ON SOME QUESTIONS RELATED TO THE MAXIMAL OPERATOR ON VARIABLE $L^p$ SPACES

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ABSTRACT. Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all exponents $p$ for which the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. A recent result by T. Kopaliani provides a characterization of $\mathcal{P}$ in terms of the Muckenhoupt-type condition $A$ under some restrictions on the behavior of $p$ at infinity. We give a different proof of a slightly extended version of this result. Then we characterize a weak type $(p(\cdot), p(\cdot))$ property of $M$ in terms of $A$ for radially decreasing $p$. Finally, we construct an example showing that $p \in \mathcal{P}(\mathbb{R}^n)$ does not imply $p(\cdot) - \alpha \in \mathcal{P}(\mathbb{R}^n)$ for all $\alpha < p_0 - 1$. Similarly, $p \in \mathcal{P}(\mathbb{R}^n)$ does not imply $\alpha p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for all $\alpha > 1/p_0$.

1. Introduction

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of functions $f$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} \, dx < \infty,$$

with norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)} \, dx \leq 1 \right\}.$$

Let $\mathcal{P}(\mathbb{R}^n)$ be the class of all functions $p$ for which the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. This class has been a focus of intense study in recent years. By the classical Hardy-Littlewood maximal theorem, any constant function $p \equiv p_0$ with $1 < p_0 < \infty$ belongs to $\mathcal{P}(\mathbb{R}^n)$. However, it has been observed quite recently that $\mathcal{P}(\mathbb{R}^n)$ consists of many nontrivial, that is, nonconstant functions. We mention briefly the key known results related to $\mathcal{P}(\mathbb{R}^n)$.

Assume that $p_- \equiv \text{ess inf}_{x \in \mathbb{R}^n} p(x) > 1$ and $p_+ \equiv \text{ess sup}_{x \in \mathbb{R}^n} p(x) < \infty$. In [5], L. Diening proved that if $p$ satisfies the log-Hölder condition

$$|p(x) - p(y)| \leq \frac{c}{\log(e + 1/|x - y|)}$$

and if $p$ is a constant outside some compact set, then $p \in \mathcal{P}(\mathbb{R}^n)$. The second condition on $p$, namely the behavior of $p$ at infinity, was improved independently

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by D. Cruz-Uribe, A. Fiorenza and C. Neugebauer [4], and A. Nekvinda [16]. It was shown in [4] that if $p$ satisfies (1.1) and
\begin{equation}
|p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)} \quad (p_\infty > 1),
\end{equation}
then $p \in \mathcal{P}(\mathbb{R}^n)$. In [16], (1.2) is replaced by a slightly more general integral condition. A new approach to these results, as well as an investigation of the limiting cases when $p_\rightarrow = 1$ and $p_\leftarrow = \infty$, can be found in the very recent works [2] [3] [7].

Conditions (1.1) and (1.2) are optimal in the pointwise sense; the corresponding examples are contained in [19] and [4]. On the other hand, they are not necessary for $p \in \mathcal{P}(\mathbb{R}^n)$. In [17] [18], A. Nekvinda constructed $p \in \mathcal{P}(\mathbb{R}^n)$ satisfying much weaker conditions at infinity than (1.2). In [13], the author established that there exist discontinuous functions $p \in \mathcal{P}(\mathbb{R}^n)$.

In [6], L. Diening showed that $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if there exists $c > 0$ such that for any family of pairwise disjoint cubes $\pi$ and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,
\begin{equation}
\| \sum_{Q \in \pi} (|f_Q| \chi_Q) \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\end{equation}
where $f_Q = \frac{1}{|Q|} \int_Q f$. This result implies, for example, that $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if $p : \mathbb{R}^n \rightarrow (0, \infty)$ satisfies (1.2).

Note that (1.3) with a single cube on the left-hand side would be a full analogue of the classical Muckenhoupt $A_p$ condition (cf. [12]) in the context of $L^{p(\cdot)}$ spaces. We give a more precise definition.

**Definition 1.1.** We say that $p$ satisfies condition $A_1^1$ ($p \in A$) if $p_\rightarrow > 1$, $p_\leftarrow < \infty$ and there exists $c > 0$ such that for any cube $Q$ and any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,
\begin{equation}
|f| Q \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{equation}

It is natural to ask whether (1.3) can be replaced by $p \in A$. In a recent work [11], T. Kopaliani gave the following partial answer: if $p$ is a constant outside some ball, then $p \in \mathcal{P}(\mathbb{R}^n)$ if and only if $p \in A$. Then this was used in [11] in order to give a new sufficient condition for $p \in \mathcal{P}(\mathbb{R}^n)$ in terms of mean oscillations of $p$.

Observe that the proof in [11] is based essentially on the above mentioned Diening characterization [6], whose proof in turn is long and complicated. In this paper we give a different, self-contained proof of an extended version of Kopaliani’s result. Our approach is based on the concept of $A_\infty$ weights and on the standard technique which, for example, can be found in the work of B. Jawerth [9].

**Theorem 1.2.** Let $p \in A$, and let $E \subset \mathbb{R}^n$ be a measurable set of positive finite measure. Then there exists a constant $c > 0$ depending on $p, n$ and $E$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,
\begin{equation}
\| (Mf)\chi_E \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{equation}

It is still unclear for us whether the class $\mathcal{P}(\mathbb{R}^n)$ can be fully characterized in terms of condition $A$. However, our next result shows that the weak type $(p(\cdot), p(\cdot))$ property of $M$ is equivalent to $p \in A$ for radially decreasing $p$.

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\footnote{Condition (1.3) is denoted in [6] by $A$, while condition (1.4) is denoted in [11] by $dx \in A_{p(\cdot)}$. We prefer to denote (1.4) by $A$, and we hope that this will not mislead the reader.
Given a function $p$, we say that $M$ is of weak type $(p(\cdot), p(\cdot))$ if there exists $c > 0$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$,
\[
\sup_{\alpha > 0} \alpha \|\chi_{\{x: |Mf(x)| > \alpha\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\]
It is easy to see that the weak type $(p(\cdot), p(\cdot))$ property of $M$ implies $p \in A$. Using Theorem 1.2, we obtain that the converse is also true for radially decreasing $p$.

Recall that a function $p$ is radially decreasing if $p(x) = p(|x|)$, where $p$ is a non-increasing function on $(0, \infty)$. The following theorem can be viewed as an analogue of Muckelnoupt’s characterization [14] of the weighted weak $L^r$ boundedness of $M$ in terms of the $A_r$ condition.

**Theorem 1.3.** Let $p$ be a radially decreasing function with $p_- > 1$ and $p_+ < \infty$. Then $M$ is of weak type $(p(\cdot), p(\cdot))$ if and only if $p \in A$.

Our next result is closely related to the author’s work [13]. It was shown there that any pointwise multiplier for $BMO(\mathbb{R}^n)$ generates a function $p \in \mathcal{P}(\mathbb{R}^n)$. A function $g$ is called a pointwise multiplier for $BMO$ if $fg \in BMO$ for any $f \in BMO$.

The main result of [13] states the following.

**Theorem A.** If $p$ is a pointwise multiplier for $BMO(\mathbb{R}^n)$ with $p_- > 0$, then there exists a constant $\alpha > 0$ such that $p(\cdot) + \alpha \in \mathcal{P}(\mathbb{R}^n)$.

Observe that conditions (1.1) and (1.2) imply that $p$ is a pointwise multiplier for $BMO(\mathbb{R}^n)$. Therefore, the following was asked in [13].

**Question 1.4.** Does any pointwise multiplier for $BMO$ with $p_- > 1$ belong to $\mathcal{P}(\mathbb{R}^n)$?

Taking into account Theorem A, this question naturally leads to the following one, which is also of some independent interest.

**Question 1.5.** Let $p \in \mathcal{P}(\mathbb{R}^n)$. Does this imply $p(\cdot) - \alpha \in \mathcal{P}(\mathbb{R}^n)$ for any $\alpha < p_- - 1$?

The following question, similar to Question 1.5, was asked in [8].

**Question 1.6.** Let $p \in \mathcal{P}(\mathbb{R}^n)$. Does this imply $\alpha p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for any $\alpha > 1/p_-$?

Our next result shows that the answers to all the above questions are negative.

**Theorem 1.7.** Let $n = 1$. Also, let $q > 1$ and $\delta > 0$. There exists a nonnegative function $p_0$ satisfying the following properties:

(i) $p_0$ is a pointwise multiplier for $BMO(\mathbb{R})$;

(ii) if $q(q - 1) \leq \delta$, then $p_{q, \delta}(x) = q + \delta p_0(x) \notin A$.

It follows immediately from this theorem that for any $q > 1$ and $\delta > 0$ such that $q(q - 1) \leq \delta$, the function $p_{q, \delta}$ yields a counterexample to Question 1.4. This, along with Theorem A, gives a counterexample to Question 1.5. Finally, applying Theorem A to $p_0(\cdot) + \varepsilon_0$, where $\varepsilon_0 > 0$, we get that there exists $\alpha_0 > 0$ such that $p_0(\cdot) + \alpha_0 \in \mathcal{P}(\mathbb{R}^n)$. Taking this function and $\alpha > 1/\alpha_0$ such that $\alpha_0(\alpha \alpha_0 - 1) \leq 1$, by Theorem 1.7, we get that $\alpha(p_0(\cdot) + \alpha_0) \notin \mathcal{P}(\mathbb{R}^n)$, which gives a counterexample to Question 1.6.

The paper is organized as follows. Section 2 contains some preliminaries. The proofs of Theorems 1.2, 1.3 and 1.7 are contained in Sections 3, 4 and 5, respectively.
2. Preliminaries

We recall that the Hardy-Littlewood maximal function is defined for \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) by

\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,
\]

where the supremum is taken over all cubes \( Q \) containing the point \( x \). We shall use the classical weak type property of \( M \) in the following form (see, e.g., [20, p. 7]):

\[
\{x \in \mathbb{R}^n : Mf(x) > \alpha\} \leq \frac{c}{\alpha} \int_{\{x : f(x) > \alpha/2\}} |f(x)|dx \quad (\alpha > 0).
\]

Recall that the conjugate function \( p' \) is defined by \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \). The following generalized Hölder inequality and a duality relation can be found in [12]:

\[
\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq 2\|f\|_{L^p(\cdot)}\|g\|_{L^{p'}(\cdot)}, \quad (2.2)
\]

\[
\|f\|_{L^p(\cdot)} \leq \sup_{\|g\|_{L^{p'}(\cdot)} \leq 1} \int_{\mathbb{R}^n} |f(x)g(x)|dx. \quad (2.3)
\]

By (2.2) and (2.3) it is easy to see that \( p \in A \) if and only if

\[
\sup_Q \frac{1}{|Q|}\|\chi_Q\|_{L^{p'}(\cdot)}\|\chi_Q\|_{L^p(\cdot)} < \infty.
\]

**Definition 2.1.** Let \( Q_0 \) be a cube. We say that a weight \( w \) (i.e., a nonnegative locally integrable function) satisfies the \( A_\infty(Q_0) \) condition if there exist constants \( \alpha, \beta \in (0, 1) \) such that for any cube \( Q \subset Q_0 \) and for any measurable subset \( E \subset Q \),

\[
|E| > \alpha|Q| \Rightarrow w(E) > \beta w(Q).
\]

It is well known that the class \( A_\infty \) can be defined in many equivalent ways. In particular, \( w \in A_\infty(Q_0) \) if and only if there exist constants \( c, \varepsilon > 0 \) such that for any cube \( Q \subset Q_0 \) and for any measurable subset \( E \subset Q \),

\[
\frac{w(E)}{w(Q)} \leq c \left( \frac{|E|}{|Q|} \right)^\varepsilon
\]

(see, e.g., [1] where this result is proved in the case \( Q_0 = \mathbb{R}^n \); the local case can be treated exactly in the same way).

3. Proof of Theorem 1.2

We start with the following lemma due to T. Kopaliani [11]. Its proof in [11] is based on some concepts from convex analysis. We give a different and simpler proof here.

**Lemma 3.1.** Let \( p \in A \). Suppose that \( |f|_Q \geq c_1 \) and \( \|f\|_{L^p(\cdot)} \leq c_2 \), where \( c_1, c_2 > 0 \). Then

\[
\int_Q (|f|_Q)^{p(x)}dx \leq c \int_Q |f(x)|^{p(x)}dx,
\]

where \( c \) depends on \( p, c_1 \) and \( c_2 \).
Proof. We consider the case $c_1 = c_2 = 1$; the same proof with trivial modifications works for general $c_1$ and $c_2$.

Let $\alpha$ be a positive constant satisfying $\int_Q \alpha^{p'(y)-1} dy = \int_Q |f|$. Then
\begin{equation}
\int_Q (|f|_Q)^{p(x)} dx = \int_Q \left( \frac{1}{|Q|} \int_Q \alpha^{p'(y)-1} dy \right)^{p(x)} \, dx 
= \left( \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \alpha^{p'(y)-p'(x)} dy \right)^{p(x)-1} \, dx \right) \int_Q \alpha^{p'(y)} dy.
\end{equation}

Since $|f|_Q \geq 1$, we have $\alpha \geq 1$. On the other hand, since $|f|_{L^p(Q)} \leq 1$, by (2.2) we get $\int_Q \alpha^{p'(y)-1} dy \leq 2 \|f\|_{L^p(Q)}$. Therefore, $\alpha \leq \frac{c}{\|f\|_{L^p(Q)}}$.

Setting $E_1(x) = \{ y \in Q : p'(y) > p'(x) \}$ and $E_2(x) = Q \setminus E_1(x)$, and using the above estimates for $\alpha$, we obtain
\begin{equation}
\int_Q \alpha^{p'(y)-p'(x)} dy = \int_{E_1(x)} \alpha^{p'(y)-p'(x)} dy + \int_{E_2(x)} \alpha^{p'(y)-p'(x)} dy
\leq c \|f\|_{L^p(Q)} + |Q|.
\end{equation}

This, along with $p \in A$, gives
\begin{equation}
\left( \frac{1}{|Q|} \int_Q \left( \frac{1}{|Q|} \int_Q \alpha^{p'(y)-p'(x)} dy \right)^{p(x)-1} \, dx \right) ^{p(x)} \leq c + c \int_Q \left( \frac{1}{\|f\|_{L^p(Q)}} \right)^{p(x)} \, dx \leq c.
\end{equation}

Further,
\begin{equation}
\int_Q \alpha^{p'(y)} dy = 2 \alpha \int_Q |f| - \int_Q \alpha^{p'(y)} dy
\leq 2 \alpha \int_{\{y \in Q : 2 \alpha |f(y)| > \alpha^{p'(y)}\}} |f(y)| dy
\leq c \int_Q |f(y)|^{p'(y)} dy.
\end{equation}

Combining (3.1) with (3.2) and (3.3) completes the proof. \hfill \Box

Corollary 3.2. Let $p \in A$. Suppose that $\xi_1 \leq t \leq \frac{\xi_2}{\|f\|_{L^p(Q)}}$, where $\xi_1, \xi_2 > 0$. Then $t^{p(x)} \in A_\infty(Q_0)$ with the $A_\infty$ constants depending only on $p, \xi_1$ and $\xi_2$.

Proof. Let $E \subset Q' \subset Q$, where $Q'$ is a cube and $|E| \geq |Q'|/2$. Then $f = t \chi_E$ satisfies the conditions of Lemma 3.1 with $Q = Q', c_1 = \xi_1/2$ and $c_2 = \xi_2$. Hence,
\begin{equation}
\frac{1}{2^{p(x)}} \int_{Q'} t^{p(x)} dx \leq c \int_E t^{p(x)} dx,
\end{equation}
which proves the $A_\infty(Q_0)$ condition. \hfill \Box

Proof of Theorem 1.2. For each integer $k$ set
\[ \Omega_k = \{ x \in \mathbb{R}^n : Mf(x) > 3^nk \} \]
and $D_k = \Omega_k \setminus \Omega_{k+1}$. Let $F_k$ be an arbitrary compact subset of $D_k$. \hfill \Box
Fix a function $\varphi \geq 0$ supported in $E$ and such that $\|\varphi\|_{L^p(E)} \leq 1$. We are going to show that
\begin{equation}
\int_{\bigcup_{k=-\infty}^{\infty} F_k} (M f) \varphi \, dx \leq c \|f\|_{L^p(E)},
\end{equation}
where $c = c(p, n, E)$. By (2.3) and by the standard limiting argument, this inequality readily gives the desired result.

By the Vitali covering lemma, there exists a finite collection of pairwise disjoint cubes $\{Q^k_j\}_{j \geq 1}$ such that $F_k \subset \bigcup_j 3Q^k_j$ and $|f|_{Q^k_j} > 3^{nk}$. Let $E^k_j = 3Q^k_j \cap F_k$ and $E^k_{j+1} = (3Q^k_j \setminus \bigcup_{k < j} 3Q^k_j) \cap F_k$, $j > 1$. Note that the sets $E^k_j$ are pairwise disjoint and $\bigcup_j E^k_j = F_k$.

Using the above definitions and (2.2), we get
\begin{align*}
\int_{\bigcup_{k=-\infty}^{\infty} F_k} (M f) \varphi \, dx &\leq 3^n \sum_{k=-\infty}^{\infty} \sum_j |f|_{Q^k_j} \int_{E^k_j} \varphi \\
&= 3^n \int_{\mathbb{R}^n} |f| T \varphi \, dx \leq 2 \cdot 3^n \|f\|_{L^p(E)} \|T \varphi\|_{L^p(E)},
\end{align*}
where
\begin{equation}
T \varphi(x) = \sum_{k=-\infty}^{\infty} \sum_j \left( \frac{1}{|Q^k_j|} \int_{E^k_j} \varphi \right) \chi_{Q^k_j}(x).
\end{equation}
Hence, in order to prove (3.4), it suffices to show that
\begin{equation}
\|T \varphi\|_{L^p(E)} \leq c(p, n, E).
\end{equation}

Let $\alpha_{j,k}(\varphi) = \frac{1}{|Q^k_j|} \int_{E^k_j} \varphi$ and
\begin{equation}
T_i \varphi(x) = \sum_{k=-\infty}^{\infty} \sum_j \alpha_{j,k}(\varphi) \chi_{Q^k_j \cap D_{k+i}}(x) \quad (l = 0, 1, \ldots).
\end{equation}

Note that $Q^k_j \subset \Omega_k = \bigcup_{l=0}^{\infty} D_{k+l}$, and hence $T \varphi(x) = \sum_{l=0}^{\infty} T_i \varphi(x)$. Also, since the sets $Q^k_j \cap D_{k+l}$ are pairwise disjoint, we have
\begin{align*}
\int_{\mathbb{R}^n} (T_i \varphi)^{p'}(x) \, dx &\geq \sum_{k=-\infty}^{\infty} \sum_j \int_{Q^k_j \cap D_{k+l}} \alpha_{j,k}(\varphi) p'(x) \, dx.
\end{align*}

We divide the last sum into two sums corresponding the indices $I_1 = \{(j, k) : \alpha_{j,k}(\varphi) > 1\}$ and $I_2 = \{(j, k) : \alpha_{j,k}(\varphi) \leq 1\}$.

Suppose first that $(j, k) \in I_1$. By (2.2) and by condition $A$,
\begin{align*}
\alpha_{j,k}(\varphi) \leq \frac{2}{|Q^k_j|} \|\chi_{E^k_j}\|_{L^{p'}} \leq \frac{2}{|Q^k_j|} \|\chi_{3Q^k_j}\|_{L^{p'}} \\
\leq \frac{c}{|Q^k_j|} \|\chi_{Q^k_j}\|_{L^{p'}} \leq \frac{c}{\|\chi_{Q^k_j}\|_{L^{p'}}}.
\end{align*}

(we have used the fact that condition $A$ implies the following “doubling” property: there exists $c > 0$ such that $\|\chi_{2Q}\|_{L^{p'}} \leq c \|\chi_{Q}\|_{L^{p'}}$ for any cube $Q$). Hence, by
Corollary 3.2 \( \alpha_{j,k}(\varphi)^{p}(x) \in A_{\infty}(Q_{j}^{k}) \). From this and from Lemma 3.1

\[
\int_{Q_{j}^{k} \cap D_{k+l}} \alpha_{j,k}(\varphi)^{p}(x) \, dx \leq c \left( \frac{|Q_{j}^{k} \cap D_{k+l}|}{|Q_{j}^{k}|} \right)^{\varepsilon} \int_{Q_{j}^{k}} \alpha_{j,k}(\varphi)^{p}(x) \, dx
\]

(3.6)

Assume now that \((j, k) \in I_{2}\). Then

\[
\int_{Q_{j}^{k} \cap D_{k+l}} \alpha_{j,k}(\varphi)^{p}(x) \, dx \leq \int_{Q_{j}^{k} \cap D_{k+l}} \alpha_{j,k}(\varphi) \, dx
\]

(3.7)

Let us show now that for each \(Q_{j}^{k}\),

\[
|Q_{j}^{k} \cap D_{k+l}| \leq 3^{n(3-l)}|Q_{j}^{k}| \quad (l \geq 4).
\]

Indeed, let \(x \in Q_{j}^{k}\) and let \(Q'\) be an arbitrary cube such that \(x \in Q'\). Observe that either \(Q' \subset 3Q_{j}^{k}\) or \(Q_{j}^{k} \subset Q'\). If the second inclusion holds, then \(3Q' \cap D_{k} \neq \emptyset\), and hence

\[
|f|_{Q'} \leq 3^{n}|f|_{3Q'} \leq 3^{n}3^{n(k+1)} \leq 3^{n(3-l)} (l \geq 2).
\]

Therefore, if \(|f|_{Q'} > 3^{n(k+1)}\), then \(Q' \subset 3Q_{j}^{k}\). From this and from the weak type \((1,1)\) property of \(M\), we get

\[
|Q_{j}^{k} \cap D_{k+l}| \leq |\{x \in Q_{j}^{k} : M(f \chi_{3Q_{j}^{k}}(x) > 3^{n(k+1)})\}|
\]

\[
\leq \frac{3^{n}}{3^{n(k+1)}} \int_{3Q_{j}^{k}} |f| \leq \frac{9^{n}|Q_{j}^{k}|}{3^{n(k+1)}} |f|_{3Q_{j}^{k}} \leq \frac{9^{n}}{3^{n(l-1)}} |Q_{j}^{k}|,
\]

proving (3.8).

Combining (3.6), (3.7) and (3.8), we get (for \(0 \leq l \leq 3\) we use a trivial estimate \(|Q_{j}^{k} \cap D_{k+l}| \leq |Q_{j}^{k}|\))

\[
\sum_{k=-\infty}^{\infty} \sum_{j} \int_{Q_{j}^{k} \cap D_{k+l}} \alpha_{j,k}(\varphi)^{p}(x) \, dx
\]

\[
\leq \sum_{(j,k) \in I_{1}} c3^{n\varepsilon(3-l)} \int_{E_{j}^{l}} \varphi(x)^{p}(x) \, dx + \sum_{(j,k) \in I_{2}} 3^{n(3-l)} \int_{E_{j}^{l}} \varphi
\]

\[
\leq c3^{n\varepsilon(3-l)} \left( \int_{\mathbb{R}^{n}} \varphi(x)^{p}(x) \, dx + \|\varphi\|_{L^{1}} \right).
\]

However, \(\int_{\mathbb{R}^{n}} \varphi(x)^{p}(x) \, dx \leq 1\), and by (2.2),

\[
\|\varphi\|_{L^{1}} = \int_{E} \varphi \leq 2\|\chi_{E}\|_{L^{p}(E)}.
\]

Therefore,

\[
\int_{\mathbb{R}^{n}} (T_{l}\varphi)^{p}(x) \, dx \leq c\|\chi_{E}\|_{L^{p}(E)} 3^{n\varepsilon(3-l)},
\]

which easily implies

\[
\|T_{l}f\|_{L^{p}(\cdot)} \leq c(p, n, E)\left(3^{n\varepsilon/(p')}\right)^{-l} \quad (l = 0, 1, \ldots).
\]
This estimate, along with
\[ \|Tf\|_{L^{p'}(\cdot)} \leq \sum_{l=0}^{\infty} \|T_{l}f\|_{L^{p'}(\cdot)}, \]
proves (3.5), and therefore the proof is complete. \( \square \)

**Remark 3.3.** The proof of Theorem 1.2 shows that
\[ \|(Mf)\chi_{E}\|_{L^{p'}(\cdot)} \leq c(p, n)c(E)\|f\|_{L^{p'}(\cdot)}, \]
where \( c(E) = \max \left( 1, \|\chi_{E}\|_{L^{1/(p')}} \right) \).

**Remark 3.4.** Theorem 1.2 easily implies the following result due to T. Kopaliani [11] mentioned in the Introduction: if \( p \) is a constant outside some ball and \( p \in A \), then \( M \) is bounded on \( L^{p(\cdot)} \). Indeed, let \( p(x) = p_{0} \) on \( B^{c} \), and let \( \|f\|_{L^{p(\cdot)}} = 1 \). Then Theorem 1.2 with \( E = 2B \) gives
\[ \int_{\mathbb{R}^{n}} (Mf)^{p(x)}dx \leq c \int_{(2B)^{c}} (Mf)^{p_{0}}dx. \]
Next, setting \( f_{1} = f\chi_{B} \) and \( f_{2} = f - f_{1} \), we get
\[ \int_{(2B)^{c}} (Mf)^{p_{0}}dx \leq 2^{p_{0}-1} \left( \int_{(2B)^{c}} (Mf_{1})^{p_{0}}dx + \int_{(2B)^{c}} (Mf_{2})^{p_{0}}dx \right). \]
By the Hardy-Littlewood maximal theorem,
\[ \int_{(2B)^{c}} (Mf_{2})^{p_{0}}dx \leq c \int_{B^{c}} |f|^{p_{0}}dx \leq c. \]
Finally, by (2.2), \( \int_{B} |f| \leq c_{B} \), and hence
\[ \int_{(2B)^{c}} (Mf_{1})^{p_{0}}dx \leq c \left( \int_{(2B)^{c}} \frac{1}{|x - x_{0}|^{p_{0}}}dx \right) \left( \int_{B} |f| \right)^{p_{0}} \leq c, \]
where \( x_{0} \) is the center of \( B \). We have proved that \( \int (Mf)^{p(x)}dx \leq c \) whenever \( \|f\|_{L^{p(\cdot)}} = 1 \), which means the boundedness of \( M \) on \( L^{p(\cdot)} \).

4. **Proof of Theorem 1.3**

The necessity part of Theorem 1.3 follows immediately from the fact that
\[ |f|_{Q\chi_{Q}(x)} \leq Mf(x) \]
for any cube \( Q \). In proving the sufficiency part it will be more convenient to work with the maximal function defined with respect to balls, so we shall assume in this section that
\[ Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B} |f(y)|dy, \]
where the supremum is taken over all balls \( B \) containing the point \( x \).

**Proof of the sufficiency part of Theorem 1.3** Let \( p \) be radially decreasing and \( p \in A \). The weak type \( (p(\cdot), p(\cdot)) \) of \( M \) means that there exists a constant \( c > 0 \) such that for any \( f \in L^{p(\cdot)} \) with \( \|f\|_{L^{p(\cdot)}} = 1 \) one has
\[ \sup_{\alpha > 0} \int_{\{x: Mf(x) > \alpha\}} \alpha^{p(x)}dx \leq c. \]
Fix an \( f \in L^p(\cdot) \) with \( \| f \|_{L^p(\cdot)} = 1 \). Observe that the case corresponding to \( \alpha \geq 1 \) follows easily from Theorem 1.2. Indeed, let \( E = \{ x : Mf(x) > 1 \} \). By (2.1),
\[
|E| \leq c \int_{\{x : |f(x)| > 1/2\}} |f(x)| \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx \leq c.
\]

Therefore, \( \| \chi_E \|_{L^p(\cdot)} \leq c \), and by (3.3) we get
\[
(4.2) \quad \int_{\{x : Mf(x) > \alpha\}} \alpha^{p(x)} \, dx \leq c \int_E (Mf)^{p(x)} \, dx \leq c \quad (\alpha \geq 1).
\]

The case when \( \alpha < 1 \) is more complicated. We are going to show that
\[
(4.3) \quad \sup_{0 < \alpha < 1} \int_{\{x : Mf(x) > 2^{\alpha+1}\}} \alpha^{p(x)} \, dx \leq c.
\]

Clearly, this estimate, along with (4.2), would imply (4.1). We start by defining several auxiliary functions.

Given a ball \( B \), denote by \( \tilde{B} \) the ball of the same radius centered at the origin. Suppose that \( |B| \geq 1 \). Then \( \| \chi_B \|_{L^{p(\cdot)}} \geq 1 \). Since \( p(x) \) is radially decreasing, \( p'(x) \) is radially increasing, and thus
\[
\int_B \left( \frac{1}{\| \chi_B \|_{L^{p(\cdot)}}} \right)^{p'(x)} \, dx \leq \int_B \left( \frac{1}{\| \chi_B \|_{L^{p(\cdot)}}} \right)^{p'(x)} \, dx = 1.
\]

Hence,
\[
\| \chi_B \|_{L^{p'(\cdot)}} \leq \| \chi_B \|_{L^{p(\cdot)}},
\]

and therefore for \( h > 1 \) we have
\[
\psi(h) \equiv \frac{1}{h} \sup_{|B|=h} \| \chi_B \|_{L^{p(\cdot)}} = \frac{1}{h} \| \chi_{\{x : |x| \leq (h/v_n)^{1/n}\}} \|_{L^{p(\cdot)}},
\]

where \( v_n \) is the volume of the unit ball. Observe that if \( |B| \geq 1 \), then
\[
|B|^{1/(p'_+)} \leq \| \chi_B \|_{L^{p(\cdot)}} \leq |B|^{1/(p'_-)}.
\]

From this and since \( (p'_+) = (p_-)' \) and \( (p'_-) = (p_+)' \), we get
\[
(4.4) \quad (1/h)^{1/p_-} \leq \psi(h) \leq (1/h)^{1/p_+} \quad (h > 1).
\]

Now, for \( 0 < \alpha < 1 \) define
\[
\varphi(\alpha) = \sup \{ h > 1 : \psi(h) > \alpha \}.
\]

By (4.4),
\[
(4.5) \quad (1/\alpha)^{p_-} \leq \varphi(\alpha) \leq (1/\alpha)^{p_+} \quad (0 < \alpha < 1).
\]

Further, setting \( B_\alpha = \{ x : |x| \leq (\varphi(\alpha)/v_n)^{1/n}\} \), we have
\[
\frac{\| \chi_{B_\alpha} \|_{L^{p(\cdot)}}}{|B_\alpha|} = \psi(\varphi(\alpha)) = \alpha.
\]

Hence, since \( p \in A \), we obtain \( \alpha \| \chi_{B_\alpha} \|_{L^{p(\cdot)}} \leq c \), or equivalently,
\[
(4.6) \quad \int_{B_\alpha} \alpha^{p(x)} \, dx \leq c
\]

(we have used an obvious fact that the definitions of class \( A \) in terms of cubes and balls are equivalent).
Combining this with (4.7) and (4.8), we have that in order to prove (4.3), it suffices

\[ \int_{\{x : Mf(x) > 2^{n+1} \alpha \}} \alpha^p(x) \, dx \leq c + \sum_{k=1}^{\infty} \int_{S_k(\alpha) \cap \{Mf > 2^{n+1} \alpha \}} \alpha^p(x) \, dx. \]

Now set \( S_k(\alpha) = (k + 1/2)B_\alpha \setminus (k - 1/2)B_\alpha \). Note that

\[ \bigcup_{x \in S_k(\alpha)} \{ B : x \in B, |B| \leq |B_\alpha/2| \} \subset S_k(\alpha). \]

Further, if \(|B| > |B_\alpha/2|\), then \( |2B| > \varphi(\alpha) \), and hence, by the definition of \( \varphi \) and by (2.2) we get

\[ \frac{1}{|B|} \int_B |f| \leq \frac{2^n}{|2B|} \int_{2B} |f| \leq 2^{n+1} \frac{\|Mf\|_{L^p(\cdot)}^p}{|2B|} \leq 2^{n+1} \alpha. \]

Therefore,

\[ S_k(\alpha) \cap \{ Mf > 2^{n+1} \alpha \} \subset \{ Mf_{S_k(\alpha)}>2^{n+1} \alpha \}. \]

Hence, setting \( \gamma_\alpha = (\varphi(\alpha)/|\alpha|)^{1/n} \) and using the fact that \( p \) is radially decreasing along with (2.1), we get (recall that \( p(x) = \rho(|x|) \))

\[ \int_{S_k(\alpha) \cap \{ Mf > 2^{n+1} \alpha \}} \alpha^p(x) \, dx \leq \alpha^{\rho((k+1)\gamma_\alpha)} \{ Mf_{S_k(\alpha)}>2^{n+1} \alpha \} \]

\[ \leq c \alpha^{\rho((k+1)\gamma_\alpha)-1} \int_{S_k(\alpha) \cap \{ |f| > 2^n \alpha \}} |f| \]

\[ \leq c \alpha^{\rho((k+1)\gamma_\alpha)-\rho((k-1/2)\gamma_\alpha)} \int_{S_k(\alpha)} |f(x)|^p(x) \, dx. \]

It is easy to see that \( \sum_{k=1}^{\infty} \chi_{S_k(\alpha)}(x) \leq 2 \), and hence

\[ \sum_{k=1}^{\infty} \int_{S_k(\alpha)} |f(x)|^p(x) \, dx \leq 2. \]

Combining this with (4.7) and (4.8), we have that in order to prove (4.3), it suffices to show that

\[ \sup_{0 < \alpha < 1} \alpha^{\rho((k+1)\gamma_\alpha)-\rho((k-1/2)\gamma_\alpha)} \leq c, \]

where \( c \) does not depend on \( k \).

Let \( \xi_1 = ((k + 3/2)\gamma_\alpha, 0, \ldots, 0) \), \( \xi_2 = ((k - 1)\gamma_\alpha, 0, \ldots, 0) \), and let \( B_1 \) and \( B_2 \) be the balls of radius \( \gamma_\alpha/2 \) centered at \( \xi_1 \) and \( \xi_2 \), respectively. Next, let \( \xi_3 = (\xi_1 + \xi_2)/2 \) and let \( B_3 \) be the ball centered at \( \xi_3 \) of radius \( 2\gamma_\alpha \). Then the balls \( B_1, B_2 \) and \( B_3 \) satisfy the following properties:

(i) \( \inf_{x \in B_1} |x| = (k + 1)\gamma_\alpha \) and \( \sup_{x \in B_2} |x| = (k - 1/2)\gamma_\alpha \);
(ii) \( |B_1| = |B_2| = \varphi(\alpha)/2^n \);
(iii) \( B_1, B_2 \subset B_3 \) and \( |B_3| = 2^n \varphi(\alpha) \).

If the supremum in (4.9) is taken over \( 2^{-n/p} - \alpha < 1 \), then the bound is trivial. Hence, one can assume that \( \alpha \leq 2^{-n/p} \). Then, by (4.5), \( |B_1| = |B_2| \geq 1 \), and therefore,

\[ \varphi(\alpha)2^{-n} \left( \frac{1}{\|\chi B_1\|_{L^p(\cdot)}} \right)^{\rho((k+1)\gamma_\alpha)} \leq \int_{B_1} \left( \frac{1}{\|\chi B_1\|_{L^p(\cdot)}} \right)^p(x) \, dx = 1. \]
Using these estimates and condition \( A \), we get
\[
\varphi(\alpha) 1^{1/p((k+1)\gamma_\alpha)+1/p((k-1/2)\gamma_\alpha)} \leq c \|\chi B_1\|_{L^p(x)} \|\chi B_2\|_{L^\infty(x)} \leq c \|\chi B_1\|_{L^p(x)} \|\chi B_2\|_{L^\infty(x)} \leq c \varphi(\alpha).
\]
Hence
\[
\varphi(\alpha) (1/p((k+1)\gamma_\alpha)-1/p((k-1/2)\gamma_\alpha)) \leq c,
\]
and thus
\[
\varphi(\alpha) \frac{1}{n/\rho(k-1/2)\gamma_\alpha} \leq c.
\]
Combining this with the left-hand side of (5.5) proves (4.9), and therefore the theorem is proved. \( \Box \)

5. Proof of Theorem 1.7

We start with the following characterization of condition \( A \) for big cubes for radially decreasing \( p \).

**Lemma 5.1.** Let \( p \) be a radially decreasing function \( (p(x) = \rho(|x|)) \) with \( p_- > 1 \) and \( p_+ < \infty \). Then

\[
(5.1) \sup_{Q \geq 1} \frac{\|\chi Q\|_{L^p(x)} \|\chi Q\|_{L^\infty(x)}}{|Q|} < \infty
\]

if and only if

\[
(5.2) \sup_{t \geq 1} t^n \int_0^1 t^{n/(\rho(t)-\rho(t))} \xi^{n-1}d\xi < \infty.
\]

*Proof.* Given \( t > 0 \), let \( B_t \) be the ball centered at the origin of radius \( t \). Observe that (5.2) is equivalent to

\[
(5.3) \sup_{t \geq 1} \|\chi B_t\|_{L^p(x)} t^{-n/p(t)} < \infty.
\]

Indeed, (5.3) holds if and only if

\[
\sup_{t \geq 1} t^n \int_{B_t} \left( \frac{1}{t^n} \right)^{\rho(t)} dx < \infty.
\]

However,

\[
\int_{B_t} \left( \frac{1}{t^n} \right)^{\rho(t)} dx = \omega_{n-1} t^{\rho(t) - 1} \int_0^1 \left( \frac{1}{t^n} \right)^{\rho(t)} \xi^{n-1}d\xi
\]

where \( \omega_{n-1} \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

Assume now that (5.1) holds. By (2.2) and (2.3), this is equivalent to

\[
|f|Q\|\chi Q\|_{L^p(x)} \leq c |f\chi Q\|_{L^p(x)}
\]

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for any locally integrable \( f \) and any cube with \(|Q| \geq 1\). In particular, setting \( f = \chi_{Q/2} \), we get
\[
\|\chi_Q\|_{L^p(Q)} \leq c \|\chi_{Q/2}\|_{L^p(Q)} \quad (|Q| \geq 1).
\]
Let \( Q_1 \) be the smallest cube containing \( B_1 \). From (5.4), for \( t \geq 1 \) we get
\[
(2^n (2^n - 1)t^n)^{1/p(t)} = \|Q_{2t} \cap Q_1\|_{P(t)^{1/p(t)}} \leq \|\chi_{Q_{2t}}\|_{L^p} \leq \|\chi_{Q_{2t}}\|_{L^p(Q_1)} \leq c \|\chi_{Q_1}\|_{L^p(Q_1)}
\]
(we use the notion \( p_-(E) = \text{ess inf}_{x \in E} p(x) \) and \( p_+(E) = \text{ess sup}_{x \in E} p(x) \)). Therefore, by (5.1),
\[
\|\chi_{B_t}\|_{L^p(Q)} \leq \|\chi_{Q_t}\|_{L^p(Q)} \leq c \frac{|Q_t|}{\|\chi_{Q_t}\|_{L^p(Q)}} \leq c t^{n/p'(t)},
\]
which proves (5.3) and so (5.2).

Suppose now that (5.2) holds. Let \( Q = \prod_{i=1}^n (a_i, a_i + h) \), where \( h \geq 1 \). Denote \( \alpha = \max_{1 \leq i \leq n} |a_i| \), and assume that \( \alpha \leq 2h \). Then it is easy to see that \( Q \subset B_{3\sqrt{n}h} \).
Next, since \( p \) is radially decreasing,
\[
\|\chi_{B_{3\sqrt{n}h}}\|_{L^p(Q)} \leq |B_{3\sqrt{n}h}|_{1/p(3\sqrt{n}h)}.
\]
From this and from (5.3),
\[
\frac{\|\chi_Q\|_{L^p(Q)}}{|Q|} \leq \frac{\|\chi_{B_{3\sqrt{n}h}}\|_{L^p(Q)} \|\chi_{B_{3\sqrt{n}h}}\|_{L^p(Q)}}{h^n} \leq c \frac{\|\chi_{B_{3\sqrt{n}h}}\|_{L^p(Q)}}{h^n} \leq c.
\]

It remains to consider the case when \( \alpha > 2h \). In this case,
\[
\sup_{x \in Q} |x| \leq \frac{3\sqrt{n}}{2} \alpha \quad \text{and} \quad \inf_{x \in Q} |x| \geq \frac{1}{2} \alpha,
\]
and therefore,
\[
p_+(Q) \leq \rho(\alpha/2) \quad \text{and} \quad p_-(Q) \geq \rho(3\sqrt{n} \alpha/2).
\]

Next, since \(|Q| \geq 1\),
\[
\|\chi_Q\|_{L^p(Q)} \leq |Q|^{1/p_-(Q)} \quad \text{and} \quad \|\chi_Q\|_{L^p(Q)} \leq |Q|^{1-1/p_+(Q)}.
\]
Combining these estimates yields
\[
\frac{\|\chi_Q\|_{L^p(Q)}}{|Q|} \leq \frac{p_+(Q) - p_-(Q)}{p_+(Q) - p_-(Q)} \leq c a^n (\rho(\alpha/2) - \rho(3\sqrt{n} \alpha/2))/p^2.
\]
But it follows from (5.2) that for \( t \geq 1 \),
\[
t^{p_-(Q) - p(t)} \leq c \quad (0 < \xi < 1),
\]
where \( c \) depends only on \( \xi \) and \( p \). Indeed, since \( \rho \) is nonincreasing, by (5.2) we get
\[
t^{\frac{1}{2} (\rho(3\sqrt{n} \alpha/2) - \rho(\alpha/2)) \xi \frac{1}{n}} \leq \int_0^\xi t^{\frac{\rho^2(\alpha/2) - \rho^2(\alpha/2)}{n}} t^{n-1} dt \leq c.
\]
Since \( \alpha > 2h \geq 2 \), we obtain that the right-hand side of (5.3) is bounded, which completes the proof. \( \square \)
Proof of Theorem 1.7. Let $E = \bigcup_{k=1}^{\infty} (e^{k^3}, e^{k^3 e^{1/k^2}})$ and 

$$p_0(x) = \int_{|x|}^{\infty} \frac{1}{\tau \log \tau} \chi_{E}(\tau) d\tau.$$ 

Let us show that $p_0$ is a pointwise multiplier for $BMO(\mathbb{R})$. This is just a combination of several known facts. First, it was proved in [15] that $p$ is a pointwise multiplier for $BMO(\mathbb{R})$ if and only if $p \in L^\infty(\mathbb{R})$ and 

$$\sup_{I} \frac{\ell(I)}{|I|^2} \int_{I} \int_{I} |p(x) - p(y)| dx dy < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$, 

$$\ell(I) = \log (e + \max(|I|, |I|^{-1}, |\text{cen}_I|))$$

and cen$_I$ denotes the center of $I$. Next, in proving [15, Proposition 4.2] it contains the proof of the fact that $g(x) = (\log log |x|) \chi_{|x| \geq \epsilon}(x)$ satisfies (5.6). But $p_0 \in L^\infty$, and it is easy to see that for all $x, y \in \mathbb{R}$, 

$$|p_0(x) - p_0(y)| \leq |g(x) - g(y)|.$$ 

Therefore, $p_0$ satisfies (5.6), which proves that $p_0$ is a pointwise multiplier for $BMO(\mathbb{R})$.

Let $p_{q, \delta}(x) = q + \delta p_0(x)$, and assume that $q(q-1) \leq \delta$. Let us show that $p_{q, \delta} \not\in A$. By Lemma 5.1 it suffices to prove that 

$$(5.7) \quad \sup_{t \geq 1} \int_{0}^{1} t^{\frac{\mu(1-t) \rho(x)}{\alpha K}} d\xi = \infty.$$ 

Denote $\alpha_k = e^{k^3}$ and $\beta_k = e^{k^3 e^{1/k^2}}$. We have 

$$\int_{0}^{1} \beta_k \rho(\beta_k) \rho(\beta_k) \rho(\alpha_k) \frac{d\xi}{\alpha_k / \beta_k} \geq \int_{0}^{1} \beta_k \rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k) \frac{d\xi}{\alpha_k / \beta_k}.$$ 

There exists a sequence $\{\xi_k\}$ such that $\xi_k \to 0$ as $k \to \infty$ and 

$$\log \beta_k - \log \log \xi_k = \log \left( \frac{\log(1/\xi)}{\log(\xi \beta_k)} + 1 \right) \geq (1 - \varepsilon_k) \log(1/\xi) \log(\xi \beta_k)$$

for any $\xi \in (\alpha_k/\beta_k, 1)$. Hence, 

$$\rho(\log \beta_k - \log \log \xi_k \beta_k) \geq \beta_k^{(1-\varepsilon_k) \log(1/\xi)} \rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k) \frac{d\xi}{\alpha_k / \beta_k} \geq (1/\xi)^{(1-\varepsilon_k)}.$$ 

From this, 

$$\int_{0}^{1} \beta_k \frac{d(\log \beta_k - \log \log \xi_k \beta_k)}{\rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k)} d\xi \geq \int_{0}^{1} \xi^{-\frac{d(1-\varepsilon_k)}{\rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k)}} d\xi.$$ 

Since $\alpha_k / \beta_k \to 0$ and $\frac{d(1-\varepsilon_k)}{\rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k)} \to \frac{\delta}{q(q-1)} \geq 1$, we have 

$$\int_{0}^{1} \xi^{-\frac{d(1-\varepsilon_k)}{\rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k)}} d\xi \to \infty \quad \text{as} \quad k \to \infty.$$ 

Indeed, for any $\varepsilon > 0$ there exists $K$ such that $\alpha_k / \beta_k < \varepsilon$ for all $k \geq K$. Hence, for all $k \geq K$ we obtain 

$$\int_{0}^{1} \xi^{-\frac{d(1-\varepsilon_k)}{\rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k)}} d\xi \geq \int_{\varepsilon}^{1} \xi^{-\frac{d(1-\varepsilon_k)}{\rho(\beta_k) \rho(\alpha_k) \rho(\alpha_k)}} d\xi,$$
and thus,
\[
\liminf_{k \to \infty} \int_{\xi_k/\beta_k}^1 \xi^{-\frac{(1-\epsilon_k)}{\rho_k(\alpha_k)+1}} d\xi \geq \int_{\xi}^1 \xi^{-\frac{\epsilon}{\rho-1}} d\xi \geq \log \frac{1}{\xi}.
\]
This proves (5.7), and therefore the proof is complete. \hfill \Box

**Remark 5.2.** It is not difficult to show that the restrictions on \( q \) and \( \delta \) in Theorem 1.7 are sharp in the sense that \( p_\alpha,\delta \in A \) if \( q(q-1) > \delta \).

**References**


