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A REDUCTION METHOD FOR NONCOMMUTATIVE L_p-SPACES AND APPLICATIONS

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Abstract. We consider the reduction of problems on general noncommutative L_p-spaces to the corresponding ones on those associated with finite von Neumann algebras. The main tool is an unpublished result of the first-named author which approximates any noncommutative L_p-space by tracial ones. We show that under some natural conditions a map between two von Neumann algebras extends to their crossed products by a locally compact abelian group or to their noncommutative L_p-spaces. We present applications of these results to the theory of noncommutative martingale inequalities by reducing most recent general noncommutative martingale/ergodic inequalities to those in the tracial case.

0. Introduction

The theory of noncommutative L_p-spaces has a long history going back to pioneering works by von Neumann and Schatten [Sc1, Sc2], Dixmier [Di], Segal [Se] and Kunze [Kun]. In the first constructions the trace of a matrix or an operator replaces the integral of a function, and the noncommutative L_p-spaces are composed of the elements whose p-th power has finite trace. Later (around 1980), generalizations to type III von Neumann algebras appeared due to the efforts of Haagerup [H2], Hilsum [Hi], Araki and Masuda [AM], Kosaki [Ko] and Terp [Te2]. These algebras have no trace, and therefore the integration theory has to be entirely redone. These generalizations were motivated and made possible by the great progress in operator algebra theory, in particular the Tomita-Takesaki theory and Connes's spectacular results on the classification of type III factors.

Since the early nineties and the arrival of new theories such as those of operator spaces and free probability, noncommutative integration has been living another period of stimulating new developments. In particular, noncommutative Khintchine and martingale inequalities have opened new perspectives. It is well known nowadays that the theory of noncommutative L_p-spaces is intimately related with many other fields such as Banach spaces, operator algebras, operator spaces, quantum probability and noncommutative harmonic analysis. Although they correspond to separate research directions, these fields present many links and interactions

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through noncommutative L_p-spaces. For instance, Pisier's works, in particular his theory on vector-valued noncommutative L_p-spaces [Pi1], have started to exhibit these interactions. Since the establishment of the noncommutative Burkholder-Gundy inequalities in [PX1], many classical martingale inequalities have been transferred to the noncommutative setting. These include the Doob inequality in [Ju1], the Burkholder/Rosenthal inequalities in [JX1, JX2], the weak type $(1,1)$ inequality for martingale transforms in [R1] and the Gundy decomposition in [PaR]. This rapid development of noncommutative inequalities is largely motivated by quantum probability and operator space theory. Notably, the latter theory has inspired many new ideas and provides numerous tools. These noncommutative inequalities have, in return, interesting applications to operator space theory, quantum probability and more generally noncommutative analysis.

The recent works on the complete embedding of Pisier's operator Hilbert space OH into a noncommutative L_1 in [Ju2] and certain Hilbertian homogeneous operator spaces into noncommutative L_p-spaces in [X3] show well how useful these L_p-spaces are as tools and examples for operator spaces. These works also illustrate the power of quantum probabilistic methods in operator spaces since Khintchine-type inequalities for free random variables play a key role there. This illustration is further shown in the works [PS] and [X4] on operator space Grothendieck theorems. It should be emphasized that type III von Neumann algebras are unavoidable in all these works, for Pisier [Pi3] showed that OH cannot completely embed in a semifinite L_1.

While the tracial noncommutative L_p-spaces are a rather transparent generalization of the usual L_p-spaces, all existing (equivalent) constructions of type III L_p-spaces are quite heavy and based on the Tomita-Takesaki theory. This makes them much less pleasant and handy. For instance, the L_p-spaces associated to a type III algebra do not form a real interpolation scale (cf. [PX2]). Another main drawback is the lack of a reasonable analogue of weak L_1 in the type III case. It is however well known that weak L_1-spaces are of paramount importance in analysis, notably through the Marcinkiewicz interpolation theorem.

It is thus desirable to reduce or approximate type III L_p-spaces to or by semifinite ones. This is exactly the objective of an unpublished work [H3] of the first-named author about three decades ago. The result there can be stated as follows. Given a σ-finite von Neumann algebra $\mathcal{M}$ equipped with a normal faithful state φ there exists another σ-finite von Neumann algebra $\mathcal{R}$ and a normal faithful state $\widehat{\varphi}$ on $\mathcal{R}$ verifying the following properties:

(i) $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{R}$, the restriction of $\widehat{\varphi}$ to $\mathcal{M}$ is equal to φ and there exists a state-preserving normal faithful conditional expectation from $\mathcal{R}$ onto $\mathcal{M}$;

(ii) there exists an increasing sequence $(\mathcal{R}_n)_n$ of finite von Neumann subalgebras of $\mathcal{R}$ such that their union is w*-dense in $\mathcal{R}$ and such that each $\mathcal{R}_n$ is the image of a state-preserving normal faithful conditional expectation.

Property (i) allows us to view $L_p(\mathcal{M})$ isometrically as a subspace of $L_p(\mathcal{R})$. (ii) insures that the sequence $(L_p(\mathcal{R}_n))_n$ is increasing and $\bigcup_n L_p(\mathcal{R}_n)$ is dense in $L_p(\mathcal{R})$ for $p < \infty$. This implies that $L_p(\mathcal{R})$, so $L_p(\mathcal{M})$, can be approximated by the $L_p(\mathcal{R}_n)$, which are based on finite algebras. This approximation theorem reduces many geometrical properties of general noncommutative L_p-spaces to the corresponding ones in the tracial case. This is indeed the case for all those properties which are of a finite-dimensional nature such as Clarkson's inequalities, uniform convexity/smoothness and type/cotype.

The preceding reduction theorem has found more and more applications since the new developments of noncommutative L_p-spaces mentioned previously. In fact, it plays a crucial role in many recent works. See, for instance, [JOS], [JP2], [JP1], [JX4], [X1] and [X2]. Moreover, in these works one needs the precise form of $\mathcal{R}$ and $\mathcal{R}_n$ as constructed in [H3]. On the other hand, applications of this theorem often require additional results such as those on the extension of maps on von Neumann algebras to their crossed products or noncommutative L_p-spaces.

The manuscript [H3], however, has been circulated only in a very limited circle of people. We feel that it would be helpful to make it accessible to the general public. This explains why we decide to include the proof of the previous reduction theorem following the presentation of [H3]. This reproduction corresponds to section 2 below. In the second part of this article we show well how useful this theorem is for noncommutative martingale and ergodic inequalities. The first part contains two extension results for maps on von Neumann algebras, one to their crossed products and another to their noncommutative L_p-spaces. These extension results are of interest for their own right.

The paper is organized as follows. In section 1 we summarize necessary preliminaries on crossed products and noncommutative L_p-spaces. For these L_p-spaces we use the construction [H2] of the first-named author. Today, they are commonly called Haagerup noncommutative L_p-spaces. In section 2 we prove the reduction theorem mentioned previously. Our presentation follows the unpublished manuscript [H3]. The tool is crossed products. Section 3 presents the first application of the reduction theorem to noncommutative L_p-spaces. The result there, already quoted previously, says roughly that a type III noncommutative L_p-space with $p < \infty$ can be approximated by tracial ones. This is in fact the original intention of [H3]. In section 4 we deal with the extension of a map between two von Neumann algebras to their crossed products by a locally compact abelian group. The extension of such a map to the corresponding L_p-spaces is treated in section 5. The second extension is much subtler than the first one. Its proof involves Kosaki-Terp's interpolation theorem. Note, however, that this extension to L_p is quite obvious in the tracial case. The last two sections contain applications to martingale/ergodic inequalities. The first one is devoted to square function type inequalities and the second to maximal inequalities. Most results in these two sections are not new and some arguments also exist in the literature. For some results there the reduction to the tracial case is not really necessary. But for some others we do not know other methods than the reduction, for example, for the maximal ergodic inequalities. We feel that it would be helpful for the reader to have a complete picture of how to reduce these inequalities to the tracial case. This is why all inequalities in consideration are properly stated and some known arguments are also included.

1. Preliminaries

In this section we collect some necessary preliminaries on crossed products and noncommutative L_p-spaces used throughout the paper.

1.1. **Crossed products.** Our references for modular theory and crossed products are [KR], [PeT], [Str], [Ta3] and [Ta2]. Throughout this paper $\mathcal{M}$ will always denote a von Neumann algebra acting on a Hilbert space H. Let G be a locally compact abelian group equipped with Haar measure dg, and $\widehat{G}$ its dual group equipped with Haar measure $d\gamma$. We choose dg and $d\gamma$ so that the Fourier inversion theorem holds. Let α be a continuous automorphic representation of G on $\mathcal{M}$. We denote by $\mathcal{R} = \mathcal{M} \rtimes_\alpha G$ the crossed product of $\mathcal{M}$ by G with respect to α. Recall that $\mathcal{R}$ is the von Neumann algebra on $L_2(G, H)$ generated by the operators $\pi_\alpha(x)$, $x \in \mathcal{M}$ and $\lambda(g)$, $g \in G$, which are defined by

$$\big(\pi_\alpha(x)\xi\big)(h) = \alpha_h^{-1}(x)\xi(h), \quad \big(\lambda(g)\xi\big)(h) = \xi(h-g), \quad \xi \in L_2(G,H),\ h \in G.$$

These operators satisfy the following commutation relation:

$$\lambda(g)\pi(x)\lambda(g)^* = \pi(\alpha_g(x)), \quad x \in \mathcal{M},\ g \in G. \tag{1.1}$$

Consequently, the family of all linear combinations of $\pi_\alpha(x)\lambda(g)$, $x \in \mathcal{M}$, $g \in G$, is a w*-dense involutive subalgebra of $\mathcal{R}$. Recall that π_α is a normal faithful representation of $\mathcal{M}$ on $L_2(G, H)$, so we can identify $\mathcal{M}$ and $\pi_\alpha(\mathcal{M})$. In the sequel, we will drop the subscript α from π_α whenever no confusion can occur.

The action α admits a dual action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on the crossed product $\mathcal{R}$ defined as follows. Let w be the following unitary representation of $\widehat{G}$ on $L_2(G, H)$:

$$\big(w(\gamma)\xi\big)(h) = \overline{\gamma(h)}\xi(h), \quad \xi \in L_2(G,H),\ h \in G,\ \gamma \in \widehat{G}.$$

Then the dual action $\widehat{\alpha}$ is implemented by w:

$$\widehat{\alpha}_\gamma(x) = w(\gamma)xw(\gamma)^*, \quad x \in \mathcal{R},\ \gamma \in \widehat{G}.$$

It is easy to check that

$$\widehat{\alpha}_\gamma(\pi(x)) = \pi(x), \quad \widehat{\alpha}_\gamma(\lambda(g)) = \overline{\gamma(g)}\lambda(g), \quad x \in \mathcal{M},\ g \in G,\ \gamma \in \widehat{G}. \tag{1.2}$$

Recall that $\pi(\mathcal{M})$ is the algebra of the fixed points of $\widehat{\alpha}$. Namely,

$$\pi(\mathcal{M}) = \{x \in \mathcal{R} \,:\, \widehat{\alpha}_\gamma(x) = x,\ \forall\, \gamma \in \widehat{G}\}. \tag{1.3}$$

There exists a normal semifinite faithful (*n.s.f.* for short) operator-valued weight Φ in the sense of [H4] from $\mathcal{R}$ to $\pi(\mathcal{M})$ defined by

$$\Phi(x) = \int_{\widehat{G}} \widehat{\alpha}_\gamma(x)\, d\gamma, \quad x \in \mathcal{R}_+. \tag{1.4}$$

Moreover, Φ is $\widehat{\alpha}$-invariant, i.e.,

$$\Phi \circ \widehat{\alpha}_\gamma = \Phi, \quad \gamma \in \widehat{G}.$$

Note that Φ takes values in the extended positive part of $\pi(\mathcal{M})$ and can be defined on the extended positive part of $\mathcal{R}$ too. We can easily determine the action of Φ on the elements in a w*-dense involutive subalgebra of $\mathcal{R}$. Indeed, let $f : G \to \mathcal{M}$ be a compactly supported w*-continuous function. Then

$$x_f = \int_G \pi(f(g))\lambda(g)\, dg$$

defines an operator in $\mathcal{R}$. One can check that the family of all such operators x_f forms a w*-dense involutive subalgebra of $\mathcal{R}$. If additionally $x_f \geq 0$, then

$$\Phi(x_f) = \pi(f(0)). \tag{1.5}$$

See [H4] for more details.

Let φ be a normal semifinite weight on $\mathcal{M}$. Then φ admits a dual weight $\widehat{\varphi}$ on the crossed product $\mathcal{R}$ given by

$$\widehat{\varphi} = \varphi \circ \pi^{-1} \circ \Phi. \tag{1.6}$$

By (1.5) for every positive x_f as above we have

$$\widehat{\varphi}(x_f) = \varphi(f(0)).$$

$\widehat{\varphi}$ is normal and semifinite; if φ is faithful, so is $\widehat{\varphi}$. Since Φ is $\widehat{\alpha}$-invariant, by (1.6), $\widehat{\varphi}$ is $\widehat{\alpha}$-invariant too:

$$\widehat{\varphi} \circ \widehat{\alpha}_\gamma = \widehat{\varphi}, \quad \gamma \in \widehat{G}.$$

Moreover, the map $\varphi \mapsto \widehat{\varphi}$ is a bijection from the set of all normal semifinite weights on $\mathcal{M}$ onto the set of all normal semifinite $\widehat{\alpha}$-invariant weights on $\mathcal{R}$ (cf. [Str, section 19.8]). The modular automorphism group of the dual weight $\widehat{\varphi}$ is uniquely determined by

$$\sigma_t^{\widehat{\varphi}}(\pi(x)) = \pi(\sigma_t^{\varphi}(x)), \quad \sigma_t^{\widehat{\varphi}}(\lambda(g)) = \lambda(g), \quad x \in \mathcal{M},\ g \in G,\ t \in \mathbb{R}. \tag{1.7}$$

Thus $\sigma_t^{\widehat{\varphi}}$ leaves all $\lambda(g)$ invariant, and if we identify $\pi(\mathcal{M})$ with $\mathcal{M}$, the restriction of $\sigma_t^{\widehat{\varphi}}$ on $\mathcal{M}$ coincides with σ_t^{φ}.

The case where G is a discrete group will play an important role later. In this case the previous construction becomes simpler. In particular, the operator-valued weight Φ defined by (1.4) becomes a conditional expectation, which is uniquely determined by

$$\Phi\big(\lambda(g)\pi(x)\big) = \begin{cases} \pi(x) & \text{if } g = 0, \\ 0 & \text{if } g \neq 0, \end{cases} \qquad x \in \mathcal{M},\ g \in G. \tag{1.8}$$

On the other hand, if φ is a normal (faithful) state on $\mathcal{M}$, $\widehat{\varphi}$ is a normal (faithful) state on $\mathcal{R}$ determined by

$$\widehat{\varphi}\big(\lambda(g)\pi(x)\big) = \begin{cases} \varphi(x) & \text{if } g = 0, \\ 0 & \text{if } g \neq 0, \end{cases} \qquad x \in \mathcal{M},\ g \in G. \tag{1.9}$$

1.2. **Noncommutative L_p-spaces.** We now introduce Haagerup noncommutative L_p-spaces following [H2] and [Te1]. All results quoted below without extra reference can be found in these two papers. Throughout this subsection, φ will be a fixed *n.s.f.* weight on $\mathcal{M}$, and $\mathcal{R}$ will denote the crossed product $\mathcal{M} \rtimes_\sigma \mathbb{R}$ of $\mathcal{M}$ by $\mathbb{R}$ with respect to the modular automorphism group $\sigma = \sigma^{\varphi}$. We will identify $\mathcal{M}$ and the subalgebra $\pi(\mathcal{M})$ of $\mathcal{R}$. It is well known that $\mathcal{R}$ is semifinite and there exists a unique *n.s.f.* trace τ on $\mathcal{R}$ such that

$$(D\widehat{\varphi} : D\tau)_t = \lambda(t), \quad t \in \mathbb{R}, \tag{1.10}$$

where $(D\widehat{\varphi} : D\tau)_t$ denotes the Radon-Nikodým cocycle of $\widehat{\varphi}$ with respect to τ. We will call this trace the canonical *n.s.f.* trace on $\mathcal{R}$. Moreover, τ is the unique *n.s.f.* trace on $\mathcal{R}$ satisfying

$$\tau \circ \widehat{\sigma}_t = e^{-t}\tau, \quad t \in \mathbb{R}. \tag{1.11}$$

Given a normal semifinite weight ψ on $\mathcal{M}$, let h_ψ denote the Radon-Nikodým derivative of the dual weight $\widehat{\psi}$ with respect to τ, which is the unique positive selfadjoint operator affiliated with $\mathcal{R}$ such that

$$\widehat{\psi}(x) = \tau(h_\psi x), \quad x \in \mathcal{R}_+.$$

Here $\tau(h_\psi x)$ is understood as $\tau(h_\psi^{1/2} x h_\psi^{1/2})$. Then by (1.11), we have

$$\widehat{\sigma}_t(h_\psi) = e^{-t} h_\psi, \quad t \in \mathbb{R}. \tag{1.12}$$

Recall that the map $\psi \mapsto h_\psi$ is a bijection from the set of all normal semifinite weights on $\mathcal{M}$ onto the set of all positive selfadjoint operators affiliated with $\mathcal{R}$ satisfying (1.12) (cf. [Te1, Proposition II.4]).

In particular, the dual weight $\widehat{\varphi}$ of our distinguished weight φ has a Radon-Nikodým derivative D_φ with respect to τ. We will call D_φ the density operator of φ and will often denote it by D whenever no confusion can occur. Then by (1.10) the regular representation $\lambda(t)$ of $\mathbb{R}$ on $L_2(\mathbb{R}, H)$ is given by

$$\lambda(t) = D^{it}, \quad t \in \mathbb{R}.$$

Now we are ready to define Haagerup noncommutative L_p-spaces. Let $L_0(\mathcal{R}, \tau)$ denote the topological involutive algebra of all operators on $L_2(\mathbb{R}, H)$ measurable with respect to $(\mathcal{R}, \tau)$ (cf. [N] and [Te1, Chapter I]). Let $0 < p \leq \infty$. Then the Haagerup L_p-space associated with $(\mathcal{M}, \varphi)$ is defined to be

$$L_p(\mathcal{M}, \varphi) = \big\{x \in L_0(\mathcal{R}, \tau) \,:\, \widehat{\sigma}_t(x) = e^{-t/p}x, \ \forall\, t \in \mathbb{R}\big\}.$$

The spaces $L_p(\mathcal{M}, \varphi)$ are closed selfadjoint linear subspaces of $L_0(\mathcal{R}, \tau)$. It is not hard to show that $L_\infty(\mathcal{M}, \varphi)$ coincides with $\mathcal{M}$. On the other hand, it is well known that the map $\omega \mapsto h_\omega$ on $\mathcal{M}_*^+$ extends to a linear homeomorphism from $\mathcal{M}_*$ onto $L_1(\mathcal{M}, \varphi)$ (equipped with the vector space topology inherited from $L_0(\mathcal{R}, \tau)$). This permits us to transfer the norm of $\mathcal{M}_*$ into a norm on $L_1(\mathcal{M}, \varphi)$, denoted by $\| \, \|_1$. Moreover, $L_1(\mathcal{M}, \varphi)$ is equipped with a distinguished contractive positive linear functional tr, the "trace", defined by

$$\mathrm{tr}\,(h_\omega) = \omega(1), \quad \omega \in \mathcal{M}_*.$$

Consequently, $\|h\|_1 = \mathrm{tr}\,(|h|)$ for every $h \in L_1(\mathcal{M}, \varphi)$.

Let $0 < p < \infty$. If $x = u|x|$ is the polar decomposition of $x \in L_0(\mathcal{R}, \tau)$, then

$$x \in L_p(\mathcal{M}, \varphi) \Leftrightarrow u \in \mathcal{M} \text{ and } |x| \in L_p(\mathcal{M}, \varphi) \Leftrightarrow u \in \mathcal{M} \text{ and } |x|^p \in L_1(\mathcal{M}, \varphi).$$

For $x \in L_p(\mathcal{M}, \varphi)$ set $\|x\|_p = \| \, |x|^p \|_1^{1/p}$. Then $\| \, \|_p$ is a norm or a p-norm according to $1 \leq p < \infty$ or $0 < p < 1$. The associated vector space topology coincides with that inherited from $L_0(\mathcal{R}, \tau)$.

Another important link between the spaces $L_p(\mathcal{M}, \varphi)$ is the external product: in fact, the product of $L_0(\mathcal{R}, \tau)$, $(x, y) \mapsto xy$, restricts to a bounded bilinear map $L_p(\mathcal{M}, \varphi) \times L_q(\mathcal{M}, \varphi) \to L_r(\mathcal{M}, \varphi)$, where $1/r = 1/p + 1/q$. This bilinear map has norm one, which amounts to saying that the usual Hölder inequality extends to Haagerup L_p-spaces. In particular, if $1/p + 1/p' = 1$, then the bilinear form $(x, y) \mapsto \mathrm{tr}\,(xy)$ defines a duality bracket between $L_p(\mathcal{M})$ and $L_{p'}(\mathcal{M}, \varphi)$, for which $L_{p'}(\mathcal{M}, \varphi)$ coincides (isometrically) with the dual of $L_p(\mathcal{M}, \varphi)$ (if $p \neq \infty$); moreover we have the tracial property:

$$\mathrm{tr}(xy) = \mathrm{tr}\,(yx), \quad x \in L_p(\mathcal{M}, \varphi), \ y \in L_{p'}(\mathcal{M}, \varphi).$$

The distinguished weight φ can be recovered from the tracial functional tr. Let

$$\mathfrak{n}_\varphi = \{x \in \mathcal{M} \ : \ \varphi(x^*x) < \infty\}, \quad \mathfrak{m}_\varphi = \mathfrak{n}_\varphi^*\mathfrak{n}_\varphi = \mathrm{span}\{y^*x \ : \ x, y \in \mathfrak{n}_\varphi\}\,.$$

It is well known that for any $x \in \mathfrak{m}_\varphi$, the operator $D^{1/2}xD^{1/2}$ is closable and its closure belongs to $L_1(\mathcal{M}, \varphi)$ (cf. e.g. [Te2] and [GL2]). Denoting the closure again by $D^{1/2}xD^{1/2}$, we then have

$$\varphi(x) = \mathrm{tr}(D^{1/2}xD^{1/2}), \quad x \in \mathfrak{m}_\varphi\,.$$

In particular, if φ is bounded, then $D \in L_1(\mathcal{M}, \varphi)$ and

$$\varphi(x) = \mathrm{tr}(D^{1/2} x D^{1/2}) = \mathrm{tr}(Dx), \quad x \in \mathcal{M}. \tag{1.13}$$

Remark 1.1. Let $\mathcal{N}$ be a w*-closed involutive subalgebra of $\mathcal{M}$ such that $\varphi\big|_{\mathcal{N}}$ is semifinite. Assume that $\mathcal{N}$ is invariant under σ^φ, i.e., $\sigma_t^\varphi(\mathcal{N}) \subset \mathcal{N}$ for all $t \in \mathbb{R}$. Then it is easy to check that $L_p(\mathcal{N}, \varphi\big|_{\mathcal{N}})$ coincides isometrically with a subspace of $L_p(\mathcal{M}, \varphi)$ for every $0 < p < \infty$.

Remark 1.2. It is proved in [H2] and [Te1] that $L_p(\mathcal{M}, \varphi)$ is independent of φ up to isometric isomorphism preserving the order and modular structure of $L_p(\mathcal{M}, \varphi)$. This independence allows us to denote $L_p(\mathcal{M}, \varphi)$ simply by $L_p(\mathcal{M})$. On the other hand, if φ is tracial, i.e., $\varphi(x^*x) = \varphi(xx^*)$ for all $x \in \mathcal{M}$, then the Haagerup space $L_p(\mathcal{M}, \varphi)$ isometrically coincides with the tracial noncommutative L_p-space associated with φ constructed by Dixmier [Di] and Segal [Se]. We refer to [PX2] for more information and historical references on noncommutative L_p-spaces.

2. The reduction theorem

This section is the core of the paper. Its result is a reduction theorem that approximates a type III von Neumann algebra by finite ones. This theorem is due to the first-named author and has never appeared in published form. Our presentation will follow the first-named author's manuscript, which has been circulated in a limited circle of people since [H3]. We will concentrate our attention mainly on the σ-finite case. Throughout this section, G will denote the discrete subgroup $\bigcup_{n\geq 1} 2^{-n}\mathbb{Z}$ of $\mathbb{R}$. Let $\mathcal{M}$ be a σ-finite von Neumann algebra equipped with a normal faithful state φ. Consider the crossed product $\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} G$. Here, the modular automorphism group σ^φ is also viewed as an automorphic representation of G on $\mathcal{M}$. Let $\widehat{\varphi}$ denote the dual weight of φ. Recall that $\widehat{\varphi}$ is a normal faithful state on $\mathcal{R}$.

Theorem 2.1. *With the notation above, there exists an increasing sequence $(\mathcal{R}_n)_{n\geq 1}$ of von Neumann subalgebras of $\mathcal{R}$ satisfying the following properties:*

(i) *each $\mathcal{R}_n$ is finite;*
(ii) *$\bigcup_{n\geq 1} \mathcal{R}_n$ is w*-dense in $\mathcal{R}$;*
(iii) *for every $n \in \mathbb{N}$ there exists a normal faithful conditional expectation Φ_n from $\mathcal{R}$ onto $\mathcal{R}_n$ such that*

$$\widehat{\varphi} \circ \Phi_n = \widehat{\varphi} \quad \textit{and} \quad \sigma_t^{\widehat{\varphi}} \circ \Phi_n = \Phi_n \circ \sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R}.$$

The remainder of this section is mainly devoted to the proof of this theorem. $\mathcal{R}_n$ will be constructed as the centralizer of a normal faithful positive functional φ_n such that the modular automorphism group $\sigma_t^{\varphi_n}$ is periodic with period 2^{-n}. We keep all the notation introduced in subsection 1.1. $\mathbb{T}$ denotes the unit circle of the complex plane equipped with normalized Haar measure dm.

Lemma 2.2. *For any $f \in L_\infty(\mathbb{T})$,*

$$\widehat{\varphi}(f(\lambda(t))) = \int_{\mathbb{T}} f(z)dm(z), \quad t \in G \setminus \{0\}. \tag{2.1}$$

In other words, the distribution measure of $\lambda(t)$ with respect to $\widehat{\varphi}$ is equal to dm.

Proof. Let $t \in G \setminus \{0\}$. Then by (1.9), for any $n \in \mathbb{Z}$,

$$\widehat{\varphi}\big(\lambda(t)^n\big) = \widehat{\varphi}(\lambda(nt)) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Thus (2.1) holds whenever f is a monomial z^n, $n \in \mathbb{Z}$, so also whenever f is a trigonometric polynomial. Then the normality of $\widehat{\varphi}$ yields (2.1) for all $f \in L_\infty(\mathbb{T})$. □

Recall that if ψ is an *n.s.f.* weight on a von Neumann algebra $\mathcal{N}$, the centralizer $\mathcal{N}_\psi$ of ψ is the fixed point algebra of σ_t^ψ:

$$\mathcal{N}_\psi = \{x \in \mathcal{N} \ : \ \sigma_t^\psi(x) = x, \ \forall\, t \in \mathbb{R}\}.$$

If additionally ψ is bounded,

$$\mathcal{N}_\psi = \{x \in \mathcal{N} \ : \ \psi x = x\psi\},$$

where for $a, b \in \mathcal{N}$, $a\psi b$ denotes the functional on $\mathcal{N}$ given by

$$a\psi b(y) = \psi(bya), \ y \in \mathcal{N}.$$

$\mathcal{Z}(\mathcal{N})$ denotes the center of $\mathcal{N}$.

Lemma 2.3. (i) *$\lambda(t) \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ for any $t \in G$.*

(ii) *For every $n \in \mathbb{N}$ there exists a unique $b_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ such that $0 \leq b_n \leq 2\pi$ and $e^{ib_n} = \lambda(2^{-n})$.*

Proof. By (1.1) and (1.7),

$$\sigma_s^{\widehat{\varphi}}(x) = \lambda(s)x\lambda(s)^*, \ x \in \mathcal{R}, \ s \in G. \tag{2.2}$$

This clearly implies (i). To prove (ii) we use the principal branch $\operatorname{Log} z$ of the logarithmic function which satisfies $0 \leq \operatorname{Im}(\operatorname{Log} z) < 2\pi, \ z \in \mathbb{C} \setminus \{0\}$. Let

$$b_n = -i\operatorname{Log}\big(\lambda(2^{-n})\big).$$

Then $0 \leq b_n \leq 2\pi, \ e^{ib_n} = \lambda(2^{-n})$; by (i) and functional calculus, $b_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$. The uniqueness of b_n follows from the fact that $\lambda(2^{-n})$ has no point spectrum by virtue of Lemma 2.2 and the faithfulness of $\widehat{\varphi}$. □

Now let $a_n = 2^n b_n$, and define a sequence $(\varphi_n)_{n \geq 1}$ of normal faithful positive functionals on $\mathcal{R}$ by

$$\varphi_n(x) = \widehat{\varphi}(e^{-a_n}x), \quad x \in \mathcal{R}, \ n \geq 1. \tag{2.3}$$

Lemma 2.4. (i) *$\sigma_t^{\varphi_n}$ is 2^{-n}-periodic for all $n \geq 1$.*

(ii) *Let $\mathcal{R}_n = \mathcal{R}_{\varphi_n}$, $n \geq 1$. There exists a unique normal faithful conditional expectation Φ_n from $\mathcal{R}$ onto $\mathcal{R}_n$ such that*

$$\widehat{\varphi} \circ \Phi_n = \widehat{\varphi} \quad \text{and} \quad \sigma_t^{\widehat{\varphi}} \circ \Phi_n = \Phi_n \circ \sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R}, \ n \geq 1.$$

(iii) *$\mathcal{R}_n \subset \mathcal{R}_{n+1}$.*

Proof. (i) By Lemma 2.3 (i), $a_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$; in particular, $a_n \in \mathcal{R}_{\widehat{\varphi}}$. Thus

$$\sigma_t^{\varphi_n}(x) = e^{-ita_n}\sigma_t^{\widehat{\varphi}}(x)e^{ita_n}, \quad x \in \mathcal{R}, \ t \in \mathbb{R}. \tag{2.4}$$

Therefore, by (2.2) and Lemma 2.3 (ii), for all $x \in \mathcal{R}$,

$$\sigma_{2^{-n}}^{\varphi_n}(x) = e^{-i\,b_n}\sigma_{2^{-n}}^{\widehat{\varphi}}(x)e^{ib_n} = \lambda(2^{-n})^*\lambda(2^{-n})x\lambda(2^{-n})^*\lambda(2^{-n}) = x,$$

whence (i).

(ii) Define Φ_n by

$$\Phi_n(x) = 2^n \int_0^{2^{-n}} \sigma_t^{\varphi_n}(x)dt, \quad x \in \mathcal{R}.$$

By the 2^{-n}-periodicity of σ^{φ_n}, we find

$$\Phi_n(x) = \int_0^1 \sigma_t^{\varphi_n}(x)dt, \quad x \in \mathcal{R}.$$

It is then a routine matter to check that Φ_n is a normal faithful conditional expectation from $\mathcal{R}$ onto $\mathcal{R}_n$. Since $a_n \in \mathcal{R}_{\widehat{\varphi}}$, by (2.4) and the $\sigma^{\widehat{\varphi}}$-invariance of $\widehat{\varphi}$, we get

$$\widehat{\varphi}\big(\sigma_t^{\varphi_n}(x)\big) = \widehat{\varphi}\big(\sigma_t^{\widehat{\varphi}}(x)\big) = \widehat{\varphi}(x), \quad x \in \mathcal{R},\ t \in \mathbb{R}.$$

Thus $\widehat{\varphi}$ is also σ^{φ_n}-invariant for all $n \geq 1$. It then follows that

$$\widehat{\varphi}(\Phi_n(x)) = \int_0^1 \widehat{\varphi}\big(\sigma_t^{\varphi_n}(x)\big)dt = \widehat{\varphi}(x), \quad x \in \mathcal{R},$$

so $\widehat{\varphi} \circ \Phi_n = \widehat{\varphi}$. This latter equality implies the uniqueness of Φ_n as well as the commutation relation between Φ_n and $\sigma_t^{\widehat{\varphi}}$ (cf., e.g., [Ta1]). Alternately, this commutation relation immediately follows from (2.4) and the definition of Φ_n.

(iii) For every natural number n, a_n and a_{n+1} commute, because they are both contained in $\mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$. Hence by (2.4), φ_{n+1} is σ^{φ_n}-invariant. Let $h_n = \dfrac{D\varphi_{n+1}}{D\varphi_n}$, the Radon-Nikodým derivative of φ_{n+1} with respect to φ_n (cf. [PeT]). Then $\varphi_{n+1}(x) = \varphi_n(h_n x)$ for all $x \in \mathcal{R}$, and by the definition of φ_n,

$$h_n = \frac{D\varphi_{n+1}}{D\widehat{\varphi}} \cdot \frac{D\widehat{\varphi}}{D\varphi_n} = e^{-(a_{n+1}-a_n)}.$$

Proving $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ is equivalent to showing $h_n \in \mathcal{Z}(\mathcal{R}_n)$ for every $n \geq 1$. By (2.4), $\mathcal{R}_{\widehat{\varphi}} \subset \mathcal{R}_{\varphi_n}$ since $a_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$; In particular, $h_n \in \mathcal{R}_{\varphi_n} = \mathcal{R}_n$. Now

$$a_n = -i2^n \mathrm{Log}\, \lambda(2^{-n}) = -i2^n \mathrm{Log}\big(\lambda(2^{-n-1})^2\big).$$

Thus

$$a_{n+1} - a_n = -i2^n \big[2\mathrm{Log}\, \lambda(2^{-n-1}) - \mathrm{Log}\, \big(\lambda(2^{-n-1})^2\big)\big].$$

However, for any $z \in \mathbb{T}$,

$$2\,\mathrm{Log}\, z - \mathrm{Log}(z^2) = \begin{cases} 0 & \text{if } 0 \leq \mathrm{Arg}\, z < \pi, \\ 2\pi i & \text{if } \pi \leq \mathrm{Arg}\, z < 2\pi. \end{cases}$$

Hence

$$a_{n+1} - a_n = 2^{n+1}\pi e_n,$$

where e_n is the spectral projection of $\lambda(2^{-n-1})$ corresponding to $\mathrm{Im}\, z < 0$. Therefore, for all $x \in \mathcal{R}$ and $t \in \mathbb{R}$,

$$\sigma_t^{\varphi_{n+1}}(x) = h_n^{it}\sigma_t^{\varphi_n}(x)h_n^{-it} = e^{-i2^{n+1}\pi t e_n}\sigma_t^{\varphi_n}(x)e^{i2^{n+1}\pi t e_n}.$$

Consequently, if $x \in \mathcal{R}_n$, then by the 2^{-n-1}-periodicity of $\sigma_t^{\varphi_{n+1}}$, we deduce

$$x = e^{-i\pi e_n} x e^{i\pi e_n} = (1 - 2e_n)x(1 - 2e_n).$$

Therefore, $1 - 2e_n \in \mathcal{Z}(\mathcal{R}_n)$, so $e_n \in \mathcal{Z}(\mathcal{R}_n)$. Thus $h_n \in \mathcal{Z}(\mathcal{R}_n)$, which in turn yields the desired inclusion $\mathcal{R}_n \subset \mathcal{R}_{n+1}$. $\square$

By Lemma 2.4, in order to complete the proof of Theorem 2.1, it remains to show the w*-density of the union of the $\mathcal{R}_n$ in $\mathcal{R}$. This is the most difficult part of the proof, and it will be done in the following lemmas. The first one is well known. Recall that an element $x \in \mathcal{N}$ is called analytic (with respect to σ^ψ) if the map $t \mapsto \sigma_t^\psi(x)$ extends to an entire function, i.e., if there exists a (necessarily unique) analytic function $F_x : \mathbb{C} \to \mathcal{M}$ such that $F_x(t) = \sigma_t^\psi(x)$ for all $t \in \mathbb{R}$. In this case we put $\sigma_s^\psi(x) = F_x(s)$ for all $s \in \mathbb{C}$. Let $\mathcal{N}_a$ denote the family of all analytic elements of $\mathcal{N}$.

Lemma 2.5. *Let $\mathcal{N}$ be a von Neumann algebra and ψ a normal faithful state on $\mathcal{N}$.*

(i) *If $x \in \mathcal{N}_a$, then there exists a constant $c \geq 0$ such that $x\psi x^* \leq c\psi$.*
(ii) *If $x \in \mathcal{N}_\psi$, then $x\psi x^* \leq \|x\|^2\psi$.*

Proof. Without loss of generality, we assume that $\mathcal{N}$ acts standardly on a Hilbert space H. Then there exists a cyclic and separating vector $\xi_0 \in H$ such that $\psi(x) = \langle x\xi_0,\ \xi_0\rangle$. Let Δ and J be the corresponding modular operator and isometric conjugation (see, for instance, [KR, section 9.2]). Then

$$\sigma_t^\psi(x) = \Delta^{it}x\Delta^{-it},\ x \in \mathcal{N},\ t \in \mathbb{R}.$$

Let $x \in \mathcal{N}_a$. Then

$$\begin{aligned} x\xi_0 &= J\Delta^{1/2}x^*\xi_0 = J\Delta^{1/2}x^*\Delta^{-1/2}\xi_0 \\ &= J\sigma_{-i/2}^\psi(x^*)\xi_0 = \big(J\sigma_{-i/2}^\psi(x^*)J\big)(\xi_0). \end{aligned}$$

Now $x' = J\sigma_{-i/2}^\psi(x^*)J \in \mathcal{N}'$. Hence, for any $y \in \mathcal{N}$ and $y \geq 0$,

$$\psi(x^*yx) = \langle yx'\xi_0,\ x'\xi_0\rangle \leq \|x'\|^2\langle y\xi_0,\ \xi_0\rangle,$$

whence $x\psi x^* \leq \|x'\|^2\psi$. If additionally $x \in \mathcal{N}_\psi$, then $\sigma_{-i/2}(x^*) = x^*$, so $x' = Jx^*J$. Thus (ii) follows. □

In the following lemma, $[x,\ y]$ denotes the commutator of two operators x and y, i.e., $[x,\ y] = xy - yx$. If ψ is a positive functional on an algebra $\mathcal{N}$, $\|x\|_\psi^2 = \psi(x^*x)$ for $x \in \mathcal{N}$.

Lemma 2.6. *Keep the notation in Lemma 2.3. Then for $x \in \mathcal{R}$,*

(i) $\lim_{n\to\infty} \|[b_n,\ x]\|_{\widehat{\varphi}} = 0$;
(ii) $\lim_{n\to\infty} \sup_{t\in[-1,\ 1]} \big\|\big[e^{itb_n},\ x\big]\big\|_{\widehat{\varphi}} = 0$;
(iii) $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \big\|\sigma_t^{\widehat{\varphi}}(x) - e^{ia_nt}xe^{-ia_nt}\big\|_{\widehat{\varphi}} = 0.$

Proof. (i) Let $x \in \mathcal{R}$ and $k \in \mathbb{Z}$. Since $\lambda(t) \in \mathcal{R}_{\widehat{\varphi}}$ because of Lemma 2.3, we have

$$\begin{aligned} \big\|[\lambda(2^{-n})^k,\ x]\big\|_{\widehat{\varphi}} &= \big\|\lambda(k2^{-n})x - x\lambda(k2^{-n})\big\|_{\widehat{\varphi}} \\ &= \big\|\lambda(k2^{-n})x\lambda(k2^{-n})^* - x\big\|_{\widehat{\varphi}} = \big\|\sigma_{k2^{-n}}^{\widehat{\varphi}}(x) - x\big\|_{\widehat{\varphi}}. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \big\|[P(\lambda(2^{-n})),\ x]\big\|_{\widehat{\varphi}} = 0 \tag{2.5}$$

for any monomial, so for any trigonometric polynomial P.

Now assume that x is analytic with respect to $\sigma_t^{\widehat{\varphi}}$. Since Log $\in L_2(\mathbb{T})$, given $\varepsilon > 0$ there exists a trigonometric polynomial P such that

$$\|P + i\ \mathrm{Log}\|_{L_2(\mathbb{T})} < \varepsilon.$$

Recalling that $b_n = -i\mathrm{Log}\big(\lambda(2^{-n})\big)$, we deduce from (2.1) that

$$\|b_n - P(\lambda(2^{-n}))\|_{\widehat{\varphi}} = \big(\widehat{\varphi}(|b_n - P(\lambda(2^{-n}))|^2)\big)^{1/2} = \| - i\mathrm{Log}\ - P\|_{L_2(\mathbb{T})} < \varepsilon.$$

On the other hand, since x is analytic, by Lemma 2.5 there exists a constant c such that for any $y \in \mathcal{R}$,

$$\|[y,\ x]\|_{\widehat{\varphi}} \le \|yx\|_{\widehat{\varphi}} + \|xy\|_{\widehat{\varphi}} \le \sqrt{c}\,\|y\|_{\widehat{\varphi}} + \|x\|\,\|y\|_{\widehat{\varphi}} = (\sqrt{c} + \|x\|)\|y\|_{\widehat{\varphi}}\,.$$

Combining the preceding inequalities, we obtain

$$\begin{aligned}\|[b_n,\ x]\|_{\widehat{\varphi}} &\le \|[P(\lambda(2^{-n})),\ x]\|_{\widehat{\varphi}} + \|[b_n - P(\lambda(2^{-n})),\ x]\|_{\widehat{\varphi}}\\ &\le \|[P(\lambda(2^{-n})),\ x]\|_{\widehat{\varphi}} + (\sqrt{c} + \|x\|)\|b_n - P(\lambda(2^{-n}))\|_{\widehat{\varphi}}\\ &\le \|[P(\lambda(2^{-n})),\ x]\|_{\widehat{\varphi}} + \varepsilon(\sqrt{c} + \|x\|).\end{aligned}$$

Therefore, by (2.5),

$$\limsup_{n\to\infty} \|[b_n,\ x]\|_{\widehat{\varphi}} \le \varepsilon(\sqrt{c} + \|x\|),$$

whence

$$\lim_{n\to\infty} \|[b_n,\ x]\|_{\widehat{\varphi}} = 0, \quad \forall\, x \in \mathcal{R}_a\,,$$

where $\mathcal{R}_a$ denotes the set of analytic elements in $\mathcal{R}$ with respect to $\sigma^{\widehat{\varphi}}$.

Next, fix any $x \in \mathcal{R}$. Since $\mathcal{R}_a$ is a σ-strongly dense involutive subalgebra of $\mathcal{R}$, for every $\varepsilon > 0$, we can choose an $x_0 \in \mathcal{R}_a$ such that $\|x - x_0\|_{\widehat{\varphi}} < \varepsilon$. Then by the fact that $b_n \in \mathcal{R}_{\widehat{\varphi}}$ and $\|b_n\| \le 2\pi$ from Lemma 2.3, we deduce, as before,

$$\lim_{n\to\infty} \|[b_n,\ x]\|_{\widehat{\varphi}} = 0.$$

(ii) For any $k \in \mathbb{N}$, we have

$$[b_n^k,\ x] = b_n[b_n^{k-1},\ x] + [b_n,\ x]b_n^{k-1}.$$

Since $b_n \in \mathcal{R}_{\widehat{\varphi}}$, an induction argument yields

$$\|[b_n^k,\ x]\|_{\widehat{\varphi}} \le k\|b_n\|^{k-1}\|[b_n,\ x]\|_{\widehat{\varphi}} \le k(2\pi)^{k-1}\|[b_n,\ x]\|_{\widehat{\varphi}}.$$

Hence for any $z \in \mathbb{C}$,

$$\begin{aligned}\|[e^{zb_n},\ x]\|_{\widehat{\varphi}} &\le \sum_{k\ge1} \frac{|z|^k}{k!}\,\|[b_n^k,\ x]\|_{\widehat{\varphi}}\\ &\le \sum_{k\ge1} \frac{|z|^k}{(k-1)!}\,(2\pi)^{k-1}\|[b_n,\ x]\|_{\widehat{\varphi}}\\ &= |z|\,e^{2\pi|z|}\|[b_n,\ x]\|_{\widehat{\varphi}}.\end{aligned}$$

Therefore

$$\sup_{t\in[-1,\ 1]} \|[e^{itb_n},\ x]\|_{\widehat{\varphi}} \le e^{2\pi}\|[b_n,\ x]\|_{\widehat{\varphi}},$$

which, together with (i), implies (ii).

(iii) Fix $x \in \mathcal{R}$ and $\varepsilon > 0$. By (ii) there exists $n_0 \in \mathbb{N}$ such that

$$\|[e^{isb_n},\ x]\|_{\widehat{\varphi}} \le \varepsilon, \quad \forall\, s \in [-1,\ 1],\ \forall\, n \ge n_0. \tag{2.6}$$

Moreover, n_0 can be chosen such that further

$$\|\sigma_s^{\widehat{\varphi}}(x) - x\|_{\widehat{\varphi}} \le \varepsilon, \quad |s| \le 2^{-n_0}. \tag{2.7}$$

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \ge n_0$. Write $t = t_1 + t_2$, where $t_1 = k\,2^{-n}$ for some $k \in \mathbb{Z}$ and $0 \le t_2 \le 2^{-n}$. Then for any $y \in \mathcal{R}$,

$$\begin{aligned} \sigma_{t_1}^{\widehat{\varphi}}(y) &= \lambda(k2^{-n})y\lambda(k2^{-n})^* = e^{ikb_n}ye^{-ikb_n} \\ &= e^{ik2^{-n}a_n}ye^{-ik2^{-n}a_n} = e^{it_1a_n}ye^{-it_1a_n}. \end{aligned}$$

Since $\|\cdot\|_{\widehat{\varphi}}$ is invariant under $\sigma_{t_1}^{\widehat{\varphi}}$ and $a_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$, we deduce

$$\begin{aligned} \|\sigma_t^{\widehat{\varphi}}(x) - e^{ia_nt}xe^{-ia_nt}\|_{\widehat{\varphi}} &= \|\sigma_{t_2}^{\widehat{\varphi}}(x) - e^{ia_nt_2}xe^{-ia_nt_2}\|_{\widehat{\varphi}} \\ &\le \|\sigma_{t_2}^{\widehat{\varphi}}(x) - x\|_{\widehat{\varphi}} + \|x - e^{ia_nt_2}xe^{-ia_nt_2}\|_{\widehat{\varphi}} \\ &= \|\sigma_{t_2}^{\widehat{\varphi}}(x) - x\|_{\widehat{\varphi}} + \|[e^{-ia_nt_2},\ x]\|_{\widehat{\varphi}}\,. \end{aligned}$$

Now $a_nt_2 = (2^nt_2)b_n$ and $2^nt_2 \le 1$. Hence from (2.6) and (2.7) it follows that

$$\|\sigma_t^{\widehat{\varphi}}(x) - e^{ia_nt}xe^{-ia_nt}\|_{\widehat{\varphi}} \le 2\varepsilon.$$

This yields (iii). $\square$

Finally, we are ready to show the w*-density of $\bigcup_{n\ge1} \mathcal{R}_n$ in $\mathcal{R}$.

Lemma 2.7. *For any $x \in \mathcal{R}$, $\Phi_n(x)$ converges to x σ-strongly as $n \to \infty$. Consequently, $\bigcup_{n\ge1} \mathcal{R}_n$ is σ-strongly dense in $\mathcal{R}$.*

Proof. By the definition of Φ_n, it suffices to show

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \left\|\sigma_t^{\varphi_n}(x) - x\right\|_{\widehat{\varphi}} = 0, \quad x \in \mathcal{R}.$$

By (2.4) and the fact that $a_n \in \mathcal{R}_{\widehat{\varphi}}$, we have

$$\left\|\sigma_t^{\varphi_n}(x) - x\right\|_{\widehat{\varphi}} = \left\|e^{-ita_n}\sigma_t^{\widehat{\varphi}}(x)e^{ita_n} - x\right\|_{\widehat{\varphi}} = \left\|\sigma_t^{\widehat{\varphi}}(x) - e^{ita_n}xe^{-ita_n}\right\|_{\widehat{\varphi}}.$$

Therefore, the desired limit follows from Lemma 2.6 (iii). Thus the proof of Theorem 2.1 is complete. $\square$

Theorem 2.1 can be extended to general von Neumann algebras (not necessarily σ-finite) as follows.

Remark 2.8. Let $\mathcal{M}$ be a von Neumann algebra. Then there exist a von Neumann algebra $\mathcal{R}$ and an increasing family $\{\mathcal{R}_i\}_{i\in I}$ of w*-closed involutive subalgebras of $\mathcal{R}$ satisfying the following properties:

(i) $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{R}$ and there exists a normal faithful conditional expectation Φ from $\mathcal{R}$ onto $\mathcal{M}$;

(ii) each $\mathcal{R}_i$ admits a normal faithful tracial state;

(iii) the union of all $\mathcal{R}_i$ is w*-dense in $\mathcal{R}$;

(iv) for every $i \in I$ there exists a normal conditional expectation Φ_i from $\mathcal{R}$ onto $\mathcal{R}_i$ such that

$$\Phi_i \circ \Phi_j = \Phi_j \circ \Phi_i = \Phi_i \quad \text{whenever} \quad i \le j\,;$$

(v) there exists an *n.s.f.* weight φ on $\mathcal{M}$ such that

$$\widehat{\varphi} \circ \Phi_i = p_i\widehat{\varphi}p_i \quad \text{and} \quad \sigma_t^{\widehat{\varphi}} \circ \Phi_i = \Phi_i \circ \sigma_t^{\widehat{\varphi}}, \quad t \in \mathbb{R},\ i \in I,$$

where $\widehat{\varphi} = \varphi \circ \Phi$ and p_i is the identity of $\mathcal{R}_i$.

It is not hard to deduce this statement from Theorem 2.1. Indeed, this is quite easy if $\mathcal{M} = \mathcal{N}\bar{\otimes}B(K)$ for some σ-finite von Neumann algebra $\mathcal{N}$ and some Hilbert space K. The general case can be reduced to the previous one by using the classical fact that any von Neumann algebra is a direct sum of algebras of the form $\mathcal{N}\bar{\otimes}B(K)$ with $\mathcal{N}$ σ-finite. Namely, any $\mathcal{M}$ admits the following direct sum decomposition:

$$\mathcal{M} = \bigoplus_{j\in J} \mathcal{N}_j\bar{\otimes}B(K_j),$$

where each $\mathcal{N}_j$ is a σ-finite von Neumann algebra. See, for instance, the proof of Theorem 2.8.1 in Sakai's book [Sa].

3. Applications to noncommutative L_p-spaces

In this section we use Theorem 2.1 to prove an approximation theorem of general noncommutative L_p-spaces by those associated with finite von Neumann algebras. This is due to the first-named author and is indeed the original intention of [H3].

Theorem 3.1. *Let $\mathcal{M}$ be a σ-finite von Neumann algebra and $0 < p < \infty$. Let $L_p(\mathcal{M})$ be the Haagerup noncommutative L_p-space associated with $\mathcal{M}$. Then there exist a Banach space X_p (a quasi-Banach space if $p < 1$), a sequence $(\mathcal{R}_n)_{n\geq 1}$ of finite von Neumann algebras, each equipped with a normal faithful finite trace τ_n, and for each $n \geq 1$ an isometric embedding $J_n : L_p(\mathcal{R}_n, \tau_n) \to X_p$ such that*

(i) *the sequence $\big(J_n\big(L_p(\mathcal{R}_n, \tau_n)\big)\big)_{n\geq 1}$ is increasing;*
(ii) *$\bigcup_{n\geq 1} J_n\big(L_p(\mathcal{R}_n, \tau_n)\big)$ is dense in X_p;*
(iii) *$L_p(\mathcal{M})$ is isometric to a subspace Y_p of X_p;*
(iv) *Y_p and all $J_n\big(L_p(\mathcal{R}_n, \tau_n)\big)$ are 1-complemented in X_p for $1 \leq p < \infty$.*

Here $L_p(\mathcal{R}_n, \tau_n)$ is the tracial noncommutative L_p-space associated with $(\mathcal{R}_n, \tau_n)$.

Proof. Fix a normal faithful state φ on $\mathcal{M}$. We will use Theorem 2.1 and keep all the notation there. The space X_p required in the statement above will be $L_p(\mathcal{R}, \widehat{\varphi})$. By Remark 1.1, $L_p(\mathcal{M}, \varphi)$ and all $L_p(\mathcal{R}_n, \widehat{\varphi}\big|_{\mathcal{R}_n})$ are naturally isometrically identified as subspaces of $L_p(\mathcal{R}, \widehat{\varphi})$ for $0 < p < \infty$. Moreover, the sequence $\big(L_p(\mathcal{R}_n, \widehat{\varphi}\big|_{\mathcal{R}_n})\big)_{n\geq 1}$ is increasing. On the other hand, by [Ju1, Lemma 2.2], $\bigcup_{n\geq 1} L_p(\mathcal{R}_n, \widehat{\varphi}_n)$ is dense in $L_p(\mathcal{R}, \widehat{\varphi})$ for $0 < p < \infty$. Finally, since each $\mathcal{R}_n$ is a finite von Neumann algebra with a finite normal faithful trace τ_n, $L_p(\mathcal{R}_n, \widehat{\varphi}_n)$ is isometric to the usual noncommutative L_p-space on $\mathcal{R}_n$ defined by τ_n (see Remark 1.2). Hence, the space $L_p(\mathcal{R}, \widehat{\varphi})$ and the sequence $\big(L_p(\mathcal{R}_n, \widehat{\varphi}_n)\big)_{n\geq 1}$ satisfy properties (i) - (iii). The complementation property for $p \geq 1$ in (iv) follows from [JX1, Lemma 2.2] thanks to the conditional expectations Φ and Φ_n. $\square$

The following two remarks show that Theorem 3.1 is general enough for most applications.

Remark 3.2. Let $\mathcal{M}$ be a general von Neumann algebra. Then for any $0 < p < \infty$,

$$L_p(\mathcal{M}) = \bigcup_e eL_p(\mathcal{M})e = \bigcup_e L_p(e\mathcal{M}e),$$

where the union runs over the directed set of all σ-finite projections of $\mathcal{M}$. Indeed, for any $x \in L_p(\mathcal{M})$, the left and right support projections of x are σ-finite, so is their union e. Thus $x \in eL_p(\mathcal{M})e$.

Remark 3.3. Theorem 3.1 (combined with Remark 3.2) reduces many geometrical properties of general noncommutative L_p-spaces to the corresponding ones in the tracial case. This is indeed true for all those properties which are of a finite-dimensional nature. These include, for instance, Clarkson's inequalities, uniform convexity, uniform smoothness, type, cotype, and UMD property. We refer to [PX2] for more information.

Remark 3.4. Using Remark 2.8, we can extend Theorem 3.1 to the general case as follows. Let $\mathcal{M}$ be a general von Neumann algebra and $0 < p < \infty$. Let $L_p(\mathcal{M})$ be the Haagerup noncommutative L_p-space associated with $\mathcal{M}$. Then there exist a Banach space X_p (a quasi-Banach space if $p < 1$), a family $\{\mathcal{R}_i\}_{i\in I}$ of finite von Neumann algebras, each equipped with a normal faithful finite trace τ_i, and for each $i \in I$ an isometric embedding $J_i : L_p(\mathcal{R}_i, \tau_i) \to X_p$ such that

(i) $J_i\big(L_p(\mathcal{R}_i, \tau_i)\big) \subset J_j\big(L_p(\mathcal{R}_j, \tau_j)\big)$ for all $i, j \in I$ such that $i \le j$;
(ii) $\bigcup_{i\in I} J_i\big(L_p(\mathcal{R}_i, \tau_i)\big)$ is dense in X_p;
(iii) $L_p(\mathcal{M})$ is isometric to a subspace Y_p of X_p;
(iv) Y_p and $J_i\big(L_p(\mathcal{R}_i, \tau_i)\big)$, $i \in I$, are 1-complemented in X_p for $1 \le p < \infty$.

4. Extensions of maps to crossed products

In noncommutative analysis we often need to extend a map between two von Neumann algebras to their corresponding noncommutative L_p-spaces. On the other hand, when applying Theorem 2.1 to concrete problems we also need to extend maps between von Neumann algebras to their crossed products. This section is devoted to the second type of extensions, while the next section is devoted to the first type. In what follows, all maps considered will be assumed linear.

Theorem 4.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras acting on the same Hilbert space H, G a locally compact abelian group, α and β two automorphic representations of G on $\mathcal{M}$ and $\mathcal{N}$, respectively. Assume that $T : \mathcal{M} \to \mathcal{N}$ is a completely bounded normal map such that*

$$T \circ \alpha_g = \beta_g \circ T, \quad g \in G. \tag{4.1}$$

Then T admits a unique completely bounded normal extension $\widehat{T}$ from $\mathcal{M} \rtimes_\alpha G$ into $\mathcal{N} \rtimes_\beta G$ such that $\|\widehat{T}\|_{cb} = \|T\|_{cb}$ and

$$\widehat{T}\big(\lambda(g)\pi_\alpha(x)\big) = \lambda(g)\pi_\beta(Tx), \quad x \in \mathcal{M},\ g \in G. \tag{4.2}$$

Moreover, $\widehat{T}$ satisfies the following properties:

(i) *Let $\mathcal{A}$ be the von Neumann subalgebra on $L_2(G, H)$ generated by all $\lambda(g)$, $g \in G$. Then*

$$\widehat{T}\big(a\pi_\alpha(x)b\big) = a\pi_\beta(Tx)b, \quad x \in \mathcal{M},\ a,\ b \in \mathcal{A}. \tag{4.3}$$

(ii) $\widehat{T} \circ \widehat{\alpha}_\gamma = \widehat{\beta}_\gamma \circ \widehat{T}$, $\gamma \in \widehat{G}$.
(iii) *If T is a homomorphism, $*$-homomorphism or completely positive map, so is $\widehat{T}$.*
(iv) *If $\mathcal{N}$ is a subalgebra of $\mathcal{M}$, $\beta = \alpha\big|_{\mathcal{N}}$ and T is a* (*faithful*) *normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$, then $\widehat{T}$ is a* (*faithful*) *normal conditional expectation from $\mathcal{M} \rtimes_\alpha G$ onto $\mathcal{N} \rtimes_\beta G$.*

(v) *Let φ (resp. ψ) be an n.s.f. weight on $\mathcal{M}$ (resp. $\mathcal{N}$) such that*

$$T \circ \sigma_t^{\varphi} = \sigma_t^{\psi} \circ T, \quad t \in \mathbb{R}. \tag{4.4}$$

Then

$$\widehat{T} \circ \sigma_t^{\widehat{\varphi}} = \sigma_t^{\widehat{\psi}} \circ \widehat{T}, \quad t \in \mathbb{R}. \tag{4.5}$$

(vi) *Assume in addition that $T \geq 0$. Let φ (resp. ψ) be an n.s.f. weight on $\mathcal{M}$ (resp. $\mathcal{N}$) such that $\psi \circ T \leq \varphi$. Then $\widehat{\psi} \circ \widehat{T} \leq \widehat{\varphi}$.*

Proof. Since T is completely bounded and normal, so is

$$T \otimes \mathrm{id}_{B(L_2(G))} : \mathcal{M}\bar{\otimes}B(L_2(G)) \to \mathcal{N}\bar{\otimes}B(L_2(G)).$$

Moreover,

$$\|T \otimes \mathrm{id}_{B(L_2(G))}\|_{cb} = \|T\|_{cb}\,.$$

We claim that $T \otimes \mathrm{id}_{B(L_2(G))}$ maps $\mathcal{M} \rtimes_\alpha G$ into $\mathcal{N} \rtimes_\beta G$. Fix an orthonormal basis $(f_i)_{i\in I}$ in $L_2(G)$. Then by the definition of $\pi_\alpha(x)$, one sees that the matrix of $\pi_\alpha(x)$ in this basis has its coefficient at the position (i,j) given by

$$\big(\pi_\alpha(x)\big)_{ij} = \int_G \alpha_h^{-1}(x)\bar{f}_i(h)f_j(h)dh.$$

Thus by the normality of T and (4.1), it follows that

$$\begin{aligned}\big[T \otimes \mathrm{id}_{B(L_2(G))}\big(\pi_\alpha(x)\big)\big]_{ij} &= T\big(\big(\pi_\alpha(x)\big)_{ij}\big) = \int_G T(\alpha_h^{-1}(x))\bar{f}_i(h)f_j(h)dh \\ &= \int_G \beta_h^{-1}(Tx)\bar{f}_i(h)f_j(h)dh = \big(\pi_\beta(Tx)\big)_{ij}\,.\end{aligned}$$

On the other hand,

$$\lambda(g) = \mathrm{id}_H \otimes \ell(g),$$

where $\ell(g) : L_2(G) \to L_2(G)$ is the translation by g. Hence, the matrix of $\lambda(g)$ in $(f_i)_{i\in I}$ is $\mathrm{id}_H \otimes (a_{ij})$, where $(a_{ij})_{i,j\in I}$ is a bounded scalar matrix. Therefore, the matrix of $\lambda(g)\pi_\alpha(x)$ is given by

$$\big(\lambda(g)\pi_\alpha(x)\big)_{ij} = \sum_{k\in I} a_{ik}\big(\pi_\alpha(x)\big)_{kj}.$$

Thus, we deduce that the coefficient at the position (i,j) of the matrix of $T \otimes \mathrm{id}_{B(L_2(G))}\big(\lambda(g)\pi_\alpha(x)\big)$ is

$$\begin{aligned}\big[T \otimes \mathrm{id}_{B(L_2(G))}\big(\lambda(g)\pi_\alpha(x)\big)\big]_{ij} &= \sum_k a_{ik}\, T\big[(\pi_\alpha(x))_{kj}\big] \\ &= \sum_k a_{ik}\,\big(\pi_\beta(Tx)\big)_{kj} \\ &= \big[\lambda(g)\pi_\beta(Tx)\big]_{ij}\,.\end{aligned}$$

Hence, it follows that

$$T \otimes \mathrm{id}_{B(L_2(G))}\big(\lambda(g)\pi_\alpha(x)\big) = \lambda(g)\pi_\beta(Tx), \quad x \in \mathcal{M},\ g \in G. \tag{4.6}$$

Recall that the family of all finite linear combinations of $\lambda(g)\pi_\alpha(x)$, $g \in G$, $x \in \mathcal{M}$, is a w*-dense involutive subalgebra of $\mathcal{M} \rtimes_\alpha G$. Thus by the normality of $T \otimes \mathrm{id}_{B(L_2(G))}$, we deduce our claim.

Now set

$$\widehat{T} = T \otimes \mathrm{id}_{B(L_2(G))}\big|_{\mathcal{M}\rtimes_\alpha G}\,.$$

Then $\widehat{T} : \mathcal{M} \rtimes_\alpha G \to \mathcal{N} \rtimes_\beta G$ is completely bounded and normal. Moreover, $\widehat{T}$ satisfies (4.2) by virtue of (4.6). Thus $\widehat{T}$ is the desired extension of T. The uniqueness of $\widehat{T}$ follows from the normality and (4.2).

Let us check the other properties of $\widehat{T}$. Let $g,\, h \in G$, $x \in \mathcal{M}$. Then by (1.1),

$$\lambda(g)\pi_\alpha(x)\lambda(h) = \lambda(g)\lambda(h)\pi_\alpha(\alpha_{-h}(x)) = \lambda(g+h)\pi_\alpha(\alpha_{-h}(x)).$$

Thus by (4.2) and (4.1),

$$\begin{aligned}\widehat{T}\big(\lambda(g)\pi_\alpha(x)\lambda(h)\big) &= \lambda(g+h)\pi_\beta\big(T(\alpha_{-h}(x))\big)\\ &= \lambda(g+h)\pi_\beta\big(\beta_{-h}(Tx)\big) = \lambda(g)\pi_\beta(Tx)\lambda(h).\end{aligned}$$

This yields (4.3) in the case where $a = \lambda(g)$ and $b = \lambda(h)$ for any $g, h \in G$. The general case then follows from the normality of $\widehat{T}$.

(ii) is a consequence of (1.2) and (4.1). If T is a homomorphism, $*$-homomorphism or completely positive map, then so is $T \otimes \mathrm{id}_{B(L_2(G))}$. Thus we get (iii).

Under the conditions of (iv), $\mathcal{N} \rtimes_\beta G$ is a subalgebra of $\mathcal{M} \rtimes_\alpha G$. If T is a (faithful) conditional expectation, so is $T \otimes \mathrm{id}_{B(L_2(G))}$. Hence $\widehat{T}$ is also a (faithful) conditional expectation.

(v) follows from (1.7) and (4.1). To prove (vi) we extend, by normality, both T and $\widehat{T}$ to the extended positive parts of $\mathcal{M}$ and $\mathcal{M} \rtimes_\alpha G$, respectively. Let Φ_α and Φ_β be the operator-valued weights from $\mathcal{M} \rtimes_\alpha G$ to $\mathcal{M}$ and from $\mathcal{N} \rtimes_\beta G$ to $\mathcal{N}$, respectively. Then by (1.4) and the normality we obtain $\Phi_\beta \circ \widehat{T} = \widehat{T} \circ \Phi_\alpha$. Thus

$$\begin{aligned}\widehat{\psi} \circ \widehat{T} &= \psi \circ \pi_\beta^{-1} \circ \Phi_\beta \circ \widehat{T} = \psi \circ \pi_\beta^{-1} \circ \widehat{T} \circ \Phi_\alpha\\ &= \psi \circ \widehat{T} \circ \pi_\alpha^{-1} \circ \Phi_\alpha = \psi \circ T \circ \pi_\alpha^{-1} \circ \Phi_\alpha\\ &\le \varphi \circ \pi_\alpha^{-1} \circ \Phi_\alpha = \widehat{\varphi}.\end{aligned}$$

Hence (vi) is proved. Therefore, the proof of the theorem is complete. □

Remark 4.2. It is easy to check that the extension in Theorem 4.1 satisfies the following functorial property. Let $\mathcal{L}$ be a third von Neumann algebra, γ an automorphic representation of G on $\mathcal{L}$ and $S : \mathcal{N} \to \mathcal{L}$ a completely bounded normal map such that $S \circ \beta_g = \gamma_g \circ S$ for all $g \in G$. Then $\widehat{S \circ T} = \widehat{S} \circ \widehat{T}$.

We end this section by specializing the extension in Theorem 4.1 to the situation described in section 2. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras on H equipped with two normal faithful states φ and ψ, respectively. Let $G = \bigcup_{n\ge1} 2^{-n}\mathbb{Z}$. We keep all the notation in section 2 for both $(\mathcal{M}, \varphi)$ and $(\mathcal{N}, \psi)$. Set $\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} G$ and $\mathcal{S} = \mathcal{N} \rtimes_{\sigma^\psi} G$. By Theorem 2.1, we have two increasing sequences $(\mathcal{R}_n)_{n\ge1}$ and $(\mathcal{S}_n)_{n\ge1}$ of finite von Neumann subalgebras of $\mathcal{R}$ and $\mathcal{S}$, respectively, which satisfy all properties there. The corresponding conditional expectations from $\mathcal{R}$ onto $\mathcal{R}_n$, respectively, from $\mathcal{S}$ onto $\mathcal{S}_n$, are denoted by Φ_n and Ψ_n.

Proposition 4.3. *Let $T : \mathcal{M} \to \mathcal{N}$ be a completely bounded normal map such that*

$$T \circ \sigma_t^\varphi = \sigma_t^\psi \circ T, \quad t \in \mathbb{R}.$$

Let $\widehat{T} : \mathcal{R} \to \mathcal{S}$ be the extension of T given by Theorem 4.1. *Then*

(i) *$\widehat{T} \circ \Phi_n = \Psi_n \circ \widehat{T}$ for every n; consequently, $\widehat{T}(\mathcal{R}_n) \subset \mathcal{S}_n$.*

(ii) *Assume in addition that* $T \geq 0$ *and* $\psi \circ T \leq \varphi$. *Then* $\psi_n \circ \widehat{T} \leq \varphi_n$ *for every* $n \in \mathbb{N}$, *where* φ_n *and* ψ_n *are, respectively, the states relative to* φ *and* ψ *defined by* (2.3).

Proof. By the definition of Φ_n in section 2 (see the proof of Lemma 2.4) and (2.4), we have

$$\Phi_n(x) = \int_0^1 e^{-ita_n}\sigma_t^{\widehat{\varphi}}(x)e^{ita_n}\,dt$$

and a similar formula for Ψ_n with φ replaced by ψ. We then deduce (i) by virtue of (4.3) and (4.5). To prove (ii) recall that $\varphi_n = e^{-a_n}\widehat{\varphi}$ and $\psi_n = e^{-a_n}\widehat{\psi}$. By the fact that $a_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\psi}})$, $a_n \in \mathcal{Z}(\mathcal{R}_{\widehat{\varphi}})$ (see Lemma 2.3) and Theorem 4.1, (i), (vi), we deduce that for $x \in \mathcal{R}_+$,

$$\psi_n \circ \widehat{T}(x) = \widehat{\psi}(e^{-a_n}\widehat{T}(x)) = \widehat{\psi}(\widehat{T}(e^{-a_n/2}xe^{-a_n/2})) \leq \widehat{\varphi}(e^{-a_n/2}xe^{-a_n/2}) = \varphi_n(x).$$

This finishes the proof. □

5. Extensions of maps to noncommutative L_p-spaces

In this section we deal with the problem of how to extend a map between two von Neumann algebras to their noncommutative L_p-spaces. We consider only the σ-finite case. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras equipped with normal faithful states φ and ψ, respectively. Let D_φ denote the Radon-Nikodým derivative of the dual weight $\widehat{\varphi}$ on $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ with respect to the canonical *n.s.f.* trace τ_φ of $\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$. D_ψ has the same meaning relative to $(\mathcal{N}, \psi)$. Consider a positive map $T : \mathcal{M} \to \mathcal{N}$ such that for some positive constant C_1,

$$\psi(T(x)) \leq C_1\varphi(x), \quad x \in \mathcal{M}_+. \tag{5.1}$$

Given $1 \leq p < \infty$ define

$$\begin{array}{rlcl} T_p : & D_\varphi^{1/2p}\,\mathcal{M}\,D_\varphi^{1/2p} & \to & D_\psi^{1/2p}\,\mathcal{N}\,D_\psi^{1/2p}, \\ & D_\varphi^{1/2p}xD_\varphi^{1/2p} & \mapsto & D_\psi^{1/2p}T(x)D_\psi^{1/2p}. \end{array}$$

Recall that by [JX1, Lemma 1.1], $D_\varphi^{1/2p}\mathcal{M}D_\varphi^{1/2p}$ is a dense subspace of $L_p(\mathcal{M}, \varphi)$. The main result of this section is the following.

Theorem 5.1. *The map* T_p *above extends to a positive bounded map from* $L_p(\mathcal{M}, \varphi)$ *into* $L_p(\mathcal{N}, \psi)$ *for all* $1 \leq p < \infty$. *Moreover,*

$$\|T_p\| \leq C_\infty^{1-1/p}C_1^{1/p}\,, \quad \textit{where } C_\infty = \|T(1)\|_\infty\,.$$

Note that there does not exist any additional factor before C_1 in the above bound $C_\infty^{1-1/p}C_1^{1/p}$ for the norm of T_p. This is very important for applications, for instance, for those applications to noncommutative ergodic theory (see section 7 below). In the tracial case, Theorem 5.1 was proved in [Y] with $4C_1$ instead of C_1 in the previous estimate on $\|T_p\|$. Theorem 5.1 was announced in [GL1]. The proof there presents, unfortunately, a serious gap.

Now we proceed to the proof of Theorem 5.1. The main difficulty lies in the extension of T_1. Once this is done, that of T_p will then follow from a rather easy interpolation argument via Kosaki's interpolation theorem. For the extension of T_1 we need the following lemma, which is a reformulation of Lemma 1.2 from [H1] into the present setting. We include a proof for the convenience of the reader. $\mathcal{M}_h$ denotes the subspace of selfadjoint elements of $\mathcal{M}$.

Lemma 5.2. *Let $x \in \mathcal{M}_h$. Then*

$$\|D_\varphi^{1/2} x D_\varphi^{1/2}\|_1 = \inf\big\{\varphi(a) + \varphi(b) \ : \ x = a - b,\ a,\ b \in \mathcal{M}_+\big\}.$$

Proof. Denote the infimum on the right-hand side by $\rho(x)$. Then $x \mapsto \rho(x)$ defines a seminorm on $\mathcal{M}_h$. By (1.13), for any $x \in \mathcal{M}_+$,

$$\rho(x) = \varphi(x) = \mathrm{tr}(D^{1/2} x D^{1/2}) = \|D^{1/2} x D^{1/2}\|_1\,.$$

Here and during this proof we denote D_φ simply by D. Thus it follows that

$$\|D^{1/2} x D^{1/2}\|_1 \le \rho(x), \quad x \in \mathcal{M}_h\,.$$

To prove the converse inequality, we fix $x_0 \in \mathcal{M}_h$. Then by the Hahn-Banach theorem there exists a linear functional $f : \mathcal{M}_h \to \mathbb{R}$ such that

$$f(x_0) = \rho(x_0) \quad \text{and} \quad |f(x)| \le \rho(x), \quad \forall\, x \in \mathcal{M}_h\,.$$

We extend f to a complex linear functional on the whole $\mathcal{M}$ by complexification, still denoted by f. Then f is Hermitian and $-\varphi \le f \le \varphi$ on $\mathcal{M}_+$. Thus by the Cauchy-Schwarz inequality, for $x,\ y \in \mathcal{M}$ we have

$$\begin{aligned}
|f(y^*x)| &\le \frac{1}{2}\big[|(\varphi+f)(y^*x)| + |(\varphi-f)(y^*x)|\big]\\
&\le \frac{1}{2}\Big[\big((\varphi+f)(y^*y)\big)^{1/2}\big((\varphi+f)(x^*x)\big)^{1/2}\\
&\qquad + \big((\varphi-f)(y^*y)\big)^{1/2}\big((\varphi-f)(x^*x)\big)^{1/2}\Big]\\
&\le \big(\varphi(y^*y)\big)^{1/2}\big(\varphi(x^*x)\big)^{1/2} = \|xD^{1/2}\|_2\,\|yD^{1/2}\|_2\,.
\end{aligned}$$

By the density of $\mathcal{M}D^{1/2}$ in $L_2(\mathcal{M},\varphi)$, we deduce that there exists a contraction B on $L_2(\mathcal{M},\varphi)$ such that

$$\langle B(xD^{1/2}),\ yD^{1/2}\rangle = f(y^*x), \quad x, y \in \mathcal{M}\,.$$

Now regarding $\mathcal{M}$ as acting standardly on $L_2(\mathcal{M},\varphi)$ by left multiplication, we claim that B belongs to the commutant of $\mathcal{M}$. Indeed, for $a \in \mathcal{M}$,

$$\begin{aligned}
\langle Ba(xD^{1/2}),\ yD^{1/2}\rangle &= \langle B(axD^{1/2}),\ yD^{1/2}\rangle = f(y^*ax)\\
&= \langle B(xD^{1/2}),\ a^*yD^{1/2}\rangle = \langle aB(xD^{1/2}),\ yD^{1/2}\rangle\,.
\end{aligned}$$

Therefore, $Ba = aB$, so our claim follows. Thus B coincides with the right multiplication on $L_2(\mathcal{M},\varphi)$ by an element $b \in \mathcal{M}$. Hence we deduce that

$$f(y^*x) = \langle xD^{1/2}b,\ yD^{1/2}\rangle = \mathrm{tr}(D^{1/2}y^*xD^{1/2}b), \quad x, y \in \mathcal{M}\,.$$

Consequently,

$$\rho(x_0) = f(x_0) = \mathrm{tr}(D^{1/2}x_0D^{1/2}b) \le \|D^{1/2}x_0D^{1/2}\|_1\|b\|_\infty \le \|D^{1/2}x_0D^{1/2}\|_1\,.$$

Therefore, the lemma is proved. □

Lemma 5.3. *T_1 extends to a positive bounded map from $L_1(\mathcal{M},\varphi)$ into $L_1(\mathcal{N},\psi)$ with norm majorized by C_1.*

Proof. Let $x \in \mathcal{M}$ and $y = D_\varphi^{1/2} x D_\varphi^{1/2}$. Assume first $x \ge 0$. Then $y \ge 0$, so $T_1(y) \ge 0$ for T is positive. Hence, by (1.13) and (5.1),

$$\begin{aligned}
\|T_1(y)\|_{L_1(\mathcal{N},\psi)} &= \mathrm{tr}(T_1(y)) = \mathrm{tr}(D_\psi^{1/2} T(x) D_\psi^{1/2})\\
&= \psi(T(x)) \le C_1\varphi(x) = C_1\|y\|_{L_1(\mathcal{M},\varphi)}.
\end{aligned}$$

Now assume that x is selfadjoint and $\varepsilon > 0$. Then by Lemma 5.2, there exist $a, b \in \mathcal{M}_+$ such that $x = a - b$ and

$$\|D_\varphi^{1/2} a D_\varphi^{1/2}\|_{L_1(\mathcal{M},\varphi)} + \|D_\varphi^{1/2} b D_\varphi^{1/2}\|_{L_1(\mathcal{M},\varphi)} \leq \|y\|_{L_1(\mathcal{M},\varphi)} + \varepsilon.$$

It follows that

$$T_1(y) = D_\psi^{1/2} T(a) D_\psi^{1/2} - D_\psi^{1/2} T(b) D_\psi^{1/2}$$

and

$$\begin{aligned} \|T_1(y)\|_{L_1(\mathcal{N},\psi)} &\leq \|D_\psi^{1/2} T(a) D_\psi^{1/2}\|_{L_1(\mathcal{N},\psi)} + \|D_\psi^{1/2} T(b) D_\psi^{1/2}\|_{L_1(\mathcal{N},\psi)} \\ &\leq C_1\big(\|D_\varphi^{1/2} a D_\varphi^{1/2}\|_{L_1(\mathcal{M},\varphi)} + \|D_\varphi^{1/2} b D_\varphi^{1/2}\|_{L_1(\mathcal{M},\varphi)}\big) \\ &\leq C_1\big(\|y\|_{L_1(\mathcal{M},\varphi)} + \varepsilon\big), \end{aligned}$$

whence again

$$\|T_1(y)\|_{L_1(\mathcal{N},\psi)} \leq C_1 \|y\|_{L_1(\mathcal{M},\varphi)}\,.$$

Finally, decomposing any $x \in \mathcal{M}$ into its real and imaginary parts, we get

$$\|T_1(y)\|_{L_1(\mathcal{N},\psi)} \leq 2\, C_1 \|y\|_{L_1(\mathcal{M},\varphi)}.$$

Therefore, T_1 is bounded relative to the L_1-norms. Since $D_\varphi^{1/2} \mathcal{M} D_\varphi^{1/2}$ is dense in $L_1(\mathcal{M},\varphi)$, T_1 extends to a bounded map from $L_1(\mathcal{M},\varphi)$ into $L_1(\mathcal{N},\psi)$ with $\|T_1\| \leq 2C_1$. Since T_1 is positive, so is its extension (which is denoted again by T_1).

Thus it remains to drop the factor 2 from the previous estimate on $\|T_1\|$. To this end, we consider the adjoint: $T_1^* : \mathcal{N} \to \mathcal{M}$. Since T_1^* is positive, T_1^* attains its norm at the identity of $\mathcal{N}$ (see [Pau]). Therefore,

$$\|T_1\| = \|T_1^*\| = \|T_1^*(1)\|_\infty\,.$$

Hence, we are reduced to showing $\|T_1^*(1)\|_\infty \leq C_1$. This is easy by duality. Indeed, let $x \in \mathcal{M}_+$ and $y = D_\varphi^{1/2} x D_\varphi^{1/2}$. Then by (1.13) and (5.1),

$$\operatorname{tr}(T_1^*(1)\, y) = \operatorname{tr}(T_1(y)) = \operatorname{tr}\big(D_\psi^{1/2} T(x) D_\psi^{1/2}\big) = \psi(T(x)) \leq C_1 \varphi(x) = C_1 \|y\|_{L_1(\mathcal{M},\varphi)}\,.$$

Since $D_\varphi^{1/2} \mathcal{M}_+ D_\varphi^{1/2}$ is dense in the positive cone of $L_1(\mathcal{M},\varphi)$, we deduce the desired estimate on $\|T_1^*(1)\|_\infty$. Hence the proof of the lemma is complete. $\square$

Proof of Theorem 5.1. To prove that T_p extends to a bounded map from $L_p(\mathcal{M},\varphi)$ to $L_p(\mathcal{N},\psi)$ for all $1 < p < \infty$, we will use interpolation. We consider the following symmetric injection of $\mathcal{M}$ into $L_1(\mathcal{M},\varphi)$:

$$\begin{array}{rccl} j_\varphi : & \mathcal{M} & \to & L_1(\mathcal{M},\varphi), \\ & x & \mapsto & D_\varphi^{1/2} x D_\varphi^{1/2}. \end{array}$$

Then j_φ turns $(\mathcal{M},\ L_1(\mathcal{M},\varphi))$ into a compatible couple, so we can consider their complex interpolation space $(\mathcal{M},\ L_1(\mathcal{M},\varphi))_{1/p}$ for any $1 < p < \infty$. By [Ko, Theorem 9.1], this space can be isometrically identified with $L_p(\mathcal{M},\varphi)$. More precisely, define $j_\varphi^p(x) = D_\varphi^{1/2p} x D_\varphi^{1/2p},\ x \in \mathcal{M}$. Then j_φ^p extends to an isometry from $(\mathcal{M},\ L_1(\mathcal{M},\varphi))_{1/p}$ onto $L_p(\mathcal{M},\varphi)$. For $(\mathcal{N},\psi)$ we use similar notation. Under the injections j_φ and j_ψ, T_1 is viewed as the same map as T (on the intersection space $\mathcal{M}$). Therefore, by Lemma 5.3 and interpolation, T is bounded from $(\mathcal{M},\ L_1(\mathcal{M},\varphi))_{1/p}$ into $(\mathcal{N},\ L_1(\mathcal{N},\psi))_{1/p}$ with norm majorized by $C_\infty^{1-1/p} C_1^{1/p}$

(recalling that $C_\infty = \|T\|_{\mathcal{M}\to\mathcal{N}}$). From this, and using the isometries $j^p_\varphi,\ j^p_\psi$ defined above, we deduce that for any $x \in \mathcal{M}$,

$$\begin{aligned}\big\|T_p\big(D_\varphi^{1/2p} x D_\varphi^{1/2p}\big)\big\|_{L_p(\mathcal{N},\psi)} &= \big\|j^p_\psi(T(x))\big\|_{L_p(\mathcal{N},\psi)} = \big\|T(x)\big\|_{(\mathcal{N},\, L_1(\mathcal{N},\psi))_{1/p}}\\ &\le C_\infty^{1-1/p}\, C_1^{1/p}\big\|x\big\|_{(\mathcal{M},\, L_1(\mathcal{M},\varphi))_{1/p}}\\ &= C_\infty^{1-1/p}\, C_1^{1/p}\big\|j^p_\varphi(x)\big\|_{L_p(\mathcal{M},\varphi)}\\ &= C_\infty^{1-1/p}\, C_1^{1/p}\big\|D_\varphi^{1/2p} x D_\varphi^{1/2p}\big\|_{L_p(\mathcal{M},\varphi)}\,.\end{aligned}$$

Therefore, by the density of $D_\varphi^{1/2p}\mathcal{M}\, D_\varphi^{1/2p}$ in $L_p(\mathcal{M},\varphi)$, this implies that T_p extends to a bounded map from $L_p(\mathcal{M},\varphi)$ to $L_p(\mathcal{N},\psi)$ with norm controlled by $C_\infty^{1-1/p}\, C_1^{1/p}$. Since T_p is positive on $D_\varphi^{1/2p}\,\mathcal{M}\, D_\varphi^{1/2p}$, so is its extension on $L_p(\mathcal{M},\varphi)$. □

Let T be as in Theorem 5.1. The extension of T_p will again be denoted by T_p. Consider the adjoint map of T_1: $S = T_1^* : \mathcal{N} \to \mathcal{M}$. We claim that S satisfies the same conditions as T. Namely, S is positive and

$$\varphi(S(y)) \le C_\infty \psi(y), \quad y \in \mathcal{N}_+\,. \tag{5.2}$$

Indeed, the positivity of S was already observed during the proof of Lemma 5.3. On the other hand, for $y \in \mathcal{N}_+$ we have

$$\begin{aligned}\varphi(S(y)) &= \mathrm{tr}(S(y)D_\varphi) = \langle T_1^*(y),\ D_\varphi\rangle = \langle y,\ D_\psi^{1/2} T(1) D_\psi^{1/2}\rangle\\ &= \mathrm{tr}\big(D_\psi^{1/2} y D_\psi^{1/2}\, T(1)\big) \le \|T(1)\|_\infty \psi(y) = C_\infty\, \psi(y).\end{aligned}$$

Therefore, applying Theorem 5.1 to S, we get the extension $S_p : L_p(\mathcal{N},\psi) \to L_p(\mathcal{M},\varphi)$. It is easy to check that $S_1^* = T$ and $S_p^* = T_{p'}$ for any $1 < p < \infty$ (p' being conjugate to p). Consequently, T is normal.

We record the discussion above in the following.

Proposition 5.4. *Let T and T_p be as in Theorem* 5.1 (*T_p also denoting the extension*). *Let $S = T_1^*$.*

(i) *The map $S : \mathcal{N} \to \mathcal{M}$ is positive and satisfies* (5.2).
(ii) *Let $S_p : L_p(\mathcal{N},\psi) \to L_p(\mathcal{M},\varphi)$ be the extension of S. Then $S_1^* = T$ and $S_p^* = T_{p'}$ for every $1 < p < \infty$.*
(iii) *T is normal.*

The extension in Theorem 5.1 is symmetric. We could also consider the left extension: $xD_\varphi^{1/p} \mapsto T(x)D_\psi^{1/p}$ ($x \in \mathcal{M}$). More generally, for any $0 \le \theta \le 1$ we can define

$$\begin{aligned}T_{p,\theta}\ :\quad D_\varphi^{(1-\theta)/p}\,\mathcal{M}\, D_\varphi^{\theta/p} &\ \to\ D_\psi^{(1-\theta)/p}\,\mathcal{N}\, D_\psi^{\theta/p},\\ D_\varphi^{(1-\theta)/p} x D_\varphi^{\theta/p} &\ \mapsto\ D_\psi^{(1-\theta)/p} T(x) D_\psi^{\theta/p}.\end{aligned}$$

Thus $T_{p,1/2}$ agrees with T_p. We do not know whether $T_{p,\theta}$ extends to a bounded map from $L_p(\mathcal{M},\varphi)$ to $L_p(\mathcal{N},\psi)$. However, if in addition T satisfies

$$T \circ \sigma_t^\varphi = \sigma_t^\psi \circ T, \quad t \in \mathbb{R}, \tag{5.3}$$

then $T_{p,\theta}$ indeed extends. Recall that $\mathcal{M}_a$ (resp. $\mathcal{N}_a$) denotes the family of all analytic elements of $\mathcal{M}$ with respect to σ^φ (resp. $\mathcal{N}$ relative to σ^ψ). By [JX1, Lemma 1.1], $D_\varphi^{(1-\theta)/p}\,\mathcal{M}_a\, D_\varphi^{\theta/p}$ is dense in $L_p(\mathcal{M},\varphi)$.

Proposition 5.5. *Let $T:\mathcal{M}\to\mathcal{N}$ satisfy* (5.3). *Then we have $T_{p,\theta}=T_p$ on $D_\varphi^{(1-\theta)/p}\,\mathcal{M}_a\,D_\varphi^{\theta/p}$. Consequently, if in addition, T is positive and satisfies* (5.1), *then $T_{p,\theta}$ extends to a bounded map from $L_p(\mathcal{M},\varphi)$ to $L_p(\mathcal{N},\psi)$ and its extension coincides with that of T_p in Theorem* 5.1.

Proof. We first note that by (5.3), T maps $\mathcal{M}_a$ into $\mathcal{N}_a$. On the other hand, it is easy to see that

$$D_\varphi^{(1-\theta)/p}\,\mathcal{M}_a\,D_\varphi^{\theta/p}=D_\varphi^{1/2p}\,\mathcal{M}_a\,D_\varphi^{1/2p}.$$

Now let $x\in\mathcal{M}_a$. Then

$$D_\varphi^{(1-\theta)/p}xD_\varphi^{\theta/p}=\sigma^\varphi_{-\frac{i(1-\theta)}{p}}(x)D_\varphi^{1/p}\,.$$

Therefore, by (5.3), we get

$$\begin{aligned}T_{p,1}\big(\sigma^\varphi_{-\frac{i(1-\theta)}{p}}(x)D_\varphi^{1/p}\big)&=T\big(\sigma^\varphi_{-\frac{i(1-\theta)}{p}}(x)\big)\,D_\psi^{1/p}=\sigma^\psi_{-\frac{i(1-\theta)}{p}}\big(T(x)\big)D_\psi^{1/p}\\&=D_\psi^{(1-\theta)/p}T(x)D_\psi^{\theta/p}=T_{p,\theta}\big(D_\varphi^{(1-\theta)/p}xD_\varphi^{\theta/p}\big)\,.\end{aligned}$$

This proves the first part of the proposition. The second simply follows from Theorem 5.1. □

Remark 5.6. Theorem 5.1 can be extended to the weighted case too. Let $\mathcal{M}$ and $\mathcal{N}$ be two general von Neumann algebras equipped with *n.s.f.* weights φ and ψ, respectively. Let $T:\mathcal{M}\to\mathcal{N}$ be a positive map such that for some positive constant C_1,

$$\psi(T(x))\le C_1\varphi(x),\quad x\in\mathcal{M}_+.$$

Consider

$$\begin{array}{rlcl}T_p: & D_\varphi^{1/2p}\,\mathfrak{m}_\varphi\,D_\varphi^{1/2p} & \to & D_\psi^{1/2p}\,\mathfrak{m}_\psi\,D_\psi^{1/2p},\\ & D_\varphi^{1/2p}xD_\varphi^{1/2p} & \mapsto & D_\psi^{1/2p}T(x)D_\psi^{1/2p}.\end{array}$$

Then T_p extends to a bounded map from $L_p(\mathcal{M},\varphi)$ into $L_p(\mathcal{N},\psi)$ for all $1\le p<\infty$.

The proof of this statement is essentially the same as that of Theorem 5.1. Now instead of Kosaki's interpolation theorem, we use that of Terp [Te2]. Note also that $D_\varphi^{1/2p}\,\mathfrak{m}_\varphi\,D_\varphi^{1/2p}$ is dense in $L_p(\mathcal{M},\varphi)$ for any $1\le p<\infty$ (see [GL2] and [Te2]).

Convention. In the sequel, we will denote, by the same symbol T, all maps T_p and $T_{p,\theta}$ as well as their extensions between the L_p-spaces in Theorem 5.1 and Proposition 5.5, whenever no confusion can occur. This is consistent with Kosaki's interpolation theorem. See the discussion in the proof of Theorem 5.1.

We end this section by three examples, which are special cases of Theorem 5.1. In all three, the map T satisfies both (5.1) and (5.3), so the extensions to the L_p-spaces can be made from T_p in Theorem 5.1 or any $T_{p,\theta}$ in Proposition 5.5.

Example 5.7. The first example is the tracial case, i.e., when both φ and ψ are tracial. Then it is well known that any positive map $T:\mathcal{M}\to\mathcal{N}$ satisfying (5.1) extends to a bounded map between the usual noncommutative L_p-spaces constructed from the traces φ and ψ. This is very easy to prove by interpolation (cf. [Y]). Note that in [Y] the estimate on $\|T_1\|$ is $4C_1$. Also note that in this case, (5.3) is trivially satisfied.

Example 5.8. The second example concerns conditional expectations. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ and $\psi = \varphi\big|_{\mathcal{N}}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be a normal faithful conditional expectation such that $\varphi \circ \mathcal{E} = \varphi$. Then it is well known that $\mathcal{E}$ commutes with the modular automorphism group σ_t^{φ} (cf. [Co, 1.4.3]). Thus $\mathcal{E}$ extends to a bounded map between the noncommutative L_p-spaces for all $1 \le p < \infty$. In fact, in this special case, Theorem 5.1 becomes [JX1, Lemma 2.2]. Note that the extension of $\mathcal{E}$ between the L_p-spaces possesses all the usual properties of a conditional expectation as in the commutative case. In particular, it is an $\mathcal{N}$-bimodular contractive projection from $L_p(\mathcal{M}, \varphi)$ onto $L_p(\mathcal{N}, \psi)$.

Example 5.9. The third example is that when T is a state-preserving isomorphism, i.e., $\psi \circ T = \varphi$. In this case, condition (5.3) is again automatically satisfied. Indeed, it is easy to check that $T^{-1} \circ \sigma_t^{\psi} \circ T$ is an automorphism group satisfying the KMS condition relative to φ, so it coincides with σ_t^{φ}. Also in this case Lemma 5.3 above admits a straightforward proof as follows. Let $x \in \mathcal{M}$ and $y \in \mathcal{N}_a$. Then

$$\begin{aligned} \operatorname{tr}\big(y\, D_\psi^{1/2} T(x) D_\psi^{1/2}\big) &= \operatorname{tr}\big(D_\psi^{1/2} \sigma_{i/2}^{\psi}(y) T(x) D_\psi^{1/2}\big) = \psi\big(\sigma_{i/2}^{\psi}(y) T(x)\big) \\ &= \varphi\big(\sigma_{i/2}^{\varphi}(T^{-1}(y))\, x\big) = \operatorname{tr}\big(T^{-1}(y)\, D_\varphi^{1/2} x D_\varphi^{1/2}\big); \end{aligned}$$

so

$$\big|\operatorname{tr}\big(y\, D_\psi^{1/2} T(x) D_\psi^{1/2}\big)\big| \le \|T^{-1}(y)\|_\infty\, \|D_\varphi^{1/2} x D_\varphi^{1/2}\|_1 \le \|y\|_\infty\, \|D_\varphi^{1/2} x D_\varphi^{1/2}\|_1.$$

This implies the boundedness of T_1 on $D_\varphi^{1/2} \mathcal{M} D_\varphi^{1/2}$.

6. Applications to noncommutative martingale inequalities

Since the establishment of the noncommutative Burkholder-Gundy inequalities in [PX1], the theory of noncommutative martingale inequalities has been rapidly developed. Many of the classical inequalities in the usual martingale theory have been transferred into the noncommutative setting. We refer, for instance, to [Ju1] for the Doob maximal inequality, to [JX1, JX2] for the Burkholder/Rosenthal inequalities, to [R1, R2, R3] for several weak type $(1,1)$ inequalities and to [PaR] for the Gundy decomposition.

The objective of this section and the next one is to show how to use Theorem 2.1 to reduce all these inequalities in the nontracial case to those in the tracial one. As a consequence, their best constants in the general case coincide with the corresponding ones in the tracial case. This section deals with square function type inequalities. The Doob maximal inequality, the only exception among those quoted previously, is postponed to the next section, where the maximal ergodic inequalities will also be considered.

6.1. **The framework.** Throughout this and the next sections, $\mathcal{M}$ will denote a von Neumann algebra equipped with a distinguished normal faithful state φ and $\sigma = \sigma^{\varphi}$ the modular automorphism group of φ. We denote $L_p(\mathcal{M}, \varphi)$ simply by $L_p(\mathcal{M})$. Assume that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and that there exists a normal faithful conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{N}$ such that

$$\varphi \circ \mathcal{E} = \varphi\,. \tag{6.1}$$

It is well known that such an $\mathcal{E}$ is unique and satisfies

$$\sigma_t \circ \mathcal{E} = \mathcal{E} \circ \sigma_t\,, \quad t \in \mathbb{R}. \tag{6.2}$$

Recall that the existence of a conditional expectation $\mathcal{E}$ satisfying (6.1) is equivalent to the σ^{φ}-invariance of $\mathcal{N}$ (see [Ta1]). The restriction of σ^{φ} to $\mathcal{N}$ is the modular automorphism group of $\varphi\big|_{\mathcal{N}}$. We will not distinguish φ and σ^{φ} from their restrictions to $\mathcal{N}$.

Let $G = \bigcup_{n\geq 1} 2^{-n}\mathbb{Z}$ (fixed throughout this and the next sections). For notational simplicity set

$$\mathcal{R}(\mathcal{M}) = \mathcal{M} \rtimes_{\sigma} G \quad \text{and} \quad \mathcal{R}(\mathcal{N}) = \mathcal{N} \rtimes_{\sigma} G\,.$$

By virtue of (6.2), $\mathcal{R}(\mathcal{N})$ is naturally viewed as a von Neumann subalgebra of $\mathcal{R}(\mathcal{M})$. Let $\widehat{\varphi}$ be the dual weight on $\mathcal{R}(\mathcal{M})$ of φ (recalling that $\widehat{\varphi}$ is a normal faithful state). Its restriction to $\mathcal{R}(\mathcal{N})$ is the dual weight of $\varphi\big|_{\mathcal{N}}$. As usual, we denote this restriction again by $\widehat{\varphi}$. The increasing sequence of the von Neumann subalgebras of $\mathcal{R}(\mathcal{M})$ constructed in Theorem 2.1 relative to $\mathcal{M}$ is denoted by $(\mathcal{R}_m(\mathcal{M}))_{m\geq 1}$, and that relative to $\mathcal{N}$ by $(\mathcal{R}_m(\mathcal{N}))_{m\geq 1}$. All $\mathcal{R}_m(\mathcal{M})$ and $\mathcal{R}_m(\mathcal{N})$ are von Neumann subalgebras of $\mathcal{R}(\mathcal{M})$. From the proof of Theorem 2.1 we easily see that

$$\mathcal{R}_m(\mathcal{N}) = \mathcal{R}_m(\mathcal{N}) \cap \mathcal{R}_m(\mathcal{M})\,, \quad m \in \mathbb{N}.$$

Let $\Phi\ :\ \mathcal{R}(\mathcal{M}) \to \mathcal{M}$ be the conditional expectation defined by (1.8). Its restriction to $\mathcal{R}(\mathcal{N})$ is the corresponding conditional expectation from $\mathcal{R}(\mathcal{N})$ onto $\mathcal{N}$, again denoted by Φ. Let $\Phi_m\ :\ \mathcal{R}(\mathcal{M}) \to \mathcal{R}_m(\mathcal{M})$ be the conditional expectation constructed in Theorem 2.1. On the other hand, by virtue of (6.2) and Theorem 4.1, $\mathcal{E}$ extends to a normal faithful conditional expectation $\widehat{\mathcal{E}}$ from $\mathcal{R}(\mathcal{M})$ onto $\mathcal{R}(\mathcal{N})$. Then by (1.9), (4.2) and (4.5), we have

$$\widehat{\varphi} \circ \widehat{\mathcal{E}} = \widehat{\varphi} \quad \text{and} \quad \sigma_t^{\widehat{\varphi}} \circ \widehat{\mathcal{E}} = \widehat{\mathcal{E}} \circ \sigma_t^{\widehat{\varphi}}\,. \tag{6.3}$$

By (1.8), (4.2) and Proposition 4.3, we find

$$\widehat{\mathcal{E}} \circ \Phi = \Phi \circ \widehat{\mathcal{E}} \quad \text{and} \quad \widehat{\mathcal{E}} \circ \Phi_m = \Phi_m \circ \widehat{\mathcal{E}}\,, \quad m \in \mathbb{N}. \tag{6.4}$$

It follows that $\mathcal{R}(\mathcal{N})$ and $\mathcal{R}_m(\mathcal{M})$ are respectively invariant under Φ_m and $\widehat{\mathcal{E}}$ for all $m \in \mathbb{N}$; moreover, $\Phi_m\big|_{\mathcal{R}(\mathcal{N})}\ :\ \mathcal{R}(\mathcal{N}) \to \mathcal{R}_m(\mathcal{N})$ and $\widehat{\mathcal{E}}\big|_{\mathcal{R}_m(\mathcal{M})}\ :\ \mathcal{R}_m(\mathcal{M}) \to \mathcal{R}_m(\mathcal{N})$ are normal faithful conditional expectations. Again, we will not distinguish these conditional expectations and their respective restrictions. By Theorem 2.1 and the discussion above, all conditional expectations Φ, Φ_m and $\widehat{\mathcal{E}}$ commute, and further commute with $\sigma^{\widehat{\varphi}}$; moreover, all these maps preserve the dual state $\widehat{\varphi}$. This commutation is shown in the following diagram:

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dl, "\mathcal{E}"'] \arrow[d, "\sigma_t^{\varphi}"'] & \mathcal{R}(\mathcal{M}) \arrow[l, "\Phi"'] \arrow[r, "\Phi_m"] \arrow[d, "\sigma_t^{\widehat{\varphi}}"'] & \mathcal{R}_m(\mathcal{M}) \arrow[d, "\sigma_t^{\widehat{\varphi}}"'] \arrow[dr, "\widehat{\mathcal{E}}"] & \\
\mathcal{N} \arrow[dr, "\sigma_t^{\varphi}"'] & \mathcal{M} \arrow[d, "\mathcal{E}"'] & \mathcal{R}(\mathcal{M}) \arrow[l, "\Phi"'] \arrow[r, "\Phi_m"] \arrow[d, "\widehat{\mathcal{E}}"'] & \mathcal{R}_m(\mathcal{M}) \arrow[d, "\widehat{\mathcal{E}}"'] & \mathcal{R}_m(\mathcal{N}) \arrow[dl, "\sigma_t^{\widehat{\varphi}}"] \\
 & \mathcal{N} & \mathcal{R}(\mathcal{N}) \arrow[l, "\Phi"'] \arrow[r, "\Phi_m"] & \mathcal{R}_m(\mathcal{N}) &
\end{tikzcd}$$

By Remark 1.1, $L_p(\mathcal{N}, \varphi) = L_p(\mathcal{N})$ is naturally identified as a subspace of $L_p(\mathcal{M})$. On the other hand, [JX1, Lemma 2.2] (or Example 5.8) implies that $\mathcal{E}$ extends to a positive contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{N})$ $(1 \leq p \leq \infty)$, again denoted by $\mathcal{E}$, which possesses the following modular property. Let $1 \le p, q, r \le \infty$ such that $1/p + 1/q + 1/r \le 1$. Then

$$\mathcal{E}(axb) = a\mathcal{E}(x)b\,, \quad a \in L_q(\mathcal{N}),\ b \in L_r(\mathcal{N}),\ x \in L_p(\mathcal{M})\,. \tag{6.5}$$

In what follows all spaces $L_p(\mathcal{R}(\mathcal{M}))$, $L_p(\mathcal{R}_m(\mathcal{M}))$, $L_p(\mathcal{R}(\mathcal{N}))$ and $L_p(\mathcal{R}_m(\mathcal{N}))$ are relative to $\widehat{\varphi}$. They are naturally identified as subspaces of $L_p(\mathcal{R}(\mathcal{M}))$. On the other hand, since $\widehat{\varphi}\big|_{\mathcal{M}} = \varphi$ (see (1.9)) and $\mathcal{M}$, $\mathcal{N}$ are $\sigma^{\widehat{\varphi}}$-invariant (see (1.7)), $L_p(\mathcal{M})$ and $L_p(\mathcal{N})$ are subspaces of $L_p(\mathcal{R}(\mathcal{M}))$ too. Thus, $L_p(\mathcal{R}(\mathcal{M}))$ is the largest space among all these noncommutative L_p-spaces. Since Φ, Φ_m and $\widehat{\mathcal{E}}$ preserve $\widehat{\varphi}$ and commute with $\sigma^{\widehat{\varphi}}$, these conditional expectations extend to positive contractive projections on $L_p(\mathcal{R}(\mathcal{M}))$ for all $1 \le p \le \infty$; moreover, their extensions satisfy the modular property (6.5). Since all these conditional expectations commute, so do their extensions. As usual, we use the same symbol to denote a map and its extensions.

Now we fix an increasing filtration $(\mathcal{M}_n)_{n\ge 1}$ of von Neumann subalgebras of $\mathcal{M}$ whose union is w*-dense in $\mathcal{M}$. Assume that for each $n \ge 1$ there exists a normal faithful conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ such that $\varphi \circ \mathcal{E}_n = \varphi$. Then

$$\mathcal{E}_n \circ \mathcal{E}_m = \mathcal{E}_m \circ \mathcal{E}_n = \mathcal{E}_{\min(n,m)}\,, \quad m,\, n \in \mathbb{N}. \tag{6.6}$$

The preceding discussion applies, of course, to each $\mathcal{M}_n$ in place of $\mathcal{N}$. Thus we have the crossed product $\mathcal{R}(\mathcal{M}_n)$ and the subalgebras $\mathcal{R}_m(\mathcal{M}_n)$. Also, each $\mathcal{E}_n$ extends to a conditional expectation $\widehat{\mathcal{E}}_n$ from $\mathcal{R}(\mathcal{M})$ to $\mathcal{R}(\mathcal{M}_n)$. By (6.6) and (4.2), we find

$$\widehat{\mathcal{E}}_n \circ \widehat{\mathcal{E}}_m = \widehat{\mathcal{E}}_m \circ \widehat{\mathcal{E}}_n = \widehat{\mathcal{E}}_{\min(n,m)}\,, \quad m,\ n \in \mathbb{N}. \tag{6.7}$$

All previous assumptions and notation will be kept fixed throughout this section.

6.2. **Martingale inequalities.** Let $\mathcal{M}$ and $(\mathcal{M}_n)$ be fixed as in the previous subsection. By definition, an L_p-martingale with respect to $(\mathcal{M}_n)_n$ is a sequence $x = (x_n) \subset L_p(\mathcal{M})$ $(1 \le p \le \infty)$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n, \quad \forall\, n \in \mathbb{N}.$$

In this case, x is adapted in the sense that $x_n \in L_p(\mathcal{M}_n)$ for all n. Define

$$\|x\|_p = \sup_n \|x_n\|_p\,.$$

If $\|x\|_p < \infty$, x is called a bounded L_p-martingale. The martingale difference sequence of x is defined to be $dx = (dx_n)_{n\ge 1}$ with $dx_n = x_n - x_{n-1}$ ($x_0 = 0$ by convention).

Remark 6.1. It is an easy exercice to check the following two properties:

(i) Let $x_\infty \in L_p(\mathcal{M})$ with $1 \le p \le \infty$, and let $x_n = \mathcal{E}_n(x_\infty)$. Then $x = (x_n)$ is a bounded L_p-martingale and x_n converges to x_∞ in $L_p(\mathcal{M})$ (in the w*-topology for $p = \infty$). Moreover, $\|x\|_p = \|x_\infty\|_p$.

(ii) Conversely, let $x = (x_n)$ be a bounded L_p-martingale with $1 < p \le \infty$. Then there exists $x_\infty \in L_p(\mathcal{M})$ such that $x_n = \mathcal{E}_n(x_\infty)$ for all n.

This remark allows us to not distinguish a martingale x and its final value x_∞ whenever the latter exists. This also explains why we use the letter x to denote sometimes an operator in $L_p(\mathcal{M})$, sometimes a martingale. We will also identify a martingale with its difference sequence. In the sequel all martingales are with respect to $(\mathcal{M}_n)$ unless explicitly stated otherwise.

Now we can begin to state the noncommutative martingale inequalities we are interested in. In the sequel, the letters $\alpha_p, \beta_p, \dots$ will denote positive constants depending only on p, and C an absolute positive constant. The simplest noncommutative martingale inequalities are the noncommutative Khintchine inequalities, which are of paramount importance in noncommutative analysis.

Khintchine inequalities. Let (ε_n) be a Rademacher sequence on a probability space $(\Omega,\ P)$, i.e., an independent sequence with $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all n. Recall that the classical Khintchine inequality asserts that for any $p < \infty$,

$$\big\|\sum_n \varepsilon_n a_n\big\|_p \sim \big(\sum_n |a_n|^2\big)^{1/2}$$

holds for all finite sequences $(a_n) \subset \mathbb{C}$, where the equivalence constants depend only on p. Using the Fubini theorem we then deduce that for any finite sequence (a_n) in a commutative L_p-space we have

$$\big(\mathbb{E}\big\|\sum_n \varepsilon_n a_n\big\|_p^p\big)^{1/p} \sim \big\|\big(\sum_n |a_n|^2\big)^{1/2}\big\|_p\,,$$

where $\mathbb{E}$ is the expectation on Ω.

The noncommutative analogue of the previous equivalence takes, unfortunately, a much less simple form due to the existence of two different absolute values of operators because of the noncommutativity. We now have two square functions:

$$\big(\sum_n a_n^* a_n\big)^{1/2} \quad \text{and} \quad \big(\sum_n a_n a_n^*\big)^{1/2}$$

for any finite sequence $(a_n) \subset L_p(\mathcal{M})$. Accordingly, we introduce the column and row L_p-spaces. The column space $L_p(\mathcal{M}; \ell_2^c)$ is the Banach space of all sequences $a = (a_n)_{n\geq 1} \subset L_p(\mathcal{M})$ such that

$$\|a\|_{L_p(\mathcal{M};\ell_2^c)} = \big\|\big(\sum_{n\geq 1} a_n^* a_n\big)^{1/2}\big\|_p < \infty.$$

The row space $L_p(\mathcal{M}; \ell_2^r)$ consists of all a such that $a^* \in L_p(\mathcal{M}; \ell_2^c)$ and is equipped with the norm $\|a\|_{L_p(\mathcal{M};\ell_2^r)} = \|a^*\|_{L_p(\mathcal{M};\ell_2^c)}$. $L_p(\mathcal{M}; \ell_2^c)$ and $L_p(\mathcal{M}; \ell_2^r)$ can be respectively regarded as the column and row subspaces of $L_p(\mathcal{M}\bar{\otimes}B(\ell_2))$ (see [PX1] for more details). Then they are 1-complemented in $L_p(\mathcal{M}\bar{\otimes}B(\ell_2))$ for all $1 \leq p \leq \infty$.

Now define the space $CR_p[L_p(\mathcal{M})]$ as follows. If $p \geq 2$,

$$CR_p[L_p(\mathcal{M})] = L_p(\mathcal{M}; \ell_2^c) \cap L_p(\mathcal{M}; \ell_2^r)$$

equipped with the intersection norm:

$$\big\|(a_n)\big\|_{CR_p[L_p(\mathcal{M})]} = \max\big\{\big\|(a_n)\big\|_{L_p(\mathcal{M};\ell_2^c)},\ \big\|(a_n)\big\|_{L_p(\mathcal{M};\ell_2^r)}\big\}.$$

If $p < 2$,

$$CR_p[L_p(\mathcal{M})] = L_p(\mathcal{M}; \ell_2^c) + L_p(\mathcal{M}; \ell_2^r)$$

equipped with the sum norm:

$$\big\|(a_n)\big\|_{CR_p[L_p(\mathcal{M})]} = \inf\big\{\big\|(b_n)\big\|_{L_p(\mathcal{M};\ell_2^c)} + \big\|(c_n)\big\|_{L_p(\mathcal{M};\ell_2^r)}\big\},$$

where the infimum runs over all decompositions $a_n = b_n + c_n$ with $b_n, c_n \in L_p(\mathcal{M})$.

We are now ready to state the noncommutative Khintchine inequalities. Recall that the vector-valued L_p-space $L_p(\Omega; L_p(\mathcal{M}))$ can be identified with $L_p(L_\infty(\Omega)\bar{\otimes}\mathcal{M})$.

Theorem 6.2. *Let $1 \leq p < \infty$ and (x_n) be a finite sequence in $L_p(\mathcal{M})$. Then*

$$(\mathrm{K}_p) \qquad \Big\| \sum_n \varepsilon_n a_n \Big\|_{L_p(\Omega; L_p(\mathcal{M}))} \sim \big\| (a_n) \big\|_{CR_p[L_p(\mathcal{M})]} .$$

More precisely, there exist two universal positive constants A and B such that

$$A^{-1} \big\| (a_n) \big\|_{CR_p[L_p(\mathcal{M})]} \leq \Big\| \sum_n \varepsilon_n a_n \Big\|_{L_p(\Omega; L_p(\mathcal{M}))} \leq \big\| (a_n) \big\|_{CR_p[L_p(\mathcal{M})]}$$

for $1 \leq p \leq 2$ and

$$\big\| (a_n) \big\|_{CR_p[L_p(\mathcal{M})]} \leq \Big\| \sum_n \varepsilon_n a_n \Big\|_{L_p(\Omega; L_p(\mathcal{M}))} \leq B\sqrt{p}\, \big\| (a_n) \big\|_{CR_p[L_p(\mathcal{M})]}$$

for $2 \leq p < \infty$.

The previous inequalities were first proved for $1 < p < \infty$ and Schatten classes (i.e., for $\mathcal{M} = B(\ell_2)$) in [LP] and then for $1 \leq p < \infty$ and tracial noncommutative L_p-spaces in [LPP] (see [Pi1] for the optimal order $\mathrm{O}(\sqrt{p})$). The arguments of [LPP] work for the type III case too. See [HM] for a very simple proof with better constants in the case $p = 1$. This last work also provides the best constants for certain other random variables instead of (ε_n), including complex Gaussians and type III Fermions. We refer the interested reader to [Ju3], [JX2], [Pi1] for more Khintchine type inequalities in the noncommutative setting.

Using Theorem 2.1, one can easily reduce the general case of (K_p) to the tracial one. See the proof of Theorem 6.8 below, notably the part concerning (S_p).

Martingale transforms. We now consider the unconditionality of noncommutative martingale difference sequences, i.e., the noncommutative martingale transforms by sequences of signs.

Theorem 6.3. *Let $1 < p < \infty$. Then for any finite L_p-martingale $x = (x_n)$,*

$$(\mathrm{MT}_p) \qquad \Big\| \sum_{n \geq 1} \varepsilon_n dx_n \Big\|_p \leq \kappa_p \, \|x\|_p, \quad \forall\, \varepsilon_n = \pm 1.$$

(MT_p) is an immediate consequence of the noncommutative Burkholder-Gundy inequalities below. It was first proved in [PX1] for the tracial case, and then extended to the general case in [JX1]. (MT_p) fails, of course, for $p = 1$. Randrianantoanina [R1] proved, however, the weak type (1,1) substitute for $p = 1$ in the tracial case, which we recall as follows. Let φ be tracial and assume that $L_p(\mathcal{M})$ is the usual tracial L_p-space associated with $(\mathcal{M}, \varphi)$. Let x be a finite L_1-martingale. Then

$$(\mathrm{MT}_1) \qquad \varphi\big(\mathbb{1}_{(\lambda,\, \infty)}(|\sum_{n \geq 1} \varepsilon_n dx_n|)\big) \leq \kappa_1 \, \frac{\|x\|_1}{\lambda}, \quad \forall\, \lambda > 0,\ \forall\, \varepsilon_n = \pm 1.$$

This weak type $(1,1)$ inequality gives an alternate simple proof of (MT_p) in the tracial case. Indeed, note that (MT_p) is trivial for $p = 2$ (with $\kappa_2 = 1$) by virtue of the orthogonality of martingale differences in $L_2(\mathcal{M})$. Then interpolating this trivial case with (MT_1) via the Marcinkiewicz interpolation theorem, we get (MT_p) for $1 < p < 2$. Finally, duality yields the case $2 < p < \infty$.

We also emphasize that Randrianantoanina's theorem provides a key to the problem of finding the optimal orders of the best constants in various noncommutative martingale inequalities. See Corollary 6.9 below and the discussion following it.

Burkholder-Gundy inequalities. To state these inequalities, we need to recall the Hardy spaces of noncommutative martingales introduced in [PX1]. Let $1 \le p < \infty$. We define $\mathcal{H}_p^c(\mathcal{M})$ to be the space of all martingales $x = (x_n)_n$ in $L_p(\mathcal{M})$ such that $dx \in L_p(\mathcal{M}; \ell_2^c)$. We equip $\mathcal{H}_p^c(\mathcal{M})$ with the norm

$$\|x\|_{\mathcal{H}_p^c} = \big\| \big(\sum_{n \ge 1} |dx_n|^2 \big)^{1/2} \big\|_p .$$

Similarly, $\mathcal{H}_p^r(\mathcal{M})$ is defined to be the space of all L_p-martingales x such that $x^* \in \mathcal{H}_p^c(\mathcal{M})$, equipped with the norm $\|x\|_{\mathcal{H}_p^r} = \|x^*\|_{\mathcal{H}_p^c}$. Finally, set

$$\mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) \cap \mathcal{H}_p^r(\mathcal{M}) \text{ for } p \ge 2 \quad \text{and} \quad \mathcal{H}_p(\mathcal{M}) = \mathcal{H}_p^c(\mathcal{M}) + \mathcal{H}_p^r(\mathcal{M}) \text{ for } p < 2 ,$$

equipped with the intersection and sum norms, respectively.

Theorem 6.4. *Let $1 < p < \infty$. Then for all finite noncommutative L_p-martingales $x = (x_n)$,*

$$(\mathrm{BG}_p) \qquad \alpha_p^{-1} \|x\|_{\mathcal{H}_p} \le \|x\|_p \le \beta_p \|x\|_{\mathcal{H}_p}.$$

These inequalities were proved in [PX1] for the tracial case and in [JX1] for the general case. The second inequality in (BG_p) remains valid for $p = 1$, while the first one has a weak type $(1, 1)$ substitute for $p = 1$ in the tracial case (see [R2]). Since the norm $\| \cdot \|_{\mathcal{H}_p}$ is unconditional on martingale difference sequences, (BG_p) immediately implies (MT_p). Conversely, by virtue of the noncommutative Khintchine inequalities (K_p), (MT_p) implies (BG_p) in the case $p \ge 2$. For $p < 2$ we further need the noncommutative Stein inequality, which is the following statement.

Theorem 6.5. *Let $1 < p < \infty$. Then for all finite sequences $(a_n)_n$ in $L_p(\mathcal{M})$,*

$$(\mathrm{S}_p) \qquad \big\| \big(\sum_n |\mathcal{E}_n(a_n)|^2 \big)^{1/2} \big\|_p \le \gamma_p \big\| \big(\sum_n |a_n|^2 \big)^{1/2} \big\|_p.$$

This result was proved in [PX1] for the tracial case and in [JX1] for the general noncommutative L_p-spaces. We emphasize that (S_p) often plays an important role when dealing with noncommutative martingales. In the tracial case, (MT_1) implies a weak type $(1, 1)$ substitute of (S_p) for $p = 1$, which, together with interpolation, provides another proof of (S_p).

Burkholder inequalities. These inequalities are closely related with (BG_p). To state them we need more notation. Let $1 \le p < \infty$. Let $x = (x_n)_{n \ge 1}$ be a finite martingale in $\mathcal{M}_a D^{1/p}$ (recalling that $\mathcal{M}_a$ denotes the family of analytic elements of $\mathcal{M}$). Define (with $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|x\|_{h_p^c} = \big\| \big(\sum_{n \ge 1} \mathcal{E}_{n-1}(|dx_n|^2) \big)^{1/2} \big\|_p .$$

This defines a norm on the vector space of all finite martingales in $\mathcal{M}_a D^{1/p}$. The corresponding completion is denoted by $h_p^c(\mathcal{M})$. Similarly, we define $h_p^r(\mathcal{M})$ by passing to adjoints. Finally, $h_p^d(\mathcal{M})$ denotes the subspace of $\ell_p(L_p(\mathcal{M}))$ consisting of martingale differences. Then we define the conditioned version $h_p(\mathcal{M})$ of $\mathcal{H}_p(\mathcal{M})$: for $2 \le p < \infty$,

$$h_p(\mathcal{M}) = h_p^d(\mathcal{M}) \cap h_p^c(\mathcal{M}) \cap h_p^r(\mathcal{M})$$

and for $1 \le p < 2$,

$$h_p(\mathcal{M}) = h_p^d(\mathcal{M}) + h_p^c(\mathcal{M}) + h_p^r(\mathcal{M}) .$$

$h_p(\mathcal{M})$ is equipped with the intersection or sum norm according to $p \geq 2$ or $p < 2$. We refer to [JX1] for more information on these spaces. The following theorem gives the noncommutative Burkholder inequalities from [JX1].

Theorem 6.6. *Let $1 < p < \infty$. Then an L_p-martingale x is bounded in $L_p(\mathcal{M})$ iff x belongs to $h_p(\mathcal{M})$; moreover, if this is the case, then*

$$(\mathrm{B}_p) \qquad \eta_p^{-1}\|x\|_{h_p} \leq \|x\|_p \leq \zeta_p \|x\|_{h_p}.$$

Inequalities (BG_p) and (B_p) are linked together through the dual form of the noncommutative Doob maximal inequality (see Remark 7.7 below). The second inequality of (B_p) remains true for $p = 1$. In the tracial case, Randrianantoanina [R3] obtained a weak type $(1,1)$ substitute for the first inequality with $p = 1$.

Rosenthal inequalities. We first recall the classical Rosenthal inequalities. Let (Ω, P) be a probability space and let (f_n) be an independent sequence of random variables in $L_p(\Omega)$ with $2 \leq p < \infty$. Then

$$\big\|\sum_n f_n\big\|_p \sim \big(\sum_n \|f_n\|_p^p\big)^{1/p} + \big(\sum_n \|f_n\|_2^2\big)^{1/2}.$$

To state the noncommutative analogue of this we need to define independence in the noncommutative setting. In contrast with the classical case, there now exist several different notions of independence. The following definition is general enough to embrace most existing independences.

Let $\mathcal{N}$ and $\mathcal{A}_n$ be σ^φ-invariant von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{N} \subset \mathcal{A}_n$ for every n. The sequence $(\mathcal{A}_n)$ can be finite. Note that the σ^φ-invariance of $\mathcal{N}$ implies that there exists a normal faithful conditional expectation $\mathcal{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ preserving the state φ. We say that $(\mathcal{A}_n)$ are independent over $\mathcal{N}$ or with respect to $\mathcal{E}_\mathcal{N}$ if for every n, $\mathcal{E}_\mathcal{N}(ab) = \mathcal{E}_\mathcal{N}(a)\mathcal{E}_\mathcal{N}(b)$ holds for all $a \in \mathcal{A}_n$ and b in the von Neumann subalgebra generated by $(\mathcal{A}_k)_{k\neq n}$. A sequence $(a_n) \subset L_p(\mathcal{M})$ is said to be independent with respect to $\mathcal{E}_\mathcal{N}$ if there exist $\mathcal{A}_n$ such that $a_n \in L_p(\mathcal{A}_n)$ and $(\mathcal{A}_n)$ is independent with respect to $\mathcal{E}_\mathcal{N}$.

Note that if $(\mathcal{A}_n)$ are independent over $\mathcal{N}$ and $a_n \in L_p(\mathcal{A}_n)$ with $\mathcal{E}_\mathcal{N}(a_n) = 0$, then (a_n) is a martingale difference sequence relative to the filtration $\big(VN(\mathcal{A}_1, ..., \mathcal{A}_n)\big)_{n\geq 1}$, where $VN(\mathcal{A}_1, ..., \mathcal{A}_n)$ is the von Neumann subalgebra generated by $\mathcal{A}_1, ..., \mathcal{A}_n$. We refer to [JX2] for more information.

Theorem 6.7. *Let $2 \leq p < \infty$ and $(a_n) \in L_p(\mathcal{M})$ be a finite independent sequence such that $\mathcal{E}_\mathcal{N}(a_n) = 0$. Then*

$$(\mathrm{R}_p) \qquad \frac{1}{2}\big\|(a_n)\big\|_{\mathcal{R}_p} \leq \big\|\sum_n a_n\big\|_p \leq \nu_p \big\|(a_n)\big\|_{\mathcal{R}_p},$$

where

$$\big\|(a_n)\big\|_{\mathcal{R}_p} = \max\big\{\big(\sum_n \|a_n\|_p^p\big)^{1/p},\ \big\|\big(\sum_n \mathcal{E}_\mathcal{N}(|a_n|^2)\big)^{1/2}\big\|_p,\ \big\|\big(\sum_n \mathcal{E}_\mathcal{N}(|a_n^*|^2)\big)^{1/2}\big\|_p\big\}.$$

This result is proved in [JX2]. Dualizing the inequality above, we get a similar one for $1 < p < 2$ (see [JX2] for more related results).

6.3. **Reduction.** In this subsection we show that all preceding inequalities can be reduced to the tracial case.

Theorem 6.8. *If the preceding inequalities* (MT_p), (BG_p), (S_p), (B_p) *and* (R_p) *hold in the tracial case, then they also hold in the general case with the same relevant best constants.*

As already noted before, the Khintchine inequalities (K_p) can also be reduced to the tracial case. The situation for (K_p) is much simpler than all other martingale inequalities. This is why (K_p) is not included in the preceding statement.

As a corollary, we deduce that the estimates on the best constants in these inequalities for the tracial case are also valid for the general case. The notation $A \approx B$ means that $C_1 A \le B \le C_2 A$ for two universal positive constants C_1 and C_2.

Corollary 6.9. *The best constants in* (MT_p), (BG_p), (S_p), (B_p) *and* (R_p) *are estimated as follows:*

(i) $\kappa_p \approx p$ *as* $p \to \infty$;
(ii) $\alpha_p \approx p$ *as* $p \to \infty$ *and* $\alpha_p \approx (p-1)^{-1}$ *as* $p \to 1$;
(iii) $\beta_p \approx p$ *as* $p \to \infty$ *and* $\beta_p \approx 1$ *as* $p \to 1$;
(iv) $\gamma_p \approx p$ *as* $p \to \infty$;
(v) $\eta_p \approx (p-1)^{-1}$ *as* $p \to 1$ *and* $\eta_p \approx p$ *as* $p \to \infty$;
(vi) $\zeta_p \approx 1$ *as* $p \to 1$ *and* $\zeta_p \le Cp$ *for* $2 \le p < \infty$;
(vii) $\nu_p \le Cp$ *for* $2 \le p < \infty$.

Using his weak type (1,1) inequality for martingale transforms, Randrianantoanina [R1] proved $\kappa_p \le C\,p$ as $p \to \infty$. This is the optimal order of κ_p for it is already so in the commutative case. The optimal order of α_p as $p \to \infty$ was established in [JX3] and is the square of its commutative counterpart. On the other hand, left open in [JX3], the case of p close to 1 was solved in [R2]. This order is the same as that in the commutative case. The optimal orders of β_p were determined in [JX1] for $p \le 2$ and in [JX3] for $p \ge 2$. They are the same as their commutative counterparts. Note that Pisier [Pi2] also showed that $\beta_p = \mathrm{O}(p)$ for even integers p. The estimate $\gamma_p \le C\,p$ was obtained in [R1]. It was proved in [JX3] that this is optimal. This optimal order of γ_p is the square of that in the commutative case. The estimates on η_p and ζ_p mainly come from [R3], although some partial results already appeared in [JX1] and [JX3]. The estimate on ν_p was obtained in [JX2]. Thus, at the time of this writing, the only undetermined optimal orders are on ζ_p and ν_p as $p \to \infty$. Recall that in the commutative case, both orders are $\mathrm{O}(p/\log p)$.

The rest of this section is devoted to the proof of Theorem 6.8. The idea is simply as follows. We first lift martingales from $L_p(\mathcal{M})$ to $L_p(\mathcal{R}(\mathcal{M}))$ and then pull them back. Remember that we embed $\mathcal{M}$ into the crossed product $\mathcal{R}(\mathcal{M})$, and the filtration $(\mathcal{M}_n)_n$ of $\mathcal{M}$ into the filtration $(\mathcal{R}(\mathcal{M}_n))_n$ of $\mathcal{R}(\mathcal{M})$. Thus any martingale x relative to $(\mathcal{M}_n)_n$ is also viewed as a martingale relative to $(\mathcal{R}(\mathcal{M}_n))_n$. This is the lifting procedure. Once we are in $\mathcal{R}(\mathcal{M})$, in order to apply results in the tracial case, we use the conditional expectations Φ_m to go to the finite algebras $\mathcal{R}_m(\mathcal{M})$. Finally, we return back to $\mathcal{M}$ via the conditional expectation Φ. This roundabout procedure is almost transparent for (MT_p) and (S_p). For the others a little more effort is needed.

Proof of Theorem 6.8. This proof is divided into several parts according to the inequalities in consideration.

(i) *Reduction of* (MT_p). Fix a finite martingale x in $L_p(\mathcal{M})$ relative to $(\mathcal{M}_n)_n$. Lifting x to $L_p(\mathcal{R}(\mathcal{M}))$, we consider x as a martingale relative to $(\mathcal{R}(\mathcal{M}_n))_n$.

Then using the conditional expectation Φ_m we compress x into a martingale in $L_p(\mathcal{R}_m(\mathcal{M}))$ for every fixed m. More precisely, put

$$x^{(m)} = \Phi_m(x) = (\Phi_m(x_n))_{n\geq 1}\,, \quad m \in \mathbb{N}.$$

Then $x^{(m)}$ is a martingale in $L_p(\mathcal{R}_m(\mathcal{M}))$ relative to $(\mathcal{R}_m(\mathcal{M}_n))_n$. Now $\mathcal{R}_m(\mathcal{M})$ admits a normal faithful finite trace φ_m. By the construction of φ_m in section 2 and (6.1), φ_m is invariant under $\widehat{\mathcal{E}}_n$ for all $n \geq 1$. On the other hand, the martingale structure determined by $(\mathcal{R}_m(\mathcal{M}_n))_n$ in $L_p(\mathcal{R}_m(\mathcal{M}))$ coincides with that in the tracial noncommutative L_p-space of $\mathcal{R}_m(\mathcal{M})$. Thus using (MT$_p$) in the tracial case, we have

$$\big\|\sum_{n\geq 1} \varepsilon_n dx_n^{(m)}\big\|_p \leq \kappa_p\, \|x\|_p\,, \quad \forall\, \varepsilon_n = \pm 1.$$

Let $y = \sum_{n\geq 1} \varepsilon_n dx_n$. By (6.4), the sum on the left-hand side is equal to $\Phi_m(y)$. Now $(\Phi_m(y))_m$ is a martingale in $L_p(\mathcal{R}(\mathcal{M}))$ with respect to $(\mathcal{R}_m(\mathcal{M}))_m$. Hence, $\Phi_m(y) \to y$ in $L_p(\mathcal{R}(\mathcal{M}))$ as $m \to \infty$ (see Remark 6.1). Therefore,

$$\big\|\sum_{n\geq 1} \varepsilon_n dx_n\big\|_p \leq \kappa_p\, \|x\|_p\,.$$

This is (MT$_p$) in the general case.
(ii) *Reduction of* (S$_p$)*.* This is also easy. The main point is the following.

Lemma 6.10. *Let $1 \leq p < \infty$. Let $a = (a_n) \in L_p(\mathcal{M}; \ell_2^c)$. Considering a as an element in $L_p(\mathcal{R}(\mathcal{M}); \ell_2^c)$, we set*

$$a^{(m)} = \Phi_m(a) = (\Phi_m(a_n))_{n\geq 1}\,.$$

Then $a^{(m)} \to a$ in $L_p(\mathcal{R}(\mathcal{M}); \ell_2^c)$ as $m \to \infty$. A similar statement holds for the row space.

Proof. We regard $L_p(\mathcal{M}; \ell_2^c)$ as the column subspace of $L_p(\mathcal{M}\bar{\otimes}B(\ell_2))$. Then $L_p(\mathcal{R}(\mathcal{M}); \ell_2^c)$ is the column subspace of $L_p(\mathcal{R}(\mathcal{M})\bar{\otimes}B(\ell_2))$. Note that

$$a^{(m)} = \Phi_m \otimes \mathrm{id}_{B(\ell_2)}(a).$$

Since $(\Phi_m)_m$ is an increasing sequence of conditional expectations, the desired result immediately follows from the martingale mean convergence in Remark 6.1. $\square$

Now as for (MT$_p$), it is easy to see why one needs only to consider (S$_p$) in the tracial case. Indeed, fix a finite sequence $a = (a_n) \subset L_p(\mathcal{M}) \subset L_p(\mathcal{R}(\mathcal{M}))$. Then $\Phi_m(a) \subset L_p(\mathcal{R}_m(\mathcal{M}))$. Applying (S$_p$) in the tracial case, we get

$$\big\|\big(\mathcal{E}_n(\Phi_m(a_n))\big)_n\big\|_{L_p(\mathcal{R}_m(\mathcal{M});\ell_2^c)} \leq \gamma_p\, \|\Phi_m(a)\|_{L_p(\mathcal{R}_m(\mathcal{M});\ell_2^c)}\,, \quad m \in \mathbb{N}.$$

By (6.4), $\mathcal{E}_n(\Phi_m(a_n)) = \Phi_m(\mathcal{E}_n(a_n))$ for all $m,\ n \in \mathbb{N}$. Therefore,

$$\big\|\big(\Phi_m(\mathcal{E}_n(a_n))\big)_n\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_2^c)} \leq \gamma_p \|\Phi_m(a)\|_{L_p(\mathcal{R}(\mathcal{M});\ell_2^c)}\,.$$

It remains to apply Lemma 6.10 to conclude the reduction argument on (S$_p$).
(iii) *Reduction of* (BG$_p$). In the following the Hardy spaces on $\mathcal{R}(\mathcal{M})$ and $\mathcal{R}_m(\mathcal{M})$ are relative to the filtrations $(\mathcal{R}(\mathcal{M}_n))_n$ and $(\mathcal{R}_m(\mathcal{M}_n))_n$, respectively.

Lemma 6.11. *Let $1 \leq p < \infty$. Then $\mathcal{H}_p(\mathcal{M})$ and $\mathcal{H}_p(\mathcal{R}_m(\mathcal{M}))$ are 1-complemented isometric subspaces of $\mathcal{H}_p(\mathcal{R}(\mathcal{M}))$.*

Proof. We consider only the part on $\mathcal{H}_p(\mathcal{M})$, that on $\mathcal{H}_p(\mathcal{R}_m(\mathcal{M}))$ being dealt with by the same arguments. It is trivial that $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$) is an isometric subspace of $\mathcal{H}_p^c(\mathcal{R}(\mathcal{M}))$ (resp. $\mathcal{H}_p^r(\mathcal{R}(\mathcal{M}))$). Let $\widehat{x} = (\widehat{x}_n)_n$ be a martingale relative to $(\mathcal{R}(\mathcal{M}_n))_n$. Put $\Phi(\widehat{x}) = (\Phi(\widehat{x}_n))_n$. By (6.4), $\Phi(\widehat{x})$ is a martingale relative to $(\mathcal{M}_n)_n$. Note that the difference sequence of $\Phi(\widehat{x})$ is given by $(\Phi(d\widehat{x}_n))_n$. Then using the tensor product argument in the proof of Lemma 6.10, one sees that the map $\widehat{x} \mapsto \Phi(\widehat{x})$ defines a contractive projection from $\mathcal{H}_p^c(\mathcal{R}(\mathcal{M}))$ (resp. $\mathcal{H}_p^r(\mathcal{R}(\mathcal{M}))$) onto $\mathcal{H}_p^c(\mathcal{M})$ (resp. $\mathcal{H}_p^r(\mathcal{M})$). Therefore, we immediately deduce the assertion on $\mathcal{H}_p(\mathcal{M})$ in the case $p \geq 2$. For the case $p < 2$ we need only to check that $\mathcal{H}_p(\mathcal{M})$ is an isometric subspace of $\mathcal{H}_p(\mathcal{R}(\mathcal{M}))$. But this is an easy consequence of the above projection. $\square$

Lemma 6.12. *Let $1 \leq p < \infty$. Let $x \in \mathcal{H}_p(\mathcal{M})$. Then $\Phi_m(x) \in \mathcal{H}_p(\mathcal{R}(\mathcal{M}))$ and $\Phi_m(x) \to x$ in $\mathcal{H}_p(\mathcal{R}(\mathcal{M}))$ as $m \to \infty$.*

Proof. By Lemma 6.10 we see that the statement above holds with $\mathcal{H}_p^c(\mathcal{R}(\mathcal{M}))$ or $\mathcal{H}_p^r(\mathcal{M})$ in place of $\mathcal{H}_p(\mathcal{R}(\mathcal{M}))$. We then deduce the assertion in the case $p \geq 2$. The case $p < 2$ is proved by virtue of the density of finite martingales in $\mathcal{H}_p(\mathcal{R}(\mathcal{M}))$ and the contractivity of Φ_m on $\mathcal{H}_p(\mathcal{R}(\mathcal{M}))$. $\square$

Using the previous two lemmas, we easily reduce the general case of (BG_p) to the tracial case, as before for (S_p). We leave the details to the reader.

(iv) *Reduction of* (B_p). The proof of this is very similar to the previous one. It suffices to apply the following lemma, which is the analogue of Lemmas 6.11 and 6.12 for the conditioned Hardy spaces.

Lemma 6.13. *Let $1 \leq p < \infty$.*

(i) *Then $h_p(\mathcal{M})$ and $h_p(\mathcal{R}_m(\mathcal{M}))$ are 1-complemented isometric subspaces of $h_p(\mathcal{R}(\mathcal{M}))$.*
(ii) *Φ_m converges to the identity of $h_p(\mathcal{R}(\mathcal{M}))$ in the point-norm topology.*

Proof. We first show that Φ projects $h_p^k(\mathcal{R}(\mathcal{M}))$ contractively onto $h_p^k(\mathcal{R}(\mathcal{M}))$ for $k \in \{d, c, r\}$. This is trivial for $h_p^d(\mathcal{R}(\mathcal{M}))$. To deal with $h_p^c(\mathcal{R}(\mathcal{M}))$ we first recall that $h_p^c(\mathcal{R}(\mathcal{M}))$ can be isometrically viewed as the column subspace of $L_p(\mathcal{R}(\mathcal{M})\bar{\otimes}B(\ell_2(\mathbb{N}^2)))$. The map realizing this is constructed by means of Kasparov's stabilization theorem for Hilbert C*-modules (see [Ju1] for more details). Using this and as in the proof of Lemma 6.11, we show that Φ is a contractive projection on $h_p^c(\mathcal{R}(\mathcal{M}))$. Passing to adjoints, we get the assertion on $h_p^r(\mathcal{R}(\mathcal{M}))$. We then deduce (i). (ii) is proved similarly as Lemma 6.12, so we omit the details. $\square$

(v) *Reduction of* (R_p). Let $(\mathcal{A}_n)$ be a sequence of von Neumann subalgebras of $\mathcal{M}$ independent over $\mathcal{N}$. We use the notation introduced in subsection 6.1. In particular, $\mathcal{R}(\mathcal{A}_n)$ and $\mathcal{R}(\mathcal{N})$ are $\sigma^{\widehat{\varphi}}$-invariant von Neumann subalgebras of $\mathcal{R}(\mathcal{M})$.

Lemma 6.14. *The algebras $\mathcal{R}(\mathcal{A}_n)$ are independent over $\mathcal{R}(\mathcal{N})$.*

Proof. We must show that

$$\mathcal{E}_{\mathcal{R}(\mathcal{N})}(\hat{a}\hat{b}) = \mathcal{E}_{\mathcal{R}(\mathcal{N})}(\hat{a})\,\mathcal{E}_{\mathcal{R}(\mathcal{N})}(\hat{b}) \tag{6.8}$$

for $\hat{a} \in \mathcal{R}(\mathcal{A}_n)$ and $\hat{b}$ in the subalgebra generated by the $\mathcal{R}(\mathcal{A}_k)$, $k \neq n$. Let $\mathcal{B}$ be the von Neumann subalgebra of $\mathcal{M}$ generated by $(\mathcal{A}_k)_{k\neq n}$. Note that $\mathcal{B}$ is σ-invariant. Using (1.1), we see that the subalgebra generated by $(\mathcal{R}(\mathcal{A}_k))_{k\neq n}$ is equal to the crossed product $\mathcal{R}(\mathcal{B}) = \mathcal{B} \rtimes_\sigma G$. On the other hand, $\mathcal{E}_{\mathcal{R}(\mathcal{N})} = \widehat{\mathcal{E}_{\mathcal{N}}}$ by Theorem 4.1. Now let $a \in \mathcal{A}_n$, $b \in \mathcal{B}$ and $g, h \in G$. Then by (1.1), (4.2) and the independence of $(\mathcal{A}_n)_n$ we find

$$\begin{aligned}
\mathcal{E}_{\mathcal{R}(\mathcal{N})}\big(\lambda(g)a\,\lambda(h)b\big) &= \widehat{\mathcal{E}_{\mathcal{N}}}\big(\lambda(g)\lambda(h)\,\sigma_{-h}(a)b\big)\\
&= \lambda(g)\lambda(h)\,\mathcal{E}_{\mathcal{N}}\big(\sigma_{-h}(a)b\big)\\
&= \lambda(g)\lambda(h)\,\mathcal{E}_{\mathcal{N}}\big(\sigma_{-h}(a)\big)\,\mathcal{E}_{\mathcal{N}}(b)\\
&= \lambda(g)\lambda(h)\,\sigma_{-h}\big(\mathcal{E}_{\mathcal{N}}(a)\big)\,\mathcal{E}_{\mathcal{N}}(b)\\
&= \lambda(g)\mathcal{E}_{\mathcal{N}}(a)\,\lambda(h)\mathcal{E}_{\mathcal{N}}(b)\\
&= \mathcal{E}_{\mathcal{R}(\mathcal{N})}\big(\lambda(g)a\big)\,\mathcal{E}_{\mathcal{R}(\mathcal{N})}\big(\lambda(h)b\big).
\end{aligned}$$

This shows (6.8) for $\hat{a} = \lambda(g)a$ and $\hat{b} = \lambda(h)b$. Since $\mathcal{R}(\mathcal{A}_n)$ (resp. $\mathcal{R}(\mathcal{B})$) is the w*-closure of all finite sums on $\lambda(g)a$ with $a \in \mathcal{A}_n, g \in G$ (resp. $\lambda(h)b$ with $b \in \mathcal{B}, h \in G$), the normality of $\mathcal{E}_{\mathcal{R}(\mathcal{N})}$ implies (6.8) for general $\hat{a}$ and $\hat{b}$. □

It is then an easy exercice to reduce the general case of (R_p) to the tracial one. We omit the details. Therefore, the proof of Theorem 6.8 is complete. □

7. Applications to noncommutative maximal inequalities

We pursue our applications of Theorem 2.1 to noncommutative inequalities. We now consider the noncommutative maximal martingale and ergodic inequalities. We should emphasize that because of the failure of the noncommutative analogue of the usual pointwise maximal function of a sequence of functions, these maximal inequalities are much subtler than those in the commutative case. This failure also explains why we are forced to work systematically with the vector-valued spaces $L_p(\mathcal{M};\ell_\infty)$. Namely, instead of a noncommutative pointwise maximal function (which does not exist now), we work with the noncommutative analogue of the usual maximal L_p-norm. In this section $\mathcal{M}$ again denotes a von Neumann algebra equipped with a normal faithful state φ.

7.1. **Positive maps on $L_p(\mathcal{M};\ell_\infty)$.** We first recall the definition of the spaces $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$ from [Ju1]. Let $1 \le p \le \infty$. A sequence (x_n) in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M};\ell_\infty)$ iff (x_n) admits a factorization $x_n = ay_nb$ with $a, b \in L_{2p}(\mathcal{M})$ and $(y_n) \in \ell_\infty(L_\infty(\mathcal{M}))$. The norm of (x_n) is then defined as

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty)} = \inf_{x_n = ay_nb} \|a\|_{2p} \sup_n \|y_n\|_\infty \|b\|_{2p}\,.$$

On the other hand, $L_p(\mathcal{M};\ell_1)$ is defined to be the space of all sequences $(x_n) \subset L_p(\mathcal{M})$ for which there exist $a_{nk}, b_{nk} \in L_{2p}(\mathcal{M})$ such that

$$x_n = \sum_k a_{nk}^* b_{nk}\,, \quad \forall\, n \ge 1.$$

$L_p(\mathcal{M};\ell_1)$ is equipped with the norm

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_1)} = \inf_{x_n = \sum_k a_{nk}^* b_{nk}} \Big\|\sum_{n,k} a_{nk}^* a_{nk}\Big\|_p^{1/2} \Big\|\sum_{n,k} b_{nk}^* b_{nk}\Big\|_p^{1/2}.$$

We refer to [Ju1] for more information (see also [JX4]). Let us note that if $\mathcal{M}$ is injective, the definition above is a special case of Pisier's vector-valued noncommutative L_p-space theory [Pi1]. The following remark is easy to check.

Remark 7.1. Let $(x_n) \subset L_p^+(\mathcal{M})$.

(i) $(x_n) \in L_p(\mathcal{M};\ell_\infty)$ iff there exists $a \in L_p^+(\mathcal{M})$ such that $x_n \le a$ for all $n \in \mathbb{N}$. In this case, we have

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty)} = \inf\big\{\|a\|_p \ :\ a \in L_p^+(\mathcal{M}) \text{ s.t. } x_n \le a,\ \forall\, n\big\}.$$

(ii) $(x_n) \in L_p(\mathcal{M};\ell_1)$ iff $\sum_n x_n \in L_p(\mathcal{M})$. If this is the case, then

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_1)} = \big\|\sum_{n\ge1} x_n\big\|_p .$$

We adopt the convention of [JX4] that the norm $\|(x_n)\|_{L_p(\mathcal{M};\ell_\infty)}$ is denoted by $\big\|\sup_n^+ x_n\big\|_p$. However, we should warn the reader that $\big\|\sup_n^+ x_n\big\|_p$ is just a notation, since $\sup_n x_n$ does not make any sense in the noncommutative setting. We should also point out that $L_p(\mathcal{M};\ell_\infty)$ and $L_p(\mathcal{M};\ell_1)$ are not closed under absolute value. In particular, $\big\|\sup_n^+ x_n\big\|_p \ne \big\|\sup_n^+ |x_n|\big\|_p$ in general.

One fundamental result on these vector-valued spaces is the duality between $L_p(\mathcal{M};\ell_1)$ and $L_{p'}(\mathcal{M};\ell_\infty)$ for $1 \le p < \infty$ established in [Ju1], where p' denotes the conjugate index of p. Namely, we have

$$L_p(\mathcal{M};\ell_1)^* = L_{p'}(\mathcal{M};\ell_\infty) \quad \text{isometrically} \tag{7.1}$$

via the duality bracket

$$\langle (x_n),\ (y_n)\rangle = \sum_{n\ge1} \mathrm{tr}(x_n y_n), \quad (x_n) \in L_p(\mathcal{M};\ell_1),\ (y_n) \in L_{p'}(\mathcal{M};\ell_\infty).$$

Moreover, if $L_p(\mathcal{M};\ell_q^n)$ denotes the ℓ_q^n-valued analogues of these spaces ($q = 1,\infty$), then

$$L_p(\mathcal{M};\ell_\infty^n)^* = L_{p'}(\mathcal{M};\ell_1^n) \quad \text{isometrically} \tag{7.2}$$

(see [JX4]). Using these duality identities and the following formula,

$$\big\|\sup_n{}^+ x_n\big\|_p = \sup_{n\ge1}\big\|\sup_{1\le k\le n}{}^+\, x_k\big\|_p \tag{7.3}$$

from [JX4], we deduce the following.

Remark 7.2. Let $(x_n) \in L_p(\mathcal{M};\ell_\infty)$, $1 \le p \le \infty$. Then

$$\big\|\sup_n{}^+ x_n\big\|_p = \sup\big\{\big|\sum_{n\ge1}\mathrm{tr}(x_n y_n)\big| \ :\ (y_n) \in L_{p'}(\mathcal{M};\ell_1),\ \big\|(y_n)\big\|_{L_{p'}(\mathcal{M};\ell_1)} \le 1\big\}.$$

Moreover, if (x_n) is positive, the supremum above can be restricted to positive (y_n) too.

Recall that a map $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is n-positive if $T_n : L_p(\mathbb{M}_n \otimes \mathcal{M}) \to L_p(\mathbb{M}_n \otimes \mathcal{M})$ is positive, where $\mathbb{M}_n$ denotes the full algebra of $n \times n$ matrices and where

$$T_n\big((x_{ij})_{1\le i,j\le n}\big) = \big(T(x_{ij})\big)_{1\le i,j\le n} .$$

Here we have viewed, as usual, the elements of $L_p(\mathbb{M}_n \otimes \mathcal{M})$ as $n \times n$ matrices with entries in $L_p(\mathcal{M})$. T is completely positive if T is n-positive for every n.

Proposition 7.3. *Let $1 \le p < \infty$ and $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive bounded map. Then $T \otimes \mathrm{id}_{\ell_\infty}$ is bounded on $L_p(\mathcal{M};\ell_\infty)$ and of norm $\le 8\|T\|$. If in addition T is 2-positive, then $\|T \otimes \mathrm{id}_{\ell_\infty}\| = \|T\|$.*

Proof. By (7.1), (7.2) and (7.3), it suffices to show that T^* extends to a bounded map on $L_{p'}(\mathcal{M};\ell_1)$. However, by [JX4], every element in the unit ball of $L_{p'}(\mathcal{M};\ell_1)$ is a sum of 8 elements in the same ball. Thus we need only to consider a positive $x=(x_n)\in L_{p'}(\mathcal{M};\ell_1)$. Then $T^*(x_n)\geq 0$ too. Therefore, by Remark 7.1,

$$\big\|T^*\otimes \mathrm{id}_{\ell_1}(x)\big\|_{L_{p'}(\mathcal{M};\ell_1)}=\big\|\sum_n T^*(x_n)\big\|_{p'}\leq \|T\|\,\big\|\sum_n x_n\big\|_{p'}\,.$$

It thus follows that $\|T^*\otimes \mathrm{id}_{\ell_1}\|\leq 8\|T\|$. This yields the first assertion.

The proof of the second assertion needs more effort. Recall that D denotes the density operator in $L_1(\mathcal{M})$ of the distinguished state φ. Let

$$\widetilde{\mathcal{M}}=\mathbb{M}_2\otimes\mathcal{M}\quad\text{and}\quad \widetilde{\varphi}=\mathrm{Tr}\otimes\varphi\,,$$

where Tr is the usual trace on $\mathbb{M}_2$. Then the density of $\widetilde{\varphi}$ is the matrix

$$\widetilde{D}=\begin{pmatrix} D & 0\\ 0 & D\end{pmatrix}.$$

Let $a,b\in L_{2p}(\mathcal{M})$ be positive operators and $\varepsilon>0$. Put

$$d=\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\quad\text{and}\quad d_\varepsilon=T_2(d^2)+\varepsilon\,\widetilde{D}^{1/p}\,.$$

Note that $d_\varepsilon\in L_p(\widetilde{\mathcal{M}})$ and d_ε is an injective positive operator affiliated with the crossed product $\widetilde{\mathcal{M}}\rtimes_{\sigma^{\widetilde{\varphi}}}\mathbb{R}$. Define

$$\widetilde{T}(\tilde z)=d_\varepsilon^{-1/2}\,T_2(d\tilde z d)\,d_\varepsilon^{-1/2}\,,\quad \tilde z\in\widetilde{\mathcal{M}}\,.$$

The 2-positivity of T implies that

$$0\leq T_2(d\tilde z d)\leq \|\tilde z\|_\infty\,T_2(d^2)\leq \|\tilde z\|_\infty\,d_\varepsilon,\quad \forall\,\tilde z\in\widetilde{\mathcal{M}}_+\,.$$

It follows that $\widetilde{T}(\tilde z)$ is bounded for every $\tilde z\geq 0$; so $\widetilde{T}(\tilde z)\in\widetilde{\mathcal{M}}\rtimes_{\sigma^{\widetilde{\varphi}}}\mathbb{R}$. On the other hand, it is easy to see that $\widetilde{T}(\tilde z)$ is invariant under the dual action of $\sigma^{\widetilde{\varphi}}$. Therefore, $\widetilde{T}(\tilde z)\in\widetilde{\mathcal{M}}$ for $\tilde z\in\widetilde{\mathcal{M}}_+$. Decomposing every element into a linear combination of 4 positive elements, we then deduce that $\widetilde{T}$ is a well-defined map on $\widetilde{\mathcal{M}}$. Moreover, $\widetilde{T}$ is positive. Thus

$$\|\widetilde{T}\|=\|\widetilde{T}(1)\|\leq 1.$$

For $z\in\mathcal{M}$ define

$$\tilde z=\begin{pmatrix} 0 & z\\ 0 & 0\end{pmatrix}.$$

Then

$$\widetilde{T}(\tilde z)=\begin{pmatrix} 0 & a_\varepsilon^{-1/2}\,T(azb)\,b_\varepsilon^{-1/2}\\ 0 & 0\end{pmatrix},$$

where $a_\varepsilon=T(a^2)+\varepsilon\,D^{1/p}$ and $b_\varepsilon=T(b^2)+\varepsilon\,D^{1/p}$. Therefore, the map $T':\mathcal{M}\to\mathcal{M}$ defined by $T'(z)=a_\varepsilon^{-1/2}\,T(azb)\,b_\varepsilon^{-1/2}$ is a contraction.

Now it is easy to finish the proof. Indeed, let $x=(x_n)\in L_p(\mathcal{M};\ell_\infty)$ and take a factorization $x_n=ay_nb$ of x with $a,b\in L_{2p}(\mathcal{M})$ and $(y_n)\in\ell_\infty(L_\infty(\mathcal{M}))$. By polar decomposition we can assume $a,b\geq 0$. Using the map T' above we get a factorization of $(T(x_n))$ as follows:

$$T(x_n) = a_\varepsilon^{1/2}\, T'(y_n)\, b_\varepsilon^{1/2} \stackrel{\text{def}}{=} a' y_n' b' \,.$$

We have

$$\|a'\|_{2p} \le \|T\|^{1/2}(\|a\|_{2p} + \mathrm{O}(\varepsilon)), \quad \|b'\|_{2p} \le \|T\|^{1/2}(\|b\|_{2p} + \mathrm{O}(\varepsilon)), \quad \|y_n'\|_\infty \le \|y_n\|_\infty .$$

It follows that $(T(x_n)) \in L_p(\mathcal{M};\ell_\infty)$ and

$$\big\| \sup_n{}^+ T(x_n)\big\|_p \le \|T\| \big(\big\| \sup_n{}^+ x_n\big\|_p + \mathrm{O}(\varepsilon)\big).$$

This implies the desired assertion. □

Let $\mathcal{N} \subset \mathcal{M}$ be a σ^φ-invariant von Neumann subalgebra and let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the normal conditional expectation preserving the state φ. Under this assumption we know that $L_p(\mathcal{N})$ is naturally identified as a subspace of $L_p(\mathcal{M})$ and $\mathcal{E}$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{N})$ (see [JX1, Lemma 1.2] and Example 5.8). Moreover, $\mathcal{E}$ is completely positive. Applying the previous proposition to this situation we immediately get the following.

Corollary 7.4. *The inclusion $L_p(\mathcal{N};\ell_\infty) \subset L_p(\mathcal{M};\ell_\infty)$ is isometric and $\mathcal{E} \otimes \mathrm{id}_{\ell_\infty}$ defines a contractive projection from $L_p(\mathcal{M};\ell_\infty)$ onto $L_p(\mathcal{N};\ell_\infty)$.*

7.2. **Doob maximal inequality.** In this subsection we keep the framework introduced in subsection 6.1. In particular, $(\mathcal{M}_n)_n$ is a filtration of von Neumann subalgebras of $\mathcal{M}$ with associated conditional expectations $(\mathcal{E}_n)_n$. The following is the noncommutative Doob maximal inequality.

Theorem 7.5. *Let $1 < p \le \infty$. Let $x = (x_n)$ be a bounded L_p-martingale with respect to $(\mathcal{M}_n)$. Then*

$$(\mathrm{D}_p) \qquad \big\| \sup_n{}^+ x_n\big\|_p \le \delta_p \|x\|_p \,.$$

Moreover, $\delta_p \le C\,(p-1)^{-2}$ for a universal constant C and this estimate is optimal as $p \to 1$.

For positive martingales, (D_p) takes the following much simpler form, from which the reader can recognize the classical Doob maximal inequality.

Remark 7.6. Let $x = (x_n)$ be a positive bounded L_p-martingale. Then there exists $a \in L_p^+(\mathcal{M})$ such that

$$(\mathrm{D}_p^+) \qquad \|a\|_p \le \delta_p^+ \,\|x\|_p \quad \text{and} \quad x_n \le a, \quad \forall\, n \in \mathbb{N}.$$

One can show that (D_p) and (D_p^+) are equivalent and that the two relevant best constants δ_p and δ_p^+ are equal (see [Ju1]). However, if we tolerate a multiple of constants, we easily see that (D_p^+) implies (D_p) with $\delta_p \le 4\delta_p^+$.

Remark 7.7. It is sometimes more convenient to work with the dual form of (D_p). Let $1 \le q < \infty$. Then for all finite sequences $(a_n)_n$ of positive elements in $L_q(\mathcal{M})$,

$$(\mathrm{D}_q') \qquad \big\| \sum_n \mathcal{E}_n(a_n)\big\|_q \le \delta_q' \,\big\| \sum_n a_n\big\|_q \,.$$

One can show that (D_p) with $1 < p \le \infty$ is equivalent to $(\mathrm{D}_{p'}')$ and $\delta_p = \delta_{p'}'$. We refer to [Ju1] for details.

Inequality (D_p) was first obtained in [Ju1], and the optimal order of δ_p was determined in [JX3]. Note that this order is the square of that in the commutative case. The proof of [Ju1] heavily relies upon Hilbert C*-module theory. An elementary proof for the tracial case was later given in [JX4]. A key ingredient of this second proof is a noncommutative Marcinkiewicz-type interpolation theorem. It is also this new proof that gives the optimal order $\delta_p = \mathrm{O}((p-1)^{-2})$ as $p \to 1$.

As for the square function type inequalities in the previous section, it suffices to show (D_p) in the tracial case.

Proposition 7.8. *If* (D_p) *holds in the tracial case, so does it in the general case with the same best constant.*

Proof. Assume that (D_p) holds in the tracial case with constant δ_p for $1 < p < \infty$. Let $x = (x_n)$ be a bounded L_p-martingale relative to $(\mathcal{M}_n)_n$. Set $x^{(m)} = (\Phi_m(x_n))_n$. Then $x^{(m)}$ is a bounded L_p-martingale relative to $(\mathcal{R}_m(\mathcal{M}_n))_n$. Now $\mathcal{R}_m(\mathcal{M})$ is a finite von Neumann algebra. Using (D_p) in the tracial case we find

$$\big\|x^{(m)}\big\|_{L_p(\mathcal{R}_m(\mathcal{M});\ell_\infty)} \le \delta_p \big\|x^{(m)}\big\|_p .$$

Consequently,

$$\big\|x^{(m)}\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_\infty)} \le \delta_p \|x\|_p .$$

In particular,

$$\big\|(x_1^{(m)}, \cdots, x_n^{(m)})\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_\infty^n)} \le \delta_p \|x\|_p , \quad \forall\, m,\ n \in \mathbb{N}.$$

Note that for a fixed n, the norm of $L_p(\mathcal{R}(\mathcal{M});\ell_\infty^n)$ is equivalent to that of $\ell_\infty^n(L_p(\mathcal{R}(\mathcal{M}))$. Since $x_k^{(m)} \to x_k$ in $L_p(\mathcal{R}(\mathcal{M}))$ as $m \to \infty$ for every $k \in \mathbb{N}$, we get

$$\lim_{m\to\infty} \big\|(x_1^{(m)}, \cdots, x_n^{(m)})\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_\infty^n)} = \big\|(x_1, \cdots, x_n)\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_\infty^n)} .$$

Thus we deduce

$$\sup_n \big\|(x_1, \cdots, x_n)\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_\infty^n)} \le \delta_p \|x\|_p .$$

Together with (7.3), this implies that

$$\big\|x\big\|_{L_p(\mathcal{R}(\mathcal{M});\ell_\infty)} \le \delta_p \|x\|_p .$$

Recall that $x = (x_n)$ is a martingale in $L_p(\mathcal{M}) \subset L_p(\mathcal{R}(\mathcal{M}))$. Applying Corollary 7.4 to the conditional expectation $\Phi : \mathcal{R}(\mathcal{M}) \to \mathcal{M}$, we finally get

$$\big\|x\big\|_{L_p(\mathcal{M};\ell_\infty)} \le \delta_p \|x\|_p .$$

This is the desired inequality (D_p) in $L_p(\mathcal{M})$. □

7.3. **Maximal ergodic inequalities.** In this subsection we deal with noncommutative maximal ergodic inequalities. In the sequel T will be assumed to be a map on $\mathcal{M}$ satisfying the following properties:

(I) T is a contraction on $\mathcal{M}$;
(II) T is completely positive;

(III) $\varphi \circ T \leq \varphi$;
(IV) $T \circ \sigma_t^\varphi = \sigma_t^\varphi \circ T$ for all $t \in \mathbb{R}$.

By Theorem 5.1, T extends to a contraction on $L_p(\mathcal{M})$ for any $p \geq 1$, the extension still being denoted by T. Note that by virtue of condition (IV) we can also use the extension given by Proposition 5.5.

We now consider the ergodic averages of T:

$$M_n \equiv M_n(T) = \frac{1}{n+1} \sum_{k=0}^{n} T^k .$$

The following theorem gives our maximal ergodic inequalities on T.

Theorem 7.9. *Let $1 < p \leq \infty$. Then*

$$\big\| \sup_n{}^+ M_n(x) \big\|_p \leq C_p \, \|x\|_p \,, \quad x \in L_p(\mathcal{M}). \tag{7.4}$$

If in addition T is symmetric in the following sense:

$$\varphi(T(y)^* x) = \varphi(y^* T(x)), \quad \forall \, x, \; y \in \mathcal{M}, \tag{7.5}$$

then

$$\big\| \sup_n{}^+ T^n(x) \big\|_p \leq C_p' \, \|x\|_p \,, \quad x \in L_p(\mathcal{M}). \tag{7.6}$$

Here C_p and C_p' are two positive constants depending only on p.

Inequality (7.4) is the noncommutative analogue of the classical Dunford-Schwartz maximal ergodic inequality for positive contractions on L_p (see [DS] for the classical case), while (7.6) extends Stein's maximal inequality [St1] (see also [St2]) to the noncommutative setting. The preceding theorem is proved in [JX4]. The order of the constant C_p obtained there is $(p-1)^{-2}$ as $p \to 1$. This order is optimal. However, we do not know the optimal order of C_p'. Recall that in the commutative case both C_p and C_p' are of optimal order $(p-1)^{-1}$ as $p \to 1$.

Like the noncommutative Doob maximal inequality, both (7.4) and (7.6) can be reduced to the tracial case. Note that if φ is tracial, condition (IV) is automatically satisfied. The proof of (7.4) in the tracial case is based on Yeadon's weak type $(1, 1)$ ergodic inequality (see [Y]) and the noncommutative Marcinkiewicz-type interpolation theorem already mentioned in the previous subsection. The reduction of (7.4) and (7.6) to the tracial case was done in [JX4]. For the convenience of the reader we include its main lines here.

Reduction of Theorem 7.9 *to the tracial case.* We will again use Theorem 2.1. Recall that $\mathcal{R}$ is the crossed product $\mathcal{M} \rtimes_{\sigma^\varphi} G$ and $(\mathcal{R}_m)_{m\geq 1}$ is the filtration of finite von Neumann subalgebras of $\mathcal{R}$ constructed in Theorem 2.1.

On the other hand, using Theorem 4.1 we extend T to a completely positive map $\widehat{T}$ on $\mathcal{R}$. By that theorem and Proposition 4.3, $\widehat{T}$ satisfies conditions (I)-(IV) relative to $(\mathcal{R}, \; \widehat{\varphi})$. Therefore, $\widehat{T}$ extends to a contraction on $L_p(\mathcal{R})$ for every $p \geq 1$. Note that if T satisfies the symmetry condition (7.5), then using the arguments in the proof of Theorem 4.1, we easily check that $\widehat{T}$ is also symmetric relative to $\widehat{\varphi}$.

Let τ_m be the trace on $\mathcal{R}_m$ corresponding to the state φ_m constructed in the proof of Theorem 2.1. By Proposition 4.3, we have $\tau_m \circ \widehat{T}\big|_{\mathcal{R}_m} \leq \tau_m$. Therefore, $\widehat{T}\big|_{\mathcal{R}_m}$ satisfies conditions (I)-(IV) relative to $(\mathcal{R}_m, \; \tau_m)$ for every m. This will allow us to apply (7.4) in the tracial case.

Now assume (7.4) in the tracial case. Fix $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. We consider x as an element in $L_p(\mathcal{R})$ and then apply the conditional expectation Φ_m to it: $x_m = \Phi_m(x) \in L_p(\mathcal{R}_m)$. Applying (7.4) to $\widehat{T}$ on $\mathcal{R}_m$, we get

$$\big\| \sup_n{}^+ M_n(\widehat{T})(x_m)\big\|_p \leq C_p \left\|x\right\|_p , \quad \forall\, m \in \mathbb{N}.$$

By the martingale convergence theorem in Remark 6.1,

$$\lim_{m\to\infty} x_m = x \quad \text{in} \quad L_p(\mathcal{R}).$$

Consequently,

$$\lim_{m\to\infty} \widehat{T}^{\,k}(x_m) = x \quad \text{in} \quad L_p(\mathcal{R}), \quad \forall\, k \geq 0.$$

It then follows that

$$\lim_{m\to\infty} \big\| \sup_{1\leq k\leq n}{}^+ M_k(\widehat{T})(x_m)\big\|_p = \big\| \sup_{1\leq k\leq n}{}^+ |M_k(\widehat{T})(x)|\big\|_p .$$

However, since $x \in L_p(\mathcal{M})$,

$$M_k(\widehat{T})(x) = M_k(T)(x).$$

Therefore, we deduce

$$\big\| \sup_{1\leq k\leq n}{}^+ M_k(T)(x)\big\|_p \leq C_p \left\|x\right\|_p , \quad \forall\, n \in \mathbb{N}.$$

Thus, by (7.3),

$$\big\| \sup_n{}^+ M_n(T)(x)\big\|_p \leq C_p \left\|x\right\|_p .$$

This shows (7.4) in the general case. The reduction of (7.6) is dealt with in a similar way.

Remark 7.10. (i) By discretization we can easily extend Theorem 7.9 to the case of semigroups. We refer to [JX4] for more details.

(ii) It is well known that a maximal ergodic theorem usually yields a corresponding individual ergodic theorem. This is indeed the case of Theorem 7.9. For instance, inequality (7.4) implies that $M_n(x)$ converges bilaterally almost surely in the sense of [Ja] to an element $\widehat{x}$ for every $x \in L_p(\mathcal{M})$ with $1 < p < \infty$. In the case $p = \infty$, $M_n(x)$ converges to $\widehat{x}$ almost uniformly in Lance's sense [La]. Thus in the latter case, we recover the individual ergodic theorems of Lance, Conze and Dang-Ngoc [CDN] and Kümmerer [Kü]. The interested reader is referred to [JX4] for more information.

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