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ON STABLE CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{S}^2 \times \mathbb{R}$ AND $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. We study the stability of immersed compact constant mean curvature (CMC) surfaces without boundary in some Riemannian 3-manifolds, in particular the Riemannian product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. We prove that rotational CMC spheres in $\mathbb{H}^2 \times \mathbb{R}$ are all stable, whereas in $\mathbb{S}^2 \times \mathbb{R}$ there exists some value $H_0 \approx 0.18$ such that rotational CMC spheres are stable for $H \geq H_0$ and unstable for $0 < H < H_0$. We show that a compact stable immersed CMC surface in $\mathbb{S}^2 \times \mathbb{R}$ is either a finite union of horizontal slices or a rotational sphere. In the more general case of an ambient manifold which is a simply connected conformally flat 3-manifold with nonnegative Ricci curvature we show that a closed stable immersed CMC surface is either a sphere or an embedded torus. Under the weaker assumption that the scalar curvature is nonnegative, we prove that a closed stable immersed CMC surface has genus at most three. In the case of $\mathbb{H}^2 \times \mathbb{R}$ we show that a closed stable immersed CMC surface is a rotational sphere if it has mean curvature $H \geq 1/\sqrt{2}$ and that it has genus at most one if $1/\sqrt{3} < H < 1/\sqrt{2}$ and genus at most two if $H = 1/\sqrt{3}$.

1. Introduction and preliminaries

Compact surfaces of constant mean curvature (CMC) in Riemannian 3-manifolds are critical points of a variational problem: minimizing the area under a volume constraint. It is therefore natural to study the *stable* ones, that is, those minimizing the area functional up to second order for *volume-preserving* deformations [3]. Let Σ be an immersed compact surface without boundary and with constant mean curvature H in a Riemannian 3-manifold $(M, \langle , \rangle)$. Suppose Σ is two-sided, i.e. there is a global unit normal field N along it. This condition is always satisfied when $H \neq 0$, and is equivalent to the orientability of Σ when the ambient manifold M is orientable. This will be the case in this article since we will consider simply connected ambient manifolds. Also we can always choose the orientation on the surface so that its (constant) mean curvature is nonnegative; therefore we can always suppose $H \geq 0$.

We consider on Σ the following quadratic form:

$$Q(u,v) = \int_\Sigma \langle \nabla u, \nabla v \rangle - (|\sigma|^2 + \mathrm{Ric}(N))uv\, dA = -\int_\Sigma uLv\, dA, \quad u, v \in \mathcal{C}^\infty(\Sigma),$$

where ∇u stands for the gradient of u, σ is the second fundamental form of the immersion and $\mathrm{Ric}(N)$ denotes the Ricci curvature of M evaluated on the field N.

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The linear operator $L = \Delta + |\sigma|^2 + \mathrm{Ric}(N)$ is the Jacobi operator of the surface, Δ being the Laplacian on Σ. The stability condition means that:

$$Q(u,u) \geq 0, \quad \text{for any } u \in \mathcal{C}^\infty(\Sigma) \quad \text{satisfying} \int_\Sigma u\, dA = 0. \tag{1.1}$$

See [3] for the details.

The Jacobi operator L is also the linearized operator of the mean curvature functional. This means that if $X : \Sigma \longrightarrow (M, \langle , \rangle)$ denotes a CMC immersion as above, then for any smooth deformation X_t through immersions of $X = X_0$, the derivative of the mean curvature functional at $t = 0$ is given by:

$$2\frac{\partial H_t}{\partial t}|_{t=0} = L(\langle \frac{\partial X_t}{\partial t}|_{t=0}, N\rangle). \tag{1.2}$$

Stability of CMC surfaces is an important notion in solving the *isoperimetric problem* in Riemannian manifolds, as the boundary of an *isoperimetric region* is a stable CMC surface. Recent advances on the isoperimetric problem can be found in the paper by Ros [14].

Stability of CMC surfaces has attracted much attention. The first results in this direction are due to Barbosa and do Carmo [2] and Barbosa, do Carmo and Eschenburg [3]. They established that in simply connected space forms, the round spheres are the only stable immersed compact CMC surfaces, a result which holds in all dimensions. In this paper, we will be concerned with stability of compact surfaces in Riemannian 3-manifolds, mainly the Riemannian product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. The theory of CMC surfaces in these spaces is currently an active research field. Abresch and Rosenberg [1] have proved that an immersed CMC sphere in any of these spaces is necessarily rotationally invariant. The rotational CMC spheres are embedded and form, in both spaces, a 1-parameter family (up to ambient isometries) parameterized by their mean curvature. They first appeared in studies of the isoperimetric problem by Hsiang and Hsiang in $\mathbb{H}^2 \times \mathbb{R}$ [7] and by Pedrosa in $\mathbb{S}^2 \times \mathbb{R}$ [12]. In $\mathbb{H}^2 \times \mathbb{R}$, the solutions to the isoperimetric problem are precisely the domains bounded by the rotational CMC spheres. In $\mathbb{S}^2 \times \mathbb{R}$, Pedrosa has shown that the rotational CMC spheres bound isoperimetric domains only when they have mean curvature $H \geq H_1 \approx 0.33$.

In section 2, we establish our first result. It concerns the stability of rotational CMC spheres (Theorem 2.2). In $\mathbb{H}^2 \times \mathbb{R}$, we prove they are all stable. In $\mathbb{S}^2 \times \mathbb{R}$, our study reveals a rather unexpected phenomenon: we show that rotational CMC spheres are stable if and only if they have mean curvature $H \geq H_0 \approx 0.18$. The value H_0 is the (unique) one at which the area of these surfaces—and also the enclosed volume—reaches its maximum, and these spheres are stable (resp. unstable) as long as the area—and also the enclosed volume—is decreasing (resp. increasing) as a function of the mean curvature. In particular, the rotational CMC spheres are still stable when their mean curvature lies in the interval $[H_0, H_1[$, although in this case they do not bound isoperimetric domains.

In section 3, we obtain some restrictions on the topology of stable CMC surfaces in some class of Riemannian 3-manifolds under positive curvature assumptions. Improving on previous results by several authors, Ros [15] proved that an immersed closed stable two-sided CMC surface in a 3-manifold with nonnegative Ricci curvature has genus at most 3. The result is sharp and cannot be improved in general since Ross [19] showed that the classical Schwarz-P surface, of genus 3 in the flat

cubic lattice, is stable. However, one can hope to improve this bound in some classes of manifolds. Recently, Ros [16] considered the case of a product manifold $M \times \mathbb{R}$ where M is a complete, orientable surface with nonnegative Gaussian curvature and proved that if Σ is a compact orientable stable and *embedded* CMC surface in $M \times \mathbb{R}$, then either M is compact and Σ a finite union of horizontal slices, or Σ is a connected surface with genus ≤ 2. We consider here the case of an ambient manifold which is simply connected and conformally flat with nonnegative Ricci curvature. We prove (Theorem 3.3) that a compact orientable stable CMC surface in such a manifold is either a finite union of totally geodesic surfaces, a sphere or an embedded torus. In particular, if we assume that the Ricci curvature is positive, our result implies that a compact isoperimetric region (when it exists) is bounded by either a sphere or a torus. Under the weaker assumption that the scalar curvature is nonnegative we prove that if such a surface is connected, then its genus is ≤ 3.

In section 3, we specialize to the case where the ambient space is $\mathbb{S}^2 \times \mathbb{R}$ and show that a closed and stable CMC surface is necessarily a rotational sphere.

In section 4, we address the problem in $\mathbb{H}^2 \times \mathbb{R}$. We obtain bounds on the genus of a stable closed CMC surface under various assumptions on the mean curvature. In particular we prove that it is a rotational sphere if it has mean curvature $\geq 1/\sqrt{2}$ (Theorem 5.1). We believe that this restriction on the mean curvature is unnecessary.

2. Stability of rotational CMC spheres in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

The rotational CMC spheres in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been studied in the works of Hsiang and Hsiang [7], Pedrosa [12] and Abresch-Rosenberg [1]. In $\mathbb{S}^2 \times \mathbb{R}$, for any value $H > 0$ there exists, up to ambient isometries, a unique rotational CMC sphere of mean curvature H. In $\mathbb{H}^2 \times \mathbb{R}$, such spheres exist only for $H > 1/2$. Moreover, for any $H > 1/2$, there exists a unique, up to ambient isometries, rotational CMC sphere of mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$. If we fix the *center of gravity* of those surfaces at some fixed point $\{p\} \times \{0\} \in \mathbb{S}^2 \times \mathbb{R}$ (resp. $\mathbb{H}^2 \times \mathbb{R}$), then they form a smooth 1-parameter family Σ_H parametrized by the mean curvature $H \in (0, +\infty)$ (resp. $H \in (\frac{1}{2}, +\infty)$). The surfaces Σ_H are embedded and invariant under the reflection τ through the horizontal slice $\mathbb{S}^2 \times \{0\}$ (resp. $\mathbb{H}^2 \times \{0\}$). Furthermore the generating curve of any of these surfaces is a convex curve in a vertical geodesic plane.

We will need the following stability criterion due to Koiso [9].

Theorem 2.1 ([9], Theorem 1.3, statement (III-B))**.** *Let Σ be a smooth compact immersed and two-sided CMC surface in a Riemannian 3-manifold M. Denote by λ_1 and λ_2 respectively the first and second eigenvalues of the Jacobi operator L on Σ. Suppose the following conditions are satisfied:*

(i) $\lambda_1 < 0$ *and* $\lambda_2 = 0$,

(ii) $\int_\Sigma g \, dA = 0$ *for any eigenfunction g associated to λ_2.*

In this case, denoting by E the eigenspace associated to $\lambda_2 = 0$, there exists a uniquely determined smooth function $u \in E^\perp$ (orthogonal to E in the L^2 sense) which satisfies $Lu = 1$. Then:

(a) *if $\int_\Sigma u \, dA \geq 0$, then Σ is stable;*

(b) *if $\int_\Sigma u \, dA < 0$, then Σ is unstable.*

Actually Koiso states the theorem for compact CMC surfaces with boundary in $\mathbb{R}^3$, but the result readily extends to compact two-sided CMC surfaces with or without boundary in any ambient Riemannian 3-manifold. The following result solves the problem of stability of rotational CMC spheres in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

Theorem 2.2. *Rotational CMC spheres in $\mathbb{H}^2 \times \mathbb{R}$ are stable.*

There exists some value $H_0 \approx 0.18$ such that rotational CMC spheres in $\mathbb{S}^2 \times \mathbb{R}$ are stable if they have mean curvature $H \geq H_0$ and unstable if $0 < H < H_0$. Moreover a finite union of horizontal slices in $\mathbb{S}^2 \times \mathbb{R}$ is stable.

It is interesting to point out (cf. the proof below) that this critical value H_0 is the mean curvature at which the area of the rotational CMC spheres reaches its maximum (it is also the value where the volume enclosed by these spheres is maximal). Pedrosa [12] has proved that, in $\mathbb{S}^2 \times \mathbb{R}$, the rotational CMC spheres of mean curvature $H \geq H_1 \approx 0.33$ bound isoperimetric domains. Our result shows that the rotational CMC spheres are still stable for $H_0 \approx 0.18 \leq H \leq H_1 \approx 0.33$ although they do not bound isoperimetric domains.

Proof. We will apply Koiso's criterion (Theorem 2.1). With the notation above, consider one of the rotational CMC spheres $\Sigma_{H_\star}$, for some $H_\star > 0$, in $\mathbb{S}^2 \times \mathbb{R}$ (resp. in $\mathbb{H}^2 \times \mathbb{R}$). $\Sigma_{H_\star}$ is therefore rotationally invariant around the axis $\{p\} \times \mathbb{R}$ and has center of gravity at the point $\{p\} \times \{0\} \in \mathbb{S}^2 \times \mathbb{R}$ (resp. $\{p\} \times \{0\} \in \mathbb{H}^2 \times \mathbb{R}$). We will first check that conditions (i) and (ii) of Theorem 2.1 are satisfied.

(i) First note that 0 is an eigenvalue of the Jacobi operator L on $\Sigma_{H_\star}$. Indeed, if X is any Killing field of $\mathbb{S}^2 \times \mathbb{R}$ (resp. of $\mathbb{H}^2 \times \mathbb{R}$), then the mean curvature of the surface $\Sigma_{H_\star}$ remains constant and equal to $H_\star$ under deformation by the flow of X and so, by (1.2), the function $v = \langle X, N \rangle$ satisfies $Lv = 0$. Recall that the isometry group of $\mathbb{S}^2 \times \mathbb{R}$ (resp. of $\mathbb{H}^2 \times \mathbb{R}$) has dimension 4 and so we have 3 independent Killing fields which do not vanish on $\Sigma_{H_\star}$. These fields induce on $\Sigma_{H_\star}$ three linearly independent functions in the kernel of L. These functions are indeed independent since otherwise the rotational surface $\Sigma_{H_\star}$ would be invariant under a second 1-parameter group of isometries of $\mathbb{S}^2 \times \mathbb{R}$ (resp. of $\mathbb{H}^2 \times \mathbb{R}$), which is easily seen to be impossible. Therefore the dimension of $E = kerL$ is at least 3. As the first eigenspace of L is one dimensional, we conclude that $\lambda_1 < 0$.

Let Ω be a regular domain in $\Sigma_{H_\star}$. Recall that $\lambda \in \mathbb{R}$ is an eigenvalue of the operator L for the Dirichlet problem on Ω if there is a nonzero function $w \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ such that $Lw + \lambda w = 0$ in Ω and $w = 0$ on $\partial\Omega$. The set of these eigenvalues is a discrete sequence converging to $+\infty$. A nonzero eigenfunction associated to the first (i.e. smallest) eigenvalue does not vanish in the interior of Ω. We will repeatedly use the standard fact that only the first eigenvalue has this property.

Given a point $q \in \mathbb{S}^2$ (resp. $q \in \mathbb{H}^2$), we denote by X_q the Killing field associated to the 1-parameter group of rotations around the vertical axis $\{q\} \times \mathbb{R}$. Consider a vertical geodesic plane Π containing the axis $\{p\} \times \mathbb{R}$, that is, Π is a product $\Gamma \times \mathbb{R} \subset \mathbb{S}^2 \times \mathbb{R}$ (resp. $\Gamma \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$) where Γ is a geodesic of $\mathbb{S}^2$ (resp. of $\mathbb{H}^2$) containing p. The plane Π is a plane of symmetry of $\Sigma_{H_\star}$ and separates it into two components $\Sigma^1_{H_\star}(\Pi)$ and $\Sigma^2_{H_\star}(\Pi)$. Take on Γ a point $q \in \mathbb{S}^2$ (resp. $q \in \mathbb{H}^2$) that is distinct from p and from its antipodal point (resp. distinct from p). Then the function $\langle X_q, N \rangle$ is a nonzero function in the kernel of L and vanishes precisely on the curve $\Pi \cap \Sigma_{H_\star}$. In particular the first eigenvalue of L for the Dirichlet problem on each of the components $\Sigma^1_{H_\star}(\Pi)$ and $\Sigma^2_{H_\star}(\Pi)$ is equal to 0. In the same way,

denoting by t the last coordinate in the product $\mathbb{S}^2 \times \mathbb{R}$ (resp. in $\mathbb{H}^2 \times \mathbb{R}$), the Killing field $\frac{\partial}{\partial t}$ induces a nonzero function $\langle \frac{\partial}{\partial t}, N\rangle$ in E whose zero set is the circle $\mathbb{S}^2 \times \{0\} \cap \Sigma_{H_\star}$ (resp. $\mathbb{H}^2 \times \{0\} \cap \Sigma_{H_\star}$). This circle separates the surface into 2 components $\Sigma^+_{H_\star}$ and $\Sigma^-_{H_\star}$ which are isometric by the reflection τ. Again we conclude that 0 is the first eigenvalue of L for the Dirichlet problem on each of the components $\Sigma^+_{H_\star}$ and $\Sigma^-_{H_\star}$.

We now prove that $\lambda_2 = 0$. Assume for contradiction that this is not the case, that is, $\lambda_2 < 0$. Denote by E_{λ_2} the eigenspace associated to λ_2. The isometric involution τ on $\Sigma_{H_\star}$ (more precisely the restriction of τ to $\Sigma_{H_\star}$) induces an involution on E_{λ_2} and so this eigenspace splits into a direct sum of the linear spaces of invariant and anti-invariant functions under the action of τ:

$$E_{\lambda_2} = \{v \in E_{\lambda_2},\, v \circ \tau = v\} \oplus \{v \in E_{\lambda_2},\, v \circ \tau = -v\}.$$

Now any anti-invariant eigenfunction vanishes on the circle $\mathbb{S}^2 \times \{0\} \cap \Sigma$. Moreover by Courant's nodal domain theorem ([4]) any nonzero eigenfunction in E_{λ_2} has exactly 2 nodal domains. Suppose there exists a nonzero anti-invariant eigenfunction in E_{λ_2}; its nodal domains are then precisely $\Sigma^+_{H_\star}$ and $\Sigma^-_{H_\star}$. It follows that λ_2 is the first eigenvalue of L for the Dirichlet problem on $\Sigma^+_{H_\star}$ (and on $\Sigma^-_{H_\star}$), and this is a contradiction since we have seen that 0 is the first eigenvalue for this problem. So all functions in E_{λ_2} are invariant under the action of τ.

Consider now as above a vertical plane $\Pi = \Gamma \times \mathbb{R}$ containing the axis $\{p\} \times \mathbb{R}$. The surface $\Sigma_{H_\star}$ is invariant under the symmetry σ_Π through Π and so as before E_{λ_2} splits into a direct sum of the spaces of invariant and anti-invariant functions with respect to the induced action of σ_Π. Again if we suppose there is a nonzero anti-invariant eigenfunction, then it has to vanish on the curve $\Pi \cap \Sigma_{H_\star}$ and as before this implies that λ_2 is an eigenvalue of L for the Dirichlet problem on $\Sigma^1_{H_\star}(\Pi)$ (and on $\Sigma^2_{H_\star}(\Pi)$), which is a contradiction. So all the functions in E_{λ_2} are invariant with respect to reflection through Π. As Π was any vertical plane containing the axis $\{p\} \times \mathbb{R}$, we conclude that each function in E_{λ_2} is invariant under rotations through this axis. In particular the zero set of such an eigenfunction is also rotationally invariant. Courant's nodal domain theorem then shows that this zero set consists of exactly one circle. As any eigenfunction in E_{λ_2} is invariant under the action of τ, this zero set is precisely the circle $\mathbb{S}^2 \times \{0\} \cap \Sigma_{H_\star}$ (resp. $\mathbb{H}^2 \times \{0\} \cap \Sigma_{H_\star}$). This is again a contradiction. Consequently $\lambda_2 = 0$.

(ii) We can view the family Σ_H, $H \in (0, \infty)$ (resp. $H \in (\frac{1}{2}, +\infty)$), as given by a family of embeddings $X_{H_\star}$ of the surface $\Sigma_{H_\star}$. Let Y be the associated deformation vector field on $\Sigma_{H_\star}$ and set $v = \langle Y, N\rangle$. Then by (1.2) we have $Lv = 2$. As L is a self-adjoint operator, it follows that $\int_{\Sigma_{H_\star}} g\, dA = 0$, for any $g \in E = kerL$. Although we will not need it, it is interesting to note that E has dimension 3 and is generated by the functions induced by the Killing fields of the ambient space. Indeed, by a result of Cheng [4], E has dimension at most 3 and so coincides with the space induced by the Killing fields since we have noted above that this latter space has dimension 3.

Write $\frac{1}{2}v = u + h$ with $u \in E^\perp$ and $h \in E$. Then $Lu = 1$ and $\int_{\Sigma_{H_\star}} u\, dA = \frac{1}{2}\int_{\Sigma_{H_\star}} v\, dA$. So to conclude the proof, it remains to study the sign of $\int_{\Sigma_{H_\star}} v\, dA$.

Let $A(H)$ denote the area of Σ_H. Then by the first variation formula for the area, $\frac{dA}{dH}(H_\star) = -2H_\star \int_{\Sigma_{H_\star}} v\, dA$. Therefore the sign of $\int_{\Sigma_{H_\star}} v\, dA$ is the opposite of that of $\frac{dA}{dH}(H_\star)$.

In the case of $\mathbb{S}^2 \times \mathbb{R}$ we have an explicit expression for $A(H)$, derived by Pedrosa [12], namely:

$$A(H) = 8\pi \left[\frac{1}{4H^2+1} + \frac{4H^2}{(4H^2+1)^{3/2}} \tanh^{-1} \frac{1}{\sqrt{4H^2+1}} \right]$$

(Pedrosa takes as definition of the mean curvature the trace of the second fundamental form, while we take half of it). A study of the derivative function $\frac{dA}{dH}$ shows that it has a unique zero on $(0, \infty)$ at some value $H_0 \approx 0.18$ and is positive on $(0, H_0)$ and negative on (H_0, ∞). Therefore the surfaces Σ_H satisfy condition (a) of Theorem 2.1 for $H \geq H_0$ and condition (b) for $0 < H < H_0$. This concludes the proof for the case of $\mathbb{S}^2 \times \mathbb{R}$.

In $\mathbb{H}^2 \times \mathbb{R}$ the expression for the area $A(H)$ was derived by Hsiang and Hsiang [7]. In this case, one has:

$$A(H) = 8\pi \left[\frac{1}{4H^2-1} + \frac{4H^2}{(4H^2-1)^{3/2}} \tan^{-1} \frac{1}{\sqrt{4H^2-1}} \right].$$

It can be checked that the derivative function $\frac{dA}{dH}$ is always negative. The conclusion follows from case (a) of Theorem 2.1.

The last statement is quite easy, and in fact a finite union of horizontal slices in $\mathbb{S}^2 \times \mathbb{R}$ is stable in a strong sense. Indeed, as a horizontal slice is totally geodesic and the Ricci curvature evaluated on its normal vanishes, its Jacobi operator is just the Laplacian on the sphere. Therefore the stability condition (1.1) is satisfied for any smooth function defined on a finite union of horizontal slices, without any condition on its integral. □

Remark 2.3. The stability of rotational CMC spheres in $\mathbb{H}^2 \times \mathbb{R}$ can be seen in an alternate way. Actually they bound isoperimetric domains [7]. We think, however, that the direct argument we gave here is interesting in its own right.

3. Stable CMC surfaces in simply connected conformally flat manifolds

We obtain in this section some control on the topology of stable compact and orientable CMC surfaces in simply connected conformally flat manifolds. Recall that a Riemannian manifold is said to be *conformally flat* if each of its points admits an open neighborhood which is conformally diffeormorphic to an open set of the Euclidean space. We will need a first result.

Proposition 3.1. *Let M be a simply connected conformally flat Riemannian 3-manifold and Σ a compact orientable surface without boundary immersed in M. Denote by K_s the sectional curvature of M evaluated on the tangent plane to Σ, by H the mean curvature of Σ and by dA its area element. Then*

$$\int_\Sigma (H^2 + K_s)\, dA \geq 4\pi, \tag{3.1}$$

and equality holds if and only if Σ is a totally umbilic sphere.

Furthermore, if Σ is not embedded, then

$$\int_\Sigma (H^2 + K_s)\, dA \geq 8\pi. \tag{3.2}$$

Proof. Denote by κ_1 and κ_2 the principal curvatures of Σ (for some choice of a unit normal field). It is known that the density $(\kappa_1 - \kappa_2)^2 dA$ is invariant under conformal changes of the metric on M. Using the Gauss equation and Gauss-Bonnet formula we get that the integral $\int_\Sigma (H^2 + K_s) dA$ is also invariant under conformal changes of the metric on M. The umbilicity of a surface in a 3-manifold is also preserved by conformal changes of the metric on the ambient manifold (see for instance [23]). Moreover for a simply connected conformally flat Riemannian 3-manifold the *developing map* $\mathrm{dev} : M \longrightarrow \mathbb{S}^3$ is a well defined conformal immersion. It is therefore enough to check the theorem when the ambient manifold is the canonical 3-sphere $\mathbb{S}^3$. The first statement is then due to Willmore [24], and the second one is a result of Li and Yau [10]. □

Remark 3.2. Proposition 3.1 applies in particular to the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ as they are conformally flat. More precisely $\mathbb{S}^2 \times \mathbb{R}$ is conformal to $\mathbb{R}^3 \setminus \{(0,0,0)\}$ and $\mathbb{H}^2 \times (0,\pi)$ is conformal to $\mathbb{H}^3$ (see [21]). The totally umbilic surfaces in these spaces and more generally in homogeneous 3-manifolds were classified recently by Toubiana and the author in [22].

Our main result in this section is:

Theorem 3.3. *Let M be a simply connected conformally flat Riemannian 3-manifold. Let Σ be a compact orientable and stable CMC surface in M with unit normal N.*

(i) Suppose the Ricci curvature of M is nonnegative; then either Σ is a finite union of totally geodesic surfaces and $Ric(N) = 0$ or Σ is a sphere or an embedded torus.

(ii) Suppose the scalar curvature of M is nonnegative and that Σ is connected. Then Σ has genus $g \leq 3$. Moreover, if $g = 3$, then Σ is embedded, and if $g = 2$ and Σ is not embedded, then it is minimal and the scalar curvature of M vanishes on Σ.

Proof. (i) We first note that if Σ is not connected, then we can take a locally constant function with vanishing integral and easily contradict stability if the function $|\sigma|^2 + \mathrm{Ric}(N)$ is not identically zero on Σ.

Assume now that Σ is a connected surface of genus g. By Brill-Noether's theory (see e.g. [5]) there exists a nonconstant conformal (i.e holomorphic) map $\phi : \Sigma \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ whose degree satisfies

$$\deg(\phi) \leq 1 + \left[\frac{g+1}{2}\right] \tag{3.3}$$

where $[x]$ denotes the integer part of x. Using an extended version of a result of Hersch [6, 10] we may assume after composing ϕ with a conformal diffeomorphism of $\mathbb{S}^2$ that its coordinate functions satisfy

$$\int_\Sigma \phi_i\, dA = 0, \quad i = 1, 2, 3. \tag{3.4}$$

We may then use these coordinate functions as test functions for the stability condition and obtain:

$$0 \leq Q(\phi_i, \phi_i) = \int_\Sigma |\nabla \phi_i|^2 - (|\sigma|^2 + \mathrm{Ric}(N))\phi_i^2, \qquad i = 1, 2, 3. \tag{3.5}$$

Summing up these inequalities and taking into account the fact that ϕ takes its values in the unit sphere, we get:

$$0 \leq \int_\Sigma |\nabla \phi|^2 - (|\sigma|^2 + \mathrm{Ric}(N)). \tag{3.6}$$

Let K_s and K_Σ denote, respectively, the sectional curvature of M evaluated on the tangent plane to Σ and the intrinsic curvature of Σ. By the Gauss equation we have $|\sigma|^2 = 4H^2 + 2K_s - 2K_\Sigma$. Using the Gauss-Bonnet theorem and the fact that for a conformal map $\int_\Sigma |\nabla \phi|^2 = \int_\Sigma 2\mathrm{Jacobian}(\phi) = 8\pi \deg(\phi)$, we transform inequality (3.6) into:

$$\int_\Sigma 4H^2 + 2K_s + \mathrm{Ric(N)} \leq 8\pi \left(2 - g + \left[\frac{g+1}{2}\right]\right). \tag{3.7}$$

Using inequality (3.1) of Proposition 3.1, we get:

$$\int_\Sigma 2H^2 + \mathrm{Ric}(N) \leq 8\pi \left(1 - g + \left[\frac{g+1}{2}\right]\right). \tag{3.8}$$

As by hypothesis the Ricci curvature is nonnegative, this implies $g \leq 3$.

If $g = 2$ or 3, then the right-hand side in (3.8) is equal to zero, and since the Ricci curvature is nonnegative there must be equality in (3.8). Furthermore all the intermediate inequalities have to be equalities, and in particular we have equality in (3.1) which by Proposition 3.1 implies that Σ is a (totally umbilic) sphere, and that is a contradiction.

Suppose now that Σ is a torus and assume it is not embedded. Then taking into account this time inequality (3.2) in Proposition 3.1 we arrive at the inequality

$$\int_\Sigma 2H^2 + \mathrm{Ric}(N) \leq 0. \tag{3.9}$$

Again, as the Ricci curvature is nonnegative there must be equality in (3.9) and so all the intermediate inequalities have to be equalities too. In particular we have equalities in (3.5). So the holomorphic map ϕ satisfies $Q(\phi_i, \phi_i) = 0$, for $i = 1, 2, 3$. As Σ is stable, for any $v \in \mathcal{C}^\infty(\Sigma)$ satisfying $\int_\Sigma v = 0$ and any $t \in \mathbb{R}$ we have $Q(\phi_i + tv, \phi_i + tv) \geq 0$ and so $Q(\phi_i, v) = 0$. It follows that each of the functions ϕ_i satisfies the equation

$$\Delta \phi_i + |\sigma|^2 \phi_i = c_i,$$

for some real constant c_i, $i = 1, 2, 3$. So ϕ satisfies an equation of the type

$$\Delta \phi + |\sigma|^2 \phi = c \tag{3.10}$$

with c a constant vector in $\mathbb{R}^3$.

Since $\phi : \Sigma \longrightarrow \mathbb{S}^2$ is holomorphic it is harmonic and therefore satisfies the equation

$$\Delta \phi + |\nabla \phi|^2 \phi = 0. \tag{3.11}$$

As ϕ takes its values in the sphere $\mathbb{S}^2$ and is nonconstant, it follows easily from (3.10) and (3.11) that necessarily $c = 0$ and $|\sigma|^2 = |\nabla \phi|^2$. So the Jacobi operator of Σ can be written as $L = \Delta + |\nabla \phi|^2$ and the stability assumption implies that

L has only one negative eigenvalue. Otherwise said, the holomorphic map ϕ has index one and such maps do not exist on tori (see [15]). So if it is a torus, Σ has to be embedded. This completes the proof of the first part of the theorem.

(ii) To prove the second part of the theorem, we proceed in the same way. Taking into account the fact that $\mathrm{Ric}(N) = S - K_s$, where S denotes the scalar curvature of M, we rewrite (3.7) as follows:

$$\int_\Sigma 4H^2 + K_s + S \leq 8\pi \left(2 - g + \left[\frac{g+1}{2}\right]\right). \tag{3.12}$$

Using inequality (3.1) again, we obtain:

$$\int_\Sigma 3H^2 + S \leq 4\pi \left(3 - 2g + 2\left[\frac{g+1}{2}\right]\right).$$

As $S \geq 0$ by hypothesis, this implies $g \leq 3$. Furthermore, suppose Σ is not embedded; then using (3.2) in (3.12) we get:

$$\int_\Sigma 3H^2 + S \leq 8\pi \left(1 - g + \left[\frac{g+1}{2}\right]\right). \tag{3.13}$$

If $g = 2$ or 3, the right-hand side in (3.13) is zero and so $H = 0$ and $S = 0$ on Σ. Moreover equality is achieved in (3.13) and therefore also in the intermediate inequalities. As above we conclude that the Jacobi operator can be written as $L = \Delta + |\nabla\phi|^2$ and thus the holomorphic map $\phi : \Sigma \longrightarrow \mathbb{S}^2$ has index one. We have $\deg(\phi) = 3$ when $g = 3$. But on a surface of genus 3, a holomorphic map of index one necessarily has degree 2 (see [15] or [13]). So if $g = 3$, the surface has to be embedded. □

In particular in case (i) if we assume the Ricci curvature is positive, then a compact stable CMC surface can be either a sphere or a torus. As a consequence a compact isoperimetric region in a simply connected locally conformally flat 3-manifold with positive Ricci curvature is bounded by either a sphere or a torus.

4. Stable CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$

Theorem 3.3 applies in particular to $\mathbb{S}^2 \times \mathbb{R}$ (see Remark 3.2). However, in this case by using the symmetries of this space we can give a complete solution to the stability problem.

Theorem 4.1. *Let Σ be a compact stable immersed CMC surface without boundary in $\mathbb{S}^2 \times \mathbb{R}$. Then Σ is either a finite union of horizontal slices or a rotational sphere of mean curvature $H \geq H_0 \approx 0.18$.*

Proof. It follows from the maximum principle that the horizontal slices $\mathbb{S}^2 \times \{t\}$, $t \in \mathbb{R}$, are the only compact minimal surfaces without boundary in $\mathbb{S}^2 \times \mathbb{R}$. In particular all compact CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are orientable. By Theorem 3.3 (and Remark 3.2), if Σ is not connected, it is a finite union of horizontal slices, and if connected, it is either a sphere or an embedded torus. We will discard the case of a torus. Assume such a stable torus Σ exists. As it has to be embedded, a simple application of Alexandrov's reflection technique through the reflections $(p,t) \in \mathbb{S}^2 \times \mathbb{R} \to (p, 2t_0 - t)$, $t_0 \in \mathbb{R}$, shows that Σ is invariant under one such reflection. Call this reflection τ and let $\Sigma_+ = \Sigma \cap \big(\mathbb{S}^2 \times [t_0, +\infty)\big)$. Note that Σ_+ is topologically an annulus.

The argument to follow uses the symmetries of the ambient space and is based on Courant's nodal domains theorem. A similar idea was used to study volume-preserving stability in some variational problems [18, 17, 20, 8]. Let $q = (p_1, t_1)$ be a point on Σ where the height function reaches its maximum. Thus $q \in \Sigma_+$. Denote by N a global unit normal on Σ. Consider the 1-parameter family of rotations f_θ of $\mathbb{S}^2 \times \mathbb{R}$ around the axis $\{p_1\} \times \mathbb{R}$. It follows from (1.2) that the associated infinitesimal rotation $u := \langle \frac{df}{d\theta}, N\rangle$ is a Jacobi function, i.e. it satisfies the equation

$$\Delta u + (|\sigma|^2 + \mathrm{Ric}(N))u = 0. \tag{4.1}$$

Also it is straightforward to check that

$$u(q) = 0 \quad \text{and} \quad \nabla u(q) = 0. \tag{4.2}$$

We will show that $u \equiv 0$ on Σ. Assume for contradiction that this is not the case. Then as u satisfies (4.1), by a result of Cheng [4] its zero set $\{u = 0\}$ is a (possibly disconnected) graph with vertices at the points where ∇u vanishes too. Moreover any sufficiently small neighborhood of a vertex is divided by the zero set of u, like a pie, into at least 4 regions such that in any two adjacent regions u has opposite signs ([4]). By (4.2), the point $q \in \Sigma_+$ is one such vertex. It is a consequence of the Jordan curve theorem that $\Sigma_+ \setminus \{u = 0\}$ has at least three connected components. Now if $\Omega \subset \Sigma_+$ is one such component, then either Ω and $\tau(\Omega)$ are two distinct connected components of $\Sigma \setminus \{u = 0\}$ or $\Omega \cup \tau(\Omega)$ is a connected component of $\Sigma \setminus \{u = 0\}$. Therefore the number of connected components of $\Sigma \setminus \{u = 0\}$ is at least as big as the number of connected components of $\Sigma_+ \setminus \{u = 0\}$. In particular $\Sigma \setminus \{u = 0\}$ has at least three connected components. Denote by $\lambda_1 < \lambda_2$ the first and second eigenvalues of the operator L on Σ. It is well known that λ_1 is simple and if u_1 is a (nonzero) associated eigenfunction, then u_1 does not vanish on Σ. Also, by Courant's nodal domain theorem (see [4]), if u_2 is a nonzero eigenfunction associated to λ_2, then the set $\Sigma \setminus \{u_2 = 0\}$ has exactly two connected components. By (4.1), 0 is an eigenvalue of L and u is an eigenfunction associated to it. We have seen that $\Sigma \setminus \{u = 0\}$ has at least three components; therefore 0 is different from λ_1 and λ_2, and so $\lambda_2 < 0$. Let $a = -\int u_2 / \int u_1$, and set $v := au_1 + u_2$. Then $\int_\Sigma v = 0$ and

$$Q(v, v) = a^2 Q(u_1, u_1) + Q(u_2, u_2) = a^2 \lambda_1 \int_\Sigma u_1^2 + \lambda_2 \int_\Sigma u_2^2 < 0,$$

and this contradicts stability.

Therefore $u \equiv 0$, which means that Σ is a surface of revolution around the axis $\{p_0\} \times \mathbb{R}$. As this axis intersects Σ, this excludes the case of a torus.

The only possibility left is therefore that Σ is a sphere. To conclude that Σ is a sphere of revolution we can use the Abresch-Rosenberg theorem [1] which asserts that an immersed CMC sphere in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ is rotationally invariant. Alternatively we can also repeat the previous argument to see that it is a sphere of revolution around the axis passing through a point where the height function reaches its maximum; the existence of the symmetry is not needed when Σ is a sphere. We know from Theorem 2.2 that the mean curvature of a stable CMC sphere is $\geq H_0 \approx 0.18$. □

In particular the theorem says there is no compact stable CMC surface in $\mathbb{S}^2 \times \mathbb{R}$ of mean curvature $0 < H < H_0 \approx 0.18$.

5. Stable CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In this section we study compact stable CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Nelli and Rosenberg [11] have proven that the surface has genus at most 3 if its mean curvature is $> 1/\sqrt{2}$. In view of the result obtained in the case of $\mathbb{S}^2 \times \mathbb{R}$, it is reasonable to conjecture that only rotational CMC spheres are stable. Our next result gives a partial answer:

Theorem 5.1. *Let Σ be a compact immersed and connected surface without boundary of constant mean curvature H and genus g in $\mathbb{H}^2 \times \mathbb{R}$. Assume Σ is stable.*

If $H \geq 1/\sqrt{2}$, then Σ is a rotational sphere.

If $1/\sqrt{3} < H < 1/\sqrt{2}$, then $g \leq 1$.

If $H = 1/\sqrt{3}$, then $g \leq 2$.

Remark 5.2. Nelli and Rosenberg [11] have proved that an immersed compact CMC surface without boundary in $\mathbb{H}^2 \times \mathbb{R}$ always has mean curvature $H > 1/2$.

Proof. First note that by the maximum principle there are no compact minimal surfaces without boundary in $\mathbb{H}^2 \times \mathbb{R}$ and so in particular Σ is orientable. We proceed as in the case of $\mathbb{S}^2 \times \mathbb{R}$. So we have a nonconstant holomorphic map $\phi : \Sigma \longrightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ satisfying conditions (3.3) and (3.4) and this leads to the inequality

$$\int_\Sigma 4H^2 + 2K_s + \mathrm{Ric}(N) \leq 8\pi \left(2 - g + \left[\frac{g+1}{2}\right]\right). \tag{5.1}$$

The Alexandrov reflection technique through planes of the form $\Gamma \times \mathbb{R}$, where Γ is any geodesic in $\mathbb{H}^2$, shows that the CMC spheres of revolution are the only embedded compact CMC surfaces without boundary in $\mathbb{H}^2 \times \mathbb{R}$. So keeping aside the genus zero case, we have that Σ is not embedded. Using then the inequality (3.2) in (5.1), we get:

$$\int_\Sigma 2H^2 + \mathrm{Ric}(N) \leq 8\pi \left(-g + \left[\frac{g+1}{2}\right]\right). \tag{5.2}$$

In $\mathbb{H}^2 \times \mathbb{R}$ we have $-1 \leq \mathrm{Ric}(V) \leq 0$ for any unit vector V and $\mathrm{Ric}(V) = -1$ if and only if V is horizontal. So:

$$\int_\Sigma 2H^2 - 1 \leq \int_\Sigma 2H^2 + \mathrm{Ric}(N) \leq 8\pi \left(-g + \left[\frac{g+1}{2}\right]\right) \leq 0. \tag{5.3}$$

Consequently if $H > 1/\sqrt{2}$ we reach a contradiction.

If now $H = 1/\sqrt{2}$, then necessarily $\mathrm{Ric}(N) = -1$ and so N is always horizontal, which is absurd as Σ is compact without boundary. So if $H \geq 1/\sqrt{2}$, the surface has to be a sphere and we can conclude as in the proof of Theorem 4.1 that it is a sphere of revolution.

To obtain the second part of the theorem, note that since the scalar curvature of $\mathbb{H}^2 \times \mathbb{R}$ is identically -1 we have $\mathrm{Ric}(N) = -1 - K_s$. So we can rewrite (5.1) as follows:

$$\int_\Sigma 4H^2 + K_s - 1 \leq 8\pi \left(2 - g + \left[\frac{g+1}{2}\right]\right). \tag{5.4}$$

Again, keeping aside the genus zero case, we know the surface is nonembedded and so we can use inequality (3.2) to obtain:

$$\int_\Sigma 3H^2 - 1 \leq 8\pi \left(1 - g + \left[\frac{g+1}{2}\right]\right). \tag{5.5}$$

This shows that $g \leq 1$ when $H > 1/\sqrt{3}$.

Suppose now that $H = 1/\sqrt{3}$; then by (5.5) we have $g \leq 3$. Moreover if $g = 2$ or 3, we have equality in (5.5) and in all the intermediate inequalities, and we conclude as in the proof of Theorem 4.1 that the Jacobi operator can be written as $L = \Delta + |\nabla\phi|^2$ and so the holomorphic map ϕ has index one. Note that $\deg(\phi) = 3$ when $g = 3$. However, on a surface of genus three a holomorphic map of index one necessarily has degree two (see [15] or [13]). This excludes the case $g = 3$. □

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