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REGULAR IDEMPOTENTS IN βS

YEVHEN ZELENYUK

Abstract. Let S be a discrete semigroup and let βS be the Stone-Čech compactification of S. We take the points of βS to be the ultrafilters on S. Being a compact Hausdorff right topological semigroup, βS has idempotents. Every idempotent $p \in \beta S$ determines a left invariant topology $\mathcal{T}_p$ on S with a neighborhood base at $a \in S$ consisting of subsets $aB \cup \{a\}$, where $B \in p$. If S is a group and p is an idempotent in $S^* = \beta S \setminus S$, $(S, \mathcal{T}_p)$ is a homogeneous Hausdorff maximal space. An idempotent $p \in \beta S$ is regular if p is uniform and the topology $\mathcal{T}_p$ is regular. We show that for every infinite cancellative semigroup S, there exists a regular idempotent in βS. As a consequence, we obtain that for every infinite cardinal κ, there exists a homogeneous regular maximal space of dispersion character κ. Another consequence says that there exists a translation invariant regular maximal topology on the real line of dispersion character $\mathfrak{c}$ stronger than the natural topology.

1. Introduction

The operation of a discrete semigroup S can be naturally extended to the Stone-Čech compactification βS of S making βS a compact right topological semigroup with S contained in its topological center. A semigroup T endowed with a topology is *right topological* if for each $p \in T$, the right translation

$$T \ni x \mapsto xp \in T$$

is continuous. The *topological center* $\Lambda(T)$ of a right topological semigroup T consists of all $a \in T$ such that the left translation

$$T \ni x \mapsto ax \in T$$

is continuous.

We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S, and $S^* = \beta S \setminus S$. The topology of βS is generated by taking as a base the subsets

$$\overline{A} = \{p \in \beta S : A \in p\},$$

where $A \subseteq S$. For $p, q \in \beta S$, the ultrafilter pq has a base of subsets

$$\bigcup \{xB_x : x \in A\},$$

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where $A \in p$ and $B_x \in q$. An elementary introduction to the semigroup βS can be found in [9].

Every compact Hausdorff right topological semigroup T has an idempotent [6, Corollary 2.10]. This implies that T has a smallest two-sided ideal $K(T)$ which is a disjoint union of minimal right ideals and a disjoint union of minimal left ideals (see [9, Section 2.2]). Every right (left) ideal contains a minimal right (left) ideal. The intersection of a minimal right ideal and a minimal left ideal is a group, all these groups are isomorphic, and the idempotents of a minimal right (left) ideal form a right (left) zero semigroup. A *right (left) zero* semigroup is one satisfying the identity $xy = y$ ($xy = x$).

It is a remarkable property of idempotents of βS that each of them, say p, determines a left invariant topology $\mathcal{T}_p$ on S with a neighborhood base at $a \in S$ consisting of subsets $aB \cup \{a\}$, where $B \in p$ [14]. A topology $\mathcal{T}$ on S is *left invariant* if for every $U \in \mathcal{T}$ and $a \in S$, both $aU \in \mathcal{T}$ and $a^{-1}U \in \mathcal{T}$, where

$$a^{-1}U = \{x \in S : ax \in U\}.$$

Equivalently, $\mathcal{T}$ is left invariant if left translations of S are continuous and open in $\mathcal{T}$. A left invariant topology on a semigroup S with identity e is completely determined by the neighborhood filter at e. For each $a \in S$, the subsets aU, where U is a neighborhood of e, form a neighborhood base at a. If S is commutative, we say *translation invariant* instead of left invariant.

If S is a group and p is an idempotent in S^*, $(S, \mathcal{T}_p)$ is a homogeneous Hausdorff maximal space. A space is *maximal* if its topology is maximal among topologies without isolated points. A Hausdorff space X is maximal if and only if for every point $x \in X$, there is exactly one nonprincipal ultrafilter on X converging to x. A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism $f : X \to X$ with $f(x) = y$. It is an old difficult question whether there exists a homogeneous regular maximal space, and more generally, a homogeneous regular maximal space of a given dispersion character $\kappa \geq \omega$. The *dispersion character* of a space is the minimum of cardinalities of nonempty open sets.

In 1950, Katětov [11] asked whether there is a regular maximal space. A countable example of such a space was constructed by van Douwen [3, 4] and that of an arbitrary dispersion character by El'kin [5]. The first example of a homogeneous regular maximal space was constructed by Malykhin [12], assuming Martin's Axiom. It was a maximal topological group, that is, a topological group that is a maximal space. Malykhin [13] also proved that every maximal topological group contains a countable open Boolean subgroup, and consequently, has countable dispersion character. Later Protasov [15] showed that the existence of a maximal topological group cannot be established in ZFC, the system of the usual axioms of set theory.

In the case where S is a group and p is an idempotent in S^*, the topology $\mathcal{T}_p$ is regular if and only if p is strongly right maximal in S^* [15] (see also [9, Theorem 9.15]). The *right* preordering on the idempotents of any semigroup T is defined by

$$p \leq_R q \text{ if and only if } qp = p.$$

Given a subset $A \subseteq T$, an idempotent $p \in A$ is *right maximal* (in A) if for every idempotent $q \in A$, $p \leq_R q$ implies $q \leq_R p$. Every compact Hausdorff right topological semigroup has a right maximal idempotent [17, Theorem 2.7]. An idempotent

$p \in A$ is *strongly right maximal* if for every $q \in A$, $qp = p$ implies $q = p$. Obviously, a strongly right maximal idempotent is right maximal.

In 1998, Hindman and Strauss [9, Theorem 9.4] showed that if S is a countable group and p is a right maximal idempotent in S^*, then

$$C(p) = \{x \in S^* : xp = p\}$$

is a finite right zero semigroup. Protasov [16] noticed that this property implies local regularity of the topology $\mathcal{T}_p$ (= each point has a neighborhood which is a regular subspace) and applying the Local Isomorphism Theorem (see Remark 6.11) deduced that there exists a strongly right maximal idempotent in S^*. Thereby it was shown that there exists a countable homogeneous regular maximal space. The proofs of both the Hindman-Strauss result and the Local Isomorphism Theorem heavily depended on the fact that S is countable. Therefore, the question whether there exists a homogeneous regular maximal space of an uncountable dispersion character, as well as the question whether there exists a uniform strongly right maximal idempotent ultrafilter on an uncountable group, remained open. An ultrafilter p on S is *uniform* if for every $A \in p$, $|A| = |S|$.

Definition 1.1. An idempotent $p \in \beta S$ is *regular* (*locally regular*) if p is uniform and the topology $\mathcal{T}_p$ is regular (locally regular).

The aim of this paper is to show that

Theorem 1.2. *For every infinite cancellative semigroup S, there exists a regular idempotent in βS.*

Recall that a semigroup is *cancellative* if both left and right translations are injective. As a consequence, we obtain from Theorem 1.2 the following.

Corollary 1.3. *For every infinite cardinal κ, there exists a homogeneous regular maximal space of dispersion character κ.*

In fact, Corollary 1.3 can be deduced from a weaker result—the existence of a locally regular idempotent (see Remark 7.4).

Corollary 1.4. *For every infinite cancellative semigroup S, there exists a uniform strongly right maximal idempotent in S^*.*

Proof. If S is a cancellative semigroup, p is an idempotent in S^* and the topology $\mathcal{T}_p$ is regular, then p is strongly right maximal in S^* [10, Theorem 4.1]. Hence, the result follows from Theorem 1.2. □

Corollary 1.5. *There exists a translation invariant regular maximal topology on the real line of dispersion character $\mathfrak{c}$ stronger than the natural topology.*

Proof. By Theorem 1.2, there exists a regular idempotent ultrafilter p on the circle group $\mathbb{T}$. Since $\mathbb{T}$ is a compact topological group, p converges to 1 [19, Lemma 3]. Identifying $\mathbb{T}$ with the subset $[-\frac{1}{2}, \frac{1}{2}) \subset \mathbb{R}$, we obtain that there exists a regular idempotent ultrafilter on $\mathbb{R}$ converging to 0. □

Note that if p is a regular idempotent ultrafilter on $\mathbb{R}$, so is $-p$. Hence, there exists a regular idempotent ultrafilter on $\mathbb{R}$ converging to 0 and containing $\mathbb{R}^+$, that is to say, converging to 0 in the Sorgenfrey topology (= with a base of half-open intervals $[a, b)$). Thus, one can replace the 'natural topology' in Corollary 1.5 by the stronger 'Sorgenfrey topology'.

The proof of Theorem 1.2 occupies the rest of the paper.

In section 2, we discuss some elementary facts about ultrafilter semigroups, left invariant topologies and local homomorphisms.

In section 3, we consider strongly discrete filters. We prove that for every strongly discrete ultrafilter p on a group S, the smallest closed subsemigroup of βS containing p admits a continuous homomorphism onto $\beta\mathbb{N}$.

In section 4, we characterize right cancellable elements in a subsemigroup $\mathbb{H}_\kappa$ of $\beta(\bigoplus_\kappa \mathbb{Z}_2)$. In particular, we prove that an ultrafilter $p \in \mathbb{H}_\kappa$ is right cancellable if and only if p is strongly discrete.

In section 5, using the results of sections 3 and 4, we show that there exists a right maximal idempotent $p \in \mathbb{H}_\kappa$ such that $C(p)$ is a finite right zero semigroup.

In section 6, we prove an analogue of the Local Isomorphism Theorem. This is the core of the whole construction.

In section 7, we deduce Theorem 1.2 from the results of sections 5 and 6. In fact we show, as was done in the case of countable groups [9, Corollary 9.12], that there are $2^{2^{|S|}}$ regular idempotents in βS.

2. Preliminaries

Definition 2.1. Let S be an infinite semigroup with identity e and let $\mathcal{T}$ be a left invariant topology on S satisfying the T_1 separation axiom. The *ultrafilter semigroup* of $\mathcal{T}$ is defined as a subset of βS by

$$\mathrm{Ult}(\mathcal{T}) = \bigcap_{U \in \mathcal{N}} \overline{U \setminus \{e\}},$$

where $\mathcal{N}$ is the neighborhood filter at e in $\mathcal{T}$.

It follows from $\bigcap \mathcal{N} = \{e\}$ that $\mathrm{Ult}(\mathcal{T}) \subseteq S^*$, so $\mathrm{Ult}(\mathcal{T})$ consists of all nonprincipal ultrafilters on S converging to e in $\mathcal{T}$. To see that $\mathrm{Ult}(\mathcal{T})$ is a subsemigroup of βS, let $p, q \in \mathrm{Ult}(\mathcal{T})$ and let $U \in \mathcal{N}$. We have to show that $U \setminus \{e\} \in pq$. Clearly one may suppose that U is open, and so is $U \setminus \{e\}$. For each $x \in U \setminus \{e\}$, there is $V_x \in \mathcal{N}$ such that $xV_x \subseteq U \setminus \{e\}$. Then $\bigcup_{x \in U \setminus \{e\}} xV_x \subseteq U \setminus \{e\}$, and since $U \setminus \{e\} \in p$ and $V_x \in q$, we obtain that $U \setminus \{e\} \in pq$.

The last argument shows also that

Lemma 2.2. *If U is an open subset of $(S, \mathcal{T})$, then $\overline{U} \cdot \mathrm{Ult}(\mathcal{T}) \subseteq \overline{U}$.*

A space is *extremally disconnected* if the closures of disjoint open sets are disjoint.

Lemma 2.3. *If $\mathrm{Ult}(\mathcal{T})$ has only one minimal right ideal, then $\mathcal{T}$ is extremally disconnected.*

Proof. Let $Q = \mathrm{Ult}(\mathcal{T})$. Assume, on the contrary, that $\mathcal{T}$ is not extremally disconnected. Then there are two disjoint open subsets U and V of $(S, \mathcal{T})$ such that

$$e \in (c\ell\ U) \cap (c\ell\ V).$$

But then by Lemma 2.2, $\overline{U} \cap Q$ and $\overline{V} \cap Q$ are two disjoint right ideals of Q, which gives a contradiction. □

Not each closed subsemigroup of S^* is the ultrafilter semigroup of a left invariant topology. However, it is so if S is a group and the subsemigroup is finite.

Lemma 2.4 ([20, Proposition 2.4])**.** *If S is a group and Q is a finite subsemigroup of S^*, then there is a left invariant T_1-topology $\mathcal{T}$ on S with $\mathrm{Ult}(\mathcal{T}) = Q$.*

We say that a subsemigroup Q of S^* is *left saturated* (in βS) if for every $x \in \beta S \setminus (Q \cup \{e\})$, one has $xQ \cap Q = \emptyset$. Note that a 1-element subsemigroup $Q = \{q\}$ of S^* is left saturated if and only if q is a strongly right maximal idempotent in S^*.

Lemma 2.5. *If $\mathcal{T}$ is a regular left invariant topology on S, then* $\mathrm{Ult}(\mathcal{T})$ *is left saturated.*

Proof. Let $Q^1 = Q \cup \{e\}$ and let $p \in \beta S \setminus Q^1$. Then there is a neighborhood U of e in $\mathcal{T}$ with $U \notin p$. Since $\mathcal{T}$ is regular, U can be chosen to be closed. Put $W = S \setminus U$. We have that $p \in \overline{W}$, W is open and $\overline{W} \cap Q^1 = \emptyset$. Then by Lemma 2.2, $\overline{W} \cdot Q^1 \subseteq \overline{W}$. Hence $pQ^1 \cap Q^1 = \emptyset$. □

Corollary 2.6. *Let p be an idempotent in S^*. If the topology $\mathcal{T}_p$ is regular, then p is strongly right maximal in S^*.*

In the case where S is a group and $\mathrm{Ult}(\mathcal{T})$ is finite, the converse of Lemma 2.5 also holds.

Lemma 2.7 ([20, Proposition 2.12])**.** *Let S be a group, let Q be a finite subsemigroup of S^*, and let $\mathcal{T}$ be a left invariant topology on S with* $\mathrm{Ult}(\mathcal{T}) = Q$. *Then $\mathcal{T}$ is regular if and only if Q is left saturated.*

The next lemma characterizes subsemigroups of a finite left saturated subsemigroup of S^* (in the case where S is a group) which determine locally regular left invariant topologies.

Lemma 2.8. *Let S be a group, let Q be a finite left saturated subsemigroup of S^*, let P be a subsemigroup of Q, and let $\mathcal{T}$ be a left invariant topology on S with* $\mathrm{Ult}(\mathcal{T}) = P$. *Then $\mathcal{T}$ is locally regular if and only if for every $q \in Q \setminus P$, $qP \cap P \neq \emptyset$ implies $Pq \cap P = \emptyset$.*

Proof. Suppose that $\mathcal{T}$ is locally regular and let $q \in Q \setminus P$ be such that $qP \cap P \neq \emptyset$. We have to show that $Pq \cap P = \emptyset$. Choose a regular open neighborhood X of $e \in S$ in $\mathcal{T}$. It suffices to show that $Xq \cap \overline{X} = \emptyset$, as this implies $\overline{X}q \cap \overline{X} = \emptyset$ and then $Pq \cap P = \emptyset$.

Assume on the contrary that $xq \in \overline{X}$ for some $x \in X$. Since $q \notin P \cup \{e\}$, xq does not converge to x, so there is a neighborhood U of $x \in X$ such that $U \notin xq$. Since X is regular, U can be chosen to be closed. We have that $X \setminus U \in xq$ and $X \setminus U$ is open. Then by Lemma 2.2, $xqP \subseteq \overline{X \setminus U}$. It follows that $xqP \cap xP = \emptyset$, and consequently, $qP \cap P = \emptyset$, a contradiction.

Conversely, let $F = \{q \in Q \setminus P : qP \cap P \neq \emptyset\}$ and suppose that for each $q \in F$, $Pq \cap P = \emptyset$. Then for each $q \in F$, there is a neighborhood X_q of e in $\mathcal{T}$ such that $\overline{X_q}q \cap \overline{X_q} = \emptyset$. Put $X = \bigcap_{q \in F} X_q$. Since F is finite, X is a neighborhood of e in $\mathcal{T}$, and for each $q \in F$, one has $Xq \cap \overline{X} = \emptyset$. We claim that X is regular.

Assume the contrary. Then there is $x \in X$ and a neighborhood U of x such that for every neighborhood V of x, $c\ell_X(V) \setminus U \neq \emptyset$. Since P is finite, it follows that there is $p \in P$, and for every neighborhood V of x, there is $y_V \in X \setminus U$ such that $V \in y_V p$. Let r be an ultrafilter on S extending the family of subsets

$$Y_V = \{y_W : W \text{ is a neighborhood of } x \text{ contained in } V\},$$

where V runs over neighborhoods of x. Then $r \in \overline{X \setminus U}$ and $rp \in xP$, so $x^{-1}rp \in P$. Put $q = x^{-1}r$. We have that (a) $qp \in P$ and $q \neq e$, and (b) $r = xq$. Since Q is left

saturated, it follows from (a) that $q \in F$. It is clear that $xq \in Xq$, and (b) gives us that $xq \in \overline{X}$. Hence $Xq \cap \overline{X} \neq \emptyset$, a contradiction. □

Given any $p \in S^*$, $C(p) \subset S^*$ is defined by

$$C(p) = \{x \in S^* : xp = p\}.$$

Note that if S is left cancellative (= left translations are injective), then $C(p)$ is a closed subsemigroup of S^*, possibly empty.

Lemma 2.9. *If S is left cancellative, $C(p)$ is left saturated and*

$$C(p) = \{x \in \beta S \setminus \{e\} : xp = p\}.$$

Proof. Let $x \in S \setminus \{e\}$. Since S is left cancellative, $xy \neq y$ for all $y \in S$. Then by [9, Lemma 6.28], $xp \neq p$. Hence $C(p) = \{x \in \beta S \setminus \{e\} : xp = p\}$.

To see that $C(p)$ is left saturated, let $xq = r$ for some $x \in \beta S \setminus \{e\}$ and $q, r \in C(p)$. Multiplying this equality by p from the right, we obtain that $xp = xqp = rp = p$. Hence $x \in C(p)$. □

Taking into account Lemma 2.9, we obtain from Lemma 2.8 the following.

Corollary 2.10. *Let S be a group and let p be a right maximal idempotent in S^*. Suppose that $C(p)$ is a finite right zero semigroup. Then the topology $\mathcal{T}_p$ is locally regular.*

Proof. To check the condition from Lemma 2.8, let $Q = C(p)$ and $P = \{p\}$. Then for every $q \in Q \setminus P$, $Pq = \{q\}$ and so $Pq \cap P = \emptyset$. □

Definition 2.11. Let S be a semigroup with identity, let $\mathcal{T}$ be a left invariant T_1-topology on S, let X be an open neighborhood of the identity, and let R be any semigroup. A mapping $f : X \to R$ is a *local homomorphism* if for every $x \in X \setminus \{e\}$, there is a neighborhood U of e such that $f(xy) = f(x)f(y)$ for all $y \in U \setminus \{e\}$. If R has an identity, we require in addition that $f(e_S) = e_R$. An injective (bijective) local homomorphism is called a *local monomorphism* (a *local isomorphism*).

Local homomorphisms are important because of the following fact.

Lemma 2.12. *If $f : X \to T$ is a local homomorphism into a compact right topological semigroup T such that $f(X) \subseteq \Lambda(T)$ and $\overline{f} : c\ell_{\beta S} X \to T$ is the continuous extension of f, then $\overline{f}|_{\mathrm{Ult}(\mathcal{T})} : \mathrm{Ult}(\mathcal{T}) \to T$ is a homomorphism.*

Proof. Apply [9, Theorem 4.21]. □

The next lemma contains another important property of local homomorphisms.

Lemma 2.13. *Let S and R be semigroups with identities, let $\mathcal{T}$ be a left invariant T_1-topology on S, let X be an open neighborhood of $e_S \in S$ in $\mathcal{T}$, and let $f : X \to R$ be a local monomorphism. Then there is a left invariant T_1-topology $\mathcal{T}^f$ on R with a neighborhood base at $e_R \in R$ consisting of subsets $f(U)$, where U is a neighborhood of $e_S \in X$. Furthermore, let $Y = f(X) \subseteq (R, \mathcal{T}^f)$ and let $\overline{f} : c\ell_{\beta S} X \to \beta R$ be the continuous extension of f. Then f homeomorphically maps X onto Y and $\overline{f}$ isomorphically maps $\mathrm{Ult}(\mathcal{T})$ onto $\mathrm{Ult}(\mathcal{T}^f)$.*

Proof. For every open neighborhood U of $e_S \in X$ and $x \in U$, there is a neighborhood V of e_S such that $xV \subseteq U$ and $f(xy) = f(x)f(y)$ for all $y \in V$. Then $f(x)f(V) = f(xV) \subseteq f(U)$. It follows that the subsets $f(U)$, where U is a neighborhood of $e_S \in X$, form a neighborhood base at e_R in a left invariant topology $\mathcal{T}^f$ on R. Since f is injective, $\mathcal{T}^f$ is a T_1-topology.

To see that f homeomorphically maps X onto Y, let $x \in X$. Choose a neighborhood U of $e_S \in X$ such that $xU \subseteq X$ and $f(xy) = f(x)f(y)$ for all $y \in U$. Then whenever V is a neighborhood of $e_S \in X$ and $V \subseteq U$, one has $f(xV) = f(x)f(V)$.

Finally, since f is injective, $\overline{f}$ is injective as well, and since $f : X \to Y$ is a homeomorphism, $\overline{f}(\mathrm{Ult}(\mathcal{T})) = \mathrm{Ult}(\mathcal{T}^f)$. Hence, by Lemma 2.12, $\overline{f}$ isomorphically maps $\mathrm{Ult}(\mathcal{T})$ onto $\mathrm{Ult}(\mathcal{T}^f)$. □

3. Strongly discrete filters

Let S be an infinite group. Given a filter $\mathcal{F}$ on S, let $\mathcal{T}[\mathcal{F}]$ denote the largest left invariant topology on S in which $\mathcal{F}$ converges to e.

Lemma 3.1 ([22, Proposition 2.2])**.** *The open neighborhoods of an element $a \in S$ in $\mathcal{T}[\mathcal{F}]$ are precisely the subsets of the form*

$$[M]_a = \{x_0 \cdots x_n : n < \omega, x_0 = a \text{ and } x_{i+1} \in M(x_0 \cdots x_i) \text{ for all } i < n\},$$

where M runs over mappings $S \to \mathcal{F}$.

Corollary 3.2. *If $\bigcap \mathcal{F} = \emptyset$, then $\mathcal{T}[\mathcal{F}]$ is a T_1-topology.*

Proof. Let $e \neq a \in S$. Define $M : S \to \mathcal{F}$ by $M(x) = S \setminus \{x^{-1}a\}$. Then $a \notin xM(x)$ for all $x \in S$. It follows that $a \neq x_0 \cdots x_n$ if $x_0 = e$ and $x_{i+1} \in M(x_0 \cdots x_i)$. Hence $a \notin [M]_e$. □

Let $\mathcal{P}(S)$ denote the power set of S. Given a mapping $M : S \to \mathcal{P}(S)$ and an element $x \in S$, an *M-decomposition* of x is any equality of the form $x = x_0 \cdots x_n$, where $x_{i+1} \in M(x_0 \cdots x_i)$ for all $i < n$. A mapping $M : S \to \mathcal{P}(S)$ is *basic* if every $x \in S$ has a unique M-decomposition $x = x_0 \cdots x_n$ with $x_0 = e$.

Lemma 3.3. *Let $M : S \to \mathcal{P}(S)$ be a basic mapping. Then the subsets $xM(x)$, where $x \in S$, are pairwise disjoint.*

Proof. Suppose that $xu = yv = z$ for some $x, y, z \in S$, $u \in M(x)$ and $v \in M(y)$. Let $x = x_0 \cdots x_n$ and $y = y_0 \cdots y_m$ be M-decompositions with $x_0 = y_0 = e$. Then $z = x_0 \cdots x_n u$ and $z = y_0 \cdots y_m v$ are M-decompositions with $x_0 = y_0 = e$. It follows that $u = v$ and $x = y$. □

A subset D of a space X is called *strongly discrete* if for each $x \in D$, there is a neighborhood U_x of $x \in X$ such that all the subsets U_x, where $x \in D$, are pairwise disjoint.

Definition 3.4. We say that a filter $\mathcal{F}$ on S is *strongly discrete* if

(1) for every $A \subseteq S$ with $|S \setminus A| < |G|$, $A \in \mathcal{F}$, and
(2) there is $M : S \to \mathcal{F}$ such that the subsets $xM(x)$, where $x \in S$, are pairwise disjoint.

The main internal property of strongly discrete filters is contained in the next proposition.

Proposition 3.5. *For every strongly discrete filter $\mathcal{F}$ on S, there is a basic mapping $M : S \to \mathcal{F}$.*

Proof. By [22, Lemma 3.1], there are $M_0 : S \to \mathcal{F}$ and $A \subset G$ with $e \in A$ such that

(i) the subsets $[M_0]_a$, where $a \in A$, form a partition of S, and
(ii) whenever $a \in A$ and $x \in [M_0]_a$, x has a unique M-decomposition $x = x_0 \cdots x_n$ with $x_0 = a$.

Define $M : S \to \mathcal{F}$ by

$$M(x) = \begin{cases} M_0(x) & \text{if } x \neq e, \\ M_0(e) \cup (A \setminus \{e\}) & \text{otherwise.} \end{cases}$$

□

Using Proposition 3.5, one can show that the topology $\mathcal{T}[\mathcal{F}]$ determined by a strongly discrete filter $\mathcal{F}$ on S possesses interesting properties. In particular, $\mathcal{T}[\mathcal{F}]$ is zero-dimensional, and if $M : S \to \mathcal{F}$ is a basic mapping, then $D = M(e)$ is a strongly discrete subset of $(S, \mathcal{T}[\mathcal{F}])$ and $c\ell D = D \cup \{e\}$. One can show also that strongly discrete filters exist in profusion on any infinite group. (See [22, Section 3].) However, in this paper, we are more interested in the following.

Proposition 3.6. *Let $\mathcal{F}$ be a strongly discrete filter on S, let $M : S \to \mathcal{F}$ be a basic mapping, and let R be a semigroup. Then every mapping $f_0 : M(e) \to R$ can be extended to a local homomorphism $f : (S, \mathcal{T}[\mathcal{F}]) \to R$.*

Proof. Given any $f_0 : M(e) \to R$, define $f : S \to R$ by

$$f(x) = f_0(x_1) \cdots f_0(x_n),$$

where $x = x_0 x_1 \cdots x_n$ is an M-decomposition with $x_0 = e$. That f is a local homomorphism follows from the next lemma. □

Lemma 3.7. *Let $\mathcal{F}$ be a strongly discrete filter on S and let $M : S \to \mathcal{F}$ be a basic mapping. Let $e \neq x \in S$ and let $x = x_0 \cdots x_n$ be an M-decomposition with $x_0 = e$. Then there is a neighborhood U of e in $\mathcal{T}[\mathcal{F}]$ such that whenever $e \neq y \in U$ and $y = y_0 \cdots y_m$ is an M-decomposition with $y_0 = e$, $xy = x_0 x_1 \cdots x_n y_1 \cdots y_m$ is an M-decomposition.*

Proof. Define $N : S \to \mathcal{F}$ by

$$N(y) = M(xy) \cap M(y)$$

and put $U = [N]_e$.

Let $e \neq y \in U$. Then there is an N-decomposition $y = y_0 \cdots y_m$ with $y_0 = e$. Clearly this is also an M-decomposition. We have that $xy = x_0 x_1 \cdots x_n y_1 \cdots y_m$. Since $x = x_0 \cdots x_n$ is an M-decomposition, $x_{i+1} \in M(x_0 \cdots x_i)$. Next, we have that $y_1 \in N(y_0) = N(e) \subseteq M(xe) = M(x_0 \cdots x_n)$ and, for $i > 1$,

$$y_{i+1} \in N(y_0 \cdots y_i) = N(y_1 \cdots y_i) \subseteq M(xy_1 \cdots y_i) = M(x_0 \cdots x_n y_1 \cdots y_i).$$

Hence, $xy = x_0 x_1 \cdots x_n y_1 \cdots y_m$ is an M-decomposition. □

We now come to the main result of this section.

For every $p \in \beta S$, let C_p denote the smallest closed subsemigroup of βS containing p.

Theorem 3.8. *Let p be a strongly discrete ultrafilter on S. Then there is a continuous homomorphism $\pi : \mathrm{Ult}(\mathcal{T}[p]) \to \beta\mathbb{N}$ with $\pi(p) = 1$ and $\pi(C_p) = \beta\mathbb{N}$.*

Proof. Let $M : S \to p$ be a basic mapping. Define $f_0 : M(e) \to \mathbb{N}$ by $f_0(x) = 1$. By Proposition 3.6, f_0 can be extended to a local homomorphism

$$f : (S, \mathcal{T}[p]) \to \mathbb{N} \subset \beta\mathbb{N}.$$

Put $\pi = \overline{f}|_{\mathrm{Ult}(\mathcal{T}[p])}$. Then $\pi : \mathrm{Ult}(\mathcal{T}[p]) \to \beta\mathbb{N}$ is a continuous homomorphism with $\pi(p) = 1$. It follows that $\pi(C_p)$ is a closed subsemigroup of $\beta\mathbb{N}$ containing 1. Hence $\pi(C_p) = \beta\mathbb{N}$. □

4. Right cancellable ultrafilters

An element p of a semigroup S is *right cancellable* if whenever $q, r \in S$ and $qp = rp$, one has $q = r$. Equivalently, p is right cancellable if the right translation by p is injective.

Throughout the rest of the paper, we will use the following notation.

Definition 4.1. Let κ be an infinite cardinal and let $G = \bigoplus_\kappa \mathbb{Z}_2$. Define functions $\theta, \phi : G \setminus \{0\} \to \kappa$ by

$$\theta(x) = \min \mathrm{supp}(x) \text{ and } \phi(x) = \max \mathrm{supp}(x)$$

and let $\overline{\theta}, \overline{\phi} : \beta G \setminus \{e\} \to \beta\kappa$ be their continuous extensions. For every $\alpha < \kappa$, put

$$G_\alpha = \{0\} \cup \{x \in G : \theta(x) \geq \alpha\}.$$

Let $\mathcal{G}$ denote the group topology on G with a neighborhood base at 0 consisting of subgroups G_α, $\alpha < \kappa$, and let $\mathbb{H}_\kappa = \mathrm{Ult}(\mathcal{G})$.

Note that $\mathbb{H}_\kappa$ is left saturated in βG. In particular, for every $p \in \mathbb{H}_\kappa$, one has $C(p) \subset \mathbb{H}_\kappa$.

In this section, we prove that

Theorem 4.2. *For every ultrafilter $p \in \mathbb{H}_\kappa$, the following statements are equivalent:*

(1) *p is right cancellable in βG,*
(2) *p is right cancellable in $\mathbb{H}_\kappa$,*
(3) *there is no idempotent $q \in G^*$ for which $p = q + p$,*
(4) *there is no $q \in G^*$ for which $p = q + p$,*
(5) *$G + p \subset \beta G$ is discrete,*
(6) *$G + p \subset \beta G$ is strongly discrete,*
(7) *p is strongly discrete.*

Proof. (1)⇒(2) is obvious.

(2)⇒(3) Assume on the contrary that there is an idempotent $q \in G^*$ for which $p = q + p$. Clearly $q \in \mathbb{H}_\kappa$. For every $\alpha < \kappa$, define $x_\alpha \in G$ by $\mathrm{supp}(x_\alpha) = \{\alpha\}$, and let $X_\alpha = \{x_\beta : \alpha \leq \beta < \kappa\}$. Pick any ultrafilter r on G extending the family of subsets X_α, where $\alpha < \kappa$. Then $r, r + q \in \mathbb{H}_\kappa$ and $r \neq r + q$. Indeed,

$$Y = \bigcup_{\alpha<\kappa} (x_\alpha + G_{\alpha+1} \setminus \{0\}) \in r + q$$

and $|\mathrm{supp}(y)| > 1$ for all $y \in Y$, but $|\mathrm{supp}(x)| = 1$ for all $x \in X_0$. On the other hand, it follows from $p = q + p$ that $r + p = r + q + p$ and, since p is right cancellable in $\mathbb{H}_\kappa$, we obtain that $r = r + q$, a contradiction.

(3)⇒(4) Assume on the contrary that there is $q \in G^*$ for which $p = q + p$. Then $C(p) \neq \emptyset$. Since $C(p)$ is a closed subsemigroup of G^*, it has an idempotent, a contradiction.

(4)⇒(5) Assume on the contrary that $G + p \subset \beta G$ is not discrete. Then there is $a \in G$ such that $a + p \in c\ell((G \setminus \{a\}) + p)$. Since

$$c\ell((G \setminus \{a\}) + p) = (\beta G \setminus \{a\}) + p,$$

we obtain that there is $r \in \beta G \setminus \{a\}$ such that $a + p = r + p$, so $-a + r + p = p$. Since $r \neq a$, $-a + r \in G^*$ by [9, Lemma 6.28]. Let $q = -a + r$. Then $q \in G^*$ and $q + p = p$, a contradiction.

(5)⇒(6) Since $G + p \subset \beta G$ is discrete, for every $x \in G$, there is $B_x \in p$ such that $y + p \notin \overline{x + B_x}$ for all $y \in G \setminus \{x\}$; that is, $x + B_x \notin y + p$ for all $y \in G \setminus \{x\}$. For every $x \in G \setminus \{0\}$, let

$$H_x = \{0\} \cup \{y \in G \setminus \{0\} : \phi(y) < \phi(x) \text{ and } \operatorname{supp}(y) \subset \operatorname{supp}(x)\}.$$

Put $A_0 = B_0$ and inductively for every $\alpha < \kappa$ and for every $x \in G$ with $\phi(x) = \alpha$, choose $A_x \in p$ such that

(i) $A_x \subseteq B_x \cap G_{\phi(x)+1}$, and
(ii) $(x + A_x) \cap (y + A_y) = \emptyset$ for all $y \in H_x$.

This can be done because H_x is finite and $y + A_y \subseteq y + B_y \notin x + p$ for all $y \in H_x$. We now claim that $(x + A_x) \cap (y + A_y) = \emptyset$ for all different $x, y \in G$.

Indeed, without loss of generality one may suppose that $x \neq 0$ and $\phi(y) \leq \phi(x)$ or $y = 0$. If $y \in H_x$, the statement holds by (ii). Otherwise $\operatorname{supp}(y) \setminus \operatorname{supp}(x) \neq \emptyset$ or $\phi(y) = \phi(x)$; in any case,

$$(y + G_{\phi(y)+1}) \cap (x + G_{\phi(x)+1}) = \emptyset,$$

so the statement holds by (i).

For every $x \in G$, $\overline{x + A_x}$ is a neighborhood of $x + p \in \beta G$, and all these neighborhoods are pairwise disjoint. Hence $G + p \subset \beta G$ is strongly discrete.

(6)⇒(7) Since $G + p \subset \beta G$ is strongly discrete, for each $x \in G$, there is $A_x \in p$ such that the subsets $\overline{x + A_x} \subset \beta G$, where $x \in G$, are pairwise disjoint. Then the subsets $x + A_x \subset G$, where $x \in G$, are pairwise disjoint. It follows that p is strongly discrete.

(7)⇒(1) Since p is strongly discrete, for each $x \in G$, there is $A_x \in p$ such that the subsets $x + A_x$ are pairwise disjoint. Let $q, r \in \beta G$ and $q \neq r$. Choose disjoint $Q \in q$ and $R \in r$ and put

$$A = \bigcup_{x \in Q} x + A_x \quad \text{and} \quad B = \bigcup_{x \in R} x + A_x.$$

Then $A \in q + p$, $B \in r + p$ and $A \cap B = \emptyset$, so $q + p \neq r + p$. Hence p is right cancellable. □

Remark 4.3. Let S be an arbitrary infinite Abelian group of cardinality κ. Then there is a κ-sequence $(F_\alpha)_{\alpha<\kappa}$ of nonempty finite subsets of S with the property that every element $x \in S$ is uniquely representable in the form $x = \sum_{\alpha \in \operatorname{supp}(x)} x_\alpha$, where $\operatorname{supp}(x)$ is a finite subset of κ and $x_\alpha \in F_\alpha$ for all $\alpha \in \operatorname{supp}(x)$ [7]. Define the left invariant topology $\mathcal{L}$ on S by taking as a neighborhood base at zero the subsets $S_\alpha = \{x \in S : \operatorname{supp}(x) \cap \alpha = \emptyset\}$, $\alpha < \kappa$. It is easy to see that Theorem 4.2 remains to be true with G and $\mathbb{H}_\kappa$ replaced by S and $\operatorname{Ult}(\mathcal{L})$, respectively.

Remark 4.4. Right cancellable ultrafilters on countable groups have been characterized by Hindman and Strauss [9, Section 8.2]. They also showed that for each such ultrafilter p, the semigroup C_p admits a continuous homomorphism onto $\beta\mathbb{N}$ [9, Theorem 8.51].

5. Right maximal idempotents

Recall that an ultrafilter u on κ is *countably complete* (or ω_1-*complete*) if whenever $\{A_n : n < \omega\}$ is a partition of κ, there is $n < \omega$ such that $A_n \in u$. An ultrafilter u on κ is κ-*complete* if whenever $\lambda < \kappa$ and $\{A_\alpha : \alpha < \lambda\}$ is a partition of κ, there is $\alpha < \lambda$ such that $A_\alpha \in u$. A cardinal κ is *measurable* (*Ulam-measurable*) if there is a κ-complete (countably complete) nonprincipal ultrafilter on κ. A cardinal is Ulam-measurable if and only if it is greater than or equal to the first uncountable measurable cardinal. It is consistent with ZFC that there is no uncountable measurable cardinal. (See [2, Section 8].)

In this section, we prove the following.

Theorem 5.1. *Let p be a right maximal idempotent in $\mathbb{H}_\kappa$. Then $C(p)$ is a compact right zero semigroup, and if $\overline{\theta}(p)$ is countably incomplete, $C(p)$ is finite.*

The next proposition is the first part of Theorem 5.1.

Proposition 5.2. *For every right maximal idempotent $p \in \mathbb{H}_\kappa$, $C(p)$ is a right zero semigroup.*

Proof. Let $C = C(p)$ and let $q \in C$. Suppose that q is not right cancellable in $\mathbb{H}_\kappa$. Then by Theorem 4.2, there is an idempotent $r \in G^*$ such that $r + q = q$. It follows that $r + q + p = q + p$ and, since $q + p = p$, we obtain that $r + p = p$, that is, $p \leq_R r$. Since p is right maximal, this implies that $r \leq_R p$, that is, $p + r = r$. But then

$$p + q = p + r + q = r + q = q$$

and

$$q + q = q + p + q = p + q = q.$$

Hence, the elements of C which are not right cancellable in $\mathbb{H}_\kappa$ form a right zero semigroup.

Now we claim that no element of C is right cancellable in $\mathbb{H}_\kappa$. Indeed, assume on the contrary that some $q \in C$ is right cancellable in $\mathbb{H}_\kappa$. Then by Theorem 4.2, q is a strongly discrete ultrafilter on G. Consequently by Theorem 3.8, C_p admits a continuous homomorphism onto $\beta\mathbb{N}$. Taking any nontrivial finite left zero semigroup in $\beta\mathbb{N}$ (see [9, Theorem 6.9]), we obtain by the next lemma that there is a nontrivial left zero semigroup in C_p, a contradiction. □

Lemma 5.3. *Let S and T be compact Hausdorff right topological semigroups, let $f : S \to T$ be a continuous surjective homomorphism, and let $Q \subseteq T$ be a finite left zero semigroup. Then there is a left zero semigroup $P \subseteq S$ such that $f|_P : P \to Q$ is an isomorphism.*

Proof. Let $C = f^{-1}(Q)$. Then C is a closed subsemigroup of S and for each $q \in Q$, $f^{-1}(q)$ is a right ideal of C, so there is a minimal right ideal $R_q \subseteq f^{-1}(q)$ of C. Also there is a minimal left ideal L of C. For each $q \in Q$, let p_q be the identity of the group $R_q \cap L$. Then $P = \{p_q : q \in Q\}$ is as required. □

The second part of Theorem 5.1 follows from the first one and the next proposition.

Proposition 5.4. *Let C be a compact right zero semigroup in $\mathbb{H}_\kappa$ and let $\overline{\theta}(C) = \{u\}$. If u is countably incomplete, then C is finite.*

Note that for every right zero semigroup $C \subset \mathbb{H}_\kappa$, $\overline{\theta}(C)$ is a singleton. Indeed, the function $\overline{\theta}$ has the property that

$$\overline{\theta}(x+y) = \overline{\theta}(x)$$

whenever $x \in \beta G \setminus \{0\}$ and $y \in \mathbb{H}_\kappa$. If $x, y \in C$, then $y = x + y$; consequently,

$$\overline{\theta}(y) = \overline{\theta}(x+y) = \overline{\theta}(x).$$

To prove Proposition 5.4, we need the following version of the Frolík lemma [9, Theorem 3.40].

Lemma 5.5. *Let $X = \bigcup_{n<\omega} X_n$ and $Y = \bigcup_{n<\omega} Y_n$ be subsets of $\beta\kappa$. Suppose that for each $n < \omega$, $c\ell X \cap c\ell Y_n = c\ell X_n \cap c\ell Y = \emptyset$. Then $c\ell X \cap c\ell Y = \emptyset$.*

Proof. Let $A = \bigcup_{n<\omega} c\ell X_n$ and $B = \bigcup_{n<\omega} c\ell Y_n$. We have that

$$X \subseteq A,\ Y \subseteq B,\ c\ell X = c\ell A \ \text{ and } \ c\ell Y = c\ell B.$$

It then follows that $c\ell A \cap B = A \cap c\ell B = \emptyset$. Since also both A and B are σ-compact, we can apply the Frolík lemma. Hence $c\ell A \cap c\ell B = \emptyset$, and so $c\ell X \cap c\ell Y = \emptyset$. □

Proof of Proposition 5.4. Assume on the contrary that C is infinite and pick any countably infinite subset $X \subset C$. Then there is $p \in c\ell X \setminus X$. Put $Y = (G \setminus \{e\})p$. Since $c\ell Y = (\beta G \setminus \{0\})p$ and $p = p + p$, $p \in c\ell Y$. Consequently $c\ell X \cap c\ell Y \neq \emptyset$. Also we have that for every $x \in X$, $x \notin c\ell Y$. Indeed, otherwise $x = y + p$ for some $y \in \beta G$ and then

$$x + p = y + p + p = y + p = x.$$

But $x + p = p \neq x$, since $x \in X \subset C$, $p \in c\ell X \setminus X \subset C$ and C is a right zero semigroup.

Hence, in order to derive a contradiction, it suffices, by Lemma 5.5, to construct a partition $\{A_n : n < \omega\}$ of $G \setminus \{0\}$ such that $c\ell X \cap c\ell Y_n = \emptyset$, where $Y_n = A_n + p$.

Since u is countably incomplete, there is a partition $\{B_n : n < \omega\}$ of κ such that $B_n \notin u$ for all $n < \omega$; equivalently, $u \notin \overline{B_n}$. Put $A_n = \theta^{-1}(B_n)$. Then for each $x \in c\ell X$, $\overline{\theta}(x) = u$, and for each $y \in Y_n$, $\overline{\theta}(y) \in B_n$, so for each $y \in c\ell Y_n$, $\overline{\theta}(y) \in \overline{B_n}$. Hence, $c\ell X \cap c\ell Y_n = \emptyset$. □

6. The Local Monomorphism Theorem

This section is the core of the whole construction. Recall that we use the notation from Definition 4.1.

Definition 6.1. Let $\mathcal{T}$ be a translation invariant topology on G such that $\mathcal{G} \subseteq \mathcal{T}$ and let X be an open neighborhood of 0 in $\mathcal{T}$. Denote by $P(X)$ the set of all $x \in X \setminus \{0\}$ which cannot be decomposed into a sum $x = y + z$, where $y, z \in X \setminus \{0\}$ and $\phi(y) < \theta(z)$. Note that $|P(X)| = \kappa$. We say that X satisfies the *P-condition* if there is a neighborhood W of $0 \in X$ such that $|P(X) \setminus W| = \kappa$.

It follows from the next lemma that $P(X)$ is a strongly discrete subset of X with at most one limit point 0.

Lemma 6.2. *Let $x \in P(X)$ and $y \in X \setminus \{0\}$. If $x \in y + G_{\phi(y)+1} \cap X$, then $x = y$.*

Proof. Otherwise $x = y + z$ for some $z \in G_{\phi(y)+1} \cap (X \setminus \{0\})$ and then $\phi(y) < \theta(z)$, which contradicts $x \in P(X)$. □

Suppose that X has the property that, whenever D is a strongly discrete subset of X with exactly one limit point 0, there is $A \subset D$ such that $|A| = \kappa$ and 0 is not a limit point of A. Then, obviously, X satisfies the P-condition. In particular, X satisfies the P-condition if $\mathrm{Ult}(\mathcal{T})$ is finite.

In fact, the P-condition is always satisfied in the following sense.

Lemma 6.3. *Let $x \in X \setminus \{0\}$, let $H = \{y \in X \setminus \{0\} : \mathrm{supp}(y) \subseteq \mathrm{supp}(x)\}$, and let $Y = X \setminus H$. Then Y satisfies the P-condition.*

Proof. Choose a subset $A \subset Y$ with $|A| = \kappa$ such that whenever $y, z \in A$ and $y \neq z$, one has $\mathrm{supp}(y) \cap \mathrm{supp}(z) = \mathrm{supp}(x)$. For each $y \in A$, there is $z_y \in P(Y)$ such that $\mathrm{supp}(z_y) \subseteq \mathrm{supp}(z)$ and $\mathrm{supp}(z_y) \cap \mathrm{supp}(x) \neq \emptyset$. Since $H \cap Y = \emptyset$, we have that $\mathrm{supp}(z_y) \setminus \mathrm{supp}(x) \neq \emptyset$ for all $y \in A$. Put $B = \{z_y : y \in A\}$. Then $B \subseteq P(Y)$, $B \cap G_{\phi(x)+1} = \emptyset$ and $|B| = \kappa$. □

The Local Monomorphism Theorem is the following result.

Theorem 6.4. *Let $\mathcal{T}$ be a translation invariant topology on G such that $\mathcal{G} \subseteq \mathcal{T}$, let X be an open neighborhood of 0 in $\mathcal{T}$, and let S be a cancellative semigroup with identity and $|S| = \kappa$. Suppose that X is zero-dimensional and satisfies the P-condition. Then there is a local monomorphism $f : X \to S$ such that the topology $\mathcal{T}^f$ is zero-dimensional. If $S = G$, then f can be chosen to be continuous with respect to $\mathcal{G}$.*

Proof. The proof consists of several lemmas.

First of all, using the P-condition, choose a clopen neighborhood W of $0 \in X$ such that $|P(X) \setminus W| = \kappa$.

Lemma 6.5. *There is a subset $A \subset X \setminus \{0\}$ and a partition $\{X(a) : a \in A\}$ of $X \setminus \{0\}$ such that*

(1) *for each $a \in A$, $X(a)$ is a clopen neighborhood of $a \in X \setminus \{0\}$,*
(2) *for each $a \in A$, $X(a) - a \subseteq W \cap G_{\phi(a)+1}$, and*
(3) *for each $a \in A \setminus W$, $X(a) \cap W = \emptyset$.*

Proof. For every $x \in X \setminus \{0\}$, choose a clopen neighborhood U_x of $0 \in X$ such that $U_x \subseteq W$, $x + U_x \subseteq X \setminus \{0\}$, and $(x + U_x) \cap W = \emptyset$ if $x \notin W$, and let

$$H_x = \{y \in X \setminus \{0\} : \phi(y) < \phi(x) \text{ and } \mathrm{supp}(y) \subset \mathrm{supp}(x)\}.$$

Note that H_x is finite. For every $\alpha < \kappa$, let

$$X_\alpha = \{x \in X \setminus \{0\} : \phi(x) = \alpha\}.$$

Now put $A_{-1} = \emptyset$ and inductively, for every $\alpha < \kappa$, define a subset $A_\alpha \subseteq X_\alpha$ and for every $a \in A_\alpha$, a clopen neighborhood $X(a)$ of $a \in X \setminus \{0\}$ by

(i) $A_\alpha = X_\alpha \setminus \bigcup_{b \in A'_\alpha} X(b)$, where $A'_\alpha = \bigcup_{\beta < \alpha} A_\beta$, and
(ii) $X(a) = (a + U_a \cap G_{\phi(a)+1}) \setminus \bigcup_{b \in H_a} X(b)$.

We put $A = \bigcup_{\alpha < \kappa} A_\alpha$.

It follows from (i) that for every $a \in A_\alpha$, $a \notin \bigcup_{b \in A'_\alpha} X(b)$. Then, since H_a is finite, we obtain from (ii) that $X(a)$ is indeed a clopen neighborhood of $a \in X \setminus \{e\}$.

It is also clear that both (2) and (3) are satisfied and that the subsets $X(a)$, $a \in A$, cover $X \setminus \{0\}$. To see that they are disjoint, let $a \in A_\alpha$, $b \in A_\alpha \cup A'_\alpha$ and $a \neq b$. If $b \in H_a$, then $X(a) \cap X(b) = \emptyset$ by (ii). Otherwise $\mathrm{supp}(b) \setminus \mathrm{supp}(a) \neq \emptyset$ or $b \in A'_\alpha$; in any case, $(a + G_{\phi(a)+1}) \cap (b + G_{\phi(b)+1}) = \emptyset$, so again $X(a) \cap X(b) = \emptyset$. □

Let F be the free semigroup on the alphabet A including the empty word $\emptyset$ and let M be the subset of F consisting of all nonempty words $a_1 \dots a_n$ such that

$$a_i + \cdots + a_n \in X(a_i)$$

for each $i = 1, \dots, n-1$. In particular,

$$a_1 + \cdots + a_n \in X(a_1) \subseteq X \setminus \{0\}.$$

Lemma 6.6. *Each $x \in X \setminus \{0\}$ can be uniquely decomposed into a sum*

$$x = a_1 + \cdots + a_n,$$

where $a_1 \dots a_n \in M$.

Proof. Let $x \in X \setminus \{0\}$. Then $x \in X(a_1)$ for some $a_1 \in A$. If $x = a_1$, we are done. Otherwise $x = a_1 + x_1$, where $x_1 = x - a_1 \in W \setminus \{0\}$, and $\phi(a_1) < \theta(x_1)$.

Suppose that we have decomposed x into a sum

$$x = a_1 + \cdots + a_i + x_i$$

with $a_1, \dots, a_i \in A$ and $x_i \in W \setminus \{0\}$ such that

(i) $\phi(a_j) < \theta(a_{j+1})$ for each $j = 1, \dots, i-1$ and $\phi(a_i) < \theta(x_i)$, and
(ii) $a_j + \cdots + a_i + x_i \in X(a_j)$ for each $j = 1, \dots, i$.

Then $x_i \in X(a_{i+1})$ for some $a_{i+1} \in A$. Since $\theta(a_{i+1}) = \theta(x_i)$, $\phi(a_i) < \theta(a_{i+1})$. If $x_i = a_{i+1}$, we are done: $x = a_1 + \cdots + a_{i+1}$ and $a_1 \dots a_{i+1} \in M$. Otherwise $x_i = a_{i+1} + x_{i+1}$, where $x_{i+1} = x_i - a_{i+1}$. Then $x = a_1 + \cdots + a_{i+1} + x_{i+1}$ and (i) and (ii) are satisfied with i replaced by $i+1$.

After $\leq |\mathrm{supp}(x)|$ steps we obtain the required decomposition.

To see that such a decomposition is unique, let $a_1 \dots a_n, b_1 \dots b_m \in M$ and let

$$a_1 + \cdots + a_n = b_1 + \cdots + b_m.$$

We show that $n = m$ and $a_i = b_i$ for each $i = 1, \dots, n$. We proceed by induction on $\min\{n, m\}$.

Suppose that $\min\{n, m\} = 1$, say $n = 1$. We have that $a_1 \in X(a_1)$ and

$$a_1 = b_1 + \cdots + b_m \in X(b_1).$$

Since $\{X(a) : a \in A\}$ is disjoint, it follows that $a_1 = b_1$. But then also $m = 1$. Indeed, otherwise $b_2 + \cdots + b_m = 0$, which contradicts $b_2 + \cdots + b_m \in X(b_2)$.

Now suppose that $\min\{n, m\} > 1$. Again we have that $a_1 + \cdots + a_n \in X(a_1)$ and

$$a_1 + \cdots + a_n = b_1 + \cdots + b_m \in X(b_1),$$

and so $a_1 = b_1$. But then $a_2 + \cdots + a_n = b_2 + \cdots + b_m$ and we can apply the inductive assumption. □

Define $h : X \to F$ by putting $h(0) = \emptyset$ and

$$h(a_1 + \cdots + a_n) = a_1 \dots a_n$$

for every $a_1 \dots a_n \in M$.

Lemma 6.7. *h is a local monomorphism.*

Proof. It suffices to check that h is a local homomorphism. Let $x \in X \setminus \{0\}$ and let $x = a_1 + \cdots + a_n$, where $a_1 \dots a_n \in M$. For each $i = 1, \dots, n$, one has $a_i + \cdots + a_n \in X(a_i)$. Define the neighborhood U of $0 \in X$ by

$$U = \bigcap_{i=1}^{n} (X(a_i) - (a_i + \cdots + a_n)) \cap X.$$

Let $y \in U \setminus \{0\}$ and let $y = a_{n+1} + \cdots + a_{n+m}$, where $a_{n+1} \dots a_{n+m} \in M$. Then for each $i = 1, \dots, n$, one has $a_i + \cdots + a_n + y \in X(a_i)$. It follows that $a_1 \dots a_n a_{n+1} \dots a_{n+m} \in M$ and we obtain that

$$\begin{aligned} h(x+y) &= h(a_1 + \cdots + a_n + a_{n+1} \cdots + a_{n+m}) \\ &= a_1 \dots a_n a_{n+1} \dots a_{n+m} \\ &= h(x)h(y). \end{aligned}$$

□

By Lemma 2.13, the local monomorphism h induces a left invariant topology $\mathcal{T}^h$ on F. We have that $Y = M \cup \{\emptyset\}$ is an open neighborhood of the identity of $(F, \mathcal{T}^h)$ and h homeomorphically maps X onto Y, so Y is zero-dimensional.

Lemma 6.8. *$\mathcal{T}^h$ is zero-dimensional.*

Proof. It suffices to show that Y is closed in $\mathcal{T}^h$. Let $a_1 \dots a_n \in F \setminus Y$. Then $a_i + \cdots + a_n \notin X(a_i)$ for some $i = 1, \dots, n-1$, so $a_{i+1} + \cdots + a_n \notin X(a_i) - a_i$. Taking the biggest such i we obtain that $a_{i+1} + \cdots + a_n \in X(a_{i+1}) \subseteq X$. It follows that there is a neighborhood U of $0 \in X$ such that $(a_{i+1} + \cdots + a_n + U) \cap (X(a_i) - a_i) = \emptyset$, so

$$(a_i + \cdots + a_n + U) \cap X(a_i) = \emptyset.$$

We claim that $(a_1 \dots a_n h(U)) \cap Y = \emptyset$.

Indeed, assume on the contrary that $a_1 \dots a_n h(y) \in M$ for some $y \in U \setminus \{0\}$. Let $h(y) = a_{n+1} \dots a_{n+m} \in M$. Since $a_1 \dots a_n h(y) = a_1 \dots a_n a_{n+1} \dots a_{n+m} \in M$, we obtain that

$$a_i + \cdots + a_n + y = a_i + \cdots + a_n + a_{n+1} + \cdots + a_{n+m} \in X(a_i),$$

which is a contradiction. □

Denote $I = A \setminus W$. Then $|I| = \kappa$ and for every $a_1 \dots a_n \in M$,

$$a \in \{a_1, \dots, a_n\} \cap I \text{ implies } a = a_1.$$

Indeed, by the construction of A (Lemma 6.5) and Lemma 6.2, $P(X) \subseteq A$, and by the choice of W, $|P(X) \setminus W| = \kappa$, so the first statement holds. Since

$$a_i + \cdots + a_n \in X(a_{i-1}) - a_{i-1} \subseteq W$$

for each $i = 2, \dots, n$, the second one holds as well.

Now let N denote the subset of F consisting of all nonempty words $a_1 \dots a_n$ such that $\phi(a_i) < \theta(a_{i+1})$ for each $i = 1, \dots, n-1$ and

$$a \in \{a_1, \dots, a_n\} \cap I \text{ implies } a = a_1.$$

Clearly $Z = N \cup \{\emptyset\}$ is a neighborhood of the identity of $(F, \mathcal{T}^h)$ containing Y. Furthermore, for every $\alpha < \kappa$,

$$Z_\alpha = \{a_1 \dots a_n \in N : \theta(a_1) \geq \alpha\} \cup \{\emptyset\}$$

is a neighborhood of the identity, and for every $a_1 \dots a_n \in N$,

$$a_1 \dots a_n Z_{\phi(a_n)+1} \subseteq Z,$$

so Z is open. In addition, and as distinguished from Y, Z has the property that, whenever $a_1 \dots a_n \in N$ and $i = 1, \dots, n-1$, one has $a_1 \dots a_i \in N$.

Lemma 6.9. *There is a local isomorphism $g : Z \to S$. If $S = G$, then g can be chosen to be continuous with respect to $\mathcal{G}$.*

Proof. We shall construct a bijection $g : Z \to S$ such that $g(\emptyset) = e$ and

$$g(a_1 \dots a_n) = g(a_1) \cdots g(a_n)$$

for each $a_1 \dots a_n \in N$. That such a g is a local homomorphism follows from the last but one sentence preceding the lemma.

It suffices to define g on A so that

(i) whenever $a_1 \dots a_n$ and $b_1 \dots b_m$ are different elements of N, $g(a_1) \cdots g(a_n)$ and $g(b_1) \cdots g(b_m)$ are different elements of $S \setminus \{e\}$, and
(ii) for each $s \in S \setminus \{e\}$, there is $a_1 \dots a_n \in N$ such that $g(a_1) \cdots g(a_n) = s$.

To this end, enumerate A without repetitions as $\{c_\alpha : \alpha < \kappa\}$ so that if $a, b \in A$, $\phi(a) < \phi(b)$, $a = c_\alpha$ and $b = c_\beta$, then $\alpha < \beta$. This defines $\nu : A \to \kappa$ by $\nu(c_\alpha) = \alpha$. Note that whenever $a_1 \dots a_n \in N$, one has $\nu(a_1) < \cdots < \nu(a_n)$. Also enumerate $S \setminus \{e\}$ as $\{s_\alpha : \alpha < \kappa\}$.

Fix $\alpha < \kappa$ and suppose that values $g(c_\beta)$ have already been defined for all $\beta < \alpha$ so that, whenever $a_1 \dots a_n$ and $b_1 \dots b_m$ are different elements of N with $\nu(a_n), \nu(b_m) < \alpha$, $g(a_1) \cdots g(a_n)$ and $g(b_1) \cdots g(b_m)$ are different elements of $S \setminus \{e\}$. Let

$$S_\alpha = \{g(a_1) \cdots g(a_n) \in S : a_1 \dots a_n \in N \text{ and } \nu(a_n) < \alpha\} \cup \{e\}.$$

Consider two cases.

Case 1. $c_\alpha \notin I$. Pick as $g(c_\alpha)$ any element of $S \setminus (S_\alpha^{-1} S_\alpha)$. This can be done because $|S_\alpha^{-1} S_\alpha| \le |S_\alpha|^2 < \kappa$, by virtue of S is left cancellative. Then whenever $a_1 \dots a_n \in N$ and $a_n = c_\alpha$, one has $g(a_1) \cdots g(a_n) \notin S_\alpha$. Indeed, otherwise

$$g(c_\alpha) = g(a_n) \in (g(a_1) \cdots g(a_{n-1}))^{-1} S_\alpha \subseteq S_\alpha^{-1} S_\alpha.$$

Also if $a_1 \dots a_n$ and $b_1 \dots b_m$ are different elements of N with $a_n = b_m = c_\alpha$, then $g(a_1) \cdots g(a_n) \neq g(b_1) \cdots g(b_m)$, by the inductive hypothesis and by virtue of the fact that S is right cancellative.

Case 2. $c_\alpha \in I$. Then whenever $a_1 \dots a_n \in N$ and $a_n = c_\alpha$, one has $n = 1$. Put $g(c_\alpha)$ to be the first element in the sequence $\{s_\beta : \beta < \kappa\} \setminus S_\alpha$.

It is clear that the mapping $g : A \to S$ so constructed satisfies (i), and since $|I| = \kappa$, (ii) is satisfied as well.

If $S = G$, the construction remains the same with the only correction in Case 1: we pick $g(c_\alpha)$ so that $\phi(g(c_\beta)) < \theta(g(c_\alpha))$ for all $\beta < \alpha$. To see that such a g is continuous, let $\alpha < \kappa$ be given. Choose $\beta < \kappa$ such that $\phi(g(c_\beta)) \ge \alpha$ and put

$$U_\alpha = \{a_1 \dots a_n \in N : \nu(a_1) > \beta \text{ and } a_1 \notin I\} \cup \{\emptyset\}.$$

Then U_α is a neighborhood of the identity of Z and $g(U_\alpha) \subseteq G_{\alpha+1}$. $\square$

The local isomorphism $g : Z \to S$ induces a left invariant topology $\mathcal{T}^g$ on S, and since Z is zero-dimensional, so is $\mathcal{T}^g$. We put $f = g \circ h$. Then $f : X \to S$ is a local monomorphism and $\mathcal{T}^f = \mathcal{T}^g$. If $S = G$ and if we choose g to be continuous with respect to $\mathcal{G}$, then so is f. □

We will use Theorem 6.4 in a very special situation when the topology $\mathcal{T}$ is maximal. But it is also interesting in the general case.

Corollary 6.10. *Let $\mathcal{T}$ be a locally zero-dimensional left invariant topology on G such that $\mathcal{G} \subseteq \mathcal{T}$ and let S be a cancellative semigroup with identity and $|S| = |G|$. Then there is a zero-dimensional left invariant topology $\mathcal{R}$ on S such that*

(1) *$(S, \mathcal{R})$ is homeomorphic to an open neighborhood of zero of $(G, \mathcal{T})$,*
(2) *$\mathrm{Ult}(\mathcal{R})$ is isomorphic to $\mathrm{Ult}(\mathcal{T})$, and*
(3) *$\mathrm{Ult}(\mathcal{R})$ is left saturated in βS.*

If $S = G$, then $\mathcal{R}$ can be chosen to be stronger than $\mathcal{G}$.

Proof. Let X be a zero-dimensional open neighborhood of zero of $(G, \mathcal{T})$. By Lemma 6.3, one may suppose that X satisfies the P-condition. Then by Theorem 6.4, there is a local monomorphism $f : X \to S$ such that the topology $\mathcal{T}^f$ on S is zero-dimensional. Put $\mathcal{R} = \mathcal{T}^f$. By Lemma 2.13, conditions (1) and (2) are satisfied, and by Lemma 2.5, (3) is satisfied as well. If $S = G$, choose f to be continuous with respect to $\mathcal{G}$. □

Remark 6.11. Theorem 6.4 was developed from [21, Proposition 3.4]. The first result of such a type was the Local Isomorphism Theorem [18, Theorem 2] (see also [9, Theorem 7.24]). It says that, whenever X and Y are countable nondiscrete regular local left topological groups with a countable base, there is a local isomorphism $f : X \to Y$ that is a homeomorphism. By a local left topological group one means an open neighborhood of the identity of a left topological group.

7. Regular idempotents

In this section, we prove that

Theorem 7.1. *For every infinite cancellative semigroup S of cardinality κ, there are 2^{2^κ} regular idempotents in βS.*

Proof. If S is cancellative, so is S^1 [1, Section 1.1, Exercise 2]. Consequently, one may suppose that S has an identity.

Let I denote the set of all right maximal idempotents $p \in \mathbb{H}_\kappa$ such that $\overline{\theta}(p)$ is countably incomplete.

Lemma 7.2. $|I| = 2^{2^\kappa}$.

Proof. Let $\{A_n : n < \omega\}$ be a partition of κ such that $|A_n| = \kappa$ for every $n < \omega$ and let U be the set of all uniform ultrafilters u on κ such that for every $m < \omega$, $\bigcup_{m \le n < \omega} A_n \in u$. It is clear that each $u \in U$ is countably incomplete. We claim that $|U| = 2^{2^\kappa}$.

Indeed, for every $n < \omega$, there are 2^{2^κ} uniform ultrafilters on κ containing A_n. Enumerate them without repetitions as $u_{n,\alpha}$, $\alpha < 2^{2^\kappa}$. For every $\alpha < 2^{2^\kappa}$, pick any ultrafilter u_α on κ such that $\bigcup_{m \le n < \omega} B_{n,\alpha} \in u_\alpha$, whenever $m < \omega$ and $B_{n,\alpha} \in u_{n,\alpha}$. Then $u_\alpha \in U$ and $u_\alpha \neq u_\beta$ if $\alpha \neq \beta$.

Now for each $u \in U$, let $J_u = \overline{\theta}^{-1}(u)$. Then whenever $x \in J_u$ and $y \in \mathbb{H}_\kappa$,

$$\overline{\theta}(x+y) = \overline{\theta}(x) = u.$$

It follows that $\{J_u : u \in U\}$ is a disjoint family of closed right ideals of $\mathbb{H}_\kappa$. For each $u \in U$, pick a right maximal idempotent p_u in J_u. We claim that $p_u \in I$.

It suffices to check that p_u is right maximal in $\mathbb{H}_\kappa$. Let q be an idempotent in $\mathbb{H}_\kappa$ such that $p_u \leq_R q$. Then $q + p_u = p_u$; consequently,

$$\overline{\theta}(q) = \overline{\theta}(q + p_u) = \overline{\theta}(p_u) = u.$$

Hence $q \in J_u$. Since p_u is right maximal in J_u, it follows that $q \leq_R p_u$. □

Now let $p \in I$. By Theorem 5.1, $C(p)$ is a finite right zero semigroup. Then by Corollary 2.10, p is locally regular. Let X_p be a regular open neighborhood of $0 \in G$ in $\mathcal{T}_p$. By Lemma 2.3, $\mathcal{T}_p$ is extremally disconnected. Since an open subspace of an extremally disconnected space is extremally disconnected and a regular extremally disconnected space is zero-dimensional, we obtain that X_p is zero-dimensional. By Theorem 6.4, there is a local monomorphism $f_p : X_p \to S$ such that the topology $\mathcal{T}^{f_p}$ on S is zero-dimensional. Let $\overline{f_p} : c\ell_{\beta G} X_p \to \beta S$ be the continuous extension of f_p and let $q_p = \overline{f_p}(p)$. Then, by Lemma 2.13, $\mathcal{T}^{f_p} = \mathcal{T}_{q_p}$ and q_p is a regular idempotent in βS.

Since $|I| = 2^{2^\kappa}$ and there are only 2^κ different mappings f_p, there exists $K \subseteq I$ with $|K| = 2^{2^\kappa}$ such that $f_p = f_r = f$ for all $p, r \in K$. Since f is injective, it follows that $q_p \neq q_r$ for all distinct $p, r \in K$. □

In the case where $S = G$, Theorem 7.1 can be strengthened. Choosing local monomorphisms $f_p : X_p \to S$ to be continuous with respect to $\mathcal{G}$, we obtain that

Theorem 7.3. *There are 2^{2^κ} regular idempotents in $\mathbb{H}_\kappa$.*

We conclude the paper with the following remark.

Remark 7.4. The existence of a regular idempotent (Theorem 1.2) is a stronger result than the existence of a homogeneous regular maximal space (Corollary 1.3). The latter can be deduced from the existence of a locally regular idempotent (Theorem 5.1), not involving the Local Monomorphism Theorem. Indeed, let p be a locally regular idempotent in $\mathbb{H}_\kappa$ and let X be a regular open neighborhood of $0 \in G$ in $\mathcal{T}_p$. Then X is maximal and zero-dimensional. To see that X is homogeneous, let $x, y \in X$. Choose a clopen neighborhood U of $0 \in X$ such that $x + U$ and $y + U$ are disjoint and contained in X. Define $f : X \to X$ by

$$f(z) = \begin{cases} y - x + z & \text{if } z \in x + U, \\ x - y + z & \text{if } z \in y + U, \\ z & \text{otherwise.} \end{cases}$$

Then f is a homeomorphism with $f(x) = y$.

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School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa

E-mail address: yevhen.zelenyuk@wits.ac.za