TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 362, Number 6, June 2010, Pages 2833–2843
S 0002-9947(10)05058-0
Article electronically published on January 21, 2010

LOCAL EQUIVALENCE OF SYMMETRIC HYPERSURFACES IN $\mathbb{C}^2$

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Abstract. The Chern-Moser normal form and its analog on finite type hypersurfaces in general do not respect symmetries. Extending the work of N. K. Stanton, we consider the local equivalence problem for symmetric Levi degenerate hypersurfaces of finite type in $\mathbb{C}^2$. The results give complete normalizations for such hypersurfaces, which respect the symmetries. In particular, they apply to tubes and rigid hypersurfaces, providing an effective classification. The main tool is a complete normal form constructed for a general hypersurface with a tube model. As an application, we describe all biholomorphic maps between tubes, answering a question posed by N. Hanges. Similar results for hypersurfaces admitting nontransversal symmetries are obtained.

1. Introduction

One of the classical problems in CR geometry concerns local biholomorphic classification of real hypersurfaces in complex space. The extrinsic approach to the problem, originating in the work of Poincaré ([17]), is to analyze directly the action of local biholomorphisms on the defining equation of the hypersurface. In the Levi nondegenerate case it was completed in the normal form construction of Chern and Moser ([2]).

In recent years, there has been substantial progress in understanding the problem on Levi degenerate manifolds (e.g. [3], [5], [12]). In particular, we mention a result of Kim and Zaitsev, which shows that the second, intrinsic approach of Cartan, Chern and Tanaka is in general not available. Following the extrinsic approach, a normal form construction for Levi degenerate hypersurfaces of finite type in dimension two was obtained in [14].

As an immediate application, the normal forms can be used for a closely related geometric problem in order to determine local symmetries of a hypersurface. In fact, except for the sphere and its blow-ups, all local automorphisms (i.e. those fixing the given point) of finite type hypersurfaces in $\mathbb{C}^2$ are linear in some normal coordinates. The symmetries are then immediately visible from the defining equation (in the nondegenerate case it follows from a result of Kruzhilin and Loboda [16], and in the degenerate case from [15]).

On the other hand, for automorphisms not fixing the point this is no longer true. The simplest example is given by rigid hypersurfaces (admitting a transversal

Received by the editors October 5, 2007.
2010 *Mathematics Subject Classification.* Primary 32V35, 32V40.
The author was supported by a grant of the GA ČR no. 201/08/0397.

infinitesimal CR automorphism). The above normal forms do not respect this symmetry.

For local analysis on domains which admit symmetries not fixing the boundary point, it is desirable to have a normalization which reflects the symmetries. This problem was first studied by N. K. Stanton, who considered rigid hypersurfaces of finite type in $\mathbb{C}^2$ and constructed a rigid normal form. The results of [19] describe all transformations preserving the rigid normal form and give a complete classification of rigid hypersurfaces, provided that the model is not a tube.

In this paper we consider real analytic Levi degenerate hypersurfaces of finite type with a tube model. In view of Stanton's results, this is the only case of further interest. On the one hand, any hypersurface which admits a transversal infinitesimal CR vector field is necessarily rigid. On the other hand, if it admits a nontransversal one, its model has to be a tube.

The case of real analytic tubes is also interesting in connection with the work of G. Francsics and N. Hanges. In [9] they analyze boundary behaviour of the Bergman kernel for Levi degenerate tubes. In relation to this work, Nicholas Hanges formulated the problem of describing all biholomorphic maps between tubes ([10]).

After introducing notation, we define in Section 3 a complete tubular normal form for a general hypersurface with a tube model. It gives the main tool for analyzing biholomorphisms of symmetric hypersurfaces. The construction is analogous to that of [14], and it is given on the level of formal power series. The fact that such a construction can be used for classification problems relies on the essential result of Baouendi, Ebenfelt and Rothschild on convergence of formal equivalences ([1]). Rigid hypersurfaces are considered in Section 4, where a rigid normal form is obtained. Then we analyze biholomorphisms preserving this normalization.

In Section 5 we consider tubes. The symmetry preserving biholomorphisms are shown to be linear, described by three real parameters. In particular, we obtain an answer to the question of Hanges. Further, we prove that the complete normal form of Section 3 is convergent for all tubes, thus providing a complete, convergent and symmetry preserving normal form.

Nontubular rigid hypersurfaces are considered in Section 6. Applying results of [7] and [19], we give a characterization of hypersurfaces for which Stanton's normal form provides a complete, symmetry preserving normalization.

Section 7 considers hypersurfaces which admit nontransversal infinitesimal CR automorphisms and obtains a complete, symmetry preserving normalization for this class of hypersurfaces.

Part of this work was done while the author was visiting the follow-up program Complex Analysis, Operator Theory, and Applications to Mathematical Physics at ESI. He would like to thank Friedrich Haslinger for the invitation and hospitality, and for the support received from ESI.

2. Preliminaries

Let $M \subseteq \mathbb{C}^2$ be a real analytic hypersurface and $p \in M$ be a point of finite type k, as defined by J. J. Kohn ([13]). Throughout the paper we assume $k > 2$; hence M is Levi degenerate at p.

We will describe M in a neighbourhood of p using local holomorphic coordinates (z, w) centered at p, where $z = x + iy,\ w = u + iv$. The hyperplane $\{v = 0\}$ is

assumed to be tangent to M at p. M is described near p as the graph of a uniquely determined real valued function

$$v = F(z, \bar{z}, u).$$

Since p is of finite type k, there exist local holomorphic coordinates such that M is given by

$$v = \sum_{j=1}^{k-1} a_j z^j \bar{z}^{k-j} + o(|z|^k, u), \tag{2.1}$$

where the leading term is a nonzero real valued homogeneous polynomial of degree k, with $a_j \in \mathbb{C}$ and $a_j = \overline{a_{k-j}}$.

The model hypersurface to M at p is defined using the leading homogeneous term,

$$M_H = \{(z, w) \in \mathbb{C}^2 \mid v = \sum_{j=1}^{k-1} a_j z^j \bar{z}^{k-j}\}. \tag{2.2}$$

We recall two basic integer valued invariants used in the normal form construction in [14]. The first one, denoted by e, is the essential type of the model hypersurface. It can be described as the lowest index in (2.1) for which $a_e \neq 0$.

When $e < \frac{k}{2}$, the second invariant is defined as follows. Let $e = m_0 < m_1 < \cdots < m_s < \frac{k}{2}$ be the indices in (2.1) for which $a_{m_i} \neq 0$. The invariant, denoted by L, is the greatest common divisor of the numbers $k - 2m_0, k - 2m_1, \ldots, k - 2m_s$.

The local automorphism group of M_H will be denoted by $Aut(M_H, 0)$. It was proved in [14] that for $e < \frac{k}{2}$ the group $Aut(M_H, 0)$ consists of linear transformations

$$z^* = \delta \exp i\theta \; z, \qquad w^* = \delta^k w,$$

where $\exp i\theta$ is an L-th root of unity and $\delta > 0$ for k even or $\delta \in \mathbb{R}^*$ for k odd.

3. A tubular normal form

We will assume that the model at $p \in M$ is a tube. By appropriate scaling and adding a harmonic term we may assume that the leading term is equal to $(\frac{z+\bar{z}}{2})^k$. In particular, e is equal to one. The model hypersurface is now

$$T_k = \{(z, w) \in \mathbb{C}^2 \mid v = x^k\}. \tag{3.1}$$

$Aut(T_k, 0)$ is isomorphic to $\mathbb{R}^*$, consisting of dilations

$$z^* = \delta z, \qquad w^* = \delta^k w, \tag{3.2}$$

where $\delta \in \mathbb{R}^*$.

A standard weight assignment will be used. The variables z, x, y are given weight one, and w and u weight k. The defining equation has form

$$v = F(x, y, u), \tag{3.3}$$

where

$$F(x, y, u) = x^k + \sum_{j+l+km \geq k+1} A_{j,l,m} \, x^j y^l u^m. \tag{3.4}$$

Hence $F(x,y,u) - x^k$ contains precisely terms of weight greater than or equal to $k+1$. Consider the partial Taylor expansion of F in x, y and write

$$F(x,y,u) = x^k + \sum_{j,l} X_{jl}(u) x^j y^l,$$

where $X_{jl}(u) = \sum_m A_{j,l,m}\, u^m$.

Following [2], we decompose F into parts containing terms of equal weight,

$$F = x^k + \sum_{\nu=k+1}^{\infty} F_\nu,$$

and subject F to a transformation of the form

$$z^* = z + f(z,w), \qquad w^* = w + g(z,w), \tag{3.5}$$

where

$$f(z,w) = \sum_{wt.>1} f_{jm} z^j w^m, \qquad g(z,w) = \sum_{wt.>k} g_{jm} z^j w^m. \tag{3.6}$$

Such transformations preserve forms (3.3), (3.4). Conversely, it's easy to verify that any transformation preserving forms (3.3), (3.4) can be written uniquely as a composition of an element of $Aut(T_k, 0)$ and a transformation of this form (this factorization will be used repeatedly in the sequel).

Again we decompose the power series into parts of the same weight

$$f = \sum_{\nu=2}^{\infty} f_\nu \qquad \text{and} \qquad g = \sum_{\nu=k+1}^{\infty} g_\nu,$$

and denote such a transformation by (f,g). Let $v^* = F^*(x^*, y^*, u^*)$ be the new defining equation, where

$$F^*(x^*, y^*, u^*) = (x^*)^k + \sum_{j+l+mk \geq k+1} A^*_{j,l,m} (x^*)^j (y^*)^l (u^*)^m.$$

We will have to also consider formal hypersurfaces and formal transformations. From now on we allow both F, F^* and f, g to be formal power series. The power series formulae are then interpreted in this sense.

Substituting (3.5) into $v^* = F^*(x^*, y^*, u^*)$ and restricting the variables to M, we get the transformation formula

$$\begin{aligned} &F^*(x + Re\, f(x+iy, u + iF(x,y,u)), y + Im\, f(x+iy, \\ &\quad u + iF(x,y,u)), u + Re\, g(x+iy, u + iF(x,y,u))) \\ &\qquad = Im\, g(x+iy, u + iF(x,y,u)) + F(x,y,u). \end{aligned} \tag{3.7}$$

In principle, by multiplying this out one can obtain equations for coefficients of F^*, expressed in terms of F, f, g. The group of formal transformations (3.5) acts on formal power series (3.4) via this transformation formula.

Definition 3.1. We say that F is in t-normal form if

$$X_{0,j} = X_{1,j} = X_{k-1,j} = X_{k,j} = 0, \tag{3.8}$$

for all $j = 0, 1, \ldots,$ and

$$X_{2k-1,0} = X_{2k-1,1} = 0. \tag{3.9}$$

Theorem 3.2. *There is a unique formal transformation* (3.5), (3.6), *which takes* M *into t-normal form.*

Proof. Using induction on weight we will prove that the normal form conditions on F^* uniquely determine all coefficients of f and g in (3.5). Let us consider the terms of weight $\mu > k$ in (3.7). We have

$$\begin{aligned} F^*_\mu(x,y,u) + kRe\ x^{k-1} f_{\mu-k+1}(x+iy, u+ix^k) \\ = F_\mu(x,y,u) + Im\ g_\mu(x+iy, u+ix^k) + \dots, \end{aligned}$$

where the dots denote the terms depending on $f_{\nu-k+1}, g_\nu, F_\nu, F^*_\nu$ for $\nu < \mu$. We denote

$$L_\mu(f,g) = Re\{ig_\mu(x+iy, u+ix^k) + kx^{k-1} f_{\mu-k+1}(x+iy, u+ix^k)\}, \tag{3.10}$$

an analog of the Chern-Moser operator. For individual monomials in (3.10) we have

$$\begin{aligned} kx^{k-1} Re\ \{f_{jm}(x+iy)^j (u+ix^k)^m\} = kx^{k-1} Re\ \{f_{jm}(x+iy)^j\} u^m \\ -mkx^{2k-1} Im\ \{f_{jm}(x+iy)^j\} u^{m-1} + O(x^{2k}) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} Re\ \{ig_{jm}(x+iy)^j (u+ix^k)^m\} = -Im\ \{g_{jm}(x+iy)^j\} u^m \\ -mx^k Re\ \{g_{jm}(x+iy)^j\} u^{m-1} + O(x^{k+1}). \end{aligned} \tag{3.12}$$

From this expansion we will collect coefficients of $x^j y^l u^m$ in (3.7). Denote $B^*_{j,l,m} = A_{j,l,m} - A^*_{j,l,m}$. First we consider $j = 0$ and $j = 1$. Since $k > 2$, all terms in (3.11), (3.12) are multiples of x^2, except for the first term in (3.12). That gives for $l = 1, 2, \dots$

$$\begin{aligned} B^*_{0,0,m} &= -Im\ g_{0,m} + \dots, \\ B^*_{0,l,m} &= -Im\ (i^l g_{l,m}) + \dots, \\ B^*_{1,l-1,m} &= -l\ Im\ (i^{l-1} g_{l,m}) + \dots, \end{aligned} \tag{3.13}$$

where the dots denote the terms depending on $f_{\nu-k+1}, g_\nu, F_\nu, F^*_\nu$ for $\nu < \mu$, which have already been determined. Hence the condition $A^*_{0,0,m} = 0$ determines $Im\ g_{0,m}$, and $A^*_{0,l,m} = 0$ determines $g_{l,m}$ for $l = 1, 2, \dots$. Further, we consider $j = k-1$ and $j = k$. For $l \geq 2$, we get a contribution from the first term in (3.11) and the two terms in (3.12). This gives

$$\begin{aligned} B^*_{k-1,l,m} &= kRe\ (i^l f_{l,m}) - C_1 Im\ (i^l g_{k+l-1,m}) + \dots, \\ B^*_{k,l-1,m} &= kl\ Re\ (i^{l-1} f_{l,m}) - C_2 Im\ (i^{l-1} g_{k+l-1,m}) \\ &\quad - (m+1) Re\ (i^{l-1} g_{l-1,m+1}) \dots, \end{aligned} \tag{3.14}$$

where $C_1 = \binom{k+l-1}{l}$ and $C_2 = \binom{k+l-1}{l-1}$, and $f_{l,m}$ for $l = 2, 3, \dots$. Next we consider $(j,l) = (k-1, 0)$, with a contribution from the first terms in (3.11) and (3.12). For $(j,l) = (2k-1, 0)$ all terms contribute, and we obtain

$$\begin{aligned} B^*_{k-1,0,m} &= kRe\ f_{0,m} - Im\ g_{k-1,m} + \dots, \\ B^*_{2k-1,0,m-1} &= kRe\ f_{k,m-1} - kmIm\ f_{0,m} \\ &\quad - Im\ g_{2k-1,m-1} - mRe\ g_{k-1,m} + \dots. \end{aligned} \tag{3.15}$$

This determines $f_{0,m}$, since all other entries have already been determined. For $(j,l) = (k-1,1)$ the first terms in (3.11) and (3.12) contribute, and for $(j,l) = (2k-1,1)$ all contribute. This gives

$$\begin{aligned} B^*_{k-1,1,m} &= -kIm\ f_{1,m} - kRe\ g_{k,m} + \dots, \\ B^*_{2k-1,1,m-1} &= -k(k+1)Im\ f_{k+1,m-1} - kmRe\ f_{1,m} \\ &\quad - 2kRe\ g_{2k,m-1} + kmIm\ g_{k,m}, \end{aligned} \tag{3.16}$$

which determines f_{1m}. For $(j,l) = (k,0)$ we get a contribution from the first term in (3.11) and the two terms in (3.12), which gives

$$B^*_{k,0,m} = kRe\ f_{1,m} - Im\ g_{k,m} - (m+1)Re\ g_{0,m+1} + \dots. \tag{3.17}$$

This determines $Re\ g_{0,m+1}$. □

Remark 3.3. Note that the same result is obtained if the conditions $X_{2k-1,0} = X_{2k-1,1} = 0$ are replaced by $X_{2k-1,0} = A$ and $X_{2k-1,1} = B$, for any fixed real numbers A, B. This remark will be used in Section 7.

From the factorization of a general map preserving form (3.3), (3.4), we obtain the following corollary.

Corollary 3.4. *The only transformations which preserve the t-normal form are the elements of $Aut(T_k, 0)$.*

4. A rigid normal form

In this section we consider rigid hypersurfaces with tube models and define a rigid t-normal form.

Consider a rigid hypersurface of finite type $k > 2$ with a tube model, given by

$$v = F(x,y), \tag{4.1}$$

where

$$F(x,y) = x^k + \sum_{j+l \geq k+1} A_{j,l} x^j y^l. \tag{4.2}$$

Definition 4.1. We say that F is in rigid t-normal form if

$$A_{0,j} = A_{1,j} = A_{k-1,j} = A_{k,j} = 0 \tag{4.3}$$

for all $j = 0, 1, \dots$.

Lemma 4.2. *There exists a unique transformation of the form*

$$z^* = z + \sum_{j=2}^{\infty} f_j z^j, \qquad w^* = w + \sum_{j=k+1}^{\infty} g_j z^j, \tag{4.4}$$

which takes F into rigid t-normal form.

Proof. We will again determine by induction the coefficients f_j, g_j in such a way that (4.3) is satisfied. Let

$$v^* = F^*(x^*, y^*) \tag{4.5}$$

in the new coordinates, where

$$F^*(x^*, y^*) = (x^*)^k + \sum_{j+l \geq k+1} A^*_{j,l} (x^*)^j (y^*)^l. \tag{4.6}$$

The transformation formula takes the form

$$F^*(x + Re\ f(x+iy), y + Im\ f(x+iy)) = Im\ g(x+iy)\ + F(x,y). \tag{4.7}$$

Terms of degree $m > k$ in this equation depend linearly on terms of degree $m-k+1$ in f and degree m in g, and nonlinearly on terms of lower degree in f. For the terms specified in (4.3) we have the following equations:

$$\begin{aligned} B^*_{0,l} &= -Im\ (i^l g_l) + \dots, \\ B^*_{1,l-1} &= -l\ Im\ (i^{l-1} g_l) + \dots, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} B^*_{k-1,l} &= kRe\ (i^l f_l) - C_1 Im\ (i^l g_{l+k-1}) + \dots, \\ B^*_{k,l-1} &= klRe\ (i^{l-1} f_l) - C_2 Im\ (i^{l-1} g_{l+k-1}) + \dots, \end{aligned} \tag{4.9}$$

where $B^*_{j,l} = A_{j,l} - A^*_{j,l}$, $C_1 = \binom{k+l-1}{l}$, $C_2 = \binom{k+l-1}{l-1}$, and the dots denote already determined numbers. The first two equations determine g_l, $l = k+1, k+2, \dots$, and the second two determine f_l, $l = 2, 3, \dots$. □

Using the complete t-normal form we will analyze biholomorphic transformations which preserve the rigid t-normal form.

5. Biholomorphic equivalence of tubes

Since a tube hypersurface automatically satisfies all t-normal form conditions except for $A_{2k-1,0,0} = 0$, this normalization can be used effectively to classify tubes.

Theorem 5.1. *Let M_1 and M_2 be two tubular hypersurfaces of finite type $k > 2$ at the origin, given by $v = F(x)$ and $v = G(x)$, respectively. If Ψ is a local biholomorphism preserving the origin, which maps M_1 to M_2, then it has the form*

$$z^* = az + ibw, \qquad w^* = cw$$

for some $a, c \in \mathbb{R}^*$ and $b \in \mathbb{R}$. In this case, F and G satisfy

$$G(ax - bF(x)) = cF(x).$$

Proof. By assumption, $F(x) = c_1 x^k + o(x^k)$ and $G(x) = c_2 x^k + o(x^k)$ for some nonzero real constants c_1, c_2. First we make those coefficients equal to one, using the dilations $w^* = c_j^{-1} w$, $j = 1, 2$. In the second step, we put M_1 into t-normal form by a transformation

$$z^* = z + ihw, \qquad w^* = w,$$

where $h \in \mathbb{R}$. In (3.7) we have $Re\ (ih(u+iF)) = -hF$, so the transformation equation becomes

$$F^*(x - hF(x)) = F(x). \tag{5.1}$$

Equating coefficients of x^{2k-1} we obtain $A^*_{2k-1,0,0} - kh = A_{2k-1,0,0}$. Hence for $h = -\frac{1}{k} A_{2k-1,0,0}$ the t-normal form is obtained. Next we perform the same normalization on M_2 and consider the two resulting hypersurfaces in t-normal form. By Corollary 3.4, a biholomorphic equivalence between them is an element of $Aut(T_k, 0)$. By composing the five linear mappings, we obtain the claimed form of the biholomorphism. The transformation formula gives $G(ax - bF(x)) = cF(x)$. □

By the same reasoning, we obtain the following corollary.

Corollary 5.2. *For any tube hypersurface the t-normal form is convergent and preserves the tubular symmetries.*

6. Nontubular hypersurfaces

The t-normal form preserves tubular symmetries, but does not in general preserve rigidity.

In this section we consider nontubular rigid hypersurfaces with a tube model and the question of completeness of Stanton's rigid normal form. We give a characterization of hypersurfaces for which there exist biholomorphisms which preserve Stanton's normal form, besides the elements of $Aut(T_k, 0)$. As a consequence, for all other hypersurfaces Stanton's normal form is complete.

Recall that Stanton's normal form for a rigid hypersurface with a tube model uses the complex Taylor expansion of F,

$$v = x^k + \sum_{j,l} A_{jl} z^j \bar{z}^l. \tag{6.1}$$

The normal form conditions are

$$A_{0,l} = A_{1,l} = 0,$$

for all $l = 1, 2, \ldots$.

Any rigid hypersurface given by (6.1) admits the infinitesimal CR automorphism $Re\, Y$, where

$$Y = \frac{\partial}{\partial w}.$$

By a result of Stanton (Proposition 3.1 in [19]), there is a transformation which preserves the rigid normal form and does not belong to $Aut(T_k, 0)$ if and only if M admits an additional linearly independent infinitesimal CR automorphism.

Note that if there is such an infinitesimal CR automorphism, $Re\, X$, we obtain the corresponding one parametric family of transformations preserving Stanton's normal form by solving a system of ODEs ([19]). Geometrically, this corresponds to straightening transversal vector fields of the form $Y + aX$, $a \in \mathbb{R}$.

On the other hand, all Levi degenerate hypersurfaces of finite type which admit a real two-parametric family of infinitesimal CR automorphisms are classified in [7]. In particular, the results show that a nontubular hypersurface with a tube model, which admits a real two-parametric family of infinitesimal CR automorphisms, belongs to one of the following two types.

The first type are hypersurfaces of the form

$$v = g(x)e^y, \tag{6.2}$$

which admit the infinitesimal CR automorphisms $Re\, X_1$, $Re\, X_2$, where

$$X_1 = \frac{\partial}{\partial w}, \qquad X_2 = i\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}.$$

The second type are hypersurfaces of the form

$$v = g(xe^{-u}), \tag{6.3}$$

with the vector fields

$$X_1 = i\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial w} + z\frac{\partial}{\partial z}.$$

In both cases, g is a smooth function with $g(0) = g'(0) = g''(0) = 0$.

The difference between the two cases is the direction of the commutator of X_1 and X_2 at the origin. For the first type, the commutator is the transversal vector field $\frac{\partial}{\partial w}$, while in the second case the commutator is the complex tangential vector field $i\frac{\partial}{\partial z}$ ([7]).

It is immediate to verify that in the second case the hypersurface is mapped into a rigid form, with the defining equation independent of u, by the transformation

$$z = e^{w^*} z^*, \qquad w = w^*,$$

where the new defining equation is given implicitly by

$$v = g(x \cos v - y \sin v).$$

Hence, by combining Proposition 3.1 in [19] and Theorem 3.2 in [7], we obtain the following result. It shows that Stanton's normal form is complete for all nontubular hypersurfaces except for the two types, given by (6.2) and (6.3).

Theorem 6.1. *Let M be a nontubular hypersurface with a tube model in Stanton's normal form, and let there exist a transformation which preserves this form and does not belong to $Aut(T_k, 0)$. Then either M is locally equivalent to a hypersurface of the form* (6.2)*, or M is locally equivalent to a hypersurface of the form* (6.3)*.*

7. Nontransversal symmetries

In this section we consider a hypersurface M which admits a nontransversal infinitesimal CR automorphism and construct a complete normal form for this class of hypersurfaces which preserves the symmetries. We consider local holomorphic coordinates (obtained by straightening the corresponding vector field) in which the defining equation has form

$$v = G(x, u), \tag{7.1}$$

where

$$G(x, u) = x^k + \sum_{j=0}^{\infty} X_j(u) x^j,$$

and the sum on the right is $o_{wt}(k)$.

Definition 7.1. M is in normal form if the defining equation has the form (7.1) and satisfies

$$X_0 = X_{k-1} = X_k = X_{2k-1} = 0. \tag{7.2}$$

Lemma 7.2. *There exists a transformation of the form*

$$z^* = z + \psi(w), \qquad w^* = w + \phi(w), \tag{7.3}$$

which takes M into normal form.

Proof. First we show that transformations (7.3) preserve (7.1). We have

$$\begin{aligned} &G^*(x + Re\ \psi(u + iG(x, u)), y + Im\ \psi(u + iG(x, u), u)) \\ &\qquad = G(x, u) + Im\ \phi(u + iG(x, u)). \end{aligned} \tag{7.4}$$

Since the right hand side is independent of y, it follows immediately that G^* is independent of y^*. By (3.13)–(3.17), the equations for $B^*_{0,0,m}$, $B^*_{k-1,0,m}$ $B^*_{k,0,m}$ and $B^*_{2k-1,0,m}$ determine the coefficients $f_{j,m}$ and $g_{j,m}$ with $j = 0$. Hence, setting all other coefficients equal to zero, the induction argument of Theorem 3.2 determines the coefficients $f_{0,m}$ and $g_{0,m}$ so that (7.2) is satisfied. □

Now we show that the conditions in Definition 7.1 define a complete normalization.

Theorem 7.3. *Let Ψ be a transformation which preserves* (7.1) *and the normal form conditions* (7.2)*. Then Ψ is an element of $Aut(T_k, 0)$.*

Proof. Again we decompose Ψ into an element of $Aut(T_k, 0)$ and a transformation (f, g) of the form (3.5), and we consider the effect of this transformation. Assume (f, g) is not the identity, and let μ be the first weight where it differs from the identity, i.e. the operator (3.10) is nontrivial. The defining equation of M satisfies all t-normal form conditions except possibly for $X_{10} = 0$. By Theorem 3.2 (as in Remark 3.3), on weight μ we have to obtain $A^*_{1,0,m} \neq A_{1,0,m}$, where $m = \frac{\mu-1}{k}$. For weight μ, consider the equation for the coefficient of $x^{2k-1}y^2u^{m-2}$. From (3.11) and (3.12) we obtain

$$\begin{aligned} 0 = &-k\binom{k+2}{2} Re\ f_{k+2,m-2} + mkIm\ f_{2,m-1} \\ &+ \binom{2k+1}{2} Im\ g_{2k+1,m-2} + m\binom{k+1}{2} Re\ g_{k+1,m-1}. \end{aligned} \tag{7.5}$$

By (3.13) and (3.14), we have

$$Re\ g_{k+1,m-1} = Im\ g_{2k+1,m-2} = Re\ f_{k+2,m-2} = 0. \tag{7.6}$$

Hence also $Im\ f_{2,m-1} = 0$. Now consider the second equation in (3.14), with $l = 2$. It gives $Im\ g_{1,m} = 0$. So by (3.13), the equation for xu^m is $B^*_{1,0,m} = 0$, which is a contradiction. Thus we proved that the only transformations preserving the normalization conditions are the elements of $Aut(T_k, 0)$. □

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