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AUTOMORPHISM GROUPS ON NORMAL SINGULAR CUBIC SURFACES WITH NO PARAMETERS

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ABSTRACT. The classification of normal singular cubic surfaces in $\mathbf{P}^3$ over a complex number field $\mathbf{C}$ was given by J. W. Bruce and C. T. C. Wall. In this paper, first we prove their results by a different way, second we provide normal forms of normal singular cubic surfaces according to the type of singularities, and finally we determine automorphism groups on normal singular cubic surfaces with no parameters.

INTRODUCTION

The study of a classification of complex projective cubic surfaces in $\mathbf{P}^3$ by their singularities was begun by L. Schläfli [16] and A. Cayley [3] in the nineteenth century. In the 1970s, J. W. Bruce and C. T. C. Wall [2] classified normal singular cubic surfaces by modern singularity terminology using the Dynkin diagram (Theorem 1). On the other hand, automorphism groups on nonsingular cubic surfaces were determined by B. Segre [17], T. Hosoh [9] and so forth. Recently, automorphisms of smooth cubic surfaces in $\mathbf{P}^3$ were used by I. V. Dolgachev and V. A. Iskovskikh [6] in their study on classification of conjugacy classes of finite subgroups of the plane Cremona group, which is the group of birational automorphisms of the projective plane.

In this paper we consider automorphism groups on normal singular cubic surfaces. First we prove the results of Bruce and Wall (Theorem 1) by a method which is different from the one of [2], second we give normal forms of normal singular cubic surfaces according to type of singularities (Theorem 2), and finally we determine automorphism groups on normal singular cubic surfaces which have no parameters in their normal forms (Theorem 3).

First we recall the classification of normal singular cubic surfaces.

Theorem 1 (J. W. Bruce, C. T. C. Wall [2])**.** *Any normal singular cubic surface in $\mathbf{P}^3$ has either rational double points or a simple elliptic singularity $\tilde{E}_6$ in Table* 1*. Moreover, the number of parameters and the number of lines on the surface, according to type of singularities, are listed in Table* 1*.*

Our main results are Theorems 2 and 3. Normal forms of singular cubic surfaces which have D_4, D_5, E_6 singularity were given in [2]. Theorem 2 provides normal forms of all normal singular cubic surfaces.

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TABLE 1

Singularities	A_1	$2A_1$	A_1A_2	$3A_1$	A_1A_3	$2A_1A_2$	$4A_1$	A_1A_4
No.of parameters	3	2	1	1	0	0	0	0
No.of lines	21	16	11	12	7	8	9	4

$2A_1A_3$	A_12A_2	A_1A_5	A_2	$2A_2$	$3A_2$	A_3	A_4	A_5	D_4	D_5	E_6	$\tilde{E}_6$
0	0	0	2	1	0	1	0	0	0	0	0	1
5	5	2	15	7	3	10	6	3	6	3	1	∞

TABLE 2

Singularities	$f_2(x_0, x_1, x_2)$	$f_3(x_0, x_1, x_2)$
A_1	$x_0x_2 - x_1^2$	$(x_0 - ax_1)(-x_0 + (b+1)x_1 - bx_2)(x_1 - cx_2)$
$2A_1$	$x_0x_2 - x_1^2$	$(x_0 - 2x_1 + x_2)(x_0 - ax_1)(x_1 - bx_2)$
A_1A_2	$x_0x_2 - x_1^2$	$(x_0 - x_1)(-x_1 + x_2)(x_0 - (a+1)x_1 + ax_2)$
$3A_1$	$x_0x_2 - x_1^2$	$x_0x_2(x_0 - (a+1)x_1 + ax_2)$
A_1A_3	$x_0x_2 - x_1^2$	$(x_0 - x_1)(-x_1 + x_2)(x_0 - 2x_1 + x_2)$
$2A_1A_2$	$x_0x_2 - x_1^2$	$x_1^2(x_0 - x_1)$
$4A_1$	$x_0x_2 - x_1^2$	$(x_0 - x_1)(x_1 - x_2)x_1$
A_1A_4	$x_0x_2 - x_1^2$	$x_0^2x_1$
$2A_1A_3$	$x_0x_2 - x_1^2$	$x_0x_1^2$
A_12A_2	$x_0x_2 - x_1^2$	x_1^3
A_1A_5	$x_0x_2 - x_1^2$	x_0^3
A_2	x_0x_1	$x_2(x_0 + x_1 + x_2)(dx_0 + ex_1 - dex_2)$
$2A_2$	x_0x_1	$x_2(x_1 + x_2)(-x_1 + dx_2)$
$3A_2$	x_0x_1	x_2^3
A_3	x_0x_1	$x_2(x_0 + x_1 + x_2)(x_0 - ux_1)$
A_4	x_0x_1	$x_0^2x_2 + x_1^3 - x_1x_2^2$
A_5	x_0x_1	$x_0^3 + x_1^3 - x_1x_2^2$
$D_4(1)$	x_0^2	$x_1^3 + x_2^3$
$D_4(2)$	x_0^2	$x_1^3 + x_2^3 + x_0x_1x_2$
D_5	x_0^2	$x_0x_2^2 + x_1^2x_2$
E_6	x_0^2	$x_0x_2^2 + x_1^3$
$\tilde{E}_6$	0	$x_1^2x_2 - x_0(x_0 - x_2)(x_0 - ax_2)$

Theorem 2. *Let X be a normal singular cubic surface in $\mathbf{P}^3$. Then X is isomorphic to the projective surface in $\mathbf{P}^3$ defined by $F = x_3f_2(x_0, x_1, x_2) - f_3(x_0, x_1, x_2) = 0$, where f_2, f_3 are in Table 2, according to type of singularities on X. In Table 2, a, b, c are three distinct elements of $\mathbf{C} \setminus \{0, 1\}$, d, e are elements of $\mathbf{C} \setminus \{0, -1\}$, and u is an element of $\mathbf{C}^\times := \mathbf{C} \setminus \{0\}$. (The polynomial F is called the normal form of X.)*

By studying automorphism groups on normal singular cubic surfaces which have no parameters in their normal forms in Table 2, we get the following.

Theorem 3. *Let X be a normal singular cubic surface in $\mathbf{P}^3$ which has no parameters in the normal form in Table 2. Then the automorphism group on X ($\operatorname{Aut} X$, for short) is given in Table 3 according to the type of singularities on X, where Σ_n is a symmetric group of degree n.*

The structure of this paper is as follows. In section 1, we will see that any rational normal singular cubic surface X in $\mathbf{P}^3$ corresponds to two effective divisors C_2, C_3 on $\mathbf{P}^2$, where $\deg C_i = i$, and we prepare the notation of *intersection symbol* for C_2 and C_3 (see Definition 1.2). Letting $\varepsilon : \tilde{X} \to \mathbf{P}^2$ be the composite of blowings-up at six (possibly infinitely near) points which are intersection points of C_2 and C_3,

TABLE 3

Singularities	$\operatorname{Aut} X$
A_1A_3	$\mathbf{Z}/2\mathbf{Z}$
$2A_1A_2$	$\mathbf{Z}/2\mathbf{Z}$
$4A_1$	Σ_4
A_1A_4	$\mathbf{C}^\times$
$2A_1A_3$	$\mathbf{C}^\times \rtimes \mathbf{Z}/2\mathbf{Z}$
A_12A_2	$\mathbf{C}^\times \rtimes \mathbf{Z}/2\mathbf{Z}$
A_1A_5	$\mathbf{C} \rtimes \mathbf{C}^\times$
$3A_2$	$(\mathbf{C}^\times)^2 \rtimes \Sigma_3$
A_4	$\mathbf{Z}/4\mathbf{Z}$
A_5	$(\mathbf{C} \rtimes \mathbf{Z}/3\mathbf{Z}) \rtimes \mathbf{Z}/2\mathbf{Z}$
$D_4(1)$	$\mathbf{C}^\times \rtimes \Sigma_3$
$D_4(2)$	Σ_3
D_5	$\mathbf{C}^\times$
E_6	$\mathbf{C} \rtimes \mathbf{C}^\times$

we obtain a birational morphism $\mu : \tilde{X} \to X$ which gives the minimal resolution of singularities, and we prove that X has only rational double points. In section 2, we will determine all (-1)-curves and (-2)-curves on $\tilde{X}$ with respect to each intersection symbol. We denote by $\Gamma_{\tilde{X}}$ the set of all $(-1), (-2)$-curves on $\tilde{X}$. It is a finite set in any case. We classify rational normal singular cubic surfaces by the Dynkin diagram which is given by the configuration of (-2)-curves, and we determine the number of lines on their surfaces (Theorem 1). In section 3, for rational normal singular cubic surfaces, we will have the normal forms and the number of parameters with respect to each type of singularity (Theorems 1, 2). In section 4, we first prove that $\operatorname{Aut} X$ is isomorphic to $\operatorname{Aut} \tilde{X}$. Finally we have automorphism groups on cubic surfaces which have no parameters in their normal forms (Theorem 3). There exists a group homomorphism $\Psi : \operatorname{Aut} \tilde{X} \to \operatorname{Aut} \Gamma_{\tilde{X}}$, where $\operatorname{Aut} \Gamma_{\tilde{X}}$ is the *automorphism group of configuration of* $\Gamma_{\tilde{X}}$ (see Definition 4.2). For a normal singular cubic surface with no parameters, we can show that Ψ is surjective with respect to each type of singularity, and we have a short exact sequence,

$$0 \longrightarrow \operatorname{Ker} \Psi \longrightarrow \operatorname{Aut} \tilde{X} \longrightarrow \operatorname{Aut} \Gamma_{\tilde{X}} \longrightarrow 0.$$

$\operatorname{Aut} \Gamma_{\tilde{X}}$ is determined by intersection relations of $(-1), (-2)$-curves on $\tilde{X}$, and $\operatorname{Ker} \Psi$ is isomorphic to a subgroup of $\operatorname{Aut} \mathbf{P}^2$. Therefore, we can determine the automorphism groups on normal singular cubic surfaces with no parameters.

Terminology and notation. For the most part, the terminology and notation of this paper agree with generally accepted usage. Throughout this paper, a *surface* will mean a projective surface over a complex number field $\mathbf{C}$.

For a smooth surface S, we denote the canonical sheaf by ω_S, the canonical divisor by K_S, the Picard group by $\operatorname{Pic} S$, and the free abelian group generated by prime divisors by $\operatorname{Div} S$. For two divisors C, D on S, we denote the intersection number of C and D by $C.D$, the intersection multiplicity of C and D at a point $P \in S$ by $(C.D)_P$, and the multiplicity of C at a point $P \in S$ by $\mu_P(C)$. We write $C \sim D$ if C and D are linearly equivalent.

1. The minimal resolution of singularities of a normal singular cubic surface

1.1. **Intersection symbol.** Let $X \subset \mathbf{P}^3$ be a normal singular cubic surface and let P be a singular point on X. $(x_0 : x_1 : x_2 : x_3)$ denotes a homogeneous coordinate in $\mathbf{P}^3$. By a projective transformation sending P to $(0:0:0:1)$, X is isomorphic to a projective surface in $\mathbf{P}^3$ defined by

$$x_3 f_2(x_0, x_1, x_2) - f_3(x_0, x_1, x_2) = 0, \tag{$\natural$}$$

where f_i is a homogeneous polynomial of degree i $(i = 2, 3)$. The rank of the quadratic form f_2 (rank f_2, for short) is three or less, and it does not depend on a choice of the transformation above.

Remark 1.1. If rank f_2 is zero, then X is projectively equivalent to a cone by a smooth cubic curve in $\mathbf{P}^2$, so X contains infinitely many lines. Moreover, X has the only simple elliptic singularity $\tilde{E}_6$.

Hereafter, let X be a normal singular cubic surface defined by the equation ($\natural$), and we suppose that rank f_2 is not zero. We regard $C_2 := \{f_2 = 0\}$ and $C_3 := \{f_3 = 0\}$ as effective divisors on the projective plane $\mathbf{P}^2$. Modifying slightly the usage of [2], we define the intersection symbol below.

Definition 1.2. If rank $f_2 = 3$, then C_2 is a smooth conic. For $i = 1, \dots, 6$, let m_i be the number of points Q with $(C_2.C_3)_Q = i$. Then the notation $6^{m_6}5^{m_5}\cdots 1^{m_1}$ is called the *intersection symbol of* C_2 *and* C_3. However, if $m_i = 0$ or $m_i = 1$, then we omit it (e.g. $6^0 5^0 4^0 3^0 2^2 1^2$ and $6^0 5^0 3^1 2^1 1^1$ denote $2^2 1^2$ and 321, respectively).

If rank $f_2 = 2$, then $C_2 = L + L'$, where L and L' are two distinct lines. Let $O := L \cap L'$. For $i = 1, 2, 3$, let m_i be the number of points Q which is different from the point O with $(L.C_3)_Q = i$, and let n_i be the number of points Q' which is different from the point O with $(L'.C_3)_{Q'} = i$. Then the notation $3^{m_3}2^{m_2}1^{m_1}.3^{n_3}2^{n_2}1^{n_1}$ is called the *intersection symbol of* $C_2 = L + L'$ *and* C_3. We identify the intersection symbol $3^{m_3}2^{m_2}1^{m_1}.3^{n_3}2^{n_2}1^{n_1}$ with $3^{n_3}2^{n_2}1^{n_1}.3^{m_3}2^{m_2}1^{m_1}$ by changing the roles of L and L'.

If rank $f_2 = 1$, then $C_2 = 2L$, where L is a line. For $i = 2, 4, 6$, let m_i be the number of points Q with $(C_2.C_3)_Q = i$. Similarly the notation $\langle 6^{m_6}4^{m_4}2^{m_2}\rangle$ is called the *intersection symbol of* $C_2 = 2L$ *and* C_3.

By an approach which is different from the one of [2], we will prove that the singularities on X are characterized by the intersection symbol of C_2 and C_3 in section 2.

1.2. **The construction of the minimal resolution of singularities of X.** For a normal singular cubic surface X defined by ($\natural$) with rank $f_2 > 0$, we will construct the minimal resolution of singularities of X. We can define a rational map Φ by

$$\Phi : \mathbf{P}^2 \ni (y_0 : y_1 : y_2) \mapsto (y_0 f_2(y_i) : y_1 f_2(y_i) : y_2 f_2(y_i) : f_3(y_i)) \in X.$$

Clearly, Φ is a birational map, so X is rational. In fact, Φ is the inverse of a projection of X from the singular point $(0:0:0:1)$ of X to the hyperplane $\{x_3 = 0\}$.

From the above, a normal singular cubic surface is rational if and only if it is projectively equivalent to a surface defined by ($\natural$) with rank $f_2 > 0$.

Any point of $C_2 \cap C_3$ is a fundamental point of Φ. Indeed, the birational morphism obtained from the elimination of indeterminacy of Φ gives the minimal resolution of singularities of X. Before showing the above, we recall the language of infinitely near points, which is convenient to describe the position of fundamental points of Φ.

Definition 1.3. Let S be a smooth surface. Then any point on any surface S', obtained from S by a finite succession of blowings-up $g : S' \to S$ at points, is called an *infinitely near point* of S. If $Q \in S'$ is a point in the open set where g is an isomorphism, then we identify Q with $g(Q)$ as infinitely near points of S.

Let $f : \tilde{S} \to S$ be the blowing-up at a point $P \in S$, E the exceptional curve, and C a curve on S. Then the points of $\tilde{C} \cap E$ are called *infinitely near points of P on C of the first order*, where $\tilde{C}$ is the strict transform of C by f. (Note that if the curve C is smooth at P, then $\tilde{C} \cap E$ is a point.) Inductively, for all $j \geqq 2$, *infinitely near points of P on C of the j-th order* are infinitely near points of Q on $\tilde{C}$ of the $(j-1)$-th order, where Q is an infinitely near point of P on C of the first order (see e.g. [1], [8]).

We eliminate fundamental points of Φ by blowings-up. Let Λ_0 be the linear system on $S_0 := \mathbf{P}^2$ associated to Φ. Then Λ_0 is contained in the complete linear system $|3E_0|$, where E_0 is a line on $\mathbf{P}^2$. Let $P_1 \in S_0$ be a base point of Λ_0, and $\varepsilon_1 : S_1 \to S_0$ the blowing-up at P_1. Then the exceptional curve $E_1 := \varepsilon_1^{-1}(P_1)$ occurs in the fixed part of the linear system $\varepsilon_1^*\Lambda_0$ with some multiplicity. In other words, the linear system $\Lambda_1 := \varepsilon_1^*\Lambda_0 - k_1E_1$ has no fixed component, where k_1 is a minimum value of $\mu_{P_1}(C)$ for any $C \in \Lambda_0$. Inductively, for $i \geqq 2$, let P_i be a base point of Λ_{i-1}, $\varepsilon_i : S_i \to S_{i-1}$ the blowing-up at P_i, and $E_i := \varepsilon_i^{-1}(P_i)$ the exceptional curve. Similarly we define a linear system on S_i by $\Lambda_i := \varepsilon_i^*\Lambda_{i-1} - k_iE_i$, where k_i is a minimum value of $\mu_{P_i}(C)$ for any $C \in \Lambda_{i-1}$. If n is a minimum value that Λ_n does not have base points, then the P_i's $(1 \leqq i \leqq n)$ are all the fundamental points of Φ.

To determine n, k_i, and to describe the positions of fundamental points P_i according to the intersection symbol, we prepare the next lemma.

Lemma 1.4. *Let S be a smooth surface, and C, C' effective divisors on S, where C and C' intersect but have no common divisors. Let $\varepsilon : \tilde{S} \to S$ be the blowing-up at a point $Q \in C \cap C'$, and E the exceptional curve. Assume that C is smooth at Q. Then, we have*

(1) *E is the only common divisor of ε^*C and ε^*C';*
(2) *let $F := \varepsilon^*C - E$ (i.e., strict transform $\tilde{C}$), $F' := \varepsilon^*C' - E$; then:*
  (a) *if $(C.C')_Q = 1$, then F does not meet F' on E;*
  (b) *if $(C.C')_Q \geqq 2$, then the only $Q' = F \cap E$ is an intersection of F and F' on E, and $(F.F')_{Q'} = (C.C')_Q - 1$.*

Proof. (1) is clear, so we prove (2). If F meets F' on the exceptional curve E, then the intersection of F and F' on E must be a point $Q' = F \cap E$, since C is smooth at Q. Clearly, $F.F' = C.C' - 1$, and the blowing-up at the point Q preserves the intersection multiplicity of C and C' at each point $Q'' \neq Q$, so we have $(F.F')_{Q'} = (C.C')_Q - 1$.

If $(C.C')_Q = 1$, then $(F.F')_{Q'} = 0$, so F does not meet F' on E. On the other hand, if $(C.C')_Q \geqq 2$, then F meets F' on E at the point Q' with $(F.F')_{Q'} = (C.C')_Q - 1$. □

Since X has only isolated singularities, we have $\operatorname{Sing} C_2 \cap \operatorname{Sing} C_3 = \emptyset$, where $\operatorname{Sing} C_i$ consists of points Q in C_i with $\mu_Q(C_i) \geqq 2$. From Lemma 1.4 and Bezout's theorem, we have the following proposition for k_i, n and the positions of fundamental points of Φ.

Proposition 1.5. *With the notation and the conditions as above, we have $k_i = 1$ for $1 \leqq i \leqq 6$, and $n = 6$. Moreover, the positions of fundamental points of Φ are as stated below:*

(1) *In the case of* $\operatorname{rank} f_2 = 3$, *let $Q_1 \in C_2 \cap C_3$ with $(C_2.C_3)_{Q_1} = m$ and Q_i be an infinitely near point of Q_1 on C_2 of the $(i-1)$-th order for $i = 2, \ldots, m$. Then $Q_1, \ldots, Q_m$ are fundamental points of Φ.*

(2) *In the case of* $\operatorname{rank} f_2 = 2$, *let $C_2 = L \cap L'$ and $O = L \cap L'$. If $Q_1 \in C_2 \cap C_3$ is not O and $(C_2.C_3)_{Q_1} = m$, then $Q_1, \ldots, Q_m$ are fundamental points of Φ, where $Q_i\,(i = 2, \ldots, m)$ is an infinitely near point of Q_1 on C_2 of the $(i-1)$-th order. If Q_1 is O and $(C_2.C_3)_{Q_1} = m$, then $Q_1, \ldots, Q_m$ are fundamental points of Φ, where $Q_i\,(i = 2, \ldots, m)$ is an infinitely near point of Q_1 on C_3 of the $(i-1)$-th order.*

(3) *In the case of* $\operatorname{rank} f_2 = 1$, *let $Q_1 \in C_2 \cap C_3$ with $(C_2.C_3)_{Q_1} = m$ and Q_i be an infinitely near point of Q_1 on C_3 of the $(i-1)$-th order for $i = 2, \ldots, m$. Then $Q_1, \ldots, Q_m$ are fundamental points of Φ.*

Proof. We apply Lemma 1.4 to divisors C_2 and C_3 on $\mathbf{P}^2$. $k_i = 1$ is from Lemma 1.4 (1), and $n = 6$ is from Bezout's theorem and Lemma 1.4 (2). In regard to the positions of fundamental points, it follows from Lemma 1.4 (2). □

Let $\varepsilon := \varepsilon_1 \circ \cdots \circ \varepsilon_6$; we have a birational morphism $\mu : S_6 \to X$ such that $\mu = \Phi \circ \varepsilon$. Since $\Lambda_0 \subset |3E_0|$, $K_{\mathbf{P}^2} = -3E_0$ and $k_i = 1$, we have $\Lambda_6 \subset |-K_{S_6}|$. Therefore, we get the following corollary.

Corollary 1.6. *With the notation and conditions as above, we have $-\omega_{S_6} = \mu^*\mathcal{O}_X(1)$.*

We recall the definition for a rational curve with a negative self-intersection number on a smooth surface.

Definition 1.7. Let S be a smooth surface. If C is a smooth rational curve on S with $C^2 = -d$ where d is a positive integer, then C is called a $(-d)$-*curve*.

We have some properties for $\mu : S_6 \to X$ as follows.

Proposition 1.8. *Let X be a normal singular cubic surface in $\mathbf{P}^3$ defined by $(\natural)$ with* $\operatorname{rank} f_2 > 0$. *Then the birational morphism $\mu : S_6 \to X$ gives the minimal resolution of singularities of X. Moreover, X has only rational double points.*

Proof. For a (-1)-curve C on S_6, we have $-K_{S_6}.C = 1$, so $\mu(C)$ is not a point. Therefore, $\mu : S_6 \to X$ gives the minimal resolution of singularities. By the adjunction formula, the dualizing sheaf ω_X° on X is isomorphic to $\mathcal{O}_X(-1)$. By Corollary 1.6, μ is crepant, i.e., $\mu^*K_X = K_{S_6}$. So X has only rational double singularities (see e.g. [14], etc.). □

Corollary 1.9. *The image of a (-1)-curve on S_6 by μ is a line on X, and the one of a (-2)-curve by μ is a singular point on X. Moreover, the strict transform of a line on X by μ is a (-1)-curve on S_6, and the inverse image of a singular point by μ is a set of connected (-2)-curves on S_6.*

Proof. Let C be a (-1)-curve on S_6. We have $\mathcal{O}_X(1).\mu(C) = (-K_{S_6}).C = 1$. So the image of C by μ is a line. Let C' be a (-2)-curve on S_6. Since $C'.K_{S_6} = 0$ and μ is the minimal resolution, $\mu(C')$ is a singular point. The latter part is clear by Proposition 1.8. □

By Proposition 1.5, we see that the six fundamental points of Φ are in an 'almost general position' in the sense of the definition of [5], in any case. So we have the next corollary.

Corollary 1.10. *With the notation as above, the anti-canonical divisor $-K_{\tilde{X}}$ is numerically effective, i.e., $-K_{\tilde{X}}.E \geqq 0$ for any effective divisor E.*

Proof. See Demazure [5], p. 39. □

Some parts mentioned in this section were known by D. F. Coray and M. A. Tsfasman [4], obtained by using the results of Bruce and Wall [2].

2. Classification of normal singular cubic surfaces

Hereafter, we denote S_6 by $\tilde{X}$. In this section, we will determine all (-1)-curves and (-2)-curves on $\tilde{X}$, according to intersection symbol. Consequently, we will prove the results of Bruce and Wall (Theorem 1) by a different approach than is used in [2].

2.1. **(-1)-curves and (-2)-curves on $\tilde{X}$.** We use the notation in section 1. Let $\pi_i := \varepsilon_{i+1} \circ \varepsilon_{i+2} \circ \cdots \circ \varepsilon_6$ $(0 \leqq i \leqq 5)$ and $\pi_6 := \mathrm{id}_{S_6}$. From section 1, we have $\operatorname{Pic} \tilde{X} \simeq \mathbf{Z}^7$, which is generated by divisor classes corresponding to the total transforms $\pi_i^* E_i$ $(0 \leqq i \leqq 6)$.

For a curve C on $\tilde{X}$ which is linearly equivalent to $a\pi_0^*E_0 - \sum_{i=1}^6 b_i \pi_i^* E_i$, where a, b_i are integers, we have the following equations:

$$C^2 = a^2 - \sum_{i=1}^{6} b_i^2,$$

$$C.K_{\tilde{X}} = -3a + \sum_{i=1}^{6} b_i.$$

If C is a (-1)-curve, then $C^2 = C.K_{\tilde{X}} = -1$. Recalling Schwarz's inequality, we have $0 \leqq a \leqq 2$. Therefore, C is linearly equivalent to one of $e_i, f_{i,j}, g_i$ below:

$$e_i := \pi_i^* E_i \ (i = 1, \dots, 6),$$
$$f_{i,j} := \pi_0^* E_0 - \pi_i^* E_i - \pi_j^* E_j \ (1 \leqq i < j \leqq 6),$$
$$g_i := 2\pi_0^* E_0 - \sum_{j \neq i} \pi_j^* E_j \ (i = 1, \dots, 6).$$

Similarly, if C is a (-2)-curve, then $C^2 = -2, C.K_{\tilde{X}} = 0$. Considering effectivity of C, we have $0 \leqq a \leqq 2$. So C is linearly equivalent to one of $p_{i,j}, q_{i,j,k}, r$ below:

$$p_{i,j} := \pi_i^* E_i - \pi_j^* E_j \ (i, j = 1, \dots, 6, i \neq j),$$
$$q_{i,j,k} := \pi_0^* E_0 - \pi_i^* E_i - \pi_j^* E_j - \pi_k^* E_k \ (1 \leqq i < j < k \leqq 6),$$
$$r := 2\pi_0^* E_0 - \sum_{i=1}^{6} \pi_i^* E_i.$$

The intersection numbers of the divisors above are as follows:

$$(*)\quad \begin{aligned}
e_i.e_j &= -\delta_{i,j},\\
e_i.f_{j,k} &= \delta_{i,j}+\delta_{i,k},\\
e_i.g_j &= 1-\delta_{i,j},\\
e_i.p_{j,k} &= \delta_{i,k}-\delta_{i,j},\\
e_i.q_{j,k,l} &= \delta_{i,j}+\delta_{i,k}+\delta_{i,l},\\
e_i.r &= 1,\\
f_{i,j}.f_{k,l} &= 1-\delta_{i,k}-\delta_{i,l}-\delta_{j,k}-\delta_{j,l},\\
f_{i,j}.g_k &= \delta_{i,k}+\delta_{j,k},\\
f_{i,j}.p_{k,l} &= \delta_{i,k}-\delta_{i,l}+\delta_{j,k}-\delta_{j,l},\\
f_{i,j}.q_{k,l,m} &= 1-\delta_{i,k}-\delta_{i,l}-\delta_{i,m}-\delta_{j,k}-\delta_{j,l}-\delta_{j,m},\\
f_{i,j}.r &= 0,\\
g_i.g_j &= -\delta_{i,j},\\
g_i.p_{j,k} &= \delta_{i,k}-\delta_{i,j},\\
g_i.q_{j,k,l} &= \delta_{i,j}+\delta_{i,k}+\delta_{i,l}-1,\\
g_i.r &= -1,\\
p_{i,j}.p_{k,l} &= \delta_{i,l}+\delta_{j,k}-\delta_{i,k}-\delta_{j,l},\\
p_{i,j}.q_{k,l,m} &= \delta_{i,k}+\delta_{i,l}+\delta_{i,m}-\delta_{j,k}-\delta_{j,l}-\delta_{j,m},\\
p_{i,j}.r &= 0,\\
q_{i,j,k}.q_{l,m,n} &= 1-\delta_{i,l}-\delta_{i,m}-\delta_{i,n}-\delta_{j,l}-\delta_{j,m}-\delta_{j,n}-\delta_{k,l}-\delta_{k,m}-\delta_{k,n},\\
q_{i,j,k}.r &= -1,\\
r^2 &= -2,
\end{aligned}$$

where $\delta_{i,j}$ is a Kronecker delta.

Definition 2.1. We denote by $M_{\tilde{X}}$ (resp. $N_{\tilde{X}}$) the set of e_i's, $f_{i,j}$'s and g_i's (resp. $p_{i,j}$'s, $q_{i,j,k}$'s and r). We denote by $I_{\tilde{X}}$ (resp. $R_{\tilde{X}}$) the set of all (-1)-curves (resp. all (-2)-curves) on $\tilde{X}$.

We recall well-known facts about the irreducibility of divisors on a smooth surface.

Lemma 2.2. *Let S be a smooth surface and let D be an effective divisor on S. If $D.C < 0$ for a prime divisor C, then $D = C + E$, where E is an effective divisor.*

Proof. It follows from the fact that the intersection number of two effective divisors without common divisors is positive or zero. □

Lemma 2.3. *Let S be a smooth surface and let C be an effective divisor on S with $C^2 < 0$. If C is a prime divisor, then the complete linear system $|C|$ consists of a unique element C. On the other hand, if C is not a prime divisor, then any member of $|C|$ is not a prime divisor.*

Proof. Let C be a prime divisor. For an effective divisor $E \in |C|$, we have $E.C = C^2 < 0$. By Lemma 2.2, we have $E = C + F$, where F is effective. So F is linearly equivalent to a zero divisor. If F is not a zero divisor, then $A.F > 0$ for an ample divisor A by the Nakai-Moishezon Criterion, so it is a contradiction. Hence, $F = 0$ and $E = C$. The latter part is clear from the first part. □

For any $C \in I_{\tilde{X}}$ (resp. $R_{\tilde{X}}$), there exists a unique divisor $D \in M_{\tilde{X}}$ (resp. $N_{\tilde{X}}$) with $C \sim D$, since any distinct two divisors of $M_{\tilde{X}}$ (resp. $N_{\tilde{X}}$) are not linearly equivalent. So we can define maps $\varphi_I : I_{\tilde{X}} \to M_{\tilde{X}}$ and $\varphi_R : R_{\tilde{X}} \to N_{\tilde{X}}$, which are injective. Using the following proposition, we can determine $I_{\tilde{X}}$ and $R_{\tilde{X}}$ according to the intersection symbol by computing intersection numbers. Furthermore we can get the configuration of (-1)-curves and (-2)-curves on $\tilde{X}$ by $(*)$.

Proposition 2.4. *With the notation above, let I' be a subset of $I_{\tilde{X}}$ and let R' be a subset of $R_{\tilde{X}}$. If there exists $C \in I' \cup R'$ with $D.C < 0$ for any $D \in M_{\tilde{X}} \setminus \varphi_I(I')$, then $I_{\tilde{X}} = I'$. Similarly, if there exists $C \in I' \cup R'$ with $D.C < 0$ for any $D \in N_{\tilde{X}} \setminus \varphi_R(R')$, then $R_{\tilde{X}} = R'$.*

Proof. We prove the first part. We assume that $I' \subsetneqq I_{\tilde{X}}$. For any curve $C' \in I_{\tilde{X}} \setminus I'$, $D := \varphi_I(C')$ is not in $\varphi_I(I')$. From the conditions, there exists $C \in I' \cup R'$ with $D.C < 0$, so we have $C'.C < 0$. Since C' is effective, there exists an effective divisor E such that $C' = C + E$. On the other hand, C' is a prime divisor, so E is zero. Therefore, we have $C' = C$, a contradiction. Hence, $I' = I_{\tilde{X}}$. The latter part is similar. □

We prepare the notation to describe curves on $\tilde{X}$.

Definition 2.5. We denote the strict transform of a curve $C \subset S_i$ via the morphism $\pi_i : \tilde{X} \to S_i$ by $\tilde{C} \subset \tilde{X}$, for short. For the fundamental points P_i of Φ, let $\wp := \{P_i (1 \leqq i \leqq 6)\}$, and $\wp' := \{P_{i_1}, \dots P_{i_m}\} \subset \wp$. If a line on $\mathbf{P}^2$ (or the strict transform on S_i) passes through all points of $\wp'$ but does not pass through all points of $\wp \setminus \wp'$, then we denote the line by $L_{i_1,\dots,i_m}$. We note that the strict transform $\tilde{L}_{i_1,\dots,i_m}$ is linearly equivalent to $\pi_0^* E_0 - \sum_{j=1}^{m} \pi_{i_j}^* E_{i_j}$.

Hereafter, we determine $I_{\tilde{X}}$ and $R_{\tilde{X}}$ by using Proposition 2.4 with respect to each intersection symbol.

2.2. **The case of** $\operatorname{rank} f_2 = 3$**.** Then $\tilde{C}_2$ is always a (-2)-curve, and there are no (-2)-curves which meet with $\tilde{C}_2$, so X has an A_1 singularity in this case.

2.2.1. *Type* 1^6. In the case of the intersection symbol 1^6, we determine the $I_{\tilde{X}}, R_{\tilde{X}}$ by using Proposition 2.4. Then the fundamental points P_i are intersection points $C_2 \cap C_3$. We can choose $I' \subset I_{\tilde{X}}$ and $R' \subset R_{\tilde{X}}$ as follows:

$$I' = \{\tilde{E}_i \ (1 \leqq i \leqq 6),\ \tilde{L}_{i,j} \ (1 \leqq i < j \leqq 6)\}, R' = \{\tilde{C}_2\}.$$

Then, we have

$$M_{\tilde{X}} \setminus \varphi_I(I') = \{g_i \ (1 \leqq i \leqq 6)\},$$
$$N_{\tilde{X}} \setminus \varphi_R(R') = \{p_{i,j} \ (1 \leqq i, j \leqq 6, i \neq j),\ q_{i,j,k} \ (1 \leqq i < j < k \leqq 6)\}.$$

For $g_i \in M_{\tilde{X}} \setminus \varphi_I(I')$ and $\tilde{C}_2 \in R'$, we have $g_i.\tilde{C}_2 = g_i.r = -1 < 0$. Therefore, we have $I_{\tilde{X}} = I'$ by Proposition 2.4. Similarly, for $p_{i,j}$ $(1 \leqq i, j \leqq 6, i \neq j)$ and $\tilde{E}_i \in I'$, we have $p_{i,j}.\tilde{E}_i = p_{i,j}.e_i = -1 < 0$. For $q_{i,j,k}$ $(1 \leqq i < j < k \leqq 6)$ and $\tilde{C}_2 \in R'$, we have $q_{i,j,k}.\tilde{C}_2 = q_{i,j,k}.r = -1 < 0$. Therefore, we have $R_{\tilde{X}} = R'$ by Proposition 2.4. In particular, X has the only A_1 singularity, and X contains exactly 21 lines by Corollary 1.9.

The calculation above is very simple, but we generally need many of them. Actually, for any $D \in (M_{\tilde{X}} \cup N_{\tilde{X}}) \setminus (\varphi_I(I') \cup \varphi_R(R'))$, we verify that there exists

$C \in \varphi_I(I') \cup \varphi_R(R')$ with $D.C < 0$ by using $(*)$ with respect to each intersection symbol. Hereafter, only the results are recorded.

2.2.2. *Type* 21^4. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 2$, P_2 an infinitely near point of P_1 on C_2 of the first order, and $P_i (i = 3, 4, 5, 6)$ a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_i \ (2 \leqq i \leqq 6),\ \tilde{L}_{i,j} \ (i, j \in \{1, 3, 4, 5, 6\}, i < j),\ \tilde{L}_{1,2}\},$$
$$R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1\}.$$

Since $\tilde{C}_2 \sim r$, $\tilde{E}_1 \sim p_{1,2}$ and $r.p_{1,2} = 0$, X has only $2A_1$ singularities and contains exactly 16 lines.

2.2.3. *Type* 31^3. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 3$, $P_i \ (i = 2, 3)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, and $P_i \ (i = 4, 5, 6)$ a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_3, \tilde{E}_4, \tilde{E}_5, \tilde{E}_6, \tilde{L}_{i,j} \ (i, j \in \{1, 4, 5, 6\}, i < j),\ \tilde{L}_{1,2}\},$$
$$R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2\},$$

so X has only A_1A_2 singularities and contains exactly 11 lines.

2.2.4. *Type* 2^21^2. Let P_1, P_3 be points of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 2$, $P_i \ (i = 2, 4)$ an infinitely near point of P_{i-1} on C_2 of the first order, and let P_5, P_6 be points of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_4, \tilde{E}_5, \tilde{E}_6, \tilde{L}_{i,j} \ (i, j \in \{1, 3, 5, 6\}, i < j),\ \tilde{L}_{1,2}, \tilde{L}_{3,4}\},$$
$$R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_3\},$$

so X has only $3A_1$ singularities and contains exactly 12 lines.

2.2.5. *Type* 41^2. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 4$, $P_i \ (i = 2, 3, 4)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, and P_5, P_6 be points of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_4, \tilde{E}_5, \tilde{E}_6, \tilde{L}_{1,5}, \tilde{L}_{1,6}, \tilde{L}_{5,6}, \tilde{L}_{1,2}\},$$
$$R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3\},$$

so X has only A_1A_3 singularities and contains exactly 7 lines.

2.2.6. *Type* 321. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 3$, $P_i \ (i = 2, 3)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, P_4 a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_4} = 2$, P_5 an infinitely near point of P_4 on C_2 of the first order, and P_6 a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_6} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_3, \tilde{E}_5, \tilde{E}_6, \tilde{L}_{1,4}, \tilde{L}_{1,6}, \tilde{L}_{4,6}, \tilde{L}_{1,2}, \tilde{L}_{4,5}\},$$
$$R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2, \tilde{E}_4\},$$

so X has only $2A_1A_2$ singularities and contains exactly 8 lines.

2.2.7. *Type* 2^3. Let P_1, P_3, P_5 be points of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 2$, and P_i $(i = 2, 4, 6)$ an infinitely near point of P_{i-1} on C_2 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_4, \tilde{E}_6, \tilde{L}_{1,3}, \tilde{L}_{1,5}, \tilde{L}_{3,5}, \tilde{L}_{1,2}, \tilde{L}_{3,4}, \tilde{L}_{5,6}\},$$
$$R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_3, \tilde{E}_5\},$$

so X has only $4A_1$ singularities and contains exactly 9 lines.

2.2.8. *Type* 51. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 5$, P_i $(i = 2, 3, 4, 5)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, and P_6 a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_6} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_5, \tilde{E}_6, \tilde{L}_{1,6}, \tilde{L}_{1,2}\}, R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4\},$$

so X has only A_1A_4 singularities and contains exactly 4 lines.

2.2.9. *Type* 42. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 4$, P_i $(i = 2, 3, 4)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, P_5 a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_5} = 2$, and P_6 an infinitely near point of P_5 on C_2 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_4, \tilde{E}_6, \tilde{L}_{1,5}, \tilde{L}_{1,2}, \tilde{L}_{5,6}\}, R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_5\},$$

so X has only $2A_1A_3$ singularities and contains exactly 5 lines.

2.2.10. *Type* 3^2. Let P_1, P_4 be points of $C_2 \cap C_3$ with $(C_2.C_3)_{P_i} = 3$, P_i $(i = 2, 3)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, and P_i $(i = 5, 6)$ an infinitely near point of P_4 on C_2 of the $(i-4)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_3, \tilde{E}_6, \tilde{L}_{1,4}, \tilde{L}_{1,2}, \tilde{L}_{4,5}\}, R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2, \tilde{E}_4, \tilde{E}_5\},$$

so X has only A_12A_2 singularities and contains exactly 5 lines.

2.2.11. *Type* 6. Let P_1 be a point of $C_2 \cap C_3$ with $(C_2.C_3)_{P_1} = 6$ and P_i $(i = 2, \ldots, 6)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_6, \tilde{L}_{1,2}\}, R_{\tilde{X}} = \{\tilde{C}_2, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5\},$$

so X has only A_1A_5 singularities and contains exactly 2 lines.

2.3. **The case of** $\operatorname{rank} f_2 = 2$**.** Let $C_2 = L + L'$ and $O = L \cap L'$, where L, L' are lines.

2.3.1. *Type* $1^3.1^3$. Let P_1, P_2, P_3 be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, and P_4, P_5, P_6 be points of $C_3 \cap L'$ with $(L'.C_3)_{P_i} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_i \ (1 \leqq i \leqq 6),\ \tilde{L}_{i,j} \ (i \in \{1, 2, 3\}, j \in \{4, 5, 6\})\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{4,5,6}\}.$$

Since $\tilde{L} \sim q_{1,2,3}$, $\tilde{L}' \sim q_{4,5,6}$ and $q_{1,2,3}.q_{4,5,6} = 1$, X has the only A_2 singularity and X contains exactly 15 lines.

2.3.2. *Type* $1^3.21$. Let P_1, P_2, P_3 be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, P_4 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_4} = 2$, P_5 an infinitely near point of P_4 on C_2 of the first order, and P_6 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_6} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_i \ (i = 1, 2, 3, 5, 6),\ \tilde{L}_{i,j} \ (i \in \{1, 2, 3\}, j \in \{4, 6\})\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{4,5,6}, \tilde{E}_4\},$$

so X has only A_2A_1 singularities and contains exactly 11 lines.

2.3.3. *Type* $1^3.3$. Let P_1, P_2, P_3 be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, P_4 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_4} = 3$, and P_i $(i = 5, 6)$ an infinitely near point of P_4 on C_2 of the $(i-4)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_6, \tilde{L}_{1,4}, \tilde{L}_{2,4}, \tilde{L}_{3,4}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{4,5,6}, \tilde{E}_4, \tilde{E}_5\},$$

so X has only $2A_2$ singularities and contains exactly 7 lines.

2.3.4. *Type* 21.21. Let P_1 be a point of $C_3 \cap L$ with $(L.C_3)_{P_1} = 2$, P_2 an infinitely near point of P_1 on C_2 of the first order, P_3 a point of $C_3 \cap L$ with $(L.C_3)_{P_3} = 1$, P_4 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_4} = 2$, P_5 an infinitely near point of P_4 on C_2 of the first order, and P_6 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_6} = 1$. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_3, \tilde{E}_5, \tilde{E}_6, \tilde{L}_{1,4}, \tilde{L}_{1,6}, \tilde{L}_{3,4}, \tilde{L}_{3,6}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{4,5,6}, \tilde{E}_1, \tilde{E}_4\},$$

so X has only $A_2 2A_1$ singularities and contains exactly 8 lines.

2.3.5. *Type* 21.3. Let P_1 be a point of $C_3 \cap L$ with $(L.C_3)_{P_1} = 2$, P_2 an infinitely near point of P_1 on C_2 of the first order, P_3 a point of $C_3 \cap L$ with $(L.C_3)_{P_3} = 1$, P_4 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_4} = 3$, and P_i $(i = 5, 6)$ an infinitely near point of P_4 on C_2 of the $(i-4)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_3, \tilde{E}_6, \tilde{L}_{1,4}, \tilde{L}_{3,4}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{4,5,6}, \tilde{E}_1, \tilde{E}_4, \tilde{E}_5\},$$

so X has only $2A_2A_1$ singularities and contains exactly 5 lines.

2.3.6. *Type* 3.3. Let P_1 be a point of $C_3 \cap L$ with $(L.C_3)_{P_1} = 3$, P_i $(i = 2, 3)$ an infinitely near point of P_1 on C_2 of the $(i-1)$-th order, P_4 a point of $C_3 \cap L'$ with $(L'.C_3)_{P_4} = 3$, and P_i $(i = 5, 6)$ an infinitely near point of P_4 on C_2 of the $(i-4)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_3, \tilde{E}_6, \tilde{L}_{1,4}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{4,5,6}, \tilde{E}_1, \tilde{E}_2, \tilde{E}_4, \tilde{E}_5\},$$

so X has only $3A_2$ singularities and contains exactly 3 lines.

2.3.7. *Type* $1^2.1^2$. Let $P_1, P_2 \neq O$ be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, $P_3, P_4 \neq O$ points of $C_3 \cap L'$ with $(L'.C_3)_{P_i} = 1$, $P_5 = O$, and let P_6 be an infinitely near point of P_5 on C_3 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_i \ (i \in \{1, 2, 3, 4, 6\}),\ \tilde{L}_{1,3}, \tilde{L}_{1,4}, \tilde{L}_{2,3}, \tilde{L}_{2,4}, \tilde{L}_{5,6}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,5}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_5\},$$

so X has the only A_3 singularity and contains exactly 10 lines.

2.3.8. *Type* $1^2.2$. Let $P_1, P_2 \neq O$ be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, $P_3 \neq O$ a point of $C_3 \cap L'$ with $(L'.C_3)_{P_3} = 2$, P_4 an infinitely near point of P_3 on C_2 of the first order, $P_5 = O$, and P_6 an infinitely near point of P_5 on C_3 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_1, \tilde{E}_2, \tilde{E}_4, \tilde{E}_6, \tilde{L}_{1,3}, \tilde{L}_{2,3}, \tilde{L}_{5,6}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,5}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_3, \tilde{E}_5\},$$

so X has only A_3A_1 singularities and contains exactly 7 lines.

2.3.9. *Type* 2.2. Let $P_1 \neq O$ be a point of $C_3 \cap L$ with $(L.C_3)_{P_1} = 2$, P_2 an infinitely near point of P_1 on C_2 of the first order, $P_3 \neq O$ a point of $C_3 \cap L'$ with $(L'.C_3)_{P_3} = 2$, P_4 an infinitely near point of P_3 on C_2 of the first order, $P_5 = O$, and P_6 an infinitely near point of P_5 on C_3 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_4, \tilde{E}_6, \tilde{L}_{1,3}, \tilde{L}_{5,6}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,5}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_1, \tilde{E}_3, \tilde{E}_5\},$$

so X has only A_32A_1 singularities and contains exactly 5 lines.

2.3.10. *Type* $1^2.1$. Let $P_1, P_2 \neq O$ be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, $P_3 \neq O$ a point of $C_3 \cap L'$ with $(L'.C_3)_{P_3} = 1$, $P_4 = O$, and P_i $(i = 5, 6)$ an infinitely near point of P_4 on C_3 of the $(i-4)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_6, \tilde{L}_{1,3}, \tilde{L}_{2,3}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,4}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_4, \tilde{E}_5\},$$

so X has the only A_4 singularity and contains exactly 6 lines.

2.3.11. *Type* 2.1. Let $P_1 \neq O$ be a point of $C_3 \cap L$ with $(L.C_3)_{P_1} = 2$, P_2 an infinitely near point of P_1 on C_2 of the first order, $P_3 \neq O$ a point of $C_3 \cap L'$ with $(L'.C_3)_{P_3} = 1$, $P_4 = O$, and P_i $(i = 5, 6)$ an infinitely near point of P_4 on C_3 of the $(i-4)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_3, \tilde{E}_6, \tilde{L}_{1,3}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,4}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_1, \tilde{E}_4, \tilde{E}_5\},$$

so X has only A_4A_1 singularities and contains exactly 4 lines.

2.3.12. *Type* $1^2.0$. Let $P_1, P_2 \neq O$ be points of $C_3 \cap L$ with $(L.C_3)_{P_i} = 1$, $P_3 = O$, and P_i $(i = 4, 5, 6)$ an infinitely near point of P_3 on C_3 of the $(i-3)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_1, \tilde{E}_2, \tilde{E}_6\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5\},$$

so X has the only A_5 singularity and contains exactly 3 lines.

2.3.13. *Type* 2.0. Let $P_1 \neq O$ be a point of $C_3 \cap L$ with $(L.C_3)_{P_1} = 2$, P_2 an infinitely near point of P_1 on C_2 of the first order, $P_3 = O$, and P_i $(i = 4, 5, 6)$ an infinitely near point of P_3 on C_3 of the $(i-3)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_6\}, R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{L}' = \tilde{L}_{3,4,5}, \tilde{E}_1, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5\},$$

so X has only A_5A_1 singularities and contains exactly 2 lines.

2.4. **The case of** rank $f_2 = 1$**.** We put $C_2 = 2L$, where L is a line.

2.4.1. *Type* $\langle 2^3 \rangle$. Let P_1, P_3, P_5 be points with $(C_2.C_3)_{P_i} = 2$, and P_i $(i = 2, 4, 6)$ an infinitely near point of P_{i-1} on C_3 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_2, \tilde{E}_4, \tilde{E}_6, \tilde{L}_{1,2}, \tilde{L}_{3,4}, \tilde{L}_{5,6}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{E}_1, \tilde{E}_3, \tilde{E}_5\},$$

so X has the only D_4 singularity and contains exactly 6 lines.

2.4.2. *Type* $\langle 42 \rangle$. Let P_1 be a point with $(C_2.C_3)_{P_1} = 4$, P_i $(i = 2, 3, 4)$ an infinitely near point of P_1 on C_3 of the $(i-1)$-th order, P_5 a point with $(C_2.C_3)_{P_5} = 2$, and P_6 an infinitely near point of P_5 on C_3 of the first order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_4, \tilde{E}_6, \tilde{L}_{5,6}\},$$
$$R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,5}, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_5\},$$

so X has the only D_5 singularity and contains exactly 3 lines.

2.4.3. *Type* $\langle 6 \rangle$. Let P_1 be a point with $(C_2.C_3)_{P_1} = 6$, and P_i $(i = 2, \ldots, 6)$ an infinitely near point of P_1 on C_3 of the $(i-1)$-th order. Then,

$$I_{\tilde{X}} = \{\tilde{E}_6\}, R_{\tilde{X}} = \{\tilde{L} = \tilde{L}_{1,2,3}, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5\},$$

so X has the only E_6 singularity and contains exactly one line.

3. Normal forms and the number of parameters

In this section, we determine normal forms of all rational normal singular cubic surfaces and the number of parameters according to the type of singularity, slightly modifying the idea of [2].

3.1. **Preliminaries.** First we prepare some lemmata for the commutativity of blowing-up and blowing-down.

Lemma 3.1. *Let $\sigma : S \to S'$ be an isomorphism between smooth surfases with $\sigma(P) = Q$, where $P \in S$ and $Q \in S'$ are points. Let $\varepsilon_P : S_P \to S$ (resp. $\varepsilon_Q : S'_Q \to S'$) be the blowing-up at P (resp. Q), and E_P (resp. E_Q) the exceptional curve. Then there exists a unique isomorphism $\tilde{\sigma} : S_P \to S'_Q$ with $\tilde{\sigma}(E_P) = E_Q$.*

Proof. It follows from the universal property of blowing-up (see e.g. [1], [8]). □

Lemma 3.2. *Let $f : X \to Y$ and $f' : X \to Y'$ be surjective morphisms between normal projective varieties, and their fibres are connected. For any curve C on X, we suppose that*

$$f(C) \text{ is a point} \Longleftrightarrow f'(C) \text{ is a point}.$$

Then there exists a unique isomorphism $g : Y \to Y'$ such that $f' = g \circ f$.

Proof. Let X_0, Y_0, Y'_0 be the topological subspaces of X, Y, Y' respectively, consisting of all closed points, and let $f_0 : X_0 \to Y_0, f'_0 : X_0 \to Y'_0$ be induced maps. For a closed point $P \in Y$, the fibre $f^{-1}(P)$ is a projective scheme which is connected. Therefore, for any two closed points on the fibre, there exists a 1-dimensional closed subscheme which contains these two points. So we have a unique bijection $g_0 : Y_0 \to Y'_0$ with $f'_0 = g_0 \circ f_0$. Since a projective morphism is a closed morphism, a subset U_0 of Y_0 is an open set if and only if $f_0^{-1}(U_0)$ is an open set of X_0. Hence, g_0 is a homeomorphism. Since we can identify Y, Y' with the set of all irreducible closed subset, on Y_0, Y_0', respectively, we have a unique induced homeomorphism $g : Y \to Y'$. Since $\mathcal{O}_Y = f_*\mathcal{O}_X, \mathcal{O}_{Y'} = f'_*\mathcal{O}_X$, g is an isomorphism. (Cf. Kawamata [12].) □

Lemma 3.3. *Let $\tilde{\sigma} : S \to S'$ be an isomorphism between smooth surfaces with $\tilde{\sigma}(E) = E'$, where E (resp. E') is a (-1)-curve on S (resp. S'). Let $\varepsilon : S \to T$ be the blowing-down of E to a point P, and $\varepsilon' : S' \to T'$ the blowing-down of E' to a point Q. Then there exists a unique isomorphism $\sigma : T \to T'$ with $\sigma(P) = Q$.*

Proof. It follows from Lemma 3.2. □

3.2. **The case of** $\operatorname{rank} f_2 = 3$**.** We determine the normal forms in the case of $\operatorname{rank} f_2 = 3$ with respect to each intersection symbol. First we recall a well-known fact for automorphisms on the projective plane.

Lemma 3.4. *Let C and C' be smooth conics in $\mathbf{P}^2$, and let Q_1, Q_2, Q_3 (resp. Q_1', Q_2', Q_3') be mutually different points on C (resp. C'). Then there exists a unique automorphism σ on $\mathbf{P}^2$ with $\sigma(C) = C', \sigma(Q_i) = Q_i'$ $(i = 1, 2, 3)$.*

Proof. It follows from an easy calculation. □

Hereafter, we put $f_2' = x_0x_2 - x_1^2$ and $C_2' = \{f_2' = 0\} \subset \mathbf{P}^2$.

3.2.1. *Type* 1^6 (A_1). For the fundamental points P_i $(i = 1, \dots, 6)$ of the birational map $\Phi : X - \to \mathbf{P}^2$ which are the points $C_2 \cap C_3$, there exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C_2'$, $\sigma_0(P_1) = (0 : 0 : 1)$, $\sigma_0(P_2) = (1 : 0 : 0)$ and $\sigma_0(P_3) = (1 : 1 : 1)$, from Lemma 3.4. Then, we have $\sigma_0(P_4) = (a^2 : a : 1)$, $\sigma_0(P_5) = (b^2 : b : 1)$ and $\sigma_0(P_6) = (c^2 : c : 1)$, where a, b, c are three distinct elements of $\mathbf{C} \setminus \{0, 1\}$. Let $X' = \{F' = x_3f_2' - f_3' = 0\} \subset \mathbf{P}^3$, where

$$f_3' = (x_0 - ax_1)(-x_0 + (b+1)x_1 - bx_2)(x_1 - cx_2),$$

and let Φ' be a birational map defined by

$$\Phi' : \mathbf{P}^2 \ni (y_0 : y_1 : y_2) \mapsto (y_0f_2'(y_i) : y_1f_2'(y_i) : y_2f_2'(y_i) : f_3'(y_i)) \in X'$$

(see section 1). For $i = 1, \dots, 6$, $P_i' := \sigma_0(P_i)$ are fundamental points of Φ'. Let $S_0' := \mathbf{P}^2$, and let $\varepsilon_i : S_i' \to S_{i-1}'$ be the blowing-up at P_i' $(i = 1, \dots, 6)$. σ_0 induces isomorphisms $\sigma_i : S_i \to S_i'$ $(i = 1, \dots, 6)$ by Lemma 3.1. As in section 1, there exists a birational morphism $\mu' : S_6' \to X'$ which gives the minimal resolution of singularities, so we have an isomorphism $\tau : X \to X'$ induced by σ_6, from Lemma 3.3. On the other hand, since 1^6 is the only case where a singular cubic surface has the only A_1 singularity, the normal form of type A_1 is the polynomial F' above.

Hereafter, we put $X' = \{x_3f_2' - f_3' = 0\} \subset \mathbf{P}^3$ and determine the polynomial f_3' for any other case. We maintain the notation and conditions (for P_i's etc.) in section 2.2, according to intersection symbol.

3.2.2. *Type* 21^4 $(2A_1)$. There exists a unique automorphism σ_0 with $\sigma_0(C_2) = C_2'$, $\sigma_0(P_1) = (1 : 1 : 1) =: P_1'$, $\sigma_0(P_3) = (0 : 0 : 1) =: P_3'$ and $\sigma_0(P_4) = (1 : 0 : 0) =: P_4'$. Then, we have $\sigma_0(P_5) = (a^2 : a : 1) =: P_5'$ and $\sigma_0(P_6) = (b^2 : b : 1) =: P_6'$, where a, b are two distinct elements of $\mathbf{C} \setminus \{0, 1\}$. Let

$$f_3' = (x_0 - 2x_1 + x_2)(x_0 - ax_1)(x_1 - bx_2),$$

and let P_2' be an infinitely near point of P_1' on C_2' of the first order. We define Φ', S_i' as above. Then, P_i' $(i = 1, \dots, 6)$ are fundamental points of Φ'. Since $\sigma_1(P_2) = P_2'$,

σ_0 induces isomorphisms $\sigma_i : S_i \to S'_i$ $(i = 1, \ldots, 6)$ by Lemma 3.1. In the same way as above, we have an isomorphism $\tau : X \to X'$ induced by σ_6.

3.2.3. *Type* 31^3 (A_1A_2). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (1:1:1)$, $\sigma_0(P_4) = (0:0:1)$ and $\sigma_0(P_5) = (1:0:0)$. Then, we have $\sigma_0(P_6) = (a^2 : a : 1)$, where a is an element of $\mathbf{C} \setminus \{0, 1\}$. Putting

$$f'_3 = (x_0 - x_1)(-x_1 + x_2)(x_0 - (a+1)x_1 + ax_2),$$

we similarly have an isomorphism between X and X'.

3.2.4. *Type* 2^21^2 $(3A_1)$. There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (0:0:1)$, $\sigma_0(P_3) = (1:0:0)$ and $\sigma_0(P_5) = (1:1:1)$. Then, $\sigma_0(P_6) = (a^2 : a : 1)$, where a is an element of $\mathbf{C} \setminus \{0, 1\}$. We have an isomorphism between X and X' by putting

$$f'_3 = x_0x_2(x_0 - (a+1)x_1 + ax_2).$$

3.2.5. *Type* 41^2 (A_1A_3). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (1:1:1)$, $\sigma_0(P_5) = (0:0:1)$ and $\sigma(P_6) = (1:0:0)$. We have an isomorphism between X and X' by putting

$$f'_3 = (x_0 - x_1)(-x_1 + x_2)(x_0 - 2x_1 + x_2).$$

3.2.6. *Type* 321 $(2A_1A_2)$. There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (0:0:1)$, $\sigma_0(P_4) = (1:0:0)$ and $\sigma_0(P_6) = (1:1:1)$. We have an isomorphism between X and X' by putting

$$f'_3 = x_1^2(x_0 - x_1).$$

3.2.7. *Type* 2^3 $(4A_1)$. There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (0:0:1)$, $\sigma_0(P_3) = (1:0:0)$ and $\sigma_0(P_5) = (1:1:1)$. We have an isomorphism between X and X' by putting

$$f'_3 = (x_0 - x_1)(x_1 - x_2)x_1.$$

3.2.8. *Type* 51 (A_1A_4). There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (0:0:1)$ and $\sigma_0(P_6) = (1:0:0)$. We have an isomorphism between X and X' by putting

$$f'_3 = x_0^2x_1.$$

3.2.9. *Type* 42 $(2A_1A_3)$. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (0:0:1)$ and $\sigma_0(P_5) = (1:0:0)$. We have an isomorphism between X and X' by putting

$$f'_3 = x_0x_1^2.$$

3.2.10. *Type* 3^2 (A_12A_2). There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$, $\sigma_0(P_1) = (0:0:1)$ and $\sigma_0(P_4) = (1:0:0)$. We have an isomorphism between X and X' by putting

$$f'_3 = x_1^3.$$

3.2.11. *Type* 6 (A_1A_5). There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(C_2) = C'_2$ and $\sigma_0(P_1) = (0:0:1)$. We have an isomorphism between X and X' by putting

$$f'_3 = x_0^3.$$

3.3. **The case of** $\operatorname{rank} f_2 = 2$**.** We determine the normal forms in the case of $\operatorname{rank} f_2 = 2$ with respect to each intersection symbol. First, we recall well-known facts for automorphisms on the projective plane.

Lemma 3.5. *Let $Q_1, \dots, Q_4$ (resp. $Q'_1, \dots, Q'_4$) be points on $\mathbf{P}^2$. Assume that no 3 of the Q_i (resp. Q'_i) are collinear. Then there exists a unique automorphism σ on $\mathbf{P}^2$ with $\sigma(Q_i) = Q'_i$ $(i = 1, 2, 3, 4)$.*

Proof. It follows from an easy calculation. □

Lemma 3.6. *Let Q_1, Q_2 and Q_3 be points on $\mathbf{P}^2$. Assume that they are on a line L. If an automorphism σ on $\mathbf{P}^2$ fixes Q_1, Q_2 and Q_3, then $\sigma|_L = \mathrm{id}_L$.*

Proof. It follows from an easy calculation. □

We put $f'_2 = x_0x_1$ and $C'_2 = \{f'_2 = 0\}$, and maintain the notation and conditions (for P_i's, etc.) in section 2.3 according to the intersection symbol.

3.3.1. *Type* $1^3.1^3$ (A_2). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0 : 1 : 0)$, $\sigma_0(P_2) = (0 : 1 : -1)$, $\sigma_0(P_4) = (1 : 0 : 0)$ and $\sigma_0(P_5) = (1 : 0 : -1)$. Then, we have $\sigma_0(P_3) = (0 : d : 1)$ and $\sigma_0(P_6) = (e : 0 : 1)$, where d, e are elements of $\mathbf{C} \setminus \{0, -1\}$. In the same way as above, X is isomorphic to $X' = \{x_3f'_2 - f'_3 = 0\}$ by putting

$$f'_3 = x_2(x_0 + x_1 + x_2)(dx_0 + ex_1 - dex_2).$$

3.3.2. *Type* $1^3.21$ (A_2A_1). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0 : 1 : 0)$, $\sigma_0(P_2) = (0 : 1 : -1)$, $\sigma_0(P_4) = (1 : 0 : 0)$ and $\sigma_0(P_6) = (1 : 0 : -1)$. Then, we have $\sigma_0(P_3) = (0 : d : 1)$, where d is an element of $\mathbf{C} \setminus \{0, -1\}$. Similarly, we have an isomorphism between X and X' by putting

$$f'_3 = x_2(x_1 - dx_2)(x_0 + x_1 + x_2).$$

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - (x_0 - x_1)(-x_1 + x_2)(x_0 - (a + 1)x_1 + ax_2) = 0\}$$

(cf. subsection 3.2.3) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (dx_3 : -dx_0 + 2dx_1 - dx_2 : x_1 - x_2 : x_1) \in X',$$

where $a = 1 + 1/d$. Therefore, the normal form of a singular cubic surface which has A_1A_2 singularities is given as in subsection 3.2.3.

3.3.3. *Type* $1^3.3$ $(2A_2)$. We must take care in this case. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0 : 1 : 0) =: P'_1$, $\sigma_0(P_2) = (0 : 1 : -1) =: P'_2$, $\sigma_0(P_4) = (1 : 0 : 0) =: P'_4$ and $\sigma_4(P_5) = P'_5$, where P'_5 is an infinitely near point of P'_4 on C'_2 of the first order. Let P'_6 be an infinitely near point of P'_4 on C'_2 of the second order. Of course, the line which passes through P'_4 and P'_5 (i.e., the line $x_1 = 0$) also passes through P'_6. We denote this line by $L'_{4,5,6}$. Since σ_0 satisfies $\sigma_0(L_{4,5,6}) = L'_{4,5,6}$, any automorphism satisfying the conditions above sends $O = L_{1,2,3} \cap L_{4,5,6}$ to $O' = L'_{1,2,3} \cap L'_{4,5,6}$, where $L'_{1,2,3}$ is a line that passes through P'_1, P'_2 and P'_3 (i.e., the line $x_0 = 0$). Therefore, σ_0 sends P_3 to $P'_3 := (0 : d : 1)$, where d is an element of $\mathbf{C} \setminus \{0, -1\}$, and d does not depend on a choice of σ_0, by Lemma 3.6. In the same way as above, we have an isomorphism between X and X' by putting

$$f'_3 = x_2(x_1 + x_2)(-x_1 + dx_2).$$

3.3.4. *Type* 21.21 ($A_2 2A_1$). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:0)$, $\sigma_0(P_3) = (0:1:-1)$, $\sigma_0(P_4) = (1:0:0)$ and $\sigma_0(P_6) = (1:0:-1)$. We have an isomorphism between X and X' by putting

$$f_3' = x_2^2(x_0 + x_1 + x_2).$$

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - x_1^2(x_0 - x_1) = 0\}$$

(cf. subsection 3.2.6) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_3 : x_0 : -x_1 : x_2) \in X'.$$

3.3.5. *Type* 21.3 ($2A_2A_1$). There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:0)$, $\sigma_0(P_3) = (0:1:-1)$, $\sigma_0(P_4) = (1:0:0) =: P_4'$ and $\sigma_4(P_5) = P_5'$, where P_5' is an infinitely near point of P_4' on C_2' of the first order. We have an isomorphism between X and X' by putting

$$f_3' = x_2^2(x_1 + x_2).$$

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - x_1^3 = 0\}$$

(cf. subsection 3.2.10) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_3 : x_1 : x_2) \in X'.$$

3.3.6. *Type* 3.3 ($3A_2$). There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:0) =: P_1'$, $\sigma_0(P_4) = (1:0:0) =: P_4'$, $\sigma_1(P_2) = P_2'$ and $\sigma_4(P_5) = P_5'$, where P_2' (resp. P_5') is an infinitely near point of P_1' (resp. P_4') on C_2' of the first order. We have an isomorphism between X and X' by putting

$$f_3' = x_2^3.$$

3.3.7. *Type* $1^2.1^2$ (A_3). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:0)$, $\sigma_0(P_2) = (0:1:-1)$, $\sigma_0(P_3) = (1:0:0)$ and $\sigma_0(P_4) = (1:0:-1)$. Then, we have $\sigma_0(P_5) = (0:0:1) =: P_5'$ and $\sigma_5(P_6) = P_6'$, where P_6' is an infinitely near point of P_5' on $x_0 - ux_1 = 0$ of the first order, and u is an element of $\mathbf{C}^\times$. We have an isomorphism between X and X' by putting

$$f_3' = x_2(x_0 + x_1 + x_2)(x_0 - ux_1).$$

3.3.8. *Type* $1^2.2$ (A_3A_1). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:1)$, $\sigma_0(P_2) = (0:1:-1)$, $\sigma_0(P_3) = (1:0:0) =: P_3'$, $\sigma_3(P_4) = P_4'$ and $\sigma_5(P_6) = P_6'$, where P_4' is an infinitely near point of P_3' on C_2' of the first order, and P_6' is an infinitely near point of $P_5' := (0:0:1)$ on

$$f_3' = x_0x_2^2 - x_1^3 + x_1x_2^2 = 0$$

of the first order. Similarly, we have an isomorphism between X and X'.

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - (x_0 - x_1)(-x_1 + x_2)(x_0 - 2x_1 + x_2) = 0\}$$

(cf. subsection 3.2.5) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (-x_3 : x_0 - 2x_1 + x_2 : x_0 - x_2 : x_0 + 2x_1 + x_2) \in X'.$$

3.3.9. *Type* 2.2 ($A_3 2A_1$). There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:0) =: P_1'$, $\sigma_1(P_2) = P_2'$, $\sigma_0(P_3) = (1:0:0) =: P_3'$, $\sigma_3(P_4) = P_4'$, $\sigma_0(P_5) = (0:0:1) =: P_5'$ and $\sigma_5(P_6) = P_6'$, where P_2' (resp. P_4') is an infinitely near point of P_1' (resp. P_3') on C_2' of the first order, and P_6' is an infinitely near point of P_5' on

$$f_3' = x_2^2(x_0 + x_1) = 0$$

of the first order. Similarly, we have an isomorphism between X and X'.

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - x_0x_1^2 = 0\}$$

(cf. subsection 3.2.9) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_3 : x_1 : x_2) \in X'.$$

3.3.10. *Type* 1^2.1 (A_4). There exists a unique automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = (0:1:1)$, $\sigma_0(P_2) = (0:1:-1)$, $\sigma_0(P_3) = (1:0:0)$, $\sigma_0(P_4) = (0:0:1) =: P_4'$, $\sigma_4(P_5) = P_5'$ and $\sigma_5(P_6) = P_6'$, where P_5' is an infinitely near point of P_4' on

$$f_3' = x_0^2x_2 + x_1^3 - x_1x_2^2 = 0$$

of the first order, and P_6' is an infinitely near point of P_4' on $f_3' = 0$ of the second order. Similarly, we have an isomorphism between X and X'.

3.3.11. *Type* 2.1 (A_4A_1). Let $P_1' := (0:1:0), P_3' := (1:0:0), P_4' := (0:0:1)$, P_2' be an infinitely near point of P_1' on C_2' of the first order, and let $P_i'(i = 5, 6)$ be an infinitely near point of P_4' on

$$f_3' = x_2(x_0^2 + x_1x_2) = 0$$

of the $(i-4)$-th order. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = P_1'$, $\sigma_0(P_3) = P_3'$, $\sigma_0(P_4) = P_4'$, and $\sigma_5(P_6) = P_6'$. Then, we have $\sigma_1(P_2) = P_2'$ and $\sigma_4(P_5) = P_5'$. Similarly, we have an isomorphism between X and X'.

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - x_0^2x_1 = 0\}$$

(cf. subsection 3.2.8) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_3 : x_1 : x_2) \in X'.$$

3.3.12. *Type* 1^2.0 (A_5). Let $P_1' := (0:1:1), P_2' := (0:1:-1), P_3' := (0:0:1)$, and let P_i' $(i = 4, 5, 6)$ be an infinitely near point of P_3' on

$$f_3' = x_0^3 + x_1^3 - x_1x_2^2 = 0$$

of the $(i-3)$-th order. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = P_1'$, $\sigma_0(P_2) = P_2'$, $\sigma_0(P_3) = P_3'$, $\sigma_3(P_4) = P_4'$, $\sigma_5(P_6) = P_6'$ and $\sigma_4(P_5) = P_5'$. Similarly, we have an isomorphism between X and X'.

3.3.13. *Type* 2.0 (A_5A_1). Let $P'_1 := (0:1:0)$, P'_2 be an infinitely near point of P'_1 on C'_2 of the first order, $P'_3 := (0:0:1)$, and let P'_i $(i = 4,5,6)$ be an infinitely near point of P'_3 on

$$f'_3 = x_0^3 + x_1x_2^2 = 0$$

of the $(i-3)$-th order. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = P'_1$, $\sigma_0(P_3) = P'_3$, $\sigma_3(P_4) = P'_4$, $\sigma_5(P_6) = P'_6$, $\sigma_1(P_2) = P'_2$ and $\sigma_4(P_5) = P'_5$. Similarly, we have an isomorphism between X and X'.

However, X' is isomorphic to

$$X'' = \{x_3(x_0x_2 - x_1^2) - x_0^3 = 0\}$$

(cf. subsection 3.2.11) by the projective transformation

$$X'' \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_3 : x_1 : x_2) \in X'.$$

3.4. **The case of** $\operatorname{rank} f_2 = 1$**.** We determine the normal forms in the case of $\operatorname{rank} f_2 = 1$ (however these are already given in [2]). We maintain the notation and condition (for P_i's, etc.) in section 2.4, according to intersection symbol. We put $f'_2 = x_0^2$ and $C'_2 = \{f'_2 = 0\}$.

3.4.1. *Type* $\langle 2^3 \rangle$ (D_4). We must take care in this case, that is, there are two normal forms according to the position of P_i.

(1) In the case where $L_{1,2}$, $L_{3,4}$ and $L_{5,6}$ meet at a point Q, we can say that P_6 is an infinitely near point of P_5 on a line, which passes through P_5 and Q. Let $P'_1 := (0:1:-1), P'_3 := (0:1:-\omega)$, and $P'_5 := (0:1:-\omega^2)$, where $\omega^3 = 1$ $(\omega \neq 1)$, and let P'_i $(i = 2,4,6)$ be an infinitely near point of P'_{i-1} on

$$f'_3 = x_1^3 + x_2^3 = 0$$

of the first order. There exists an automorphism σ_0 satisfying $\sigma_0(P_1) = P'_1$, $\sigma_0(P_3) = P'_3$, $\sigma_0(P_5) = P'_5$, $\sigma_1(P_2) = P'_2$ and $\sigma_3(P_4) = P'_4$. Then we have $\sigma_0(Q) = (1:0:0)$. Since $L'_{5,6}$ passes through $(1:0:0)$, where $L'_{5,6}$ is a line that passes through P'_5 and P'_6 (i.e., the line $x_1 + \omega x_2 = 0$), we have $\sigma_0(L_{5,6}) = L'_{5,6}$. Hence, σ_0 sends P_6 to P'_6. In the same way as above, X is isomorphic to X'.

(2) In the case where $L_{1,2}$, $L_{3,4}$ and $L_{5,6}$ do not meet at a point, we put $P'_1 := (0:1:-1), P'_3 := (0:1:-\omega), P'_5 := (0:1:-\omega^2)$. Also let P'_i $(i = 2,4,6)$ be an infinitely near point of P'_{i-1} on

$$f'_3 = x_1^3 + x_2^3 + x_0x_1x_2 = 0$$

of the first order. There exists a unique automorphism σ_0 satisfying $\sigma_0(P_1) = P'_1$, $\sigma_0(P_3) = P'_3$, $\sigma_0(P_5) = P'_5$, $\sigma_1(P_2) = P'_2$, $\sigma_3(P_4) = P'_4$ and $\sigma_5(P_6) = P'_6$. Similarly, X is isomorphic to X'.

3.4.2. *Type* $\langle 42 \rangle$ (D_5). We put

$$f'_3 = x_0x_2^2 + x_1^2x_2$$

and $C'_3 = \{f'_3 = 0\}$. Let $P'_1 := (0:0:1)$ and let P'_i $(i = 2,3,4)$ be an infinitely near point of P'_1 on C'_3 of the $(i-1)$-th order, $P'_5 := (0:1:0)$, and let P'_6 be an infinitely near point of P'_5 on C'_3 of the first order. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = P'_1$, $\sigma_0(P_5) = P'_5$, $\sigma_2(P_3) = P'_3$, $\sigma_3(P_4) = P'_4$ and $\sigma_5(P_6) = P'_6$. Then, we have $\sigma_1(P_2) = P'_2$. Similarly, X is isomorphic to X'.

3.4.3. *Type* $\langle 6 \rangle$ (E_6). We put

$$f_3' = x_0 x_2^2 + x_1^3$$

and $C_3' = \{f_3' = 0\}$. Let $P_1' := (0 : 0 : 1)$, and let P_i' $(i = 2, \ldots, 6)$ be an infinitely near point of P_1' on C_3' of the $(i-1)$-th order. There exists an automorphism σ_0 on $\mathbf{P}^2$ with $\sigma_0(P_1) = P_1'$, $\sigma_1(P_2) = P_2'$, $\sigma_3(P_4) = P_4'$, $\sigma_4(P_5) = P_5'$ and $\sigma_5(P_6) = P_6'$. Then, we have $\sigma_2(P_3) = P_3'$. Similarly, X is isomorphic to X'.

4. Proof of Theorem 3

In this section, we determine automorphism groups on normal singular cubic surfaces with no parameters.

4.1. **Preliminaries.**

Proposition 4.1. *Let X be a rational normal singular cubic surface and let $\mu : \tilde{X} \to X$ be the minimal resolution of singularities. Then, there exists a group isomorphism $\gamma : \operatorname{Aut} X \simeq \operatorname{Aut} \tilde{X}$.*

Proof. We define a group homomorphism $\gamma : \operatorname{Aut} X \to \operatorname{Aut} \tilde{X}$ as follows. For any $\sigma \in \operatorname{Aut} X$, we show that the birational map $\tilde{\sigma} := \mu^{-1} \circ \sigma \circ \mu$ is an automorphism on $\tilde{X}$. If $\tilde{\sigma}$ has some fundamental points, then we consider the elimination of indeterminacy of $\tilde{\sigma}$. We note that σ sends the singular points of X to the singular points of X and that the fundamental points of $\tilde{\sigma}$ are on (-2)-curves. We assume that $\tilde{\sigma}$ has m fundamental points (possibly infinitely near). We can eliminate the indeterminacy of σ by m blowings-up. Let $g : \tilde{X}' \to \tilde{X}$ be a composition of their blowings-up. Then we have a birational morphism $\varphi := \tilde{\sigma} \circ g$. For the exceptional curve E of the m-th blowing-up, $\varphi(E)$ is one of (-2)-curves. However, the self-intersection number does not decrease by a birational morphism, so it is a contradiction. Therefore, $\tilde{\sigma}$ is an automorphism on $\tilde{X}$. We define a map $\gamma : \operatorname{Aut} X \to \operatorname{Aut} \tilde{X}$ by this correspondence. Obviously, γ is a group homomorphism.

It is clear that γ is injective. We prove that γ is surjective. For $\tilde{\sigma} \in \operatorname{Aut} \tilde{X}$, there exists a unique automorphism $\sigma \in \operatorname{Aut} X$ with $\mu \circ \tilde{\sigma} = \sigma \circ \mu$ by Lemma 3.2, and we have $\gamma(\sigma) = \tilde{\sigma}$. So γ is surjective. □

Definition 4.2. For a subset $\Gamma_{\tilde{X}} := R_{\tilde{X}} \cup I_{\tilde{X}}$ of $\operatorname{Div} \tilde{X}$, we define the *automorphism group of configuration of* $\Gamma_{\tilde{X}}$, $\operatorname{Aut} \Gamma_{\tilde{X}}$, to be the group of permutations of $\Gamma_{\tilde{X}}$ preserving the relation of intersection. There exists a group homomorphism $\Psi : \operatorname{Aut} \tilde{X} \to \operatorname{Aut} \Gamma_{\tilde{X}}$, which sends $\tilde{\sigma} \in \operatorname{Aut} \tilde{X}$ to a permutation of $\Gamma_{\tilde{X}}$ induced by the pullback $(\tilde{\sigma}^{-1})^* : \operatorname{Div} \tilde{X} \to \operatorname{Div} \tilde{X}$.

Indeed, for any normal singular cubic surface X with no parameters, we can show that Ψ is surjective, so we have an exact sequence

$$(\sharp) \qquad 0 \longrightarrow \operatorname{Ker} \Psi \longrightarrow \operatorname{Aut} \tilde{X} \longrightarrow \operatorname{Aut} \Gamma_{\tilde{X}} \longrightarrow 0.$$

$\operatorname{Ker} \Psi$ is determined by the geometry on a projective plane $\mathbf{P}^2$, and $\operatorname{Aut} \Gamma_{\tilde{X}}$ is a finite group determined by the intersection relations of $\Gamma_{\tilde{X}}$. Hereafter, we determine $\operatorname{Aut} \tilde{X}$ with respect to each type of singularity by investigating the properties of Ψ and the group structures of $\operatorname{Ker} \Psi$ and $\operatorname{Aut} \Gamma_{\tilde{X}}$.

First we recall a well-known fact below. It is useful to determine automorphism groups on normal singular cubic surfaces which have A_1 singularity.

Lemma 4.3. *Let C be a smooth conic on $\mathbf{P}^2$. Then, the subgroup $G := \{\tau \in \operatorname{Aut}\mathbf{P}^2 | \tau(C) = C\}$ of $\operatorname{Aut}\mathbf{P}^2$ is isomorphic to $\operatorname{Aut}\mathbf{P}^1$.*

Proof. See Harris [7], pp. 117–118. □

4.2. **Type A_1A_3.** We use the notation and conditions in subsection 2.2.5. First we determine $\operatorname{Aut}\Gamma_{\tilde{X}}$. For any $s \in \operatorname{Aut}\Gamma_{\tilde{X}}$, we have $s(\tilde{C}_2) = \tilde{C}_2, s(\tilde{E}_i) = \tilde{E}_i$ $(i = 1,2,3,4), s(\tilde{L}_{1,2}) = \tilde{L}_{1,2}, s(\tilde{L}_{5,6}) = \tilde{L}_{5,6}$. On the other hand, $s(\tilde{E}_5)$ is either $\tilde{E}_5$ or $\tilde{E}_6$. If $s(\tilde{E}_5) = \tilde{E}_5$, then $s(\tilde{L}_{1,5}) = \tilde{L}_{1,5}$, $s(\tilde{L}_{1,6}) = \tilde{L}_{1,6}$ and $s(\tilde{E}_6) = \tilde{E}_6$. Hence, $s = \mathrm{id}_{\Gamma_{\tilde{X}}}$. On the other hand, if $s(\tilde{E}_5) = \tilde{E}_6$, then $s(\tilde{L}_{1,5}) = \tilde{L}_{1,6}, s(\tilde{L}_{1,6}) = \tilde{L}_{1,5}$ and $s(\tilde{E}_6) = \tilde{E}_5$. Hence, $s \neq \mathrm{id}_{\Gamma_{\tilde{X}}}$ and $s^2 = \mathrm{id}_{\Gamma_{\tilde{X}}}$. Therefore, we have $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \mathbf{Z}/2\mathbf{Z}$.

Let s_1 be the element of order 2 in $\operatorname{Aut}\Gamma_{\tilde{X}}$. We secondly prove that Ψ is surjective, by finding $\tilde{\sigma} \in \operatorname{Aut}\tilde{X}$ with $\Psi(\tilde{\sigma}) = s_1$. There exists a unique automorphism σ on $\mathbf{P}^2$ with $\sigma(C_2) = C_2, \sigma(P_1) = P_1, \sigma(P_5) = P_6, \sigma(P_6) = P_5$ by Lemma 4.3. σ induces $\tilde{\sigma} \in \operatorname{Aut}\tilde{X}$ such that $\Psi(\tilde{\sigma}) = s_1$ by Lemma 3.1, hence Ψ is surjective.

We finally prove that Ψ is injective, i.e., $\operatorname{Ker}\Psi = \{\mathrm{id}\}$. For $\tilde{\sigma} \in \operatorname{Ker}\Psi$, $\tilde{\sigma}$ induces $\sigma \in \operatorname{Aut}\mathbf{P}^2$ via $\varepsilon : \tilde{X} \to \mathbf{P}^2$ by Lemma 3.3. Then, $\sigma(C_2) = C_2, \sigma(P_1) = P_1, \sigma(P_5) = P_5$ and $\sigma(P_6) = P_6$, so $\sigma = \mathrm{id}_{\mathbf{P}^2}$ by Lemma 4.3, and we have $\tilde{\sigma} = \mathrm{id}_{\tilde{X}}$. Therefore, Ψ is injective. Hence, $\operatorname{Aut}\tilde{X} \simeq \mathbf{Z}/2\mathbf{Z}$.

For $X = \{(x_0x_2 - x_1^2)x_3 - (x_0 - x_1)(-x_1 + x_2)(x_0 - 2x_1 + x_2) = 0\}$ in subsection 3.2.5, $\operatorname{Aut}X$ is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_2 : x_1 : x_0 : x_3) \in X.$$

4.3. **Type $2A_1A_2$.** We use the notation and conditions in subsection 2.2.6. In the same way as above, $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \mathbf{Z}/2\mathbf{Z}$, and Ψ is injective. We prove that Ψ is surjective. Let s_1 be the element of order 2 in $\operatorname{Aut}\Gamma_{\tilde{X}}$. Then, we have $s_1(\tilde{C}_2) = \tilde{E}_4$, $s_1(\tilde{E}_4) = \tilde{C}_2$, $s_1(\tilde{E}_5) = \tilde{E}_5$, $s_1(\tilde{L}_{4,5}) = \tilde{L}_{4,5}$, $s_1(\tilde{L}_{1,4}) = \tilde{E}_3$, $s_1(\tilde{E}_3) = \tilde{L}_{1,4}$, $s_1(\tilde{E}_1) = \tilde{E}_2$, $s_1(\tilde{E}_2) = \tilde{E}_1$, $s_1(\tilde{L}_{1,6}) = \tilde{L}_{1,2}$, $s_1(\tilde{L}_{1,2}) = \tilde{L}_{1,6}$, $s_1(\tilde{E}_6) = \tilde{L}_{4,6}$ and $s_1(\tilde{L}_{4,6}) = \tilde{E}_6$. We have a birational morphism $p : \tilde{X} \to \mathbf{P}^2$ which is a succession of blowings-down of the curves $\tilde{L}_{4,6}$, $\tilde{E}_5$, $\tilde{C}_2$, $\tilde{L}_{1,4}$, $\tilde{E}_1$ and $\tilde{E}_2$, in that order. Let $C' := p(\tilde{E}_4)$, $Q_1 := p(\tilde{L}_{1,4}) = p(\tilde{E}_1) = p(\tilde{E}_2)$, $Q_4 := p(\tilde{C}_2) = p(\tilde{E}_5)$ and $Q_6 := p(\tilde{L}_{4,6})$. Then, C' is a smooth conic which contains three points Q_1, Q_4, Q_6. By Lemma 3.4, there exists a unique automorphism σ on $\mathbf{P}^2$ with $\sigma(C_2) = C'$, $\sigma(P_1) = Q_1$, $\sigma(P_4) = Q_4$ and $\sigma(P_6) = Q_6$. σ induces $\tilde{\sigma} \in \operatorname{Aut}\tilde{X}$ by Lemma 3.1, and we have $\Psi(\tilde{\sigma}) = s_1$. Hence, $\operatorname{Aut}\tilde{X} \simeq \mathbf{Z}/2\mathbf{Z}$.

For $X = \{(x_0x_2 - x_1^2)x_3 - x_1^2(x_0 - x_1) = 0\}$ in subsection 3.2.6, $\operatorname{Aut}X$ is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_3 : x_1 : x_2 : x_0) \in X.$$

4.4. **Type $4A_1$.** We use the notation and conditions in subsection 2.2.7. In the same way as above, $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \Sigma_4$, and Ψ is injective. Let $F_1 := \tilde{C}_2$, $F_2 := \tilde{E}_1$, $F_3 := \tilde{E}_3$ and $F_4 := \tilde{E}_5$. $s \in \operatorname{Aut}\Gamma_{\tilde{X}}$ induces $\bar{s} \in \Sigma_4$ satisfying $s(F_i) = F_{\bar{s}(i)}$ for $i = 1,2,3,4$. For each $i = 2,3,4$, there exists a unique (-1)-curve G_i meeting both $F_{\bar{s}(1)}$ and $F_{\bar{s}(i)}$. We have a birational morphism $p : \tilde{X} \to \mathbf{P}^2$ which is a succession of blowings-down of G_4, $F_{\bar{s}(4)}$, G_3, $F_{\bar{s}(3)}$, G_2 and $F_{\bar{s}(2)}$, in that order. Let $C' := p(F_{\bar{s}(1)})$ and $Q_i := p(G_i) = p(F_{\bar{s}(i)})$ $(i = 2,3,4)$. Then, C' is a smooth conic which contains three points, Q_2, Q_3, Q_4. There exists a unique automorphism

σ on $\mathbf{P}^2$ with $\sigma(C_2) = C'$, $\sigma(P_1) = Q_2$, $\sigma(P_3) = Q_3$ and $\sigma(P_5) = Q_4$. σ induces $\tilde{\sigma} \in \operatorname{Aut}\tilde{X}$ by Lemma 3.1, and $\tilde{\sigma}$ satisfies $\Psi(\tilde{\sigma}) = s$. Hence, $\operatorname{Aut}\tilde{X} \simeq \Sigma_4$.

For $X = \{(x_0x_2 - x_1^2)x_3 - (x_0 - x_1)(x_1 - x_2)x_1 = 0\}$ in subsection 3.2.7, $\operatorname{Aut} X$ is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_0 - x_1 : x_0 - 2x_1 + x_2 : x_3) \in X,$$
$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 - 2x_1 + x_2 : -x_1 + x_2 : x_2 : x_3) \in X,$$

and

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_1 + x_3 : -x_1 + x_2) \in X.$$

4.5. **Type A_1A_4.** We use the notation and conditions in subsection 2.2.8. In the same way as above, we have $\operatorname{Aut}\Gamma_{\tilde{X}} = \{\mathrm{id}\}$. So $\operatorname{Aut}\tilde{X}$ is isomorphic to $\operatorname{Ker}\Psi$. $\tilde{\sigma} \in \operatorname{Ker}\Psi$ induces an automorphism σ on $\mathbf{P}^2$ with $\sigma(P_1) = P_1$ and $\sigma(P_6) = P_6$, and we have a group homomorphism $\psi : \operatorname{Ker}\Psi \to \operatorname{Aut}\mathbf{P}^2$ which is injective. It is easy to see that $\operatorname{Im}\psi = \{\sigma \in \operatorname{Aut}\mathbf{P}^2 | \sigma(C_2) = C_2, \sigma(P_1) = P_1, \sigma(P_6) = P_6\}$. This subgroup of $\operatorname{Aut}\mathbf{P}^2$ is isomorphic to a subgroup of $\operatorname{Aut}\mathbf{P}^1$ which consists of all automorphisms leaving two points fixed, and it is isomorphic to $\mathbf{C}^\times$. Hence, $\operatorname{Aut}\tilde{X} \simeq \mathbf{C}^\times$.

For $X = \{(x_0x_2 - x_1^2)x_3 - x_0^2x_1 = 0\}$ in subsection 3.2.8, any element of $\operatorname{Aut} X$ is written by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (ax_0 : a^2x_1 : a^3x_2 : x_3) \in X,$$

where $a \in \mathbf{C}^\times$.

4.6. **Type $2A_1A_3$.** We use the notation and conditions in subsection 2.2.9. Similarly, we have $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \mathbf{Z}/2\mathbf{Z}$. Let s_1 be an element of order 2 in $\operatorname{Aut}\Gamma_{\tilde{X}}$. In the same way as above, we get $\tilde{\sigma} \in \operatorname{Aut}\tilde{X}$ with $\Psi(\tilde{\sigma}) = s_1$, so Ψ is surjective. Therefore, we have the short exact sequence of groups ($\sharp$). For $X = \{(x_0x_2 - x_1^2)x_3 - x_0x_1^2 = 0\}$ in subsection 3.2.9, $\tau \in \operatorname{Aut} X$ induced by $\tilde{\sigma}$ is written by

$$\tau : X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_3 : x_1 : x_2 : x_0) \in X.$$

We define a group homomorphism $\theta : \operatorname{Aut}\Gamma_{\tilde{X}} \to \operatorname{Aut}\tilde{X}$ by $\theta(\mathrm{id}) = \mathrm{id}$ and $\theta(s_1) = \tilde{\sigma}$. Then, the exact sequence ($\sharp$) splits by a section θ. In the same way as section 4.5, we have $\operatorname{Ker}\Psi \simeq \mathbf{C}^\times$. Hence, $\operatorname{Aut}\tilde{X} \simeq \mathbf{C}^\times \rtimes \mathbf{Z}/2\mathbf{Z}$, and $\operatorname{Aut} X$ is generated by τ and

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : ax_1 : a^2x_2 : x_3) \in X,$$

where $a \in \mathbf{C}^\times$.

4.7. **Type A_12A_2.** We use the notation and conditions in subsection 2.2.10. In the same way as section 4.6, we have $\operatorname{Aut}\tilde{X} \simeq \mathbf{C}^\times \rtimes \mathbf{Z}/2\mathbf{Z}$. For $X = \{(x_0x_2 - x_1^2)x_3 - x_1^3 = 0\}$ in subsection 3.2.10, $\operatorname{Aut} X$ is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_2 : x_1 : x_0 : x_3) \in X$$

and

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (a^2x_0 : ax_1 : x_2 : ax_3) \in X,$$

where $a \in \mathbf{C}^\times$.

4.8. **Type A_1A_5.** We use the notation and conditions in subsection 2.2.11. Clearly, $\mathrm{Aut}\,\Gamma_{\tilde{X}} = \{\mathrm{id}\}$, so we have

$$\mathrm{Aut}\,\tilde{X} \simeq \mathrm{Ker}\,\Psi \simeq \{\sigma \in \mathrm{Aut}\,\mathbf{P}^1 | \sigma(P) = P\},$$

where $P \in \mathbf{P}^1$ is a point. The right hand is isomorphic to

$$H := \left\{ \begin{pmatrix} 1 & 0 \\ s & t \end{pmatrix} \in GL_2(\mathbf{C}) \right\},$$

where $GL_2(\mathbf{C})$ is the general linear group of dimension 2. We have a natural split exact sequence for H, so $H \simeq \mathbf{C} \rtimes \mathbf{C}^\times$. Hence, $\mathrm{Aut}\,\tilde{X} \simeq \mathbf{C} \rtimes \mathbf{C}^\times$. For $X = \{(x_0x_2 - x_1^2)x_3 - x_0^3 = 0\}$ in subsection 3.2.11, $\mathrm{Aut}\,X$ is generated by

$$\begin{aligned} X \ni & (x_0 : x_1 : x_2 : x_3) \\ & \mapsto (a^2x_0 : a^2bx_0 + a^3x_1 : a^2b^2x_0 + 2a^3bx_1 + a^4x_2 : x_3) \in X, \end{aligned}$$

where $a, b \in \mathbf{C}$ and $a \neq 0$.

4.9. **Type $3A_2$.** We use the notation and conditions in subsection 2.3.6. Clearly, $\mathrm{Aut}\,\Gamma_{\tilde{X}} \simeq \Sigma_3$ and

$$\mathrm{Ker}\,\Psi \simeq \{\sigma \in \mathrm{Aut}\,\mathbf{P}^2 | \sigma(Q_i) = Q_i \ (i = 1, 2, 3)\},$$

where Q_1, Q_2, Q_3 are points and are not collinear. The right hand is isomorphic to $(\mathbf{C}^\times)^2$. For $X = \{x_0x_1x_3 - x_2^3 = 0\}$ in subsection 3.3.6,

$$\begin{aligned} \tau_1 : X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_1 : x_0 : x_2 : x_3) \in X, \\ \tau_2 : X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_3 : x_2 : x_1) \in X \end{aligned}$$

are elements of $\mathrm{Aut}\,X$. Let $\tilde{\sigma}_i$ be the elements of $\mathrm{Aut}\,\tilde{X}$ corresponding to τ_i, and $s_i := \Psi(\tilde{\sigma}_i)$. Then s_1, s_2 generate $\mathrm{Aut}\,\Gamma_{\tilde{X}}$, so Ψ is surjective. On the other hand, we can define a natural group homomorphism $\theta : \mathrm{Aut}\,\Gamma_{\tilde{X}} \to \mathrm{Aut}\,\tilde{X}$, which is injective. So the exact sequence ($\sharp$) splits by a section θ. Hence, $\mathrm{Aut}\,X \simeq (\mathbf{C}^\times)^2 \rtimes \Sigma_3$. $\mathrm{Aut}\,X$ is generated by τ_1, τ_2 and

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (a^2bx_0 : ab^2x_1 : abx_2 : x_3) \in X,$$

where $a, b \in \mathbf{C}^\times$.

4.10. **Type A_4.** We use the notation and conditions in subsection 2.3.10. Clearly, $\mathrm{Aut}\,\Gamma_{\tilde{X}} \simeq \mathbf{Z}/2\mathbf{Z}$, and Ψ is surjective. So we have the short exact sequence ($\sharp$). We can put $C_2 = \{f_2 = x_0x_1 = 0\}, L = \{x_0 = 0\}, L' = \{x_1 = 0\}, C_3 = \{f_3 = x_0^2x_2 + x_1^3 - x_1x_2^2 = 0\}$ and $X = \{x_3f_2 - f_3 = 0\}$ by subsection 3.3.10. For

$$\iota : X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : ix_1 : -ix_2 : -x_3) \in X,$$

where $i = \sqrt{-1}$, $\Psi(\iota)$ is an element of order 2 of $\mathrm{Aut}\,\Gamma_{\tilde{X}}$. On the other hand, $\mathrm{Ker}\,\Psi$ is isomorphic to a subgroup of $\mathrm{Aut}\,\mathbf{P}^2$ which fixes $L = L_{1,2,4}$, $L' = L_{3,4,5}$, $P_1 = (0 : 1 : 1)$, $P_2 = (0 : 1 : -1)$, $P_3 = (1 : 0 : 0)$ and P_6. We lift up automorphisms on $\mathbf{P}^2$ to $\tilde{X}$, and apply conditions above. We have $\mathrm{Ker}\,\Psi \simeq \mathbf{Z}/2\mathbf{Z}$, and it is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : -x_1 : -x_2 : x_3) \in X,$$

via $\gamma : \mathrm{Aut}\,X \simeq \mathrm{Aut}\,\tilde{X}$. Therefore, $\mathrm{Aut}\,X$ is a finite group of order 4. Since ι is an element of order 4 in $\mathrm{Aut}\,X$, we have $\mathrm{Aut}\,X \simeq \mathbf{Z}/4\mathbf{Z}$.

4.11. **Type A_5.** We use the notation and conditions in subsection 2.3.12. It is easy to see that $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \mathbf{Z}/2\mathbf{Z}$. We can put $C_2 = \{f_2 = x_0x_1 = 0\}, L = \{x_0 = 0\}, L' = \{x_1 = 0\}, C_3 = \{f_3 = x_0^3 + x_1^3 - x_1x_2^2 = 0\}$ and $X = \{x_3f_2 - f_3 = 0\}$ by subsection 3.3.12. For

$$\tau : X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : -x_2 : x_3) \in X,$$

$\Psi(\tau)$ is an element of order 2 in $\operatorname{Aut}\Gamma_{\tilde{X}}$, so we have the short exact sequence ($\sharp$) which splits. On the other hand, $\operatorname{Ker}\Psi$ is isomorphic to a subgroup of $\operatorname{Aut}\mathbf{P}^2$ which fixes $L = L_{1,2,3}$, $L' = L_{3,4,5}$, $P_1 = (0 : 1 : 1)$, $P_2 = (0 : 1 : -1)$ and P_6. In the same way as above, $\operatorname{Ker}\Psi$ is generated by

$$\sigma : X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \omega^i x_1 : ax_0 + \omega^i x_2 : -a^2x_0 - 2\omega^i a x_2 + \omega^{-i}x_3) \in X,$$

via $\gamma : \operatorname{Aut}X \simeq \operatorname{Aut}\tilde{X}$, where $a \in \mathbf{C}, \omega^3 = 1$ $(\omega \neq 1), i = 0, 1, 2$. It is isomorphic to $\mathbf{C} \rtimes \mathbf{Z}/3\mathbf{Z}$. Hence, $\operatorname{Aut}X \simeq (\mathbf{C} \rtimes \mathbf{Z}/3\mathbf{Z}) \rtimes \mathbf{Z}/2\mathbf{Z}$.

4.12. **Type D_4.** We use the notation and conditions in subsection 2.4.1. It is easy to see that $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \Sigma_3$. There are two normal forms in this case (cf. subsection 3.4.1).

(1) In the case of $C_2 = \{f_2 = x_0^2 = 0\}$ and $C_3 = \{f_3 = x_1^3 + x_2^3 = 0\}$, $\operatorname{Ker}\Psi$ is isomorphic to a subgroup of $\operatorname{Aut}\mathbf{P}^2$ which fixes P_1, P_3, P_5, $L_{1,2}$, $L_{3,4}$ and $L_{5,6}$, where $L_{1,2}$, $L_{3,4}$ and $L_{5,6}$ meet at a point. Hence, $\operatorname{Ker}\Psi \simeq \mathbf{C}^\times$. Similarly, we have the exact sequence ($\sharp$) which splits, so we have $\operatorname{Aut}X \simeq \mathbf{C}^\times \rtimes \Sigma_3$, and it is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_2 : x_1 : x_3) \in X,$$
$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \omega x_2 : \omega^2 x_1 : x_3) \in X$$

and

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (a^3x_0 : a^2x_1 : a^2x_2 : x_3) \in X,$$

where $a \in \mathbf{C}^\times$.

(2) In the case of $C_2 = \{f_2 = x_0^2 = 0\}$ and $C_3 = \{f_3 = x_1^3 + x_2^3 + x_0x_1x_2 = 0\}$, we have $\operatorname{Ker}\Psi = \{\mathrm{id}\}$. Since automorphisms

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_2 : x_1 : x_3) \in X$$

and

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \omega x_2 : \omega^2 x_1 : x_3) \in X$$

correspond to generators of $\operatorname{Aut}\Gamma_{\tilde{X}} \simeq \Sigma_3$, Ψ is surjective. Hence, $\operatorname{Aut}X \simeq \Sigma_3$.

4.13. **Type D_5.** We use the notation and conditions in subsection 2.4.2. We can put $C_2 = \{x_0^2 = 0\}, L = \{x_0 = 0\}, C_3 = \{x_0x_2^2 + x_1^2x_2 = 0\}$ and $X = \{x_3f_2 - f_3 = 0\}$ by subsection 3.4.2. Clearly, $\operatorname{Aut}\Gamma_{\tilde{X}} = \{\mathrm{id}\}$, and $\operatorname{Ker}\Psi$ is isomorphic to a subgroup of $\operatorname{Aut}\mathbf{P}^2$ which fixes $L = L_{1,2,5}$, $L_{5,6}$, $P_1 = (0 : 0 : 1)$, P_3 and P_4. In the same way as above, we have $\operatorname{Ker}\Psi \simeq \mathbf{C}^\times$. Hence, $\operatorname{Aut}X \simeq \mathbf{C}^\times$, and it is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (a^4x_0 : a^3x_1 : a^2x_2 : x_3) \in X,$$

where $a \in \mathbf{C}^\times$.

4.14. **Type E_6.** We use the notation and conditions in subsection 2.4.3. We can put $C_2 = \{x_0^2 = 0\}, L = \{x_0 = 0\}, C_3 = \{x_0 x_2^2 + x_1^3 = 0\}$ and $X = \{x_3 f_2 - f_3 = 0\}$ by subsection 3.4.3. Clearly, $\mathrm{Aut}\,\Gamma_{\tilde{X}} = \{\mathrm{id}\}$, and $\mathrm{Ker}\,\Psi$ is isomorphic to a subgroup of $\mathrm{Aut}\,\mathbf{P}^2$ which fixes $L = L_{1,2,3}$, $P_1 = (0:0:1)$, P_4, P_5 and P_6. In the same way as above, we have $\mathrm{Ker}\,\Psi \simeq \mathbf{C} \rtimes \mathbf{C}^\times$. Hence, $\mathrm{Aut}\,X \simeq \mathbf{C} \rtimes \mathbf{C}^\times$, and it is generated by

$$X \ni (x_0 : x_1 : x_2 : x_3) \mapsto (a^6 x_0 : a^4 x_1 : a^3 b x_0 + a^3 x_2 : b^2 x_0 + 2b x_2 + x_3) \in X,$$

where $a \in \mathbf{C}^\times, b \in \mathbf{C}$.

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