline{U}_0^{l_0}$  relevant to the ring extension  $B^*(\overline{U}_0^{l_0})$  over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ . This is done in Lemma 3.1. We do this by restricting to  $D_h$  and by induction on the degree  $d$ . Note that the map

$$\overline{M}_{0,2}(\mathbb{P}^n, k) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d - k) \rightarrow \overline{M}_{0,1}(\mathbb{P}^n, d),$$

a regular local embedding of stacks, is in general not an embedding, which makes induction in  $H^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  more difficult than in the ring  $B^*$ . Another useful ingredient was the known ring structure of  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  ([MM1]), in particular of the annihilator of  $D_h$ .

In the case of the blow-ups  $\overline{U}_m^{k-1} \rightarrow \overline{U}_m^k$ , we find that the pullback morphism induced by the embedding of the blow-up locus in  $\overline{U}_m^k$  is surjective at the level of  $B^*$  rings. Based on this, further calculations employing known codimension 1 relations on moduli spaces lead to the complete characterization of  $B^*(\overline{M}_{0,m+1}(\mathbb{P}^n, d))$  as an algebra over  $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$  (see Theorems 3.3 and 3.4).

Based on the above mentioned theorems and on the ring structure of  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  computed in [MM1], one may calculate the ring structure of  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ . At the end of this paper we illustrate this procedure in the case  $d = 2$ . The case of general degree  $d$  is the subject of [MM3], based on the results of the present paper.

The plan of the paper is as follows: In Section 1 we introduce the spaces  $\overline{U}_m^k$  and describe the morphisms between them. In the first part of Section 2 we offer an intuitive motivation for the introduction of the extended  $B^*$ -rings and discuss a simple example intended to familiarize the reader with the nature of these rings. A more rigorous definition and properties are contained in the second part. Section 3 contains the main calculations of  $B^*(\overline{M}_{0,m+1}(\mathbb{P}^n, d))$  as an algebra over  $B^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ , having as a simple application the structure of  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, 2))$  as in [BO].

The sequence of intermediate spaces birational to  $\overline{M}_{0,0}(\mathbb{P}^n, d)$  mentioned above is also constructed by Parker ([Par]) via GIT quotients under the  $SL_2$ -action. Indeed, he regards  $\overline{M}_{0,0}(\mathbb{P}^n, d)$  as a GIT quotient of the graph space  $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^1, (d, 1))$  and its intermediate spaces as GIT quotients of the intermediate spaces for  $\overline{M}_{0,0}(\mathbb{P}^n \times \mathbb{P}^1, (d, 1))$  constructed in [MM1]. The existence of this contraction of  $\overline{M}_{0,0}(\mathbb{P}^n, d)$  was proved by different methods in [CHS].

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### 1. A FACTORIZATION OF THE FORGETFUL MAP

1.1. Let  $m, n, d$  be natural numbers. We recall the various compactifications of  $M_{0,m}(\mathbb{P}^n, d)$  constructed in [MM2]:

Fix a rational number  $a > 0$  and an  $m$ -tuple  $\mathcal{A} = (a_1, \dots, a_m) \in \mathbb{Q}^m$  such that  $0 \leq a_j \leq 1$  for all  $j = 1, \dots, m$  and such that  $\sum_{i=1}^m a_i + da > 2$ .

**Definition 1.1.** An  $(\mathcal{A}, a)$ -stable family of degree  $d$  nodal maps from rational curves with  $m$  marked sections to  $\mathbb{P}^n$  consists of the following data:

$$(\pi : C \rightarrow S, \{p_i\}_{1 \leq i \leq m}, \mathcal{L}, e),$$

where  $\mathcal{L}$  is a line bundle on  $C$  of degree  $d$  on each fiber  $C_s$  and  $e : \mathcal{O}^{n+1} \rightarrow \mathcal{L}$  is a morphism of fiber bundles (specified up to isomorphisms of the target) such that

- (1)  $\omega_{C|S}(\sum_{i=1}^m a_i p_i) \otimes \mathcal{L}^a$  is relatively ample over  $S$ ,
- (2)  $\mathcal{G} := \text{coker } e$ , restricted over each fiber  $C_s$ , is a skyscraper sheaf supported only on smooth points of  $C_s$ , and
- (3) for any  $p \in S$  and for any  $I \subseteq \{1, \dots, m\}$  (possibly empty) such that  $p = p_i$ , for all  $i \in I$  the following holds:

$$\sum_{i \in I} a_i + a \dim \mathcal{G}_p \leq 1.$$

**Proposition 1.2** ([MM2]). *The moduli problem of  $(\mathcal{A}, a)$ -stable degree  $d$  nodal maps with  $m$  marked points into  $\mathbb{P}^n$  is finely represented by a smooth Deligne–Mumford stack  $\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a)$ .*

*For any other pair  $(\mathcal{A}', a')$  as above such that  $a_i \geq a'_i$  for all  $i = 1, \dots, m$  and  $a \geq a'$ , there is a natural birational morphism*

$$\overline{M}_{0,\mathcal{A}}(\mathbb{P}^n, d, a) \rightarrow \overline{M}_{0,\mathcal{A}'}(\mathbb{P}^n, d, a'),$$

*a weighted blow-up along regular local embeddings.*

As a corollary, we construct a sequence of spaces and morphisms posed between  $\overline{M}_{0,m+1}(\mathbb{P}^n, d)$  and  $\overline{M}_{0,m}(\mathbb{P}^n, d)$ . The case  $m = 0$  is somewhat special, and we treat it separately.

**Corollary 1.3.** *There is a sequence of smooth Deligne–Mumford stacks  $\{\overline{U}^k\}_{0 \leq k \leq \lfloor (d-1)/2 \rfloor}$  and of morphisms*

$$\overline{M}_{0,1}(\mathbb{P}^n, d) = \overline{U}^0 \rightarrow \dots \rightarrow \overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$$

*such that the morphism  $\overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$  has 1 dimensional fibers, and for each of the morphisms  $f_k^{k-1} : \overline{U}^{k-1} \rightarrow \overline{U}^k$  there is one boundary divisor  $\overline{U}_h^k$  of  $\overline{U}^k$  with the property that  $f_k^{k-1}$  restricted to  $\overline{U}_h^k$  has 1 dimensional fibers.*

*Proof.* Let  $k$  be any natural number such that  $0 \leq k \leq \lfloor (d-1)/2 \rfloor$ . Fix a rational number  $\epsilon$  such that  $0 < \epsilon < 1$ . Consider the space  $\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k+\epsilon))$  constructed according to Definition 1.1, and its universal family  $\tilde{U}^k$ . Alternatively,  $\tilde{U}^k$  may be obtained directly by Definition 1.1, as the space  $\overline{M}_{0,(0)}(\mathbb{P}^n, d, 1/(k+\epsilon))$  of maps with one marked point of weight zero. Indeed, Definition 1.1 still makes sense for weights zero, although more than one marked point of zero weight may create singularities of the moduli space ([Has]).

By Proposition 1.2, the objects above are smooth Deligne–Mumford stacks, and there are natural birational morphisms

$$\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k'+\epsilon)) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k+\epsilon)) \text{ and } \tilde{U}^{k'} \rightarrow \tilde{U}^k$$

for  $0 \leq k' < k \leq \lfloor (d-1)/2 \rfloor$ . Note that  $\overline{M}_{0,0}(\mathbb{P}^n, d) = \overline{M}_{0,0}(\mathbb{P}^n, d, 1/\epsilon)$ , and thus  $\overline{M}_{0,1}(\mathbb{P}^n, d) \cong \tilde{U}^0$ .

The space  $\overline{U}^k$  is defined as the fiber product

$$\overline{U}^k = \overline{M}_{0,0}(\mathbb{P}^n, d) \times_{\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k+\epsilon))} \tilde{U}^k.$$

The existence of natural morphisms  $\overline{U}^{k'} \rightarrow \overline{U}^k$  for  $k' < k$  is implied by the universality property of fiber products. The morphism from  $\overline{M}_{0,1}(\mathbb{P}^n, d)$  to  $\overline{U}^k$  contracts the components of degree no larger than  $k$  in each fiber over  $\overline{M}_{0,0}(\mathbb{P}^n, d)$ .  $\square$

*Notation.* For any  $0 \leq k' \leq k \leq \lfloor (d-1)/2 \rfloor$ , we denote the natural morphisms constructed above by  $f_k^{k'} : \overline{U}^{k'} \rightarrow \overline{U}^k$  and  $f^k : \overline{U}^k \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$ .

Consider a homogeneous coordinate system  $\bar{t} = (t_0 : \dots : t_n)$  on  $\mathbb{P}^n$ . An étale cover  $\bigsqcup_{\bar{t}} \overline{M}_{0,0}(\mathbb{P}^n, d, \bar{t})$  for the stack  $\overline{M}_{0,0}(\mathbb{P}^n, d)$  was constructed in [FP].  $\overline{M}_{0,0}(\mathbb{P}^n, d, \bar{t})$  parametrizes stable maps together with  $(n+1)d$  sections obtained by pullback of the hyperplanes  $(t_i = 0)$  in  $\mathbb{P}^n$ .

Pullback of  $\overline{M}_{0,0}(\mathbb{P}^n, d, \bar{t})$  to  $\overline{U}^k$  yields analogous étale covers  $\bigsqcup_{\bar{t}} \overline{U}^k(\bar{t})$ . The morphisms  $f_{k+1}^k : \overline{U}^k \rightarrow \overline{U}^{k+1}$  may be well understood at the level of these covers.

**Lemma 1.4.** *For  $k < \lfloor (d-1)/2 \rfloor$ ,  $f_{k+1}^k : \overline{U}^k \rightarrow \overline{U}^{k+1}$  is a blow-up along a codimension 2 regular embedding.*

*When  $d$  is odd, the morphism  $f^{\lfloor (d-1)/2 \rfloor} : \overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$  is a  $\mathbb{P}^1$ -bundle. When  $d$  is even, let  $\overline{M}(d/2)$  denote the image of*

$$\overline{M}_{0,1}(\mathbb{P}^n, d/2) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d/2)$$

*in  $\overline{M}_{0,0}(\mathbb{P}^n, d)$ . Over the complement of  $\overline{M}(d/2)$  in  $\overline{M}_{0,0}(\mathbb{P}^n, d)$ ,  $\overline{U}^{\lfloor (d-1)/2 \rfloor}$  is a  $\mathbb{P}^1$ -fibre bundle. Over  $\overline{M}(d/2)$ , it is the  $\mathbb{Z}_2$ -quotient of a stack obtained by gluing two  $\mathbb{P}^1$ -bundles along a common section.*

*Proof.* Consider a subset  $h \subset \{1, \dots, d\}$  of cardinality  $k$ . Let  $\bar{h}$  denote its complement and let  $\overline{M}_{h\bar{h}} := \overline{M}_{0,1}(\mathbb{P}^n, |h|) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, |\bar{h}|)$ . We denote by  $\overline{U}_{h\bar{h}}^{k-1}$  the fiber product  $\overline{M}_{h\bar{h}} \times_{\overline{M}_{0,0}(\mathbb{P}^n, d)} \overline{U}^{k-1}$ , and by  $\overline{U}_{h\bar{h}}^k$  the fiber product  $\overline{M}_{h\bar{h}} \times_{\overline{M}_{0,0}(\mathbb{P}^n, d)} \overline{U}^k$ . The latter admits a canonical section  $s_h : \overline{M}_{h\bar{h}} \rightarrow \overline{U}_{h\bar{h}}^k$ . Indeed, we recall from Definition 1.1 the existence of a bundle  $\mathcal{L}$  over  $\overline{U}^k$  and of a morphism  $e : \mathcal{O}^{n+1} \rightarrow \mathcal{L}$ . Moreover, each fiber  $C_x$  of  $f^k$  over a point  $x \in \overline{M}_{h\bar{h}}$  contains a special point  $p_x$  in the support of  $\text{coker } e_x$ , such that  $\dim(\text{coker } e_x)_{p_x} = k$ . Set theoretically, we may say  $s_h(x) := p_x$ . This extends canonically to a stack-theoretical definition.

A look at the étale covers establishes  $\overline{U}^{k-1}$  as the blow-up of  $\overline{U}^k$  along the image of  $s_h$ . Indeed, consider a homogeneous coordinate system  $\bar{t} = (t_0 : \dots : t_n)$  on  $\mathbb{P}^n$  as in the discussion preceding Lemma 1.4. By our construction, the morphism  $\overline{U}^{k-1}(\bar{t}) \rightarrow \overline{U}^k(\bar{t})$  is the pullback of the morphism

$$\tilde{U}^{k-1}(\bar{t}) \rightarrow \tilde{U}^k(\bar{t}) \times_{\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k+\epsilon), \bar{t})} \overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k-1+\epsilon), \bar{t}).$$

We recall (see [MM2], Proposition 1.6) that the space  $\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k + \epsilon), \bar{t})$  is the total space of a  $(\mathbb{C}^*)^n$ -bundle over a locally closed subset in the moduli space of weighted curves  $\overline{M}_{0,\mathcal{A}_k}$ , where  $\mathcal{A}_k$  is the  $(n + 1)d$ -tuple of weights  $(1/(k + \epsilon), \dots, 1/(k + \epsilon))$ . Indeed, the  $(n + 1)d$  marked points come from the hyperplane sections of  $\bar{t}_i$ . Therefore, as in [Has], Proposition 4.5 and Remark 4.6, the map

$$\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k - 1 + \epsilon), \bar{t}) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k + \epsilon), \bar{t})$$

is a blow-up along the image of  $\overline{M}_{h\bar{h}}(\bar{t})$  in  $\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k + \epsilon), \bar{t})$ . Similarly,  $\tilde{U}^k(\bar{t})$  is the total space of a  $(\mathbb{C}^*)^n$ -bundle over a locally closed subset in the moduli space of weighted curves  $\overline{M}_{0,(0,\mathcal{A}_k)}$ . Therefore the map  $\tilde{U}^{k-1}(\bar{t}) \rightarrow \tilde{U}^k(\bar{t})$  is a composition of blow-ups contracting the image of  $\bar{U}_{h\bar{h}}^{k-1}(\bar{t})$  in  $\tilde{U}^{k-1}(\bar{t})$ . We note that  $\bar{U}_{h\bar{h}}^{k-1}(\bar{t})$  consists of two components,  $\bar{U}_h^{k-1}(\bar{t})$  and  $\bar{U}_{\bar{h}}^{k-1}(\bar{t})$ . The rigid maps parametrized by  $\bar{U}_h^{k-1}(\bar{t})$  are from split curves having components of degrees  $k$  and  $d - k$ , respectively, and a marked point on the degree  $k$  component, whereas for  $\bar{U}_{\bar{h}}^{k-1}(\bar{t})$ , the marked point is on the degree  $d - k$  component. From our examination of the morphism

$$\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k - 1 + \epsilon), \bar{t}) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k + \epsilon), \bar{t})$$

above, it follows that the image of  $\bar{U}_h^{k-1}(\bar{t})$  is the divisor contracted by the map

$$\tilde{U}^k(\bar{t}) \times_{\overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k + \epsilon), \bar{t})} \overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k - 1 + \epsilon), \bar{t}) \rightarrow \tilde{U}^k(\bar{t}).$$

Thus

$$\tilde{U}^{k-1}(\bar{t}) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d, 1/(k - 1 + \epsilon), \bar{t})$$

is a blow-up contracting the image of  $\bar{U}_h^{k-1}(\bar{t})$ . Clearly the image of  $\bar{U}_h^{k-1}$  in  $\bar{U}_{h\bar{h}}^k$  coincides with the image of  $s_h$ . We note that while  $\bar{U}_{h\bar{h}}^k \rightarrow \bar{U}^k$  is only a local regular embedding, the image of  $s_h$  is regularly embedded in  $\bar{U}^{k+1}$ .

When  $d$  is odd, the existence of enough distinct sections makes the morphism  $f^{[(d-1)/2]}(\bar{t}) : \bar{U}^{[(d-1)/2]}(\bar{t}) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d, \bar{t})$  locally trivial. The sections come from among the marked sections of the universal curve

$$\bar{U}^{[(d-1)/2]}(\bar{t}) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d, 1/([(d-1)/2] + \epsilon), \bar{t})$$

given by the hyperplanes  $\bar{t}_i$ . When  $d$  is even, the same reasoning applies on the complement of  $\overline{M}(d/2)$ . Now let  $h \subset \{1, \dots, d\}$  be a cardinal  $(d/2)$ -subset, and let  $\overline{M}_{h\bar{h}}$  be defined as above. Consider two copies  $\overline{M}_h$  and  $\overline{M}_{\bar{h}}$  of  $\overline{M}_{0,2}(\mathbb{P}^n, d/2) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d/2)$ . There is a fiber square diagram

$$\begin{array}{ccc} \overline{M}_h \cup_s \overline{M}_{\bar{h}} & \longrightarrow & \overline{M}_{0,1}(\mathbb{P}^n, d) \times_{\overline{M}_{0,0}(\mathbb{P}^n, d)} \overline{M}(d/2) \\ \downarrow & & \downarrow \\ \overline{M}_{h\bar{h}} & \longrightarrow & \overline{M}(d/2), \end{array}$$

where the two fiber products  $\overline{M}_h$  and  $\overline{M}_{\bar{h}}$  are glued along the canonical section

$$s : \overline{M}_{0,1}(\mathbb{P}^n, d/2) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d/2) \rightarrow \overline{M}_{0,2}(\mathbb{P}^n, d/2) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d/2).$$

There is a natural  $\mathbb{Z}_2$ -action on the two spaces at the left side of the diagram, and the quotients are on the right side. The same symmetry is preserved after  $\overline{M}_h$  and  $\overline{M}_{\bar{h}}$  are contracted to two  $\mathbb{P}^1$ -bundles over  $\overline{M}_{h\bar{h}}$  by the successive blow-downs down to the  $((d-2)/2)$ -th step. (The precise moduli problems for these contractions are defined in Section 2.)  $\square$

*Notation.* Let  $A, B$  be two disjoint sets such that  $A \cup B = \{1, \dots, m + 1\}$  and let  $i$  be a natural number,  $0 \leq i \leq d$ . We denote by  $D(A, i|B, d - i)$ , the divisor representing split curves, of a degree  $i$ -component containing the set  $A$  of marked points and a degree  $(d - i)$ -component containing the set  $B$  of marked points.

Similar to the previous construction, there is a sequence of morphisms

$$\overline{M}_{0,m+1}(\mathbb{P}^n, d) = \overline{U}_m^0 \rightarrow \overline{U}_m^1 \rightarrow \dots \rightarrow \overline{U}_m^{d+m-2} \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$$

defined as follows. For  $k = 1, \dots, d + m - 2$ , let  $a_k = 1/(k + \epsilon)$ , where  $\epsilon$  is a small positive rational number, and let  $\mathcal{A}_k$  be the  $m$ -tuple consisting of one copy of 1 and  $m - 1$  copies of  $a_k$ . The space  $\overline{U}_m^k$  is the pullback to  $\overline{M}_{0,m}(\mathbb{P}^n, d)$  of the universal family over  $\overline{M}_{0,\mathcal{A}_k}(\mathbb{P}^n, d, a_k)$ . Let 1 be the marked point of weight 1 on the generic curve over  $\overline{M}_{0,\mathcal{A}_k}(\mathbb{P}^n, d, a_k)$ . Thus the morphism  $\overline{M}_{0,m+1}(\mathbb{P}^n, d) \rightarrow \overline{U}_m^1$  is a blow-up with exceptional divisor  $D(\{m + 1\}, 1|\{1, \dots, m\}, d - 1)$ .

The components  $C$  of the fibers of  $\overline{M}_{0,m+1}(\mathbb{P}^n, d) \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$  which are contracted by the morphism  $\overline{M}_{0,m+1}(\mathbb{P}^n, d) \rightarrow \overline{U}_m^k(\mathbb{P}^n)$  will be called  $k$ -unstable whenever  $k \geq 1$ . They are such that the sum of the degree of  $C$  with the number of marked points on  $C$  is no larger than  $k$ .

The morphism  $\overline{U}_m^{k-1} \rightarrow \overline{U}_m^k$  factors into a sequence of blow-ups along disjoint codimension two loci. Pullbacks of the exceptional divisors to  $\overline{M}_{0,m+1}(\mathbb{P}^n, d)$  are  $D(A, i|B, d - i)$  such that  $m + 1 \in A$ ,  $1 \in B$ , and  $|A| + i = k$ . The corresponding blow-up locus is isomorphic to the support of  $D(A \setminus \{m + 1\}, i|B, d - i)$ . The intermediate blow-ups posed between  $\overline{U}_m^{k-1}$  and  $\overline{U}_m^k$  are also moduli spaces in their own right. Indeed, such a space is obtained as a pullback of the universal family over  $\overline{M}_{0,\mathcal{A}'_k}(\mathbb{P}^n, d, a_k)$ , where the  $m$ -tuple  $\mathcal{A}'_k$  consists of one copy of 1, some copies of  $a_k$  and other copies of  $a_{k-1}$ , in the desired order. The universal family is in itself a moduli space of weighted pointed stable maps, where the generic point of its fiber is assigned weight 0.

Finally,  $\overline{U}_m^{d+m-2} \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$  is a  $\mathbb{P}^1$ -bundle. Indeed, let  $C_x$  denote the fiber of the forgetful morphism  $\overline{M}_{0,m+1}(\mathbb{P}^n, d) \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$  over the point  $x$ , with the  $m$  marked sections  $p_1, \dots, p_m$ . The fiber of  $\overline{U}_m^{d+m-2} \rightarrow \overline{M}_{0,m}(\mathbb{P}^n, d)$  consists of the irreducible component of  $C_x$  containing  $p_1$ . On rigid covers, the above morphism always has three disjoint sections:  $p_1$ , the universal section, and at least one hyperplane section.

**1.2. The case of a general target.** Let  $X$  be a smooth complex projective variety, and let  $d \in H_2(X)$  be a class curve. A factorization of the forgetful morphism  $\overline{M}_{0,m+1}(X, d) \rightarrow \overline{M}_{0,m}(X, d)$  may be induced from any projective embedding of  $X$ . From the point of view of the boundary strata however, it is more natural to consider the embedding of  $X$  in a product of projective spaces as follows. Consider  $\mathcal{L}_1, \dots, \mathcal{L}_s \in \text{Pic}(X)$  very ample, such that their first Chern classes generate the algebraic part of  $H^2(X, \mathbb{Z})$ . We consider the embedding of  $X$  given by all  $\mathcal{L}_i$ .

To any curve class  $d \in H_2(X)$  we assign an  $s$ -tuple  $d = (d_1, \dots, d_s)$ , such that  $\int_d c_1(\mathcal{L}_i) = d_i$ .

Let  $n_i := h^0(\mathcal{L}_i) - 1$ . Definition 1.1 above may be extended to the case when the target is  $\prod_i \mathbb{P}^{n_i}$  by considering an  $s$ -tuple of weights  $a = (a^1, \dots, a^s)$ , an  $s$ -tuple  $\mathcal{L} := \{\mathcal{L}'_i\}_i$  of line bundles with morphisms  $e^i : \mathcal{O}_C^{n_i+1} \rightarrow \mathcal{L}'_i$ , and defining  $\mathcal{L}^a := \otimes_i \mathcal{L}'_i^{\otimes a^i}$ . The spaces  $\overline{M}_{0,\mathcal{A}}(\prod_i \mathbb{P}^{n_i}, d, a)$  are still smooth Deligne–Mumford stacks, by the same reasons as for target  $\mathbb{P}^n$ .

For each tuple of non-negative integers  $k = (k_1, \dots, k_s, k'_2, \dots, k'_m)$ , define a system of weights  $a_k := (\frac{1}{k_1+\epsilon}, \dots, \frac{1}{k_s+\epsilon})$  and  $\mathcal{A}_k := (1, \frac{1}{k'_2+\epsilon}, \dots, \frac{1}{k'_m+\epsilon})$ . To these we assign the moduli spaces  $\overline{U}_m^k(\prod_i \mathbb{P}^{n_i})$  set between  $\overline{M}_{0,m+1}(\prod_i \mathbb{P}^{n_i}, d)$  and  $\overline{M}_{0,m}(\prod_i \mathbb{P}^{n_i}, d)$ , constructed by the same method as above. We define

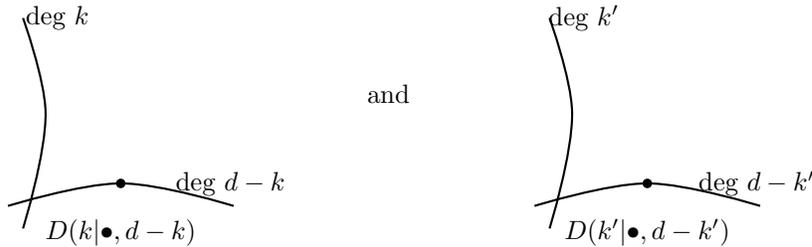
$$\overline{U}_m^k(X) := \overline{U}_m^k(\prod_i \mathbb{P}^{n_i}) \times_{\overline{M}_{0,m}(\prod_i \mathbb{P}^{n_i}, d)} \overline{M}_{0,m}(X, d).$$

The morphism  $\overline{M}_{0,m+1}(X, d) \rightarrow \overline{U}_m^k(X)$  contracts components  $C$  of the fibers of  $\overline{M}_{0,m+1}(X, d) \rightarrow \overline{M}_{0,m}(X, d)$  such that the degree of  $C$  is  $(l_1, \dots, l_s)$  and  $\sum_{i=1}^s \frac{l_i}{k_i+\epsilon} + \sum_{p_j \in C} \frac{1}{k'_j+\epsilon} \leq 1$ . Such a component will be called  $k$ -unstable.

2. THE EXTENDED COHOMOLOGY AND CHOW RINGS

The factorization of the forgetful map facilitates understanding the structure of the ring  $H^*(\overline{M}_{0,m+1}(\mathbb{P}^n, d))$  as an algebra over  $H^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$ . Boundary classes are adjoined at each blow-up step, and codimension two relations among them exist. Additionally,  $\overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$  admits a relative cotangent class  $\psi'$ , whose pullback is related to the canonical class  $\psi$  on  $\overline{M}_{0,1}(\mathbb{P}^n, d)$ . The relation between intersection rings becomes truly transparent when we regard our spaces together with the networks generated by their canonical closed strata. Then relations among cohomology classes may be deduced by simple induction.

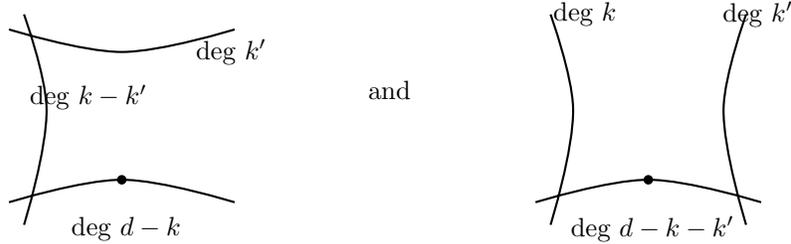
The cohomology ring  $H^*(\overline{M}_{0,m}(\mathbb{P}^n, d))$  contains a wealth of classes in all degrees coming from boundary strata, via Poincaré duality. The generic point of each such stratum represents a curve in  $\mathbb{P}^n$  of a specific split type. Such strata have intricate intersection patterns. One of the simplest examples involves the classes  $D(k|\bullet, d-k)$  and  $D(k'|\bullet, d-k')$ , representing curves of split types



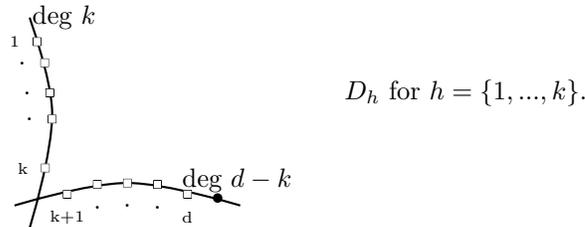
when  $m = 1$ , and satisfying the degree two relation

$$D(k|\bullet, d - k)D(k'|\bullet, d - k') = D(k'|k - k'|\bullet, d - k) + D(k|\bullet, d - k - k'|k')$$

whenever  $k' \leq k$  and  $k + k' \leq d$ . Here  $\bullet$  denotes the marked point, while  $D(k'|k - k'|\bullet, d - k)$  and  $D(k|\bullet, d - k - k'|k')$  represent curves of split types



respectively. More complex relations arise in higher codimension. These strata are usually encoded by a system of decorated stable trees. However, the relations between intersections of various strata cannot be easily tracked down in this system. An alternative route is motivated by the local structure of the stable map spaces (given by  $\bar{t}$ -covers like those presented in the proof of Lemma 1.4.): we extend the cohomology ring of the moduli spaces by a set of new, degree 1 variables  $\{D_h\}_h$ , such that all boundary strata are symmetric polynomials of these variables. For example when  $m = 1$ , intuitively  $D_h$  represents curves  $C$  in  $\mathbb{P}^n$  of a certain split type  $(k|\bullet, d - k)$  as above, together with a partition  $h \cup \bar{h} = \{1, \dots, d\}$  of  $d$  special points (thought of as coming from a generic hyperplane section of  $C$ ), such that the points in  $\bar{h}$  lie on the component containing the marked point  $\bullet$ .



Thus, for example

$$\begin{aligned}
 D(k|\bullet, d - k) &= \sum_{|h|=k} D_h, \\
 D(k|\bullet, d - k - k'|k') &= \sum_{|h|=k, |h'|=k', h' \cap h = \emptyset} D_h D_{h'}, \\
 D(k'|k - k'|\bullet, d - k) &= \sum_{|h|=k, |h'|=k', h' \subset h} D_h D_{h'}
 \end{aligned}$$

are polynomials invariant with respect to the natural action of the symmetric group  $S_d$  on the power set  $\mathcal{P}(\{1, \dots, d\})$ .

In the following paragraph we proceed to a more rigorous definition of the strata and of the above mentioned extension of the cohomology ring of  $\bar{M}_{0,m}(X, d)$ , for a general smooth projective variety  $X$ . We refer the interested reader to [MM1], Lemma 3.9–Lemma 3.14 for detailed proofs.

Consider a general smooth projective target  $X \hookrightarrow \prod_{i=1}^s \mathbb{P}^{n_i}$  as in the preceding subsection, and  $d = (d_1, \dots, d_s)$ . We denote by  $G = S_{d_1} \times \dots \times S_{d_s}$  the product of  $s$

groups of permutations  $S_{d_i}$ . We recall succinctly the construction of a  $G$ -network of morphisms associated to  $\overline{U}_m^k(X)$  and its extended cohomology ring (see [MM1] and [MM2] for a more detailed motivation). We keep notation from the preceding section. Here we work in cohomology although the same construction works for Chow rings.

Let  $I$  be a set whose elements are of the form  $h = h_1 \sqcup \dots \sqcup h_s \sqcup M_h$  such that each  $h_i \subset \{1, \dots, d_i\}$  and  $M_h \subset \{2, \dots, m\}$ . Assume that  $h \cap h' = h, h'$  or  $\emptyset$  for all  $h, h' \in I$ .

**Definition 2.1.** The space  $\overline{U}_{m,I}^k(X)$  parametrizes degree  $d$  stable maps

$$\varphi : (C, \{p_j\}_{j=1, \dots, m}) \rightarrow X$$

together with a smooth point  $p_{m+1} \in C$  and marked closed curves  $\{C_h\}_{h \in I}$  such that:

- (1)  $p_{m+1} \notin C'$  for any  $k$ -unstable curve  $C' \subset C$ .
- (2)  $\forall h \in I, p_1 \notin C_h \subset C$ , and the degree of the map  $\varphi|_{C_h}$  is  $(|h_1|, \dots, |h_s|)$ .
- (3) The incidence relations among the elements of  $I$  translate into analogous incidence relations among the curves  $C_h$ :
  - $\forall h \in I, \forall i \in \{2, \dots, m\}, p_i \in C_h$  iff  $i \in M_h$ .
  - $C_h \subset C_{h'}$  iff  $h \subset h'$  and  $C_h \cap C_{h'} = \emptyset$  iff  $h \cap h' = \emptyset$ .

The space  $\overline{U}_{m,I}^k(X)$  is locally embedded in  $\overline{U}_m^k(X)$ . In fact,  $\overline{U}_m^k(X)$  corresponds to the choice  $I = \emptyset$ . For any two subsets  $I \subset J$  as above, there is natural local regular embedding  $\phi_J^I : \overline{U}_{m,J}^k(X) \rightarrow \overline{U}_{m,I}^k(X)$ . The spaces  $\overline{U}_{m,I}^k(X)$  with the morphisms  $\phi_J^I$  form a network on which the group  $G$  acts naturally.

In the case of  $\overline{M}_{0,0}(X, d)$  we employ slightly different notation. Here we take  $I$  to be a set of 2-partitions  $h \cup \bar{h} = \bigsqcup_i \{1, \dots, d_i\}$  such that for any pair  $(h, \bar{h}), (h', \bar{h}') \in I$ , the set  $\{h \cap h', h \cap \bar{h}', \bar{h} \cap h', \bar{h} \cap \bar{h}'\}$  has exactly three nonempty elements. Then  $\overline{U}_{0,I}^k(X)$  parametrizes stable maps  $\varphi : C \rightarrow X$ , together with one point  $p_1 \in C$  and splittings  $C_h \cup C_{\bar{h}} = C$  for all  $(h, \bar{h}) \in I$  satisfying all the relevant properties from Definition 2.1.

Let  $\mathcal{I}$  be the set of all sets  $I$  as above.  $G$  acts on  $\mathcal{I}$  by permutations. For each  $I \in \mathcal{I}$ , let  $G_I \subset G$  be the subgroup which fixes all elements of  $I$ . Any  $g \in G$  induces a canonical isomorphism  $g : \overline{U}_{m,I}^k(X) \rightarrow \overline{U}_{m,g(I)}^k(X)$  such that the following diagram is commutative:

$$\begin{CD} \overline{U}_{m,J}^k(X) @>g>> \overline{U}_{m,g(J)}^k(X) \\ @V\phi_J^I VV @VV\phi_{g(J)}^{g(I)} V \\ \overline{U}_{m,I}^k(X) @>g>> \overline{U}_{m,g(I)}^k(X) \end{CD}$$

whenever  $I \subset J$ .

In this paper all cohomology is considered with rational coefficients. The extended cohomology ring  $B^*(\overline{U}_m^k(X))$  is constructed as follows.

**Definition 2.2.** The graded  $\mathbb{Q}$ -vector space  $B^*(\overline{U}_m^k(X); \mathbb{Q})$  is

$$B^*(\overline{U}_m^k(X); \mathbb{Q}) := \bigoplus_{l=0}^{\dim(\overline{U}_m^k)} B^l(\overline{U}_m^k(X)),$$

where the extended cohomology groups are

$$B^l(\overline{U}_m^k(X)) := \bigoplus_I H^{l - \text{codim}_{\overline{U}_m^k(X)} \overline{U}_{m,I}^k(X)}(\overline{U}_{m,I}^k(X)) / \sim,$$

the sum taken after all subsets  $I$  as above with  $\text{codim}_{\overline{U}_m^k(X)} \overline{U}_{m,I}^k(X) \leq l$ . The equivalence relation  $\sim$  is generated by

$$\phi_{J*}^I(\alpha) \sim \sum_{[g] \in G_I/G_J} g_*(\alpha)$$

for any  $\alpha \in H^*(\overline{U}_{m,J}^k(X))$  and  $J \supset I$ . Here  $\phi_{J*}^I$  is the Gysin pushforward, as all  $\phi_J^I : \overline{U}_{m,J}^k(X) \rightarrow \overline{U}_{m,I}^k(X)$  are local complete intersections. Multiplication is defined as follows:

$$\alpha \cdot_r \beta := \overline{\phi}_{I \cup J}^{I*}(\alpha) \cdot \overline{\phi}_{I \cup J}^{J*}(\beta) \cdot c_{\text{top}}(\overline{\phi}_{I \cup J}^{I \cap J*} \mathcal{N}_{\overline{U}_{m,I \cap J}^k(X) | \overline{U}_m^k(X)})$$

for  $\alpha \in H^*(\overline{U}_{m,I}^k(X))$  and  $\beta \in H^*(\overline{U}_{m,J}^k(X))$ , where  $\mathcal{N}_{\overline{U}_{m,I \cap J}^k(X) | \overline{U}_m^k(X)}$  denotes the normal bundle of  $\overline{U}_{m,I \cap J}^k(X)$  in  $\overline{U}_m^k(X)$ .

The extended rings  $B^*$  satisfy all the good properties of the usual intersection rings, under suitable conditions as described below. Thus pullback is well defined for morphisms compatible with the  $G$ -network. An example is provided by the morphisms  $f_{k,m}^{k'} : \overline{U}_m^{k'}(X) \rightarrow \overline{U}_m^k(X)$ . For each  $I$  as in Definition 2.1, consider the fiber square

$$\begin{array}{ccc} \overline{U}_m^{k'}(X) \times_{\overline{U}_m^k(X)} \overline{U}_{m,I}^k(X) & \longrightarrow & \overline{U}_m^{k'}(X) \\ \downarrow f_{k,m,I}^{k'} & & \downarrow f_{k,m}^{k'} \\ \overline{U}_{m,I}^k(X) & \longrightarrow & \overline{U}_m^k(X) \end{array}$$

where  $\overline{U}_m^{k'}(X) \times_{\overline{U}_m^k(X)} \overline{U}_{m,I}^k(X)$  is a union of normal strata in  $\overline{U}_m^{k'}(X)$ . The pullback  $f_{k,m}^{k'*} : B^*(\overline{U}_m^k(X)) \rightarrow B^*(\overline{U}_m^{k'}(X))$  is constructed by concatenating all pullbacks  $f_{k,m,I}^{k'*}$ . These are obviously compatible with the equivalence relation of Definition 2.2.

When the target  $X$  is convex, all the moduli spaces  $\overline{U}_{m,I}^k(X)$  are smooth Deligne–Mumford stacks, and there is an isomorphism between  $B^*(\overline{U}_{m,I}^k(X))$  and the analogously defined homology  $B_*(\overline{U}_{m,I}^k(X))$ . Then pushforward is well defined for morphisms compatible with the  $G$ -network.

As a central feature,  $B^*(\overline{U}_m^k(X))$  contains the usual ring  $H^*(\overline{U}_m^k(X))$  as an invariant subalgebra under the natural action of the group  $G$ .

Working in  $B^*(\overline{M}_{0,m+1}(X, d))$  is very convenient because all boundaries may be decomposed as polynomials of divisor classes. Indeed, for  $h = h_1 \sqcup \dots \sqcup h_s \sqcup M_h$

as above, let the fundamental class of  $\overline{U}_{m,h}^0(X)$  in  $B^*(\overline{M}_{0,m}(X,d))$  be denoted by  $D_h$ . Then  $[\overline{U}_{m,I}^0(X)] = \prod_{h \in I} D_h$ , while the image of  $\overline{U}_{m,I}^0$  in  $\overline{M}_{0,m+1}(X,d)$  has class  $\sum_{g \in G} \prod_{h \in I} D_{g(h)}$ .

Another important feature is that the product  $D_h^2$  has a well known geometric significance. Indeed, the stratum of class  $D_h$  is a fiber product of the form  $\overline{U}_{m,h}^0 = \overline{M}_h \times_X \overline{M}_{\bar{h}}$ , where  $\overline{M}_h$  and  $\overline{M}_{\bar{h}}$  are themselves moduli spaces and the product is along evaluation morphisms  $ev_h : \overline{M}_h \rightarrow X$ ,  $ev_{\bar{h}} : \overline{M}_{\bar{h}} \rightarrow X$ . There are canonical  $\psi$ -classes associated to  $ev_h$  and  $ev_{\bar{h}}$ . Their pullbacks  $\psi_h$  and  $\psi_{\bar{h}}$  are known to satisfy the relation

$$(2.1) \quad \psi_h + \psi_{\bar{h}} = c_1(\mathcal{N}_{\overline{U}_{m,h}^0/\overline{M}_{0,m+1}(X,d)}^\vee)$$

on  $\overline{U}_{m,h}^0$ , which is  $-D_h^2$  in  $B^*(\overline{M}_{0,m+1}(X,d))$ .

### 3. UNIVERSAL RELATIONS IN COHOMOLOGY AND CHOW RINGS

We will denote by  $D_h$  the fundamental classes of boundary divisors in  $B^*(\overline{M}_{0,m+1}(X,d))$ , where  $h = h_1 \sqcup \dots \sqcup h_s \sqcup M_h$  are as above. In  $B^*(\overline{M}_{0,0}(X,d))$ , the fundamental classes of boundary divisors will be denoted by  $D_{h\bar{h}}$ , where  $h = h_1 \sqcup \dots \sqcup h_s$ . For most of this section we will focus on the case  $X = \mathbb{P}^n$ .

Consider the forgetful map  $f : \overline{M}_{0,m+1}(\mathbb{P}^n,d) \rightarrow \overline{M}_{0,m}(\mathbb{P}^n,d)$  with  $m$  canonical sections  $s_i$  and the evaluation maps  $ev_i : \overline{M}_{0,m+1}(\mathbb{P}^n,d) \rightarrow \mathbb{P}^n$ , for  $i = 1, \dots, m+1$ . Let  $\psi_i := c_1(L_i)$ , where the tautological line bundle  $L_i$  is a pullback by  $s_i$  of the relative cotangent bundle of  $f$ . For any class  $\alpha \in H^*(\mathbb{P}^n)$ , we define the kappa class  $k(\alpha) := f_* ev_{m+1}^* \alpha$ .

$H$  will denote the hyperplane divisor on  $\mathbb{P}^n$ .

We recall the following comparison formula on  $\overline{M}_{0,m+1}(\mathbb{P}^n,d)$ :

$$(3.1) \quad \psi_i = f^* \psi_i + D_{i,m+1},$$

where  $D_{i,m+1}$  is the Cartier divisor associated to the canonical section  $s_i$  of  $f$  (see for example [W]).

The notation introduced at the beginning of Section 2 will be employed in the following lemma.

**Lemma 3.1.** *The following codimension two relation holds in the cohomology ring  $H^*(\overline{M}_{0,1}(\mathbb{P}^n,d))$ :*

$$\begin{aligned} \frac{4}{d^3} k(H^3) - \frac{3}{d^4} k(H^2)^2 &= \psi^2 - \sum_{k \leq d} N_k^d \psi D(k|\bullet, d-k) \\ &+ \sum_{k \leq k'} M_{kk'}^d D(k|\bullet, d-k) D(k'|\bullet, d-k') + \sum_{k \leq k'} P_{kk'}^d D(k|\bullet, d-k-k'|k'), \end{aligned}$$

where

$$\begin{aligned} N_k^d &= \frac{k^2}{d^2} (6 - 4 \frac{k}{d}), \\ M_{kk'}^d &= \frac{k^2}{d^2} (6 \frac{k'}{d} - 2 \frac{k}{d} - 3 \frac{k'^2}{d^2}), \\ P_{kk'}^d &= 2 \frac{(k+k')(k^2 - 4kk' + k'^2)}{d^3}. \end{aligned}$$

Written in the ring  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ , the relation above is equivalent to the following class being zero:

$$R := \psi^2 - \sum_h N_h^d \psi D_h + \sum_{(h,h')} N_{hh'}^d D_h D_{h'} + \frac{3}{d^4} k(H^2)^2 - \frac{4}{d^3} k(H^3),$$

where

$$N_h^d := N_{|h|}^d,$$

$$N_{h'h}^d = N_{hh'}^d = \begin{cases} M_{|h||h'|}^d = \frac{|h|^2}{d^2} (6 \frac{|h'|}{d} - 2 \frac{|h|}{d} - 3 \frac{|h'|^2}{d^2}) & \text{if } h' \supseteq h, \text{ and} \\ -3 \frac{|h|^2 |h'|^2}{d^4} & \text{if } h' \cap h = \emptyset. \end{cases}$$

*Proof.* The formula can be checked by increasing induction on  $d$ . When  $d = 1$ ,

$$\frac{\psi^2}{4} + \frac{3}{4} k(H^2)^2 - k(H^3) = 0$$

describes the usual relation in the cohomology of the flag space  $F(*, \mathbb{P}^1, \mathbb{P}^n)$  over that of the Grassmannian  $G(\mathbb{P}^1, \mathbb{P}^n)$ .

Assuming the formula is true for all degrees less than  $d$ , we can derive a codimension two relation on any divisor  $D_h$ . Indeed,  $D_h$  is the class of a normal stratum  $\overline{U}_h := \overline{M}_{0,1}(\mathbb{P}^n, |h|) \times_{\mathbb{P}^n} \overline{M}_{0,2}(\mathbb{P}^n, |\bar{h}|)$ , where the fiber product is via evaluation maps at the first marked points of  $\overline{M}_{0,1}(\mathbb{P}^n, |h|)$  and  $\overline{M}_{0,2}(\mathbb{P}^n, |\bar{h}|)$ . Let  $\pi_h$  and  $\pi_{\bar{h}}$  be the projections from  $\overline{U}_h$  on the first and second factors; let  $f_h : \overline{M}_{0,1}(\mathbb{P}^n, |h|) \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, |h|)$ ,  $f_{\bar{h}} : \overline{M}_{0,2}(\mathbb{P}^n, |\bar{h}|) \rightarrow \overline{M}_{0,1}(\mathbb{P}^n, |\bar{h}|)$  be the morphisms forgetting the first marked points. The evaluation maps at these points are  $ev_h$  and  $ev_{\bar{h}}$ . The following  $\psi$ -classes on  $\overline{U}_h$  will play a role in the computation:  $\psi_h := \pi_h^* \psi_1$  and  $\psi$ , pullback of the  $\psi$ -class from  $\overline{M}_{0,1}(\mathbb{P}^n, d)$ .

We define  $k_h(H^2) := \pi_h^* f_h^* f_{h*} ev_h^*(H^2)$ ,  $k_h(H^3) := \pi_h^* f_h^* f_{h*} ev_h^*(H^3)$ , and  $k_{\bar{h}}(H^2) := \pi_{\bar{h}}^* f_{\bar{h}}^* f_{\bar{h}*} ev_{\bar{h}}^*(H^2)$ ,  $k_{\bar{h}}(H^3) := \pi_{\bar{h}}^* f_{\bar{h}}^* f_{\bar{h}*} ev_{\bar{h}}^*(H^3)$ . All the following arguments take place on  $\overline{U}_h$ . By the additivity of kappa classes ([KK], Lemma 3.3),

$$k_h(H^2) + k_{\bar{h}}(H^2) = k(H^2) \text{ and } k_h(H^3) + k_{\bar{h}}(H^3) = k(H^3).$$

The induction hypothesis gives a formula for  $k_h(H^3)$  as a quadratic expression in  $k_h(H^2)$ ,  $\psi_h$  and divisors  $\{D_{h'}\}_{h' \subset h}$ :

$$\psi_h^2 - \sum_{h' \subset h} N_{h'}^{|h|} \psi_h D_{h'} + \sum_{(h',h'')} N_{h'h''}^{|h|} D_{h'} D_{h''} = \frac{4}{|h|^3} k_h(H^3) - \frac{3}{|h|^4} k_h(H^2)^2,$$

where  $h', h'' \subset h$ . Simultaneously, pullback via  $f_{\bar{h}}$  of the analogous relation on  $\overline{M}_{0,1}(\mathbb{P}^n, |\bar{h}|)$  expresses  $k_{\bar{h}}(H^3)$  as a quadratic function of  $k_{\bar{h}}(H^2)$ ,  $\pi_{\bar{h}}^* f_{\bar{h}}^* \psi$ , and  $\pi_{\bar{h}}^* f_{\bar{h}}^* D_{h'} = D_{h'} + D_{h' \cup h}$ , for sets  $h'$  such that  $h' \cap h = \emptyset$ :

$$\begin{aligned} & \pi_{\bar{h}}^* f_{\bar{h}}^* \psi^2 - \sum_{h' \subset \bar{h}} N_{h'}^{|\bar{h}|} \pi_{\bar{h}}^* f_{\bar{h}}^* (\psi D_{h'}) + \sum_{(h',h'')} N_{h'h''}^{|\bar{h}|} \pi_{\bar{h}}^* f_{\bar{h}}^* (D_{h'} D_{h''}) \\ & = \frac{4}{|\bar{h}|^3} k_{\bar{h}}(H^3) - \frac{3}{|\bar{h}|^4} k_{\bar{h}}(H^2)^2, \end{aligned}$$

where  $h', h'' \subset \bar{h}$ . Furthermore, by formula (3.1) and by Theorem 1 in [LP],  $\psi_h$  and  $\pi_{\bar{h}}^* f_{\bar{h}}^* \psi$  may be written as

$$\psi_h = \psi - \sum_{h' \supseteq h} D_{h'} \text{ and } \pi_{\bar{h}}^* f_{\bar{h}}^* \psi = \psi - D_{\bar{h}}.$$

Summation of  $k_h(H^3)$  and  $k_{\bar{h}}(H^3)$  yields  $k(H^3)$  as an expression in  $\psi$ ,  $k_h(H^2)$ ,  $k_{\bar{h}}(H^2)$ , and boundary divisors.

The classes  $k_h(H^2)$  and  $k_{\bar{h}}(H^2)$  can be written in terms of  $k(H^2)$ ,  $\psi$  and boundary divisors via the divisorial relation in Lemma 2.2.2 of [Pan]: on  $\overline{M}_{0,1}(\mathbb{P}^n, d)$ ,

$$(3.2) \quad \psi + \frac{2}{d}ev^*H - \frac{1}{d^2}k(H^2) - \sum_{h'} \frac{|h'|^2}{d^2}D_{h'} = 0.$$

Pullback by  $\pi_{\bar{h}} \circ f_{\bar{h}}$  of the equivalent relation on  $\overline{M}_{0,1}(\mathbb{P}^n, |\bar{h}|)$  is

$$\psi - D_{\bar{h}} + \frac{2}{|\bar{h}|}ev^*H - \frac{1}{|\bar{h}|^2}k_{\bar{h}}(H^2) - \sum_{h' \subset \bar{h}} \frac{|h'|^2}{|\bar{h}|^2}(D_{h'} + D_{h' \cup h}) = 0,$$

and thus, after eliminating  $ev^*H$ ,

$$k_{\bar{h}}(H^2) = \frac{|\bar{h}|}{d}k(H^2) - |\bar{h}||h|\psi - \sum_{h' \subset \bar{h}} \frac{|h'|^2|h|}{d}D_{h'} - \sum_{h' \supseteq h} |h|(\frac{|\bar{h}'|^2}{d} - |\bar{h}|)D_{h'},$$

while  $k_h(H^2) = k(H^2) - k_{\bar{h}}(H^2)$ .

Substituting the expressions for  $k_h(H^2)$  and  $k_{\bar{h}}(H^2)$  into the expression for  $k(H^3)$ , we obtain the desired formula modulo the codimension two annihilator of  $D_{\bar{h}}$  in  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ . By Theorem 3.23 in [MM1], this annihilator is generated by  $\psi D_{\bar{h}}$ ,  $D_{h'}D_{h''}$ ,  $D_{h'}\psi$ ,  $D_{h'}ev^*H$ , where  $h'$  is such that  $h' \cap h \neq h', h$  or  $\emptyset$ , and  $\{D_{h'}(\psi - \sum_{h'' \supseteq h \cup h'} D_{h''})\}_{h' \cap h = \emptyset}$ . Thus in  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ ,  $R = a(h)$ , where  $a(h)$  is a linear combination of the terms above. Of necessity,  $a(h) = a(h')$  for any  $h, h' \subset \{1, \dots, d\}$ . But there are no codimension two elements that annihilate all boundary divisors in  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ . Indeed, the only codimension two relations in  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  are linear combinations of monomials  $D_{h'}D_{h''}$  for  $h' \cap h'' \neq h', h'', \emptyset$ . It is enough to consider  $h, h'$  such that  $|h| = |h'| = d - 1$ . The only common annihilators for these are of the form  $D_{h''}D_{\bar{h}''}$ , where  $h'' \supset h$  or  $h'' \supset h'$ . But these do not annihilate  $D_{h''}$ .  $\square$

*Notation.* Let  $\overline{M}_{h\bar{h}} := \overline{M}_{0,1}(\mathbb{P}^n, |h|) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, |\bar{h}|)$ . Consider the gluing map  $\overline{M}_{h\bar{h}} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$  and its class  $D_{h\bar{h}}$  in  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ . Let  $\pi_h$  and  $\pi_{\bar{h}}$  denote the projections from  $\overline{M}_{h\bar{h}}$  on the two components and let  $\psi_h := \pi_h^*\psi_1$ . The image of the class  $\psi_h \in H^*(\overline{M}_{h\bar{h}})$  in  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  is a degree 2 class denoted by  $F_h$ .

Based on the structure of  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  found in [MM1], Theorem 3.23, it is not hard to see that  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  is generated over  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  by the set of classes  $F_h, D_{h\bar{h}}$  for  $h \subset \{1, \dots, d\}$ . (In [MM1] we have worked with Chow rings; the entire argument works identically for cohomology.)

*Notation.* Choose  $I := \{h \subset \{1, \dots, d\}, |h| > d/2\}$  if  $d$  is odd, and let  $I$  additionally contain half of the sets  $h$  with  $|h| = d/2$  if  $d$  is even, under the condition that no two sets  $h, \bar{h}$  are simultaneously in  $I$ . We define the following classes in  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$ :

$$\psi'_I := \psi - \sum_{h \in I} D_h,$$

$$D_I(h) := \sum_{h' \in I, h' \subset h} D_{h'} \text{ and } D_I(\bar{h}) := \sum_{h' \in I, h' \subset \bar{h}} D_{\bar{h}'}$$

for any  $h \in I$ . The class  $\psi_I(h)$  is defined as  $\psi_I(h) := \psi'_I + D_I(h)$ .

When  $d$  is odd,  $\psi'_I$  is the pullback of the relative cotangent class for the morphism  $\overline{U}^{\lfloor (d-1)/2 \rfloor} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$ .

*Remark 3.2.* The following relation between  $\psi$  classes on  $\overline{M}_{0,m}(\mathbb{P}^n, d)$  by Y.P. Lee and R. Pandharipande ([LP], Theorem 1) is instrumental in our computations:

$$\psi_i + \psi_j = D(i|j).$$

Here  $m \geq 2$ ,  $i$  and  $j \leq m$ , and  $D(i|j)$  is the divisor representing split curves, such that the marked points  $i$  and  $j$  lie in different components. With the notation from the previous sections, we will employ an analogous relation existing on intermediate spaces  $\overline{U}_m^k$ , where  $m > 0$  and  $k = (k_1, \{a_i\}_{i=2, \dots, m})$  consists of a positive integer  $k_1 < d + m - 1$  and weights  $a_i$  on the marked points. Indeed, the class  $\psi_1$  on  $\overline{M}_{m+1}(\mathbb{P}^n, d)$  descends to  $\psi'_1$  on each  $\overline{U}_m^k$ , while for any  $j = 2, \dots, m + 1$ , the class  $\psi_j - \sum_{j \in h} D_h$  descends to a class  $\psi'_j$  on  $\overline{U}_m^k$ , where the sum above is after all divisors  $D_h$  contracted by the morphism  $\overline{M}_{m+1}(\mathbb{P}^n, d) \rightarrow \overline{U}_m^k$ , i.e. for all  $h = h_1 \sqcup M_h$  such that

$$|h|_k := \frac{|h|}{k_1 + \epsilon} + \sum_{i \in M_h} a_i \leq 1.$$

Thus on  $\overline{U}_m^k$  the following relation holds:

$$(3.3) \quad \psi'_1 + \psi'_j = \sum_{j \in h} D_h,$$

where the sum is taken after all  $h$  with  $|h|_k > 1$ .

Next we show how the algebra  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  may be constructed starting from  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ , by adjoining a divisor for each intermediate space  $\overline{U}^k$ . Each divisor comes with a natural quadratic relation. Thus, relation (1) below is pulled back from  $\overline{U}^{\lfloor (d-1)/2 \rfloor}$ , while relations (2) and (3) are pulled back from  $B^*(\overline{U}^{\lfloor h \rfloor - 1})$ . We keep the notation from Lemma 3.1 throughout.

**Theorem 3.3.** *The algebra  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, d))$  over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  is generated by divisors  $\psi'_I$  and  $\{D_h\}_{h \in I}$ . The ideal of relations is generated by the following:*

- (1)  $\psi_I'^2 - \sum_{h \notin I} \frac{1}{2} N_h^d (F_h - F_{\bar{h}}) + \sum_{h \cap h' = \emptyset} N_{hh'}^d D_{h\bar{h}} D_{h'\bar{h}'} + \frac{3}{d^2} k(H^2)^2 - \frac{4}{d^2} k(H^3)$  for a choice of the set  $I$  as above.
- (2)  $D_h^2 - D_h(D_{h\bar{h}} - \psi_I(h)) - \frac{1}{2} [D_{h\bar{h}}(\psi_I(h) - D_I(\bar{h})) + F_h]$  for all  $h \in I$ .
- (3)  $D_h \ker\{B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))[\psi'_I, \{D_{h'}\}_{h' \in I; |h'| \leq |h|}] \rightarrow B^*(\overline{M}_{h\bar{h}})\}$ , where the above morphism sends  $\psi'_I$  to  $\psi_h + \sum_{h' \in I, h' \subset h} D_{h'\bar{h}'}$ ; it sends  $D_{h'}$  to  $D_{h'\bar{h}'}$  if  $h' \subset h$  and to 0 otherwise.

*Note.* We consider by convention  $D_{\bar{h}h} = D_{h\bar{h}}$ , such that both terms  $D_{\bar{h}h} D_{h\bar{h}}$  and  $D_{h\bar{h}} D_{\bar{h}h}$  appear in relation (1) above.

*Proof. Part I.* We study  $B^*(\overline{U}^{\lfloor (d-1)/2 \rfloor})$  as an algebra over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ . When  $d$  is odd,  $\overline{U}^{\lfloor (d-1)/2 \rfloor}$  is a projective bundle over  $\overline{M}_{0,0}(\mathbb{P}^n, d)$ . Its cohomology ring is thus generated over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  by the cotangent class  $\psi'$ , which satisfies a degree 2 relation over the base ring. Pullback of  $\psi'$  to  $\overline{M}_{0,1}(\mathbb{P}^n, d)$  is  $\psi + \sum_{h \in I} D_h$ . Substituting this in the relation of Lemma 3.1 yields (1).

When  $d$  is even, we refer to the description of  $\overline{U}^{(d-2)/2}$  from Lemma 1.4. Keeping the same notation, let  $\overline{U}^{(d-2)/2}(d/2) := \overline{U}^{(d-2)/2} \times_{\overline{M}_{0,0}(\mathbb{P}^n, d)} \overline{M}(d/2)$ . We may apply the open-closed exact sequence on extended cohomology rings to  $\overline{M}_{0,0}(\mathbb{P}^n, d) = \overline{M}(d/2) \cup (\overline{M}_{0,0}(\mathbb{P}^n, d) \setminus \overline{M}(d/2))$  and

$$\overline{U}^{(d-2)/2} = \overline{U}^{(d-2)/2}(d/2) \cup (\overline{U}^{(d-2)/2} \setminus \overline{U}^{(d-2)/2}(d/2)).$$

Over the complement of  $\overline{M}(d/2)$ , the cohomology ring of  $\overline{U}^{(d-2)/2} \setminus \overline{U}^{(d-2)/2}(d/2)$  is generated by the relative cotangent class  $\psi' = \psi - \sum_{|h|>d/2} D_h$ .

Let  $h \subset \{1, \dots, d\}$  be a subset of cardinality  $d/2$ . Consider the fiber product  $\overline{U}_{h\bar{h}}^{|h|-1} := \overline{U}^{|h|-1} \times_{\overline{M}_{0,0}(\mathbb{P}^n, d)} \overline{M}_{h\bar{h}}$ . It is the union of two  $\mathbb{P}^1$  bundles  $\overline{U}_h^{|h|-1} \xrightarrow{f_h^{|h|-1}} \overline{M}_{h\bar{h}}$  and  $\overline{U}_{\bar{h}}^{|h|-1} \xrightarrow{f_{\bar{h}}^{|h|-1}} \overline{M}_{h\bar{h}}$  sharing a common section  $S$ . Let  $s$  denote the class of  $S$  in  $B^*(\overline{U}^{|h|-1})$ , as well as in  $H^*(\overline{U}_h^{|h|-1})$ ,  $H^*(\overline{U}_{\bar{h}}^{|h|-1})$ .

The classes of  $\overline{U}_h^{|h|-1}$  and  $\overline{U}_{\bar{h}}^{|h|-1}$  in  $B^*(\overline{U}^{|h|-1})$  will be denoted by  $D_h$  and  $D_{\bar{h}}$ . Then  $D_h + D_{\bar{h}} = D_{h\bar{h}}$  and  $s = D_h D_{\bar{h}}$ . Thus,  $\psi'$  and  $\{D_h\}_{|h|=d/2}$  are generators for the algebra  $B^*(\overline{U}^{|h|-1})$  over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ .

On each of the bundles  $\overline{U}_h^{|h|-1}$  and  $\overline{U}_{\bar{h}}^{|h|-1}$ , the class  $s$  may be written in terms of  $\psi'$ .

Indeed,  $\overline{U}_h^{|h|-1} = \overline{U}_1^{|h|-1}(\mathbb{P}^n, d/2) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, d/2)$ , where  $\overline{U}_1^{|h|-1}(\mathbb{P}^n, d/2)$  is a contraction of  $\overline{M}_{0,2}(\mathbb{P}^n, d/2)$  as described in Subsection 1.2, such that the marked points both have weight 1 and the map to  $\mathbb{P}^n$  has weight  $\frac{1}{d/2+\epsilon}$ . The fiber product is along the evaluation maps at the first marked point. Let  $\psi_{\bar{h}}$  denote the  $\psi$  class of  $\overline{U}_1^{|h|-1}(\mathbb{P}^n, d/2)$  at the first marked point, as in Remark 3.2. Then, by comparison formula (3.1) in conjunction with formula (3.3),

$$(3.4) \quad -s = f_h^{|h|-1*} \psi_{\bar{h}} - \psi_{\bar{h}} = f_h^{|h|-1*} \psi_{\bar{h}} + \psi' \text{ on } \overline{U}_h^{|h|-1}.$$

Moreover, as  $f_{h\bar{h}}^{|h|-1} : \overline{U}_{h\bar{h}}^{|h|-1} \rightarrow \overline{M}_{h\bar{h}}$  is the sum of  $f_h^{|h|-1}$  and  $f_{\bar{h}}^{|h|-1}$  on components, then

$$f_h^{|h|-1*} \psi_{\bar{h}} = f_{h\bar{h}}^{|h|-1*} \psi_{\bar{h}} - f_{\bar{h}}^{|h|-1*} \psi_{\bar{h}}$$

and, by formula (2.1),

$$(3.5) \quad f_h^{|h|-1*} \psi_{\bar{h}} = f^{|h|-1*} F_{\bar{h}} + f_h^{|h|-1*} \psi_h + f_{\bar{h}}^{|h|-1*} D_{h\bar{h}}.$$

After comparing formulas (3.4) and (3.5) and their analogues for  $\bar{h}$ , we obtain the following relation in the algebra  $B^*(\overline{U}^{|h|-1})$  over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ :

$$(3.6) \quad F_h - F_{\bar{h}} = (D_h - D_{\bar{h}})(2\psi' - D_{h\bar{h}}).$$

This formula accounts for the different versions of relation (1) depending on the choice of the set  $I$  above. Indeed, choose  $I$  such that  $h \in I$ . Then formula (3.6) is equivalent to

$$F_h - F_{\bar{h}} = \psi'_{I \setminus \{h\} \cup \{\bar{h}\}} - \psi'_I,$$

which is exactly the difference between relation (1) applied to  $I$  and to  $I \setminus \{h\} \cup \{\bar{h}\}$ . Here we assumed the compatibility condition  $D_h D_{h'\bar{h}'} = 0$  for all  $h, h'$  such that  $|h| = |h'| = d/2$  (relation (3) in the theorem). Note that in  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  there is also a compatibility condition  $D_{h\bar{h}} D_{h'\bar{h}'} = 0$  for  $h, h'$  as above.

Equation (3.6) may be recast into relation (2) of the theorem via formula (2.1). We note that the same equation may be obtained in Part II of the proof by formally (at the level of étale covers) decomposing  $\overline{U}^{(d-2)/2} \rightarrow \overline{M}_{0,0}(\mathbb{P}^n, d)$  into a blow-down along the section  $S$  above, and a  $\mathbb{P}^1$ -bundle. Moreover, the reasoning employed in Part II of the proof also guarantees that there are no other relations besides (1), (2), and (3) (for  $|h| = d/2$ ) in  $B^*(\overline{U}^{(d-2)/2})$ .

**Part II: The blow-down**  $f_{k+1}^k : \overline{U}^k \rightarrow \overline{U}^{k+1}$ . Once a choice of the set  $I$  has been fixed, we will drop the subscript  $I$ .

Consider a nonnegative integer  $k < \lfloor (d-1)/2 \rfloor$ . We choose  $h \subset \{1, \dots, d\}$  such that  $|h| = d - k - 1$ . With the notation from Lemma 1.4, consider the commutative diagram

$$\begin{array}{ccccc}
 & & & & j_h^k \\
 & & & & \curvearrowright \\
 \overline{U}_h^k & & \overline{U}_h^k & \xrightarrow{j_h^k} & \overline{U}^k \\
 \downarrow f_{k+1,h}^k & & \downarrow f_{k+1,\bar{h}}^k & & \downarrow f_{k+1}^k \\
 S_h & \xrightarrow{j} & \overline{U}_{h\bar{h}}^{k+1} & \xrightarrow{j_{h\bar{h}}^{k+1}} & \overline{U}^{k+1} \\
 \searrow f_h^{k+1} & & \downarrow f_{h\bar{h}}^{k+1} & & \downarrow f_{k+1}^k \\
 & & \overline{M}_{h\bar{h}} & \xrightarrow{j_{h\bar{h}}} & \overline{M}_{0,0}(\mathbb{P}^n, d)
 \end{array}$$

where  $f_{h\bar{h}}^{k+1}$  admits a section  $s_h : \overline{M}_{h\bar{h}} \rightarrow \overline{U}_{h\bar{h}}^{k+1}$ ,  $S_h$  is the image of  $s_h$ ,  $j$  is the embedding of  $S_h$  into  $\overline{U}_{h\bar{h}}^{k+1}$ , and  $j_h^{k+1} := j_{h\bar{h}}^{k+1} \circ j$ . The space  $\overline{U}^k$  is the blow-up of  $\overline{U}^{k+1}$  along  $S_h$ , and  $\overline{U}_h^k$  is its exceptional divisor, with regular embedding  $j_h^k : \overline{U}_h^k \rightarrow \overline{U}^k$ . Let  $f_h^k$  be the composition  $f_h^k := f_{h\bar{h}}^{k+1} \circ f_{k+1,h}^k$ . In addition,  $\overline{U}_{h\bar{h}}^k$  is the strict transform of  $\overline{U}_{h\bar{h}}^{k+1}$  in  $\overline{U}^k$  and  $f_{h\bar{h}}^k := f_{h\bar{h}}^{k+1} \circ f_{k+1,\bar{h}}^k$ .

The fundamental class of  $\overline{U}_{h\bar{h}}^{k+1}$  in  $B^*(\overline{U}^{k+1})$  is denoted by  $D_{h\bar{h}}$  and is the pullback of the analogous class on  $\overline{M}_{0,0}(\mathbb{P}^n, d)$ . The classes of  $\overline{U}_h^k$  and  $\overline{U}_{h\bar{h}}^k$  in  $\overline{U}^k$  are  $D_h$  and  $D_{\bar{h}}$ , respectively.

Let  $\tilde{X} \rightarrow X$  be a blow-up along a regularly embedded locus  $Y$ . Following [Ke] (Theorem 2 in the Appendix),  $H^*(\tilde{X})$  may be written explicitly as an algebra over  $H^*(X)$ , provided that the pullback morphism  $H^*(X) \rightarrow H^*(Y)$  is surjective. In the presence of compatible  $G$ -networks in  $X$ ,  $Y$  and  $\tilde{X}$ , the analogous statement holds for extended cohomology rings ([MM1]). Note that in the present case  $H^*(\overline{U}^{k+1}) \rightarrow H^*(S_h)$  is not surjective, while  $j_h^{k+1*} : B^*(\overline{U}^{k+1}) \rightarrow B^*(S_h)$  is. We verify the surjectivity condition here.

By induction, the algebra  $B^*(\overline{U}^{k+1})$  has generators  $\psi'$  and  $\{D_{h'}\}_{h'}$  over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$ , for all  $h' \in I$  such that  $|h'| < d - k - 1$ . On the other hand, the algebra  $B^*(S_h) \cong B^*(\overline{M}_{h\bar{h}})$  over  $B^*(\overline{M}_{0,0}(\mathbb{P}^n, d))$  is generated by the divisor class  $f_h^{k+1*} \psi_h$ . By formulas (3.7) and (3.8) below,

$$f_h^{k+1*} \psi_h = -j_h^{k+1*} \psi(h),$$

while  $j_h^{k+1*} D_{h'} = D_{h'\bar{h}'}$  if  $h' \subset h$  and 0 otherwise. This proves that  $j_h^{k+1*}$  is surjective.

By [Ke], the ideal of relations in  $B^*(\bar{U}^k)$  is made of two parts:  $D_h \ker j_h^{k+1*}$  and  $D_h^2 - aD_h + b$ , where  $a, b \in B^*(\bar{U}^{k+1})$  are such that  $j_h^{k+1*}a = c_1(\mathcal{N}_{S_h|\bar{U}^{k+1}})$  and  $b = j_{h*}^{k+1}[S_h]$ . We proceed to find  $a$  and  $b$  in terms of the generators of  $B^*(\bar{U}^{k+1})$ .

Let  $\psi(k+1)$  denote the first Chern class of the relative dualizing sheaf for  $\bar{U}^{k+1} \rightarrow \bar{M}_{0,0}(\mathbb{P}^n, d)$ . Thus pullback of  $\psi(k+1)$  to  $\bar{M}_{0,1}(\mathbb{P}^n, d)$  is  $\psi' + \sum_{h' \in I, |h'| < |h|} D_{h'}$ .

The space  $\bar{U}_{h\bar{h}}^{k+1}$  is a fiber product  $\bar{U}_1^{k'}(\mathbb{P}^n, |h|) \times_{\mathbb{P}^n} \bar{M}_{0,1}(\mathbb{P}^n, |\bar{h}|)$ , where  $\bar{U}_1^{k'}(\mathbb{P}^n, |h|)$  is a contraction of  $\bar{M}_{0,2}(\mathbb{P}^n, |h|)$  as described in Subsection 1.2, such that both marked points have weight 1 and the map to  $\mathbb{P}^n$  has weight  $\frac{1}{k+\epsilon}$ . The fiber product is taken along the first marked point. Let  $\psi_h$  and  $\tilde{\psi}$  on  $\bar{U}_{h\bar{h}}^{k+1}$  be pullbacks of the two  $\psi$ -classes from  $\bar{U}_1^{k'}(\mathbb{P}^n, |h|)$ . We note that  $\tilde{\psi}$  differs from  $j_{h\bar{h}}^{k+1*}\psi(k+1)$  by the class of the section  $j_*[S_h]$ . Thus by formula (3.3),

$$\psi_h = -\psi(k+1) - j_*[S_h] + \sum_{h' \supset \bar{h}, |h'| < |h|} D_{h'},$$

which may be reformulated as

$$(3.7) \quad \psi_h = -\psi(h) + D(\bar{h}) - j_*[S_h]$$

on  $\bar{U}_{h\bar{h}}^{k+1}$ . On the other hand, by comparison formula (3.1)

$$-j_*[S_h] = f_{h\bar{h}}^{k+1*}\psi_h - \psi_h.$$

Putting these two equations together, we obtain

$$(3.8) \quad j_{h*}^{k+1}[S_h] = - \sum_{|h|=d-k-1} \frac{1}{2} [F_h + (\psi(h) - D(\bar{h}))D_{h\bar{h}}].$$

Note that  $j^*\psi_h = 0$  on  $S_h$ . Thus equation (3.7) implies that the first Chern class  $c_1(\mathcal{N}_{S_h|\bar{U}^{k+1}})$  is  $D_{h\bar{h}} - \psi(h)$ , as  $D(\bar{h})$  is also null on  $S_h$ . Relation (2) follows.  $\square$

**The case  $m \geq 1$ .** We have shown in Section 1 how the forgetful morphism  $f_{m+1} : \bar{M}_{0,m+1}(\mathbb{P}^n, d) \rightarrow \bar{M}_{0,m}(\mathbb{P}^n, d)$  factors out into a series of blow-downs and the projection of a  $\mathbb{P}^1$ -bundle over  $\bar{M}_{0,m}(\mathbb{P}^n, d)$ . The first marked point was chosen to play a special role in our construction. The analysis done in the proof of Theorem 3.3 carries out to this case, with the extra simplification provided by the existence of sections in the intermediate spaces. Indeed, the  $\mathbb{P}^1$ -bundle over  $\bar{M}_{0,m}(\mathbb{P}^n, d)$  admits a canonical section  $\sigma_1$ . This determines a Cartier divisor  $D_{1,m+1}$  on the bundle, whose class generates the cohomology of the bundle as an algebra over  $H^*(\bar{M}_{0,m}(\mathbb{P}^n, d))$ . Note that the subscript  $\{1, m+1\}$  is a distinct convention from the subscripts  $h$  employed elsewhere. The same notation will be used for the pullback of the above divisor to  $\bar{M}_{0,m+1}(\mathbb{P}^n, d)$ . The following well-known relation holds on  $\bar{M}_{0,m}(\mathbb{P}^n, d)$  (see for example [Pan]):

$$\sigma_1^* D_{1,m+1} = f_{m+1*}(D_{1,m+1}^2) = -\psi_1,$$

while  $\psi_1 = f_{m+1}^*\psi_1 + D_{1,m+1}$  on  $\bar{M}_{0,m+1}(\mathbb{P}^n, d)$ .

To every  $h \subset \{1, \dots, d\} \sqcup \{2, \dots, m+1\}$  such that  $m+1 \in h$  and either  $h \cap \{1, \dots, d\} \neq \emptyset$  or  $|h \cap \{2, \dots, m+1\}| > 2$ , there corresponds a blow-down in the factorization

of  $f_{m+1}$ . The blow-up locus is a section  $S_h$  over

$$\begin{aligned} \overline{M}_{h \setminus \{m+1\}} \\ := \overline{M}_{h \cap \{2, \dots, m\}, \bullet}(\mathbb{P}^n, |h \cap \{1, \dots, d\}|) \times_{\mathbb{P}^n} \overline{M}_{\{1, \dots, m\} \setminus h, \bullet}(\mathbb{P}^n, |\{1, \dots, d\} \setminus h|), \end{aligned}$$

whose class may be computed as above. With the notation from Section 1, the exceptional divisor  $D_h$  is paired with a strict transform  $D_{h \setminus \{m+1\}}$  of the homonymous divisor by the relation  $f_{m+1}^* D_{h \setminus \{m+1\}} = D_h + D_{h \setminus \{m+1\}}$ . The following theorem follows by the same arguments as those for Theorem 3.3.

**Theorem 3.4.** *The algebra  $B^*(\overline{M}_{0, m+1}(\mathbb{P}^n, d))$  over  $B^*(\overline{M}_{0, m}(\mathbb{P}^n, d))$  is generated by the divisor classes  $D_{1, m+1}$  and  $\{D_h\}_{m+1 \in h}$ , where the sets*

$$h \subset \{1, \dots, d\} \sqcup \{2, \dots, m+1\}$$

are such that  $h \cap \{1, \dots, d\} \neq \emptyset$  or  $|h \cap \{2, \dots, m+1\}| > 2$ . The ideal of relations is generated by

- (1)  $D_{1, m+1}^2 + f_{m+1}^* \psi_1 \cdot D_{1, m+1}$ ;
- (2)  $(D_h - f_{m+1}^* D_{h \setminus \{m+1\}})(D_h + f_{m+1}^* \psi_1 + D_{1, m+1} - \sum_{\{m+1\} \cup h \subset h'} D_{h'})$ , for  $h$  as above;
- (3)  $D_h \ker(B^*(\overline{M}_{0, m}(\mathbb{P}^n, d)) \rightarrow B^*(\overline{M}_{h \setminus \{m+1\}}))$ ,  $D_h D_{1, m+1}$ , and  $D_h D_{h'}$  whenever  $h \cap h' \neq h, h'$  or  $\emptyset$ .

**3.1. The case of a general target.** The relations in Theorems 3.3 and 3.4 characterize at least partially the role of the forgetful map in the cohomology of  $\overline{M}_{0, m}(X, d)$  for a general smooth projective target  $X$ . We recall the embedding of  $X$  in a product of projective spaces  $\prod_i \mathbb{P}^{n_i}$ , as defined in Subsection 1.2. We keep the notation from Subsection 1.2 and Section 3. By construction, any embedding  $X \hookrightarrow \mathbb{P}^N$ , given by a choice of  $N+1$  sections in a very ample line bundle  $\mathcal{L}$ , is the composition of the map  $X \hookrightarrow \prod_i \mathbb{P}^{n_i}$  above with another embedding of  $\prod_i \mathbb{P}^{n_i}$  in  $\mathbb{P}^N$ , as  $\mathcal{L} = \bigotimes_i \mathcal{L}_i^{\otimes a_i}$  for some nonnegative integers  $a_i$ . Then a curve of class  $d = (d_1, \dots, d_s)$  in  $X$  is sent to a curve of degree  $\sum_i a_i d_i$  in  $\mathbb{P}^N$ . For any  $h = (h_1, \dots, h_s) \subset \{1, \dots, d_1\} \sqcup \dots \sqcup \{1, \dots, d_s\} \sqcup M_h$ , there exists a subset  $\tilde{h}$  of the general set of  $(\sum_i a_i d_i + m - 1)$  elements, such that the divisor  $D_h$  on  $\overline{M}_{0, m}(X, d)$  is the pullback from  $\overline{M}_{0, m}(\mathbb{P}^N, \sum_i a_i d_i)$  of  $D_{\tilde{h}}$ . Indeed,  $\tilde{h}$  may be constructed as a disjoint union of  $a_i$  copies of each  $h_i$ , for all  $i$ . Thus by pullback of the quadratic relation in  $D_{\tilde{h}}$  from  $\overline{M}_{0, m}(\mathbb{P}^N, \sum_i a_i d_i)$ , we obtain a quadratic relation in  $D_h$ , generalizing equation (2) in Theorem 3.3 or 3.4. Similarly, relation (1) in Theorem 3.3 admits a straightforward generalization, where  $H$  is replaced by  $\mathcal{L}$  and  $d$  is replaced by  $\int_d c_1(\mathcal{L}) = \sum_i a_i d_i$ . Also, relation (1) in Theorem 3.4 induces an identical relation in  $\overline{M}_{0, m+1}(X, d)$ .

However, for an arbitrary convex target, the above theorems do not exhaust the list of generators for  $B^*(\overline{M}_{0, m+1}(X, d))$  as an algebra over  $B^*(\overline{M}_{0, m}(X, d))$ . Let us consider for example the case  $m = 0$ . With the notation from Theorem 3.3, the issue is that in general the morphism  $j_{h\tilde{h}}^{k+1*} : B^*(\overline{U}^{k+1}(X)) \rightarrow B^*(\overline{M}_{h\tilde{h}}(X))$  may not be surjective; for example, when the cohomology ring of  $X$  is not generated by divisors. A class  $\alpha_h \in B^*(\overline{M}_{h\tilde{h}}(X))$  which is not in the image of  $j_{h\tilde{h}}^{k+1*}$  contributes the classes  $f_{k+1, h}^{k*} \alpha_h$  and  $f_{k+1, h}^{k*} \alpha_h D_h$  to  $B^*(\overline{U}^k(X))$ , while classes of the type  $f_{k+1, h}^{k*} \alpha_h D_h^a$  with  $a \geq 2$  can be written in terms of the above via relation (2) in Theorem 3.3. Still, we remark that for any blow-up, even though the pullback of the restriction map to

the blow-up locus may not be surjective, this does not affect the proof that the only relations exclusively concerning the exceptional divisor over the cohomology ring of the base are those known in [Ke]. Thus on  $B^*(\overline{M}_{0,m+1}(X, d))$  over  $B^*(\overline{M}_{0,m}(X, d))$ , the only relations among the boundary and  $\psi$ -classes are the generalizations of those found in Theorems 3.3 and 3.4. In this sense we call the relations above universal.

3.2. Here we present a simple application to the case  $X = \mathbb{P}^n, d = 2$ .

**Example 3.5.** The ring  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, 2))$  was computed in [BO].

By Theorem 3.3,  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, 2))$  is the algebra extension of  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, 2))$  obtained by adjoining classes  $D_1, D_2, \psi, f$ , where  $f := \frac{1}{2}(F_1 - F_2)$  and the following quadratic equations hold:

$$(3.9) \quad (\psi - D_i)^2 - \frac{3}{16}D^2 + \frac{3}{16}k(H^2)^2 - \frac{1}{2}k(H^3) \pm f,$$

$$(3.10) \quad f^2 + \frac{1}{2}D^4 + \frac{3}{4}k(H^2)^2D^2 - 2k(H^3)D^2.$$

Here  $D = D_1 + D_2$  is the class of  $\overline{M}_{1,2} := \overline{M}_{0,1}(\mathbb{P}^n, 1) \times_{\mathbb{P}^n} \overline{M}_{0,1}(\mathbb{P}^n, 1)$ , and the second equation is obtained by writing  $F_1$  and  $F_2$  in the two flag varieties  $\overline{M}_{0,1}(\mathbb{P}^n, 1)$ .

Thus  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, 2))$  may be recovered as the subring of invariants of  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, 2))$  under an action of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Extracting the invariant relations is a fun exercise.

In [MM1],  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, 2))$  is computed as the  $\mathbb{Q}$ -algebra generated by divisors  $ev^*H, \psi, D_1$  and  $D_2$ , with the ideal of relations generated by

$$ev^*H^{n+1}, D_1D_2\psi, D_i(ev^*H + \psi)^{n+1}, D_iQ(\psi - D_i),$$

$$\sum_{s=\psi}^2 Q(s) \Big|_{s=\psi}^{s=\psi-D_i} + \frac{(ev^*H+2\psi)^{n+1}}{\psi},$$

where

$$Q(t) := \frac{(ev^*H + \psi + t)^{n+1} - (ev^*H + \psi)^{n+1}}{t}.$$

Let  $D := D_1 + D_2, t := \frac{1}{2}(k(H^2) - D)$  and  $b := ev^*H + \psi = \frac{1}{4}(k(H^2) + D)$ , such that  $Q(s) = \sum_{l=0}^n (b + s)^l b^{n-l}$ . Let  $S_N := \sum_{l=0}^N a_l b^{N-l}$ , where

$$a_l := (b + \psi - D_1)^l + (b + \psi - D_2)^l - (b + \psi)^l + (b - \psi)^l,$$

such that the last relation in the ring becomes  $S_n = 0$  and the first and fourth imply  $a_{n+1} = 0$ . Equivalently,  $S_n = S_{n+1} = 0$ . A third invariant relation is  $Db^{n+1} = (t - 2b)b^{n+1} = 0$ , obtained from the third relation in  $B^*(\overline{M}_{0,1}(\mathbb{P}^n, 2))$ .

The following simple observation will come into play:

(\*) If  $a_1, \dots, a_m$  are variables such that  $\prod_{i=1}^m a_i = 0$ , then for any polynomial  $P$ , the following relation holds:

$$\sum_{l=0}^m (-1)^l \sum_{i_1, \dots, i_l \in \{1, \dots, m\}} P\left(\sum_{j \in \{i_1, \dots, i_l\}} a_j\right) = 0,$$

where  $i_1, \dots, i_l$  are distinct elements of  $\{1, \dots, m\}$ .

This helps us write  $S_n$  invariantly. Indeed, by (\*) applied to  $\psi, D_1$  and  $D_2$ ,

$$a_l := (b - D_1)^l + (b - D_2)^l + (b + \psi - D)^l + (b - \psi)^l - b^l - (b - D)^l.$$

After applying recurrence relations for pairs of summands,

$$a_{l+2} - ta_{l+1} + (b^2 - bD)a_l + D_1D_2[(b - D_1)^l + (b - D_2)^l] - (\psi - D)\psi[(b + \psi - D)^l - (b - \psi)^l] = 0$$

for all  $l \geq 0$ , which, via relation  $D_1D_2\psi = 0$  and observation (\*), yields  $a_{l+2} - ta_{l+1} + (b^2 - bD - D_1D_2 - (\psi - D)\psi)a_l = 0$ . Furthermore, formulas (3.9) and (3.10) imply the following invariant formulation:

$$k := b^2 - bD + D_1D_2 - (\psi - D)\psi = \frac{1}{4}k(H^2)^2 + \frac{1}{8}k(H^2)D + \frac{1}{8}D^2 - \frac{1}{2}k(H^3),$$

and  $a_{l+2} - ta_{l+1} + ka_l = 0$ , or, summing up,  $S_{l+2} - tS_{l+1} + kS_l = b^l(2b - t)$ . The relations in  $H^*(\overline{M}_{0,0}(\mathbb{P}^n, 2))$  found above can thus be written as  $Y_{n+1} = 0$ , where

$$Y_{l+1} = \begin{pmatrix} b^{l+1}(2b - t) \\ S_{l+1} - tS_l \\ S_l \end{pmatrix} = \begin{pmatrix} b & 0 & 0 \\ 1 & 0 & k \\ 0 & 1 & t \end{pmatrix}^l \begin{pmatrix} b(2b - t) \\ 2b - t \\ 2 \end{pmatrix}.$$

This is consistent with the ring structure computed in [BO].

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