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# DISCRETE GROUPS AND THE COMPLEX CONTACT GEOMETRY OF $Sl(2,\mathbb{C})$

BRENDAN FOREMAN

Abstract. We investigate the vertical foliation of the standard complex contact structure on $\Gamma \setminus Sl(2,\mathbb{C})$, where $\Gamma$ is a discrete subgroup. We find that, if $\Gamma$ is nonelementary, the vertical leaves on $\Gamma \setminus Sl(2,\mathbb{C})$ are holomorphic but not regular. However, if $\Gamma$ is Kleinian, then $\Gamma \setminus Sl(2,\mathbb{C})$ contains an open, dense set on which the vertical leaves are regular, complete and biholomorphic to $\mathbb{C}^*$. If $\Gamma$ is a uniform lattice, the foliation is nowhere regular, although there are both infinitely many compact and infinitely many nonclosed leaves.

## 1. Introduction

A complex contact structure on a $(2n+1)$-dimensional complex manifold $(M, J)$ is a maximally nonintegrable $n$-dimensional holomorphic subbundle $\mathcal{H}$ of maximal rank; i.e., $\mathcal{H}$ is a global holomorphic subbundle of the holomorphic tangent bundle defined as the kernel of local holomorphic 1-forms $\eta$ such that $\eta \wedge (d\eta)^n$ is nowhere zero. These spaces were first investigated in [8]. They include the twistor spaces of quaternionic Kähler manifolds with nonzero curvature, most notably $\mathbb{C}P^{2n+1}$, the complex Heisenberg group, the projectivized cotangent bundle of an arbitrary complex manifold, and $Sl(2,\mathbb{C})$.

Let $\mathcal{H}^R$ be the real subbundle of $T^RM$ induced from $\mathcal{H}$ in $T^{1,0}M$. Then it is possible to construct a particular $J$-invariant subbundle of $T^RM$, $\mathcal{V}$, such that $T^RM = \mathcal{H} \oplus \mathcal{V}$. $\mathcal{V}$ is called the *vertical subbundle* of $\mathcal{H}$ and acts somewhat analogously as the Reeb vector field does for a real contact manifold. If there is a global holomorphic 1-form $\eta$ that defines $\mathcal{H}$, then $\mathcal{H}$ is called a *strict* complex contact structure, and there is a well-defined holomorphic vector field $\xi$ transverse to $\mathcal{H}$ given by the equations $\eta(\xi) = 1$ and $\iota(\xi)d\eta = 0$. $\xi$ is called the *complex Reeb vector field* or the *vertical vector field*. The vertical subbundle in this case is the span of the real and imaginary parts of $\xi$ and hence induces a holomorphic line bundle in $T^{1,0}M$. In particular, $\mathcal{V}$ is a foliation.

A great deal is still yet unknown about the vertical subbundles of general complex contact structures, particularly with regard to integrability and regularity issues. Examples of complex contact manifolds with vertical subbundles satisfying all of the possible degrees of integrability and regularity have yet to be found. This paper fills in some of the gaps in knowledge by investigating the vertical foliations of the complex contact manifolds of the form $\Gamma \setminus Sl(2,\mathbb{C})$, where $\Gamma$ is a discrete subgroup. The main objective is to come to understand how the leaves of the vertical foliation are embedded in $\Gamma \setminus Sl(2,\mathbb{C})$. A nice consequence of this work is

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a clearer understanding of the manifolds of the form $\Gamma \setminus Sl(2,\mathbb{C})$. In particular, we find that the vertical leaves on $\Gamma \setminus Sl(2,\mathbb{C})$ are holomorphic but not regular for any nonelementary discrete subgroup $\Gamma \subset Sl(2,\mathbb{C})$. However, if $\Gamma$ is Kleinian, then there is an open dense subset of $\Gamma \setminus Sl(2,\mathbb{C})$ for which the vertical leaves are regular and complete. On the other hand, if $\Gamma$ is a uniform lattice, the foliation is nowhere regular, although there are both infinitely many compact and infinitely many nonclosed leaves.

In order to accomplish this work, we will take advantage of the rich geometric structure of $Sl(2,\mathbb{C})$, particularly its role as the simply connected cover of the space of Möbius transformations on the extended complex plane, $\mathcal{M}$, which acts as both the space of isometries of hyperbolic three-space, $\mathcal{H}yp^3$, and the space of conformal mappings of $\hat{\mathbb{C}}$. For this reason, all of the work will be done within the context of $\mathcal{M}$ rather than $Sl(2,\mathbb{C})$. The pertinent aspects of this latter structure are reviewed in Section 2. In the third section, we explore the possible configurations of the vertical leaves in terms of regularity and closure. Finally, in the last section, we compare these configurations with those of other known complex contact manifolds.

For more information on complex contact geometry, see [3]. For more details regarding hyperbolic geometry and its isometries, see [1] and [12]. Finally, for the structure of Kleinian subgroups, see [10] and [7].

## 2. The geometry of $\mathcal{M}$

The space $\mathcal{M}$ has a very wide and very deep geometric structure. For purposes of brevity and coherence, only the pertinent aspects of $\mathcal{M}$ are discussed in this section. Specific references are given throughout for those readers who would like to learn more details and extensions of the results listed here.

2.1. **$\mathcal{M}$ as a space of isometries.** Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane, which will often be identified with $\mathbb{C}P^1$ by

$$z \cong \begin{bmatrix} z \\ 1 \end{bmatrix} \text{ for } z \in \mathbb{C} \text{ and } \infty \cong \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let $\mathcal{M}$ be the space of Möbius transformations on $\hat{\mathbb{C}}$. Then the homomorphism on $Sl(2,\mathbb{C})$ defined by

$$p : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \left( z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta} \right)$$

establishes $Sl(2,\mathbb{C})$ as the simply connected cover of $\mathcal{M}$. Since $p^{-1}(id) = \{\pm I\} \subset Sl(2,\mathbb{C})$, $\mathcal{M} \cong Sl(2,\mathbb{C})/\{\pm I\}$, and so elements of $\mathcal{M}$ will often be denoted as elements of $Sl(2,\mathbb{C})$ in brackets, e.g., $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ corresponding to the transformation $g(z) = \frac{z}{z+1}$. It is well known that by a certain extension of domain, $\mathcal{M}$ is the space of orientation-preserving isometries of $Hyp^3$, using the upper-half-space model. These transformations can be classified (up to conjugacy) into four categories. An element of $\mathcal{M}$, $g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, is called

(1) *parabolic* if $g$ has exactly one fixed point on $\hat{\mathbb{C}}$,
(2) *elliptic* if $g$ has a fixed point in $\mathcal{H}yp^3$,
(3) *loxodromic* if $g$ is neither parabolic nor elliptic,
(4) *hyperbolic* if $g$ is loxodromic and preserves an open disc in $\hat{\mathbb{C}}$.

This is the standard method of categorizing Möbius transformations, and it extends to higher dimensions (see Section IV.C of [10]). For further details regarding this categorization on the extended plane, see Chapter IV of [1].

For $z \in \hat{\mathbb{C}}$, let $N_z$ be the space of all parabolic transformations in $\mathcal{M}$ fixing $z$ plus the identity transformation. For distinct points $z_1,\ z_2 \in \hat{\mathbb{C}}$, let $F_{z_1,z_2}$ be the space of all transformations fixing $z_1$ and $z_2$. Let $\mathfrak{n}_z$ and $\mathfrak{f}_{z_1,z_2}$ denote the corresponding Lie algebras.

For $z_1, z_2 \in \hat{\mathbb{C}}$, let the mapping $z_{1_{z_2}} : \hat{\mathbb{C}} - \{z_2\} \to \hat{\mathbb{C}}$ be given by $z_{1_{z_2}}(z) = z_1$ for all $z \in \hat{\mathbb{C}} - \{z_2\}$. Set

$$\mathcal{QC} = \{z_{1_{z_2}} : z_1, z_2 \in \hat{\mathbb{C}}\},$$

called the set of *quasiconstant transformations* on $\hat{\mathbb{C}}$. This set is naturally identified with $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, which gives it a topology. Furthermore, we say that a sequence $\{g_j\} \subset \mathcal{M}$ converges to $z_{1_{z_2}}$ if $\{g_j\}$ converges to the constant mapping $z_1$ uniformly on compact subsets of $\hat{\mathbb{C}} - \{z_2\}$. These topologies combined with the topology of $\mathcal{M}$ induced from $Sl(2,\mathbb{C})$ give a topology on $\mathcal{M} \cup \mathcal{QC}$. The proof of the following proposition is given in Section 3.6 of [7].

**Proposition 2.1.** *The space $\mathcal{M} \cup \mathcal{QC}$ is compact.*

Alternative approaches to the topology of $\mathcal{M}$, in particular, its sequential limits, can be found in Section B of Chapter IV of [10] as well as Sections 4.7 and 4.9 of [7].

2.2. **Properties of discrete groups of $\mathcal{M}$.** Let $\Gamma \subset \mathcal{M}$ be a discrete subgroup. $\Gamma$ is called a *lattice* if $\Gamma \setminus \mathcal{M}$ has finite volume, and is called a *uniform lattice* if $\Gamma \setminus \mathcal{M}$ is compact. Since the isotropy group of the action of $\mathcal{M}$ on a given point in $Hyp^3$ is compact, it is not difficult to see that $\Gamma$ is a lattice if and only if $\Gamma \setminus Hyp^3$ has finite volume; and it is a uniform lattice if and only if $\Gamma \setminus Hyp^3$ is compact.

Let ${}^\circ\Omega = {}^\circ\Omega\,(\Gamma)$ be the set of all $z \in \hat{\mathbb{C}}$ at which $\Gamma$ acts properly discontinuously, i.e., all $z \in \hat{\mathbb{C}}$ about which there is an open neighborhood $U$ such that

$$\gamma(U) \cap U = \emptyset \text{ for all } \gamma \in \Gamma - \{id\}.$$

$\Gamma$ is called *Kleinian* if ${}^\circ\Omega$ is nonempty.

Let $\Lambda = \Lambda(\Gamma)$ be the set of all $z \in \hat{\mathbb{C}}$ such that there is a sequence $\{\gamma_j\} \subset \Gamma$ and a $v \in \mathcal{H}yp^3$ such that $\lim_{j\to\infty} \gamma_j(v) = z$ in the topology of the one-point compactification of the Euclidean upper half-space $\{(a,b,c) \in R^3 : c \geq 0\}$. Set $\Omega = \hat{\mathbb{C}} - \Lambda$. $\Gamma$ is called *elementary* if $\Lambda$ has cardinality at most two.

The following theorem consists of several well-known facts. Details and proofs of all of these statements except Statement (5) can be found in Sections C, D and E of Chapter II of [10]. The statement and proof of Statement (5) can be found in Section 6 of Chapter 3 in [7].

**Theorem 2.2.** *Let $\Gamma \subset \mathcal{M}$ be a discrete subgroup with $\Lambda$, ${}^\circ\Omega$ and $\Omega$ as defined above.*

(1) *If $\Gamma$ is nonelementary, then the cardinality of $\Lambda$ is infinite.*
(2) *$\Lambda$ is closed in $\hat{\mathbb{C}}$ and is the set of accumulation points of the set $\Gamma\left(\hat{\mathbb{C}}\right)$ in $\hat{\mathbb{C}}$.*
(3) *Both ${}^\circ\Omega$ and $\Omega$ are open in $\hat{\mathbb{C}}$. Furthermore, if $\Gamma$ is Kleinian, then ${}^\circ\Omega$ is dense in $\hat{\mathbb{C}}$.*

(4) *$\Omega$ is the set of all $z \in \hat{\mathbb{C}}$ about which there is an open neighborhood $U$ such that $\gamma(U) \cap U = \emptyset$ for all but finitely many $\gamma \in \Gamma$. In particular, if $z \in \Omega - {}^\circ\Omega$, then there is an elliptic element of $\Gamma$ fixing $z$.*

(5) *If $\Gamma$ is nonelementary, then, for any two distinct $z_1,\ z_2 \in \Lambda$, there exists a sequence of loxodromic elements of $\Gamma$, $\{\gamma_j\}$, with fixed points $p_j$ and $q_j$ such that*
   (a) *$p_j \neq z_1$ and $q_j \neq z_2$ for each $j \in \mathbb{N}$,*
   (b) *$\lim_{j\to\infty} p_j = z_1$, and*
   (c) *$\lim_{j\to\infty} q_j = z_2$.*

(6) *If $\gamma_1 \in \Gamma$ is loxodromic and $\gamma_2 \in \Gamma$, then either $\gamma_2$ has the same fixed points as $\gamma_1$ or $\gamma_1$ and $\gamma_2$ have no fixed points in common.*

2.3. **The left-invariant complex contact structures of $\mathcal{M}$.** If $\mathcal{H}$ is a left-invariant complex contact structure of $\mathcal{M}$, i.e., preserved by left multiplication of elements in $\mathcal{M}$, then it will correspond to a two-dimensional subspace of the Lie algebra of $\mathcal{M}$, $\mathfrak{sl}(2,\mathbb{C})$, which is not preserved by the Lie bracket. In [6], it is shown that all such subspaces are conjugate to each other and of the form $\mathfrak{n}_{z_1} + \mathfrak{n}_{z_2}$ for two distinct $z_1, z_2 \in \hat{\mathbb{C}}$ so that the corresponding complex Reeb vector field spans the subalgebra $\mathfrak{f}_{z_1,z_2}$ (recall from Subsection 2.1 that this is the Lie algebra of the subgroup of $\mathcal{M}$ fixing $z_1$ and $z_2$). We call $\mathcal{H} = \mathfrak{n}_0 + \mathfrak{n}_\infty$ the *standard complex contact structure* on $Sl(2,\mathbb{C})$. If we set $\{E_1,\ E_2,\ E_3\}$ to be the left-invariant basis of $\mathfrak{sl}(2,\mathbb{C})$ given by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with dual basis $\{E_1^*,\ E_2^*,\ E_3^*\}$, then $\mathcal{H} = \ker(E_1^*)$. Set $\eta = E_1^*$ so that the complex Reeb vector field $\xi$ of $\eta$ is given by $\xi = E_1$ and $\langle\xi\rangle_{\mathbb{C}} = \mathfrak{f}_{0,\infty}$. Let $\mathcal{V}$ be the holomorphic foliation of $T^{1,0}\mathcal{M}$ induced by the vector field $\xi$. The leaves of this foliation are called the *vertical leaves*.

Let $S$ be the complex surface given by $S = \mathbb{C}P^1 \times \mathbb{C}P^1 - \Delta$, where $\Delta = \{([\zeta],[\zeta]) : \zeta \in \mathbb{C}^*\}$. Define the mapping $\pi : \mathcal{M} \to S$ by

$$\pi \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \left( \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}, \begin{bmatrix} \beta \\ \delta \end{bmatrix} \right).$$

By identifying $\hat{\mathbb{C}}$ with $\mathbb{C}P^1$, $\pi$ can also be defined by $\pi(g) = (g(\infty), g(0))$ for all $g \in \mathcal{M}$.

$\pi$ is surjective and holomorphic and, for $A \in Sl(2,\mathbb{C})$,

$$\pi\left( L_{[A]} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \left( \left[ A \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \right], \left[ A \begin{pmatrix} \beta \\ \delta \end{pmatrix} \right] \right).$$

Hence, for any discrete group $\Gamma \subset \mathcal{M}$, left equivalence modulo $\Gamma$ on $S$ is well-defined, and $\pi$ induces a well-defined mapping from $\Gamma \setminus \mathcal{M}$ to $\Gamma \setminus S$, also denoted by $\pi$.

Furthermore, for $[A],[B] \in \mathcal{M}$, $\pi([A]) = \pi([B])$ if and only if there is an element $[C] \in F_{0,\infty}$, i.e., $[C] = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$ for some $\zeta \in \mathbb{C}^*$, such that $[B] = [A][C]$. Therefore, $S$ and $\mathcal{M}/F_{0,\infty}$ are biholomorphic, and each $\pi^{-1}(p)$ for $p \in S$ is a vertical leaf, which is denoted by $\mathcal{L}_p$.

Recall that a foliation $\mathcal{F}$ of a manifold $M$ is called *regular* at a point $x \in M$ if there is an open neighborhood of $x$, $U$, such that, for any leaf $\mathcal{L}$ of $\mathcal{F}$, the intersection

$\mathcal{L} \cap U$ is a connected open subset of $\mathcal{L}$, and that the foliation itself is called *regular* if each point of $M$ is a regular point of $\mathcal{F}$. If the foliation is regular, then the space of leaves is a well-defined manifold (see [9]). This combined with the above work yields the following proposition.

**Proposition 2.3.** *Let $\mathcal{H}$ be the standard left-invariant complex contact subbundle on $\mathcal{M}$ given as the kernel of the left-invariant form $\eta = {E_1}^*$. Let $\xi$ be the Reeb vector field of $\eta$ with corresponding foliation $\mathcal{V} \subset T^{1,0}\mathcal{M}$. Then $\mathcal{V}$ is a regular foliation of $\mathcal{M}$, each leaf of which is biholomorphic to $\mathbb{C}^*$. Furthermore, the space of vertical leaves of $\mathcal{V}$ is biholomorphic to the complex surface $S$ via the $\mathbb{C}^*$-fibre bundle map $\pi : \mathcal{M} \to S$.*

## 3. Main results

For this section, $\Gamma$ is a nonelementary discrete subgroup of $\mathcal{M}$. Let $\Lambda$, $^\circ\Omega$, and $\Omega$ be its limit set, set of proper discontinuity, and set of discontinuity, respectively. Denote the projection $\mathcal{M} \to \Gamma \setminus \mathcal{M}$ by ${}_\Gamma[*]$ or $[*]$.

Let $\tilde{\Lambda} = \{p = (z_1, z_2) \in S : z_1,\ z_2 \in \Lambda\}$, $\tilde{\Omega} = S - \tilde{\Lambda}$ and $\Omega' = ({}^\circ\Omega \times {}^\circ\Omega) \cap S$. In particular, $\Omega' \subset \tilde{\Omega}$. However, it is important to note that $\Omega \times \Lambda$ is also a subset of $\tilde{\Omega}$. Nevertheless, if $\Gamma$ is Kleinian, then $^\circ\Omega$ is dense in $\hat{\mathbb{C}}$ so that $\Omega'$ is dense in $S$.

The first subsection addresses the issue of the regularity of $\mathcal{V}$ at points on certain vertical leaves of $\Gamma \setminus \mathcal{M}$. The second subsection describes when a given vertical leaf is closed or compact in $\Gamma \setminus \mathcal{M}$ and also contains the culminating results regarding Kleinian groups and uniform lattices.

### 3.1. Regularity of the vertical foliation in $\Gamma \setminus \mathcal{M}$.

**Proposition 3.1.** *Let $\Gamma \subset \mathcal{M}$ be a nonelementary discrete subgroup. For $p \in S$, $\Gamma$ acts properly discontinuously at $p$ if and only if $p \in \tilde{\Omega}$ and $p$ is not fixed by an elliptic element of $\Gamma$.*

*Proof.* Let $p = (z_1, z_2) \in S$ such that $p \in \tilde{\Omega}$ and $p$ is not fixed by elliptic elements in $\Gamma$. Then exactly one of the following two cases must occur:

(1) one of the points, $z_1$ or $z_2$, is an element of $^\circ\Omega$,
(2) one of the points is an element of $\Omega - {}^\circ\Omega$, the other is an element of $(\Omega - {}^\circ\Omega) \cup \Lambda$, and no elliptic element of $\Gamma$ fixes both points.

If either $z_1$ or $z_2$, say $z_1$, is an element of $^\circ\Omega$, then there is an open neighborhood $U$ of $z_1$, not containing $z_2$, such that $\gamma(U) \cap U = \emptyset$ for all nontrivial elements $\gamma \in \Gamma$. By choosing any open neighborhood $V$ of $z_2$ not intersecting $U$, we have an open neighborhood of $p$, $U \times V$, such that $\gamma\,(U \times V) \cap (U \times V) = \emptyset$ for all nontrivial $\gamma \in \Gamma$. Thus, $\Gamma$ acts properly discontinuously at $p$.

Now, suppose that $z_1 \in \Omega - {}^\circ\Omega$, $z_2 \in (\Omega - {}^\circ\Omega) \cup \Lambda$, and no elliptic element of $\Gamma$ fixes both points. By definition of $\Omega$, there is a neighborhood $U$ of $z_1$ such that, for all but finitely many $\gamma \in \Gamma$, $\gamma(U) \cap U = \emptyset$. Let $\hat{\Gamma} = \{\gamma_1, \ldots, \gamma_k\}$ be the set of all nontrivial $\gamma \in \Gamma$ such that $\gamma(U) \cap U \neq \emptyset$. The elements of $\hat{\Gamma}$ are necessarily elliptic and fix $z_1$. By the assumption that no elliptic element fixes both $z_1$ and $z_2$, no elements of $\hat{\Gamma}$ fix $z_2$. Thus, there is a neighborhood of $z_2$, $V$, such that $\gamma(V) \cap V = \emptyset$ for all nontrivial $\gamma \in \hat{\Gamma}$, which means that $\gamma\,(U \times V) \cap (U \times V) = \emptyset$ for all nontrivial $\gamma \in \Gamma$.

Conversely, there are two possibilities to consider: either $p$ is fixed by an elliptic element of $\Gamma$ or $p \in \tilde{\Lambda}$. If $p$ is fixed by an elliptic element of $\Gamma$, then clearly $\Gamma$ does

not act properly discontinuously at $p$. If $p \in \tilde{\Lambda}$, then, since $\Gamma$ is nonelementary, there is a sequence $\{\gamma_j\} \subset \Gamma$ with fixed points $\{p_j, q_j\}$ such that

(1) $\lim_{j\to\infty} p_j = z_1$,
(2) $\lim_{j\to\infty} q_j = z_2$,
(3) for each $j \in \mathbb{N}$, the cardinality of the set $\{p_1, \ldots, p_j\}$ is $j$, and
(4) for each $j \in \mathbb{N}$, the cardinality of the set $\{q_1, \ldots, q_j\}$ is $j$.

Let $\tilde{U} \subset S$ be any open set containing $p$. Then there are infinitely many $j \in \mathbb{N}$ such that $(p_j, q_j) \in \tilde{U}$, which implies that, for those same values of $j$, $\gamma_j\left(\tilde{U}\right) \cap \tilde{U} \neq \emptyset$. This proves the proposition.

**Proposition 3.2.** *Let $\Gamma \subset \mathcal{M}$ be a nonelementary discrete subgroup of $\mathcal{M}$. Let $[g] \in \Gamma \setminus \mathcal{M}$ lying in the vertical leaf ${}_\Gamma[\mathcal{L}_p]$. Then the vertical foliation is regular at $[g]$ if and only if $\Gamma$ acts properly discontinuously at $p$.*

*Proof.* First, suppose that $\Gamma$ acts properly discontinuously at $p$. Then there is an open neighborhood $\tilde{U}$ of $p$ such that $\gamma(\tilde{U}) \cap \tilde{U} = \emptyset$ for all nontrivial $\gamma \in \Gamma$. Let $U = \pi^{-1}(\tilde{U}) \subset \mathcal{M}$ and $U' = {}_\Gamma[U] \subset \Gamma \setminus \mathcal{M}$. Note that, due to the properly discontinuous condition at $p$, any two distinct vertical leaves passing through $U$ (actually, they'll be completely contained in $U$) are also distinct modulo left multiplication by $\Gamma$. Furthermore, also due to the condition at $p$, the projection from $U$ to $U'$ is a diffeomorphism. Therefore, the open neighborhood $U'$ of $[g]$ is a regular neighborhood of the vertical foliation.

Second, suppose that $\Gamma$ does not act properly discontinuously at $p$. Let $g$ be a representative in $\mathcal{M}$ of $[g]$ such that $p = \pi(g)$, and let $U'$ and $U$ be open neighborhoods of $[g]$ and $g$, respectively, such that the projection $U \to U'$ is a diffeomorphism. Finally, set $\tilde{U} = \pi(U) \subset S$, an open neighborhood of $p$. Since $\Gamma$ does not act properly discontinuously at $p$, there is a nontrivial $\gamma \in \Gamma$ such that $\gamma(\tilde{U}) \cap \tilde{U} \neq \emptyset$. In particular, there are at least two distinct vertical leaves in $\mathcal{M}$ passing through $U$ that are equivalent modulo left multiplication by $\Gamma$. The images of these leaves in $U'$ are thus two distinct connected components of the same vertical leaf in $\Gamma \setminus \mathcal{M}$. Therefore, the vertical foliation is not regular at $[g]$.

Careful examination of the above proof reveals that Proposition 3.2 is true in a much broader setting, namely, the following proposition.

**Proposition 3.3.** *Let $G$ be a Lie group, $H \subset G$ a Lie subgroup, and $\Gamma \subset G$ a discrete subgroup. Let $\mathcal{V}$ be the vertical foliation induced from the projection $\pi : G \to G/H$, which also induces a foliation $\mathcal{V}$ on $\Gamma \setminus G$. Let $[g] \in \Gamma \setminus G$ be represented by $g \in G$, and let $p = \pi(g)$. Then ${}_\Gamma[\mathcal{V}]$ is regular at $[g]$ if and only if $\Gamma$ acts properly discontinuously at $p$.*

Combining Propositions 3.1 and 3.2 yields the following corollary.

**Corollary 3.4.** *Let $\Gamma \subset \mathcal{M}$ be a nonelementary discrete subgroup of $\mathcal{M}$. Let $[g] \in \Gamma \setminus \mathcal{M}$. Then the vertical foliation is regular at $[g]$ if and only if $[g]$ is an element of a vertical leaf ${}_\Gamma[\mathcal{L}_p]$ such that $p \in \tilde{\Omega}$ and $p$ is not fixed by an elliptic element of $\Gamma$.*

3.2. **Closure of the vertical leaves in $\Gamma \setminus \mathcal{M}$.** Before proceeding, we return to the general situation of the fiberings of Lie groups to which Proposition 3.3 is applied. The following proposition and corollary, the proofs of which are very

straightforward, are generalizations of the concept of *subgroups with good* $\Gamma$*-heredity* (see pp. 20-25 of [11]).

**Proposition 3.5.** *Let $G$ be a Lie group, $H \subset G$ a Lie subgroup and $\Gamma \subset G$ a discrete subgroup. Let $\pi : G \to G/H$ be the canonical projection and denote by ${}_\Gamma[*]$ the projection from $G$ to $\Gamma \setminus G$. For $p \in G/H$, the following statements are equivalent.*

(1) ${}_\Gamma\left[\pi^{-1}(p)\right]$ *is closed in* $\Gamma \setminus G$.
(2) $\Gamma\pi^{-1}(p)$ *is a closed subset of* $G$.
(3) *The mapping* $j_p : \Gamma_p \backslash \pi^{-1}(p) \to \operatorname{clos}\left({}_\Gamma\left[\pi^{-1}(p)\right]\right)$ *given by* $j_p\left({}_{\Gamma_p}[g]\right) = {}_\Gamma[g]$, *where* $\Gamma_p$ *is the stabilizer of* $p$ *in* $\Gamma$, *is a homeomorphism.*

**Corollary 3.6.** *If the space $\Gamma_p \setminus \pi^{-1}(p)$ is compact, then ${}_\Gamma\left[\pi^{-1}(p)\right]$ is closed in $\Gamma \setminus G$.*

We now relate these results to the case at hand, namely, $G = \mathcal{M}$, $H = \mathcal{F}_{0,\infty}$, and $\Gamma$ is a discrete subgroup of $\mathcal{M}$. Note that, for $p \in S$, the vertical leaf $\mathcal{L}_p$ is given by $\pi^{-1}(p) = \{g \in \mathcal{M} : g(\infty, 0) = p\}$ so that the stabilizer of $p$ in $\Gamma$ is given by $\Gamma_p = \{\gamma \in \Gamma : \gamma(p) = p\}$.

**Proposition 3.7.** *For $p \in S$,*

(1) $\Gamma\mathcal{L}_p = \{g \in \mathcal{M} :$ *there is a* $\gamma \in \Gamma$ *such that* $g(\infty, 0) = \gamma(p)\}$;
(2) $\pi\left(\Gamma\mathcal{L}_p\right) = \Gamma p$, *the orbit of* $p$ *under the action of* $\Gamma$;
(3) $\Gamma\mathcal{L}_p$ *is closed in* $\mathcal{M}$ *if and only if* $\Gamma p$ *is closed in* $S$.

*Proof.* The second statement follows immediately from the first, and the third follows from the continuity of $\pi$ and the second statement. Hence, we need only to prove the first statement.

Let $g \in \Gamma\mathcal{L}_p$. Then there is $\gamma \in \Gamma$ and $h \in \mathcal{L}_p$ such that $g = \gamma h$. In particular, $g(\infty, 0) = \gamma(h(\infty, 0)) = \gamma(p)$.

Now, suppose $g \in \mathcal{M}$ such that there is a $\gamma \in \Gamma$ with $g(\infty, 0) = \gamma(p)$. Then $h = \gamma^{-1}g \in \mathcal{L}_p$. So, $g \in \Gamma\mathcal{L}_p$. The first statement is proven, and so the proposition has been proven.

This proposition combined with Proposition 3.5 gives a criterion to determine when a given vertical leaf in $\Gamma \setminus \mathcal{M}$ is closed.

**Corollary 3.8.** *Let $\mathcal{L}$ be a vertical leaf in $\Gamma \setminus \mathcal{M}$ corresponding to the point $p \in S$, i.e., $\mathcal{L} = {}_\Gamma[\mathcal{L}_p]$. Then $\mathcal{L}$ is closed in $\Gamma \setminus \mathcal{M}$ if and only if $\Gamma p$ is closed in $S$.*

**Theorem 3.9.** *Let $\Gamma \subset \mathcal{M}$ be a discrete subgroup. Suppose $p = (z_1, z_2) \in S$. Let $\mathcal{L}_p = \pi^{-1}(p)$ be its corresponding vertical leaf in $\mathcal{M}$.*

(1) *If $p \in \Omega'$, then the vertical leaf ${}_\Gamma[\mathcal{L}_p]$ is closed in $\Gamma \setminus \mathcal{M}$ and biholomorphic to $\mathbb{C}^*$.*
(2) *The vertical leaf ${}_\Gamma[\mathcal{L}_p]$ is compact in $\Gamma \setminus \mathcal{M}$ if and only if the stabilizer $\Gamma_p$ is infinite (or, equivalently, $\Gamma_p$ contains a loxodromic element).*
(3) *If there is a loxodromic element $\gamma \in \Gamma$ fixing $z_1$ but not $z_2$ (or vice versa), then the vertical leaf ${}_\Gamma[\mathcal{L}_p]$ is not closed in $\Gamma \setminus \mathcal{M}$.*

*Proof of Statement* (1). Let $p \in \Omega'$. Let $\{\gamma_j\} \subset \Gamma$ be an arbitrary sequence. Any subsequence $\{\gamma_{j_k}\}$ of this sequence convergent in $\mathcal{M} \cup \mathcal{QC}$ will converge to a quasiconstant mapping $w_{1_{w_2}}$ for some $w_1,\ w_2 \in \Lambda$. Since $z_1,\ z_2 \in {}^\circ\Omega$, $w_2 \neq z_j$ for either $j = 1$ or $j = 2$. In particular, $\lim_{k\to\infty} \gamma_{j_k}(p) = (w_1, w_1)$. Thus, $\Gamma p$ has no

accumulation points in $S$. This implies that $\Gamma p$ is closed in $S$. Thus, by Corollary 3.8, the vertical leaf ${}_{\Gamma}[\mathcal{L}_p]$ is closed.

Since ${}_{\Gamma}[\mathcal{L}_p]$ is closed in $\Gamma \setminus \mathcal{M}$, the leaf itself is diffeomorphic to $\Gamma_p \setminus \mathcal{L}_p$, where $\Gamma_p$ is the stabilizer of $p$ in $\Gamma$. But $\Gamma_p = \{id\}$. So, ${}_{\Gamma}[\mathcal{L}_p]$ is diffeomorphic (actually biholomorphic) to $\mathbb{C}^*$. Statement (1) is proven.

*Proof of Statement* (2). Clearly, if $\Gamma_p \backslash \mathcal{L}_p$ is compact, then $\Gamma_p$ is necessarily infinite.

To prove the converse, we assume without loss of any generality that $p = (\infty, 0)$ so that the vertical leaf in $\mathcal{M}$, $\mathcal{L}_p$, is given by

$$\mathcal{L}_p = \{g(z) = \alpha z : \alpha \in \mathbb{C}^*\}.$$

We identify $\mathcal{L}_p$ with $\mathbb{C}^*$ by the mapping $(g(z) = \alpha z) \mapsto \alpha$. Since $\Gamma_p$ is infinite and hence has accumulation points in $\mathcal{M} \cup \mathcal{QC}$ (namely, the quasiconstant mappings $0_\infty$ and $\infty_0$), we can view its action on $\mathbb{C}^*$ (i.e., $\mathcal{L}_p$) as that of an elementary Kleinian group with exactly two limit points, 0 and $\infty$.

By definition, each element $\gamma$ of $\Gamma_p$ satisfies $\gamma(0) = 0$ and $\gamma(\infty) = \infty$. In Section V.F of [10], we see that there are two possible cases, namely,

(1) $\Gamma_p$ is generated by a nonhyperbolic loxodromic transformation, or
(2) $\Gamma_p$ is generated by a loxodromic transformation and an elliptic transformation.

In both cases, $\Gamma_p \setminus \mathbb{C}^*$ is diffeomorphic to a torus; i.e., it is compact. This proves Statement (2).

*Proof of Statement* (3). Suppose $\gamma \in \Gamma$ is loxodromic and $z_0 \in \hat{\mathbb{C}} - \{z_1, z_2\}$ such that $\gamma$ fixes both $z_0$ and $z_1$. Assume that $z_0$ is the attracting fixed point of $\gamma$ and $z_1$ is the repelling fixed point. Then $\lim_{n\to\infty} \gamma^n(z_2) = z_0$ so that $(z_1, z_0)$ is an accumulation point of $\Gamma p$. If $(z_1, z_0)$ is an actual element of $\Gamma p$, then there would be a $\gamma' \in \Gamma$ such that $\gamma'(z_1) = z_1$ and $\gamma'(z_2) = z_0$, i.e., an element of $\Gamma$ that fixes $z_1$ but not $z_0$. This would contradict Statement (6) of Theorem 2.2. Thus, $\Gamma p$ is not closed in $S$, and, by Corollary 3.8, Statement (3) is proven.

The following two corollaries combine the results from Proposition 3.1, Corollary 3.4, and Theorem 3.9. Since $Sl(2, \mathbb{C})$ is the 2-to-1 cover of $\mathcal{M}$, analogous results hold for $Sl(2, \mathbb{C})$.

**Corollary 3.10.** *Let $\Gamma \subset \mathcal{M}$ be a Kleinian subgroup with limit set $\Lambda$. Then the vertical foliation of $\Gamma \backslash \mathcal{M}$ is not regular, but there is an open dense subset of $\Gamma \backslash \mathcal{M}$ on which the foliation is regular and fibres over a complex symplectic complex surface as a holomorphic $\mathbb{C}^*$-bundle.*

*Proof.* Since $\Gamma$ is Kleinian, its limit set is nonempty, which implies by Corollary 3.4 that the vertical foliation is not globally regular. However, $\Omega'$ is an open, dense subset of $S$ so that $\mathcal{M}' = \pi^{-1}(\Omega')$ is an open dense subset of $\mathcal{M}$. Hence $\Gamma \setminus \mathcal{M}'$, on which the vertical foliation is regular, is an open dense subset of $\Gamma \setminus \mathcal{M}$. By Proposition 3.1, $\Gamma$ acts discontinuously on $\Omega'$ so that $\Gamma \setminus \Omega'$ is a manifold. Furthermore, by Statement (1) of Theorem 3.9, each vertical leaf of $\Gamma \setminus \mathcal{M}'$ is closed and biholomorphic to $\mathbb{C}^*$. So, $\Gamma \setminus \mathcal{M}'$ is a $\mathbb{C}^*$-bundle over the four-manifold $\Gamma \setminus \Omega'$. In addition, the Lie derivative of the complex structure on $\mathcal{M}$ satisfies $L_\xi J = 0$ so that $J$ projects to a complex structure on $\Gamma \setminus \Omega'$. Finally, the covariant derivative of the complex contact form on $\Gamma \setminus \mathcal{M}'$, $d\eta$, is the pullback of a complex symplectic form $\omega$ on $\Gamma \setminus \Omega'$.

**Corollary 3.11.** *Let $\Gamma \subset \mathcal{M}$ be a uniform lattice of $\mathcal{M}$. Then there are no regular vertical leaves of $\Gamma \setminus \mathcal{M}$. Furthermore, infinitely many of the vertical leaves are nonclosed, and infinitely many are compact.*

*Proof.* Since $\Gamma$ is a uniform lattice, $\Lambda = \hat{\mathbb{C}}$. So, by Corollary 3.4, the vertical foliation is nowhere regular. Since the points in $\hat{\mathbb{C}}$ fixed by a loxodromic element of $\Gamma$ are dense in $\Lambda$, there are infinitely many $z \in \hat{\mathbb{C}}$ fixed by a loxodromic element of $\Gamma$. Let $z_1 \in \hat{\mathbb{C}}$ be one such point with loxodromic $\gamma \in \Gamma$ such that $\gamma(z_1) = z_1$. Let $z_2$ be the other fixed point of $\gamma$, and set $p = (z_1, z_2)$. By Statement (2) of Theorem 3.9, the vertical leaf ${}_\Gamma[\mathcal{L}_p]$ is compact. Let $z_3 \in \hat{\mathbb{C}} - \{z_1, z_2\}$ and $q = (z_1, z_3)$. By Statement (3) of Theorem 3.9, the vertical leaf ${}_\Gamma[\mathcal{L}_q]$ is not closed. This proves the corollary.

## 4. Final remarks

The primary motivation of this paper is to describe an example of a compact complex contact manifold with a holomorphic but nonregular vertical subbundle. For a general complex contact manifold $(M, J, \mathcal{H})$, the vertical subbundle $\mathcal{V}$ is constructed as follows. Let $L$ be the complex subbundle given by $L = T^{1,0}M/\mathcal{H}$. Then $L$ can be identified with the local span of 1-forms $\pi$ such that $\mathcal{H} = \ker\ \pi$. Choose a Hermitian bundle metric on $L$, and let $\{\pi\}$ be a set of local unit 1-forms of $L$ defined on an atlas, $\{\mathcal{O}\}$, of $M$. Then, each $\pi = u - iv$, where $u$ and $v = u \circ J$ are local real-valued 1-forms such that, for any two intersecting elements of the atlas, $\mathcal{O}$ and $\mathcal{O}'$, with corresponding 1-forms, $\pi$ and $\pi'$, respectively, there is a mapping $s : \mathcal{O} \cap \mathcal{O}' \to S^1$ such that $\pi' = s\pi$ on $\mathcal{O} \cap \mathcal{O}'$. The *vertical subbundle* $\mathcal{V} \subset T^R M$ is defined as the span of the local vector fields $U$ and $V = -JU$ given by $u(U) = 1,\ v(U) = 0$, and $\iota(U)du = 0$. $\mathcal{V}$ then is a particular $J$-invariant subbundle transverse to $\mathcal{H}^R$ that generalizes the complex Reeb vector field on strict complex contact manifolds. For details, see Chapter 12 of [3].

As mentioned in the Introduction, there is a lot yet to be discovered regarding the vertical subbundle on a general complex contact manifold, particularly regarding its topology. The most studied of the compact complex contact manifolds have integrable, regular vertical subbundles. For example, there are lattices $\Gamma \subset \mathcal{H}eis_{\mathbb{C}}{}^{2n+1}$ of the complex Heisenberg group such that the vertical leaves of $\Gamma \setminus \mathcal{H}eis_{\mathbb{C}}{}^{2n+1}$ are tori, embedded as complex submanifolds, and fibre over a rectangular complex torus (see [5]). Also, the vertical subbundle of the twistor space of a compact quaternionic Kähler manifold is regular, but its leaves, which are embedded spheres, are nonholomorphic (see [3]). There is some indication that, if the vertical subbundle is integrable and the leaves are embedded spheres, then $\mathcal{V}$ will necessarily be nonholomorphic.

The purpose of this paper was to describe a compact example where $\mathcal{V}$ is integrable but not regular, as we did in Corollary 3.11. Besides providing the desired example, this result also relates to an open conjecture in real contact geometry, namely, the Weinstein conjecture. In [14], A. Weinstein conjectured that on any compact real contact manifold with trivial first homology, the Reeb vector field has a closed orbit, and, in [3], D. Blair expanded the conjecture to all compact real contact manifolds, noting that there were no known examples of a compact contact manifold with nontrivial first homology and no closed orbits of the Reeb vector field. Recently, this conjecture was proven for all three-dimensional compact contact manifolds by C. Taubes [13]. For a uniform lattice $\Gamma \subset Sl(2,\mathbb{C})$, Corollary 3.11 shows that $\Gamma \setminus Sl(2,\mathbb{C})$ is a compact complex contact manifold with infinitely many

closed orbits of the complex Reeb vector field. Note that the first homology group of $\Gamma \setminus Sl(2,\mathbb{C})$ is isomorphic to the first homology group of the compact hyperbolic manifold $\Gamma \setminus \mathcal{H}yp^3$, which is not necessarily nontrivial (see [2]).

It should be noted however that for a general complex contact manifold the vertical subbundle is not necessarily integrable. Recently, D. Blair in [4] showed that the standard complex contact structure of the projectivized holomorphic tangent bundle of $n$-dimensional complex hyperbolic space, $\mathbb{C}H^n$, does not have an integrable vertical subbundle. For a compact example, let $\Gamma \subset PU(n,1)$ be a discrete subgroup such that $\Gamma \setminus \mathbb{C}H^n$ is compact. Then the projectivized holomorphic tangent bundle of $\Gamma \setminus \mathbb{C}H^n$ will inherit both the complex contact structure and vertical subbundle from $\mathbb{C}H^n$. Thus, a straight analogue of Weinstein's conjecture to the category of all compact complex contact manifolds is false. Rather it seems that it would need to be restricted to the category of compact complex contact manifolds for which $\mathcal{V}$ is integrable.

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Department of Mathematics, John Carroll University, University Heights, Ohio 44118