RESULTS ON A WEIGHTED POINCARÉ INEQUALITY
OF COMPLETE MANIFOLDS

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Abstract. We study manifolds satisfying a weighted Poincaré inequality, which was first introduced by Li and Wang. We generalized their result by relaxing the Ricci curvature bound condition only being satisfied outside a compact set and established a finitely many ends result. We also proved a vanishing result for an $L^2$ harmonic 1-form provided that the weight function $\rho$ is of sub-quadratic growth of the distance function, which generalized the Li-Wang result on manifolds with a positive spectrum.

Introduction

In a work of Witten and Yau [13], the authors proved that if $M^n$ is a conformally compact, Einstein, $n$-dimensional manifold ($n \geq 3$) whose boundary has positive Yamabe constant, then $M$ must have only one end. Shortly after that, Cai and Galloway [1] relaxed the assumptions to allow the boundary of $M$ to have nonnegative Yamabe constant. In [12], X. Wang proved the following:

Theorem (X. Wang). Let $M^n$ be an $n$-dimensional ($n \geq 3$), conformally compact manifold with Ricci curvature bounded from below by

$$\text{Ric}_M \geq -(n-1).$$

Let $\lambda_1(M)$ be the lower bound of the spectrum of the Laplacian on $M$. If

$$\lambda_1(M) \geq n - 2,$$

then either

(a) $H^1(L^2(M)) = 0$; or

(b) $M = \mathbb{R} \times N$ with the warped product metric $ds^2 = dt^2 + \cosh^2 t \, ds_N^2$, where $N$ is a compact manifold with $\text{Ric}_N \geq n - 2$. In particular, $M$ either has only one end or it must be a warped product given as above.

The above result generalized the results of Witten and Yau and Cai and Galloway since a theorem of Mazzeo [10] identifies the $L^2$ cohomology group $H^1(L^2(M))$ with the relative cohomology group $H^1(M, \partial M)$ for conformally compact manifolds and
a theorem of Lee [4] asserts that $\lambda_1 (M) = \frac{(n-1)^2}{4}$ for an $n$-dimensional Einstein, conformally compact manifold with nonnegative Yamabe constant for its boundary. In [8], Li and Wang considered a complete $n$-dimensional Riemannian manifold $M^n$ with Ricci curvature bounded from below by $\text{Ric}_M \geq -\frac{n-1}{n-2} \lambda_1 (M)$, where $\lambda_1 (M)$, the greatest lower bound of the spectrum of the Laplacian acting on $L^2$ functions, is assumed to be positive. They proved the following.

**Theorem** (Li-Wang). Let $M^n$ be a complete Riemannian manifold of dimension $n \geq 3$. Suppose $\lambda_1 (M) > 0$ and $\text{Ric}_M \geq -\frac{n-1}{n-2} \lambda_1 (M)$.

Then either

1. $M$ has only one end with infinite volume; or
2. $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \cosh^2 \left( \sqrt{\frac{\lambda_1 (M)}{n-2}} t \right) ds_N^2,$$

where $N$ is a compact manifold with Ricci curvature bounded from below by $\text{Ric}_N \geq -\lambda_1 (M)$.

Since all the ends of a conformally compact manifold must have infinite volume, the above theorem thus generalized the work of [1], [12] and [13] to complete manifolds with positive spectrum. Since $\lambda_1 (M) > 0$, the variational principle for $\lambda_1 (M)$ implies the following Poincaré inequality:

$$\lambda_1 (M) \int_M \phi^2 \leq \int_M |\nabla \phi|^2,$$

for any compactly supported smooth function $\phi \in C^\infty_c (M)$. In [9], the authors considered manifolds satisfying a weighted Poincaré inequality and generalized most of their results in [8] to manifolds satisfying a weighted Poincaré inequality. A manifold $M^n$ is said to satisfy a weighted Poincaré inequality with a nonnegative weight function $\rho (x)$ if

$$\int_M \rho (x) \phi^2 (x) dV \leq \int_M |\nabla \phi|^2 dV,$$

for any compactly supported smooth function $\phi \in C^\infty_c (M)$. In particular, when $\rho (x) = \lambda_1 (M)$ is a positive constant, $M$ is a manifold with positive spectrum. We say that a manifold $M$ has property $(P_{\rho})$ if a weighted Poincaré inequality is valid on $M$ with some nonnegative weight function $\rho$ and the $\rho$-metric, defined by

$$ds_\rho^2 = \rho \, ds_M^2,$$

is complete. Define

$$S (R) = \Sup_{B_{\rho} (R)} \sqrt{\rho},$$
where $B_{\rho}(R)$ is the ball of radius $R$ (with respect to some fixed point $p$) under the $\rho$-metric. In [9], the authors proved the following.

**Theorem 0.1** (Li-Wang). Let $M^n$ be a complete manifold with dimension $n \geq 3$. Assume that $M$ satisfies property $(\mathcal{P}_\rho)$ for some nonzero weight function $\rho \geq 0$. Suppose

$$\text{Ric}_M(x) \geq -\frac{n-1}{n-2} \rho(x)$$

for all $x \in M$. If $\rho$ satisfies the growth estimate

$$\liminf_{R \to \infty} \frac{S(R)}{F(R)} = 0,$$

where

$$F(R) = \begin{cases} \exp\left(\frac{n-3}{n-2} R\right) & \text{for } n \geq 4, \\ R & \text{for } n = 3, \end{cases}$$

then either

1. $M$ has only one nonparabolic end; or
2. $M$ has two nonparabolic ends and is given by $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \eta^2(t)ds_N^2,$$

for some positive function $\eta(t)$, and some compact manifold $N$. Moreover, $\rho(t)$ is a function of $t$ alone satisfying

$$\eta'' \eta^{-1} = \rho$$

and

$$\liminf_{x \to \infty} \rho(x) > 0; \text{ or}$$

3. $M$ has one parabolic end and one nonparabolic end and is given by $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \eta^2(t)ds_N^2,$$

for some positive function $\eta(t)$, and some compact manifold $N$. Moreover, $\rho(t)$ is a function of $t$ alone satisfying

$$\eta'' \eta^{-1} = \rho$$

and

$$\liminf_{x \to \infty} \rho(x) > 0 \text{ on the nonparabolic end.}$$

It is interesting to see if a similar theorem holds by relaxing the above assumptions to be only satisfied outside a compact set of $M$. In this article, the following theorem has been established:

**Theorem 0.2.** Let $M^n$ be a complete manifold with dimension $n \geq 3$. Assume that $M$ has property $(\mathcal{P}_\rho)$. Also assume that the Ricci curvature of $M$ satisfies the lower bound

$$\text{Ric}_{M \setminus K}(x) \geq -\frac{n-1}{n-2} \rho(x) + \tilde{\varepsilon}$$
for some \( \varepsilon > 0 \), compact set \( K \subseteq M \). If \( \rho \) satisfies the growth estimate

\[
\liminf_{R \to \infty} \frac{S(R)}{F(R)} = 0,
\]

where

\[
F(R) = \begin{cases} 
\exp\left(\frac{n-3}{n-2} R\right) & \text{for } n \geq 4, \\
R & \text{for } n = 3,
\end{cases}
\]

then \( M \) has only finitely many nonparabolic ends. If in addition \( \liminf_{x \to \infty} \rho(x) > 0 \), then \( M \) has only finitely many infinite volume ends.

Remark 0.3. One may assume \( \rho \) is positive somewhere outside \( K \); otherwise \( M \) will have positive Ricci curvature outside a compact set and the conclusion is well known (see [9], for instance).

On the other hand, Li and Wang also proved a vanishing theorem for \( L^2 \) integrable harmonic 1-forms on \( M \) in [8]:

**Theorem 0.4** (Li-Wang). Let \( M \) be an \( n \)-dimensional complete Riemannian manifold with \( \lambda_1(M) > 0 \) and

\[
\text{Ric}_M \geq -\frac{n}{n-1} \lambda_1(M) + \varepsilon,
\]

for some \( \varepsilon > 0 \). Then \( H^1(L^2(M)) = 0 \).

In this article, we generalized the above theorem to manifolds satisfying a weighted Poincaré inequality:

**Theorem 0.5.** Let \( M \) be an \( n \)-dimensional complete Riemannian manifold satisfying a weighted Poincaré inequality with a nonnegative weight function \( \rho(x) \). Assume the Ricci curvature satisfies

\[
\text{Ric}_M(x) \geq -\frac{n}{n-1} \rho(x) + \varepsilon,
\]

for some \( \varepsilon > 0 \). Let \( r(x, p) \) be the distance function from \( x \) to some fixed point \( p \). If \( \rho(x) = O(r^{2-\alpha}(x, p)) \), for some \( 0 < \alpha < 2 \), then \( H^1(L^2(M)) = 0 \).

1. **Some lemmas**

The following lemma is modified from [9] to suit our situation.

**Lemma 1.1.** Assume that

\[
\text{Ric}_M(x) \geq -\frac{n-1}{n-2} l(x)
\]

for some function \( l(x) \). If \( f \) is a positive harmonic function on \( M \), then

\[
|\nabla f|(x) \leq \left( (n-1) \sup_{B_{r_1}(x_1)} \sqrt{l(y)} + C_1 \sup_{B_{r_2}(x_2)} \sqrt{\rho(y)} \right) f(x),
\]

where \( C_1 \) is a constant only depending on \( n \). In particular, if the lower bound of the Ricci curvature of \( M \) satisfies

\[
\text{Ric}_{M \setminus K}(x) \geq -\frac{n-1}{n-2} \rho(x) + \tilde{\varepsilon},
\]

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where $K$ is a compact subdomain of $M$ and for some $\tilde{\epsilon} > 0$, then
\begin{equation}
|\nabla f|(x) \leq C(\sup_{B_{\rho}(x,1)} \sqrt{\rho})f(x),
\end{equation}
where $C = C(n)$, provided that $B_{\rho}(x,1) \cap K = \emptyset$.

**Proof.** Cheng and Yau’s [2] (see also [8]) local gradient estimate for positive harmonic functions implies that for any $R > 0$,
\begin{equation}
|\nabla f|(x) \leq (n-1) \left( \begin{array}{c}
\sup_{B(x,R)} \sqrt{\rho} \\
\end{array} \right) \left( \begin{array}{c}
L + CR^{-1} \\
\end{array} \right) f(x),
\end{equation}
where $C = C(n)$. Consider the function $g(r) = r - (\sup_{B(x,r)} \sqrt{\rho})^{-1}$. Since $g$ is negative when $r \to 0$ and $g \to +\infty$ as $r \to +\infty$, hence we can choose $R_0 > 0$ such that $g(R_0) = 0$, that is, $R_0 = (\sup_{B(x,R_o)} \sqrt{\rho})^{-1}$. For any point $y \in B(x,R_0)$, let $\gamma$ be a minimizing geodesic (with respect to $ds_M^2$) joining $x,y$. Then
\begin{align*}
  r_\rho(x,y) &\leq \int_{\gamma} \sqrt{\rho(\gamma(t))} \, dt \\
  &\leq \left( \begin{array}{c}
\sup_{B(x,R_0)} \sqrt{\rho(y)} \\
\end{array} \right) R_0 \\
  &\leq 1;
\end{align*}
hence $B(x,R_0) \subseteq B_{\rho}(x,1)$. Combining with (1.2) by choosing $R = R_0$, the result follows. \hfill \square

The following version of the Bochner formula is well known and was first used by Yau [14].

**Lemma 1.2** ([14] (see also [9])). Let $M^n$ be a complete Riemannian manifold of dimension $n \geq 2$. Assume that the Ricci curvature of $M$ satisfies the lower bound
\begin{equation}
\text{Ric}_M(x) \geq -(n-1)\tau(x),
\end{equation}
for some function $\tau$. Assume that $f$ is a nonconstant harmonic function on $M$. Then the function $h = |\nabla f|$ satisfies the following differential inequality:
\begin{equation}
h \Delta h \geq -(n-1)\tau h^2 + \frac{|\nabla h|^2}{n-1}
\end{equation}
in the weak sense. In addition, if we write $g = h^{\frac{n-2}{n-1}}$, the above inequality becomes
\begin{equation}
\Delta g \geq -(n-2)\tau g.
\end{equation}

**Proof.** For the sake of completeness, we outline the proof here. We choose a local orthonormal frame $\{e_1,e_2,\ldots,e_n\}$ such that $e_1f = |\nabla f|$ and $e_\alpha f = 0$, for $\alpha = 2,\ldots,n$ at a point $x$,
\begin{align*}
h^2 &= |\nabla f|^2, \\
hh_j &= \sum_{i=1}^{n} f_{ij} f_i.
\end{align*}
Hence
\[ h_j = f_{1j}, \]
\[ |\nabla h|^2 = \sum_{j=1}^{n} h_j^2 = \sum_{j=1}^{n} f_{1j}^2, \]
when evaluated at \( x \). Using the fact that \( f \) is harmonic and the Ricci formula, we compute
\[ |\nabla h|^2 + h \Delta h = \sum_{j=1}^{n} (h_j^2 + hh_{jj}) \]
\[ = \sum_{i,j=1}^{n} (f_{ij}^2 + f_{ij}f_{ij}) \]
\[ = \sum_{i,j=1}^{n} (f_{ij}^2 + f_{ij}f_{ij} + R_{ij}f_{ij}) \]
\[ = \sum_{i,j=1}^{n} (f_{ij}^2 + R_{ij}f_{ij}) \]
\[ \geq -(n-1)\tau h^2 + \sum_{i,j=1}^{n} f_{ij}^2 \]
and
\[ \sum_{i,j=1}^{n} f_{ij}^2 = 2 \sum_{\alpha=2}^{n} f_{1\alpha}^2 + f_{11}^2 + \sum_{\beta=2}^{n} f_{\beta\beta}^2 \]
\[ \geq 2 \sum_{\alpha=2}^{n} f_{1\alpha}^2 + f_{11}^2 + \frac{1}{n-1} \left( \sum_{\beta=2}^{n} f_{\beta\beta} \right)^2 \]
\[ = 2 \sum_{\alpha=2}^{n} f_{1\alpha}^2 + \frac{n}{n-1} f_{11}^2 \]
\[ \geq \frac{n}{n-1} \sum_{j=1}^{n} f_{ij}^2. \]
Combining the above inequality with (1.3), (1.4) and evaluating at \( x \), the result follows.

A theory of Li and Tam [7] allows us to count the number of nonparabolic ends of \( M \) by counting the dimension of \( K^0(M) \), a subspace of the space of all harmonic functions on \( M \). We outline the construction of Li and Tam here. Assuming that \( M \) has at least two nonparabolic ends, \( E_1, E_2 \), for \( R > 0 \), we solve the following equation:
\[
\begin{cases}
\triangle f_R = 0 & \text{in } B(R), \\
f_R = 1 & \text{on } \partial B(R) \cap E_1, \\
f_R = 0 & \text{on } \partial B(R) \setminus E_1.
\end{cases}
\]
By passing to a convergent subsequence, the sequence \( \{f_R\} \) converges to a non-constant harmonic function \( f_1 \) with finite Dirichlet integral, satisfying \( 0 \leq f_1 \leq 1 \).
Clearly for each nonparabolic end $E_i$, we can construct a corresponding $f_i$ by the above process. Let $K^0(M)$ be the linear space containing all the $f_i$’s constructed as above. By the construction, the number of nonparabolic ends of $M$ is given by the dimension of $K^0(M)$. The following lemma of Li is useful in proving finiteness type theorems.

**Lemma 1.3** ([5]). Let $H$ be a finite dimensional subspace of $L^2 p$-forms defined over a set $D \subseteq M^n$. If $V(D)$ denotes the volume of the set $D$, then there exists $\omega_0 \in H$ such that

$$\dim H \int_D |\omega_0|^2 \leq V(D) \cdot \sup_D |\omega_0|^2 \cdot \min \left\{ \left( \frac{n}{p} \right), \dim H \right\}.$$ 

2. **Finitely many ends**

**Theorem 2.1.** Let $M^n$ be a complete manifold with dimension $n \geq 3$. Assume that $M$ has property $(P_\rho)$. Also assume that the Ricci curvature of $M$ satisfies the lower bound

$$\text{Ric}_{M \setminus K}(x) \geq -\frac{n-1}{n-2} \rho(x) + \tilde{\epsilon}$$

for some $\tilde{\epsilon} > 0$, compact set $K \subseteq M$. If $\rho$ satisfies the growth estimate

$$\liminf_{R \to \infty} \frac{S(R)}{F(R)} = 0,$$

where

$$F(R) = \left\{ \begin{array}{ll} \exp\left( \frac{n-3}{n-2} R \right) & \text{for } n \geq 4, \\
R & \text{for } n = 3, \end{array} \right.$$ 

then $M$ has only finitely many nonparabolic ends. If in addition $\liminf_{x \to \infty} \rho(x) > 0$, then $M$ has only finitely many infinite volume ends.

**Proof.** By the discussion preceding Lemma 1.3, it is sufficient to estimate $\dim K^0(M)$. We may assume that $M$ has at least two nonparabolic ends and hence there exists a nonconstant bounded harmonic function $f \in K^0(M)$ with finite Dirichlet integral. By the maximal principle, we may assume that $\inf f = 0$ and $\sup f = 1$ (see [7]). Using the assumption on the Ricci curvature and Lemma 1.2, we have

$$\triangle |\nabla f| \geq -\frac{n-1}{n-2} |\nabla f| (\rho - \varepsilon) + \frac{|\nabla |\nabla f||^2}{(n-1)||\nabla f||}$$

in $M \setminus K$, where $\varepsilon = \frac{n-2}{n-1} \tilde{\epsilon}$. Applying Lemma 1.2, the above equation becomes

$$\triangle g + \rho g \geq \varepsilon g$$

in $M \setminus K$, where $g = |\nabla f|^{\frac{n-2}{n-1}}$. Let $\phi \in C_0^\infty(M \setminus K)$ be a nonnegative smooth function with compact support in $M \setminus K$. Using the property $(P_\rho)$ of $M$ and
integration by parts, we have

\[(2.2) \quad \int_M \phi^2 \rho g^2 \leq \int_M |\nabla (\phi g)|^2\]

\[= \int_M |\nabla \phi|^2 \rho g^2 + \int_M \phi^2 |\nabla g|^2 + 2 \int_M \phi g \langle \nabla \phi, \nabla g \rangle\]

\[= \int_M |\nabla \phi|^2 \rho g^2 + \int_M \phi^2 |\nabla g|^2 + \frac{1}{2} \int_M \langle \nabla (\phi)^2, \nabla (g)^2 \rangle\]

\[= \int_M |\nabla \phi|^2 \rho g^2 - \int_M \phi g \Delta g.\]

Combining (2.1) and (2.2), we have

\[\varepsilon \int_M \phi^2 \rho g^2 \leq \int_M |\nabla \phi|^2 \rho g^2, \forall \phi \in C^\infty_c (M \setminus K).\]

Let \(B(t)\) be the geodesic ball of radius \(t\) with respect to some fixed point \(p\). Since \(K\) is compact, we may choose \(R_0 > 0\) such that

\[K \subseteq \bigcup_{x \in K} B_{\rho}(x, 1) \subseteq B(R_0 - 1),\]

where we have used the facts that for any \(\varepsilon > 0\), \(\rho + \varepsilon\) is strictly positive, the metric \(d_{\rho + \varepsilon}^2\) is complete and \(B_{\rho}(x, 1) \subseteq B_{\rho + \varepsilon}(x, 1)\). Therefore \(B_{\rho}(x, 1)\) is precompact. Let \(R > 0\) be such that \(B(R_0) \subseteq B_{\rho}(R - 1)\). The above inequality implies

\[(2.3) \quad \varepsilon \int_{B_{\rho}(R) \setminus B(R_0 - 1)} \phi^2 \rho g^2 \leq \int_{B_{\rho}(R) \setminus B(R_0 - 1)} |\nabla \phi|^2 \rho g^2,\]

for any \(\phi \in C^\infty_c (B_{\rho}(R) \setminus B(R_0 - 1))\). If we write \(\phi = \psi \cdot \chi\), the right hand side of (2.3) becomes

\[(2.4) \quad \int_{B_{\rho}(R) \setminus B(R_0 - 1)} |\nabla \phi|^2 \rho g^2 \leq 2 \int_{B_{\rho}(R) \setminus B(R_0 - 1)} |\nabla \psi|^2 \rho g^2 \phi^2 + 2 \int_{B_{\rho}(R) \setminus B(R_0 - 1)} |\nabla \chi|^2 \rho g^2 \phi^2.\]

Now we choose \(\chi, \psi\) as in (9):

\[\chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0, \delta \bar{\tau}) \cup \mathcal{L}(1 - \delta \bar{\tau}, 1), \\ (-\log \delta)^{-1}(\log f - \log(\delta \bar{\tau})) & \text{on } \mathcal{L}(\delta \bar{\tau}, \bar{\tau}) \cap (M \setminus E_1), \\ (-\log \delta)^{-1}(\log(1 - f) - \log(\delta \bar{\tau})) & \text{on } \mathcal{L}(1 - \tau, 1 - \delta \bar{\tau}) \cap E_1, \\ 1 & \text{otherwise} \end{cases}\]

for some \(0 < \delta < 1\) and \(0 < \tau < \frac{1}{2}\) to be determined later, where

\[\mathcal{L}(a, b) = \{x \in M : a < f(x) < b\},\]

\[\psi(x) = \begin{cases} 0 & \text{on } B(R_0 - 1), \\ 1 & \text{on } B_{\rho}(R - 1) \setminus B(R_0), \\ R - r_\rho & \text{on } B_{\rho}(R) \setminus B_{\rho}(R - 1), \\ 0 & \text{on } M \setminus B_{\rho}(R). \end{cases}\]
The first term on the right hand side of (2.4) can be estimated by

\begin{equation}
(2.5) \quad 2 \int_{B_ρ(R)\setminus B(R_0-1)} |\nabla \psi|^2 \chi^2 g^2 \leq 2 \int_{B_ρ(R)\setminus B(R_0-1)} |\nabla \psi|^2 \chi^2 g^2 + 2 \int_{B(R_0)\setminus B(R_0-1)} |\nabla \psi|^2 \chi^2 g^2.
\end{equation}

Now consider

\begin{equation}
(2.6)
\int_{E_1} |\nabla \chi|^2 \psi^2 g^2
\leq (\log \delta)^{-2} \int_{\mathcal{L}(1-\tau,1-\delta \tau) \cap E_1 \cap \{B_ρ(R) \setminus B(R_0-1)\}} |\nabla f|^2 + \frac{2(n-1)}{n-2} (1-f)^{-2}
\leq CS \frac{2(n-2)}{n-2} (R+1) (\log \delta)^{-2} \int_{\mathcal{L}(1-\tau,1-\delta \tau) \cap E_1 \cap \{B_ρ(R) \setminus B(R_0-1)\}} |\nabla f|^2 (1-f)^{\frac{2(n-2)}{n-2} - 2}
\leq CS \frac{2(n-2)}{n-2} (R+1) (\log \delta)^{-2} \int_{\mathcal{L}(1-\tau,1-\delta \tau) \cap E_1 \cap \{B_ρ(R) \setminus B(R_0-1)\}} |\nabla f|^2 (1-f)^{\frac{2(n-2)}{n-2} - 2},
\end{equation}

where the second inequality follows from (1.1) by replacing \( f \) by \( 1-f \) and the choice of \( R_0 \) so that \( B_ρ(x,1) \) does not intersect \( K \) for any \( x \in B_ρ(R) \setminus B(R_0-1) \).

We can now follow the arguments as in [9] to yield

\[ \int_{B_ρ(R)\setminus B_ρ(R_0-1)} |\nabla \psi|^2 \chi^2 g^2 \leq CS \frac{2(n-2)}{n-2} (R)(\delta \tau)^{-\frac{n-2}{n-1}} \exp(-2R) \]

and

\[ \int_{M} |\nabla \chi|^2 \psi^2 g^2 \leq \left\{ \begin{array}{ll} CS \frac{2(n-2)}{n-2} (R+1)(-\log \delta)^{-2}(1-\delta \frac{n-2}{n-1}) \frac{\delta \tau^\frac{n-2}{n-1}}{\tau^{n-2}}, & n \geq 4, \\ CS(R+1)(-\log \delta)^{-1}, & n = 3, \end{array} \right. \]

where \( C \) is a constant independent of \( R \). Continuing to follow the arguments in [9], we choose

\[ \delta = \frac{1}{2}, \tau = \exp(-2R) \quad \text{when} \quad n \geq 4, \]
\[ \delta = \tau = \exp(-Rq(R)) \quad \text{when} \quad n = 3, \]

where \( q(R) = \sqrt{\frac{S(R+1)}{R}} \). Combining with the growth assumption of \( \rho \), we can now conclude as in [9] that

\[ \int_{M} |\nabla \chi|^2 \psi^2 g^2 \to 0 \quad \text{as} \quad R \to +\infty, \quad \text{and} \]
\[ \int_{B_ρ(R)\setminus B_ρ(R_0-1)} |\nabla \psi|^2 \chi^2 g^2 \to 0 \quad \text{as} \quad R \to +\infty. \]

Letting \( R \to +\infty \) and combining the above results with (2.4) and (2.5), (2.3) becomes

\[ \varepsilon \int_{M\setminus B(R_0)} g^2 \leq \tilde{C} \int_{B(R_0)\setminus B(R_0-1)} g^2, \]
where $\tilde{C}$ is a constant depending only on $n$, which implies

$$\int_{B(2R_0)} g^2 \leq C \int_{B(R_0)} g^2,$$

where $C = C(\varepsilon, n)$. The function $g$ satisfies the differential inequality

$$\Delta g \geq -\alpha g$$

on $B(p, 2R_0)$, where $\alpha = \inf_{x \in B(p, 2R_0)} \text{Ric}_M$, and the mean value inequality of Li and Tam \[6\] implies that

$$g^2(x) \leq C_1 \int_{B(p, 2R_0)} g^2$$

for any $x \in B(p, R_0)$, where $\nu = \inf_{x \in B(p, R_0)} \text{Ric}_M$. Combining the above equation with (2.7), we have

$$\sup_{B(p, R_0)} g^2 \leq C_2 \int_{B(p, R_0)} g^2,$$

where $C_2 = C_2(\varepsilon, n, \alpha, \nu)$. On the other hand, Schwarz’s inequality implies that

$$\int_{B(R_0)} g^2 \leq \left( \int_{B(R_0)} |\nabla f|^2 \right)^{\frac{n-2}{n-4}} V_p(R_0) \frac{n}{n-4}.$$

Therefore we have

$$\sup_{B(R_0)} |\nabla f|^2 \leq C_3 \int_{B(R_0)} |\nabla f|^2,$$

where $C_3 = C_3(\varepsilon, n, \alpha, \nu, R_0)$ is a constant independent of $f \in K^0(M)$. By unique continuation,

$$\int_{B(R_0)} |\nabla f|^2 \neq 0,$$

provided that $f$ is not a constant function. Therefore,

$$\int_{B(R_0)} \langle \nabla f, \nabla g \rangle$$

defines a nondegenerate bilinear form on the space of 1-forms $K = \{df : f \in K^0(M)\}$.

Lemma 1.3 asserts that there exists $f_0 \in K^0(M)$ such that

$$\dim K \int_{B(R_0)} |df_0|^2 \leq nV_p(R_0) \sup_{B(R_0)} |df_0|^2.$$

Combining the above with (2.8) implies

$$\dim K^0(M) = \dim K + 1 \leq C_4$$

for some fixed constant $C_4 = C_4(C_3, V_p(R_0))$, which completes the proof. The second part of the theorem follows from \[9\]; an end is nonparabolic if and only if it has infinite volume, provided that $\liminf_{x \to \infty} \rho > 0$. \[\square\]
Remark 2.2. We would like to point out that the growth condition is not too restrictive. It is satisfied if the weight function \( \rho \) does not grow too fast, for instance, if \( \rho \) has only polynomial growth \( (n \geq 4) \).

### 3. Vanishing Theorems

In this section we study the space of \( L^2 \) integrable harmonic 1-forms, \( H^1(L^2(M)) \). If \( f \) is a harmonic function with finite Dirichlet integral, then the exterior derivative of \( f \), \( df \), is an \( L^2 \) integrable harmonic 1-form. By the theory of Li and Tam [7] (see also [8]), we have

\[
\dim H^1(L^2(M)) + 1 \geq \dim K^0(M) \\
\geq \text{number of nonparabolic ends of } M.
\]

If we further assume that \( \lambda_1(M) > 0 \), then

\[
\dim H^1(L^2(M)) + 1 \geq \text{number of infinite volume ends of } M.
\]

Therefore, an estimate on \( \dim H^1(L^2(M)) \) is, in general, a stronger estimate than an estimate on the number of nonparabolic ends (infinite volume ends if, in addition, \( \lambda_1(M) > 0 \)). It is known that if \( \omega \) is an \( L^2 \) harmonic 1-form, then it is both closed and co-closed. In particular, \( h = |\omega| \) satisfies a Bochner type formula

\[
\triangle h \geq \frac{\text{Ric}_M(\omega, \omega)}{h} + \frac{|\nabla h|^2}{(n-1)h}.
\]

We start by proving an estimate for functions that satisfy the above Bochner type formula. We believe the estimate is of independent interest and will be useful in many other situations.

**Lemma 3.1.** Let \( b > -1 \). Assume that \( h \) is a nonnegative function and satisfies the differential inequality

\[
h \triangle h \geq -ah^2 + b|\nabla h|^2
\]

in the weak sense, for some constant \( a \). For any \( \varepsilon > 0 \), we have the following estimate:

\[
(b(1-\varepsilon) + 1) \int_M |\nabla (\phi h)|^2 \leq \left(b\left(\frac{1}{\varepsilon} - 1\right) + 1\right) \int_M h^2 |\nabla \phi|^2 + a \int_M \phi^2 h^2,
\]

for any compactly supported smooth function \( \phi \in C_c^\infty(M) \). In addition, if

\[
\int_{B_r(R)} h^2 = o(R^2),
\]

then

\[
\int_M |\nabla h|^2 \leq \frac{a}{b+1} \int_M h^2.
\]

In particular, \( h \) has finite Dirichlet integral if \( h \in L^2(M) \).
Proof. Let \( \phi \in C_\infty^c(M) \) be a smooth function with compact support. Integration by parts implies

\[
\int_M \phi^2 \Delta h = - \int_M \langle \nabla (\phi^2 h), \nabla h \rangle
\]

\[
= \int_M \langle h \nabla \phi + \nabla (\phi h), h \nabla \phi - \nabla (\phi h) \rangle
\]

\[
= \int_M h^2 |\nabla \phi|^2 - \int_M |\nabla (\phi h)|^2.
\]

By the differential inequality satisfied by \( h \), we have

\[
\int_M \phi^2 \Delta h \geq -a \int_M \phi^2 h^2 + b \int_M \phi^2 |\nabla h|^2.
\]

Combining the above inequality with (3.1) gives

\[
\int_M h^2 |\nabla \phi|^2 + a \int_M \phi^2 h^2 \geq b \int_M \phi^2 |\nabla h|^2 + \int_M |\nabla (\phi h)|^2.
\]

On the other hand, the Schwarz inequality implies

\[
\int_M \phi^2 |\nabla h|^2 = \int_M \langle \nabla (\phi h) - h \nabla \phi, \nabla (\phi h) - h \nabla \phi \rangle
\]

\[
= \int_M (|\nabla (\phi h)|^2 - 2 \langle \nabla (\phi h), h \nabla \phi \rangle + h^2 |\nabla \phi|^2)
\]

\[
\geq (1 - \varepsilon) \int_M |\nabla (\phi h)|^2 + (1 - \frac{1}{\varepsilon}) \int_M h^2 |\nabla \phi|^2,
\]

for any \( \varepsilon > 0 \). Combining the above with (3.2) gives

\[
\int_M \phi^2 h \Delta h \geq -a \int_M \phi^2 h^2 + b \int_M \phi^2 |\nabla h|^2.
\]

Using \( \int_B \phi^2 h^2 = o(R^2) \), and letting \( R \to +\infty, \varepsilon \to 0 \), the second part of the lemma is achieved. \( \square \)

Remark 3.2. By the proof of the above lemma, the above conclusions are still valid if we assume that \( a = a(x) \) is a function of \( x \), in the following forms:

\[
b(1 - \varepsilon) + 1 \int_M |\nabla (\phi h)|^2 \leq b(\frac{1}{\varepsilon} - 1) + 1 \int_M h^2 |\nabla \phi|^2 + \int_M a(x) \phi^2 h^2
\]

and

\[
\int_M |\nabla h|^2 \leq \frac{1}{b + 1} \int_M a(x) h^2.
\]

The following theorem is an immediate application of the above lemma.
Corollary 3.3. Let \( h \) be a function satisfying the assumptions in Lemma 3.1. Assume that \( \lambda_1(M) > 0 \) and the Ricci curvature of \( M \) satisfies
\[
\text{Ric}_M \geq -(b+1)\lambda_1(M) + \delta,
\]
for some \( \delta > 0 \). If
\[
\int_{B_\varepsilon(R)} h^2 = o(R^2),
\]
then \( h \equiv 0 \). In particular, we recover a theorem of Li and Wang in [8]: Let \( M^n \) be a complete noncompact \( n \)-dimensional Riemannian manifold with \( \lambda_1(M) > 0 \), and the Ricci curvature satisfies
\[
\text{Ric}_M \geq -\frac{n}{n-1} \lambda_1(M) + \delta,
\]
for some \( \delta > 0 \). Then \( H^1(L^2(M)) = 0 \).

Proof. Combining Lemma 3.1 with the variational principle of \( \lambda_1(M) \), we have
\[
(b(1-\varepsilon) + 1)\lambda_1(M) \int_M \phi^2 h^2 \leq (b(1-\varepsilon) + 1) \int_M (|\nabla \phi h|^2 + a \int_M \phi^2 h^2 + \left(b \frac{1}{\varepsilon} - 1 + 1\right) \int_M h^2 |\nabla \phi|^2,
\]
for any \( \varepsilon > 0 \) and any compactly supported smooth function \( \phi \in C^\infty_c(M) \). The above inequality implies
\[
\delta \int_M \phi^2 h^2 \leq b \varepsilon \lambda_1(M) \int_M \phi^2 h^2 + \left(b \frac{1}{\varepsilon} - 1 + 1\right) \int_M h^2 |\nabla \phi|^2.
\]
Let
\[
\phi = \begin{cases} 1 & \text{on } B(R), \\ 0 & \text{on } M \setminus B(2R), \end{cases}
\]
such that \( |\nabla \phi|^2 \leq C/R^2 \) on \( B(2R) \setminus B(R) \). The above inequality becomes
\[
\delta \int_{B(R)} h^2 \leq b \varepsilon \lambda_1(M) \int_{B(2R)} h^2 + R^{-2} \left(b \frac{1}{\varepsilon} - 1 + 1\right) \int_{B(2R) \setminus B(R)} h^2.
\]
Combining the above inequality with the growth assumption on \( h \), we let \( R \to +\infty \), then take \( \varepsilon \to 0 \) and conclude that \( \int_M h^2 \leq 0 \). Thus \( h \equiv 0 \). For the second part, let \( \omega \in H^1(L^2(M)) \) and \( h = |\omega| \in L^2(M) \). Then it is well known that \( h \) satisfies the differential inequality
\[
h \Delta h \geq -ah^2 + b|h|^2
\]
in the weak sense with \( a = (b+1)\lambda_1(M) - \delta \) and \( b = 1/(n-1) \) for any Riemannian manifold of dimension \( n \).

Theorem 3.4. Let \( M^n \) be a complete noncompact manifold of dimension \( n \) satisfying the weighted Poincaré inequality with a nonnegative weight function \( \rho(x) \). Assume the Ricci curvature satisfies
\[
\text{Ric}_M(x) \geq -\frac{n}{n-1} \rho(x) + \delta,
\]
for some \( \delta > 0 \). If \( \rho(x) \) is of subquadratic growth of the distance function, that is,
\[
\rho(x) = O(r^{2-\alpha}(x)),
\]
Theorem 3.5. Let $\omega \in H^1(L^2(M))$ and $h = |\omega| \in L^2(M)$. Applying Lemma 3.1 with $b = \frac{1}{n-1}$, $a = (b+1)\rho - \delta$ and using the weighted Poincaré inequality, we have

$$
(b(1-\varepsilon)+1) \int_M \rho \phi^2 h^2 \leq (b(1-\varepsilon)+1) \int_M |(\nabla \phi h)|^2 \leq \int_M \phi^2 h^2 + \left( (b(1-\varepsilon)+1) \int_M h^2 |\nabla \phi|^2 \right).
$$

Thus

$$
\delta \int_M \phi^2 h^2 \leq \left( (1-b) + b \varepsilon^{-1} \right) \int_M h^2 |\nabla \phi|^2 + b \varepsilon \int_M \rho \phi^2 h^2,
$$

for any $\varepsilon > 0$ and any compactly supported smooth function $\phi \in C_c^\infty(M)$. Let $\varepsilon = R^{\alpha/2-2}$ and

$$
\phi = \begin{cases} 
1 & \text{on } B(R), \\
0 & \text{on } M \setminus B(2R) 
\end{cases}
$$

such that $|\nabla \phi|^2 \leq C/R^2$ on $B(2R) \setminus B(R)$. The above inequality becomes

$$
\delta \int_{B(R)} h^2 \leq b R^{-\alpha/2} \int_{B(2R)} h^2 + \left( (1-b) R^{-2} + b R^{-\alpha/2} \right) \int_{B(2R)} h^2.
$$

Combining the above inequality with the assumption $h \in L^2(M)$ and letting $R \to +\infty$, we conclude that $\int_M h^2 \leq 0$ and thus $h \equiv 0$. 

Theorem 3.5. Let $M^n$ be a complete noncompact manifold of dimension $n$ satisfying the weighted Poincaré inequality with weight function $\rho > 0$. Assume the Ricci curvature satisfies $\text{Ric}_M \geq - \left( \frac{n-1}{n-2} - \delta \right) \rho$, for some $\delta > 0$, and $\rho = O(r_p^{2-\alpha}(x))$ for some $0 < \alpha < 2$. Then $H^1(L^2(M)) = 0$.

Proof. Let $\omega \in H^1(L^2(M))$ and $h = |\omega| \in L^2(M)$. Applying Lemma 3.1 with $b = \frac{1}{n-1}$, $a = (b+1-\delta)\rho$, and weighted Poincaré inequality, we have

$$
(b(1-\varepsilon)+1) \int_M \rho \phi^2 h^2 \leq (b(1-\varepsilon)+1) \int_M |(\nabla \phi h)|^2 \leq \int_M \phi^2 h^2 + \left( (b(1-\varepsilon)+1) \int_M h^2 |\nabla \phi|^2 \right).
$$

Thus

$$
\delta \int_M \phi^2 h^2 \leq b \varepsilon \int_M \rho \phi^2 h^2 + (b \varepsilon^{-1} - b + 1) \int_M h^2 |\nabla \phi|^2.
$$

If we choose

$$
\phi = \begin{cases} 
1 & \text{on } B(R), \\
0 & \text{on } M \setminus B(2R), 
\end{cases}
$$

using $\rho(x) = O(r_p^{2-\alpha}(x))$ and letting $\varepsilon = R^{\alpha/2-2}$, the above inequality implies

$$
\delta \int_{B(R)} \rho h^2 \leq CR^{-\alpha/2} \int_{B(2R)} h^2 + (b R^{-\alpha/2} + (1-b) R^{-2}) \int_{B(2R)} h^2,
$$

where $r_p(x)$ is the distance function from $x$ to some fixed point $p$, for some $0 < \alpha < 2$, then $H^1(L^2(M)) = 0$.
for some constant $C$. Using $h \in L^2(M)$ and letting $R \to +\infty$, we conclude that $\int_M \rho h^2 = 0$. By Lemma 3.1, we have
\[
\int_M |\nabla h|^2 \leq \left(1 - \frac{\delta}{b+1}\right) \int_M \rho h^2 = 0.
\]
Therefore $|\nabla h| \equiv 0$. Hence $h \equiv c \in L^2(M)$ for some constant $c$. Since $M$ is nonparabolic, it must have infinite volume and thus $h \equiv 0$. □

4. Manifolds with parallel forms

The following theorem in [3] will allow us to obtain some improved vanishing theorems for manifolds with parallel forms when combining with the result in the previous section.

**Theorem 4.1** ([3]). Let $M^n$ be an $n$-dimensional complete Riemannian manifold with a parallel $p$-form $\omega$. Assume that $\alpha$ is an $L^2$ harmonic 1-form, that is, $\triangle \alpha = 0$ and $\int_M |\alpha|^2 < +\infty$. Then $\alpha$ satisfies
\[
d^*(\alpha \wedge \omega) = 0.
\]

**Proof.** For the sake of completeness, we outline the proof here. First, we claim that
\[
* d^*(\alpha \wedge \omega) = (-1)^{n-1} d^* (\alpha \wedge * \omega).
\]
For any $x \in M$, we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ such that $\nabla e_i e_j(x) = 0$. Let $\{\theta^i\}_{i=1}^n$ be the coframe. For any $p$-form $\omega$, we have
\[
d \omega = \varepsilon(\theta^i) \nabla e_i \omega
\]
at $x$ and $\omega$ is parallel if and only if $\nabla e_i \omega = 0, \forall i$. Let $\alpha = \sum_{i=1}^n a_i e_i$, and hence $\overline{\alpha} = \sum_{i=1}^n a_i e_i$ is the dual of $\alpha$. We use the notation $d\alpha = \sum_{i,j=1}^n a_i j \theta^j \wedge \theta^i$, where $a_{i,j} = \nabla e_i a_j$. Since $\alpha$ is $L^2$ harmonic, it is both closed and co-closed, which are equivalent to the conditions that $a_{i,j} = a_{j,i}$ and $\sum_{i=1}^n a_{i,i} = 0$. The following calculations are all evaluated at $x$:
\[
d^*(\alpha \wedge * \omega) = d \varepsilon(\alpha) \wedge \omega
\]
\[
= (-1)^{(p-1)(n-p)} d[l(\overline{\alpha}) \omega]
\]
\[
= (-1)^{(p-1)(n-p)} \sum_{i=1}^n \varepsilon(\theta^i) \nabla e_i (l(\overline{\alpha}) \omega)
\]
\[
= (-1)^{(p-1)(n-p)} \sum_{i,j=1}^n \varepsilon(\theta^i) a_{j,i} (l(e_j) \omega),
\]
where the third equality follows from \( \nabla e_i e_j(x) = 0 \) and the last equality follows from \( \nabla \omega = 0 \). On the other hand,

\[
* d * (\alpha \wedge \omega) = * d * \varepsilon(\alpha) \omega
\]

(4.3) 

\[
= * \left( \sum_{i=1}^{n} \varepsilon(\theta^i) \nabla e_i \left( * \varepsilon \left( \sum_{j=1}^{n} a_j \theta^j \right) \omega \right) \right)
\]

\[
= * \left( \sum_{i,j=1}^{n} a_{i,j} \varepsilon(\theta^i) * (\varepsilon(\theta^j) \omega) \right)
\]

\[
= (-1)^{p(n-p-1)} \sum_{i,j=1}^{n} a_{i,j} l(e_i) \varepsilon(\theta) \omega
\]

\[
= (-1)^{p(n-p-1)} \left( \sum_{i=1}^{n} a_{i,i} [\omega - \varepsilon(\theta^i) l(e_i) \omega] - \sum_{i \neq j}^{n} a_{i,j} \varepsilon(\theta^i) l(e_i) \omega \right)
\]

\[
= (-1)^{p(n-p-1)+1} \sum_{i,j=1}^{n} a_{i,j} \varepsilon(\theta^i) (l(e_j) \omega),
\]

where the last equality follows from \( a_{i,j} = a_{j,i} \) and \( \sum_{i=1}^{n} a_{i,i} = 0 \). (4.1) now follows from (4.2) and (4.3). Let

\[
\phi(x) = \begin{cases} 
1 & \text{on } B_p(R), \\
0 & \text{on } M \setminus B_p(2R)
\end{cases}
\]

such that \( |\nabla \phi| \leq C_1 R^{-1} \). Consider

(4.4) 

\[
\int_M \phi^2 |d * (\alpha \wedge \omega)|^2 = \int_M \phi^2 d * (\alpha \wedge \omega) \wedge * d * (\alpha \wedge \omega)
\]

\[
= \int_M \phi^2 d * (\alpha \wedge \omega) \wedge d * (\alpha \wedge \omega)
\]

\[
= \int_M d\phi^2 \wedge *(\alpha \wedge \omega) \wedge d * (\alpha \wedge \omega)
\]

\[
\leq 2 \left( \int_M |d\phi|^2 * (\alpha \wedge \omega) \right)^{1/2} \left( \int_M \phi^2 |d * (\alpha \wedge \omega)|^2 \right)^{1/2}
\]

\[
= 2 \left( \int_M |d\phi|^2 * (\alpha \wedge \omega) \right)^{1/2} \left( \int_M \phi^2 |d * (\alpha \wedge \omega)|^2 \right)^{1/2},
\]

where the second and the last equality follow from (4.4), the third equality follows from integration by parts and the fact that \( d^2 = 0 \). \( \omega \) is parallel implies

\[
| * (\alpha \wedge \omega) | \leq C_2 |a|,
\]
for some constant $C_2$. Combining the above with (4.4), we have
\[
\int_{B_p(R)} |d \ast (\alpha \wedge \omega)|^2 \leq C_3 R^{-2} \int_{B_p(2R)} |\alpha|^2.
\]
Letting $R \to +\infty$, the result follows from the assumption that $\alpha$ is $L^2$ integrable.

\[\square\]

**Theorem 4.2.** Let $M^{2n}$ be a real $2n$-dimensional Kähler manifold with $\lambda_1(M) > 0.$ Assume the Ricci curvature of $M$ satisfies
\[
\text{Ric}_M \geq -2\lambda_1(M) + \delta,
\]
for some $\delta > 0.$ Then $H^1(L^2(M)) = 0.$

**Proof.** Let $\omega \in H^1(L^2(M))$ and $h = |\omega|.$ We claim that $h$ satisfies the Bochner formula of the following form:
\[
\Delta h \geq \frac{\text{Ric}_M(\omega, \omega)}{h} + \frac{|
abla h|^2}{h}.
\]
Applying Corollary 3.3 with $b = 1$, the result follows. To prove the claim, we let \[\{e_i\}_{i=1}^{2n} = \{\tau_1, \ldots, \tau_n, I\tau_1, \ldots, I\tau_n\}\] be a local orthonormal frame, where $I$ is the complex structure and let \[\{\theta^i\}_{i=1}^{2n} = \{\overline{\theta}_1, \ldots, \overline{\theta}_n, I\overline{\theta}_1, \ldots, I\overline{\theta}_n\}\] be the orthonormal coframe. The Kähler form, satisfying $\Omega(X, Y) = g(X, IY)$, is then given by
\[
\Omega = -\sum_{i=1}^{n} \theta^i \wedge I\overline{\theta}^i = -\sum_{i=1}^{n} \theta^i \wedge \theta^{n+i}.
\]
With the above notation, we can write $\omega = \sum_{i=1}^{2n} a_i \theta^i.$ By Theorem 4.1
\[
d \ast (\omega \wedge \Omega) = 0,
\]
which is equivalent to, by (4.3),
\[
(4.5) \quad \sum_{i,j=1}^{2n} a_{ij} \theta^i \wedge l(e_j)\Omega = 0,
\]
where we have used the notation $a_{ij} = a_{i,j}.$ Since
\[
l(e_j)\Omega = \begin{cases} 
-\theta^{j+n} & 1 \leq j \leq n, \\
\theta^{j-n} & n + 1 \leq j \leq 2n,
\end{cases}
\]
hence (4.5) becomes
\[
(4.6) \quad \sum_{i=1}^{2n} \left( -\sum_{j=1}^{n} a_{ij} \theta^i \wedge \theta^{j+n} + \sum_{j=1}^{n} a_{i+n,j+n} \theta^i \wedge \theta^j \right) = 0.
\]
The coefficient of $\theta^i \wedge \theta^{i+n}$ in the above equation is zero, and thus we conclude that
\[
a_{ii} + a_{i+n,i+n} = 0,
\]
for any \(1 \leq i \leq n\). Now we go back to studying the form \(\omega\). Let \(\{e_i\}_{i=1}^{2n}\) be as described above with \(e_1\) such that \(\omega(e_1) = |\omega|\) and \(\omega(e_j) = 0\) for any \(j \neq 1\) at a fixed point \(p\). Then

\[
|\nabla \theta|^2 = \sum_{i,j=1}^{2n} a_{ij}^2
\]

\[
\geq a_{11}^2 + a_{n+1,n+1}^2 + 2 \sum_{j=2}^{2n} a_{1j}^2
\]

\[
= 2 \left( a_{11}^2 + \sum_{j=2}^{2n} a_{1j}^2 \right)
\]

\[
= 2 |\nabla h|^2
\]

at \(p\), where the third equality follows from (4.6). Combining the above inequality with the Bochner formula gives us

\[
\frac{1}{2} \Delta (h^2) = \text{Ric}(\omega, \omega) + |\nabla \theta|^2
\geq 2 |\nabla h|^2 + \text{Ric}(\omega, \omega).
\]

Hence

\[
\Delta h \geq \frac{\text{Ric}_M(\omega, \omega)}{h} + \frac{|\nabla h|^2}{h},
\]

and the claim is justified. \(\square\)

**Theorem 4.3.** Let \(M^{4n}\) be a \(4n\)-dimensional quaternionic Kähler manifold. Assume that \(\lambda_1(M) > 0\) and that the Ricci curvature of \(M\) satisfies

\[
\text{Ric}_M \geq -\frac{4}{3} \lambda_1(M) + \delta,
\]

for some \(\delta > 0\). Then \(H^1(L^2(M)) = 0\).

**Proof.** We follow the notation in [11]. \(M\) has a rank 3 vector bundle \(V \subseteq \text{End}(TM)\) satisfying

1. In a local coordinate neighborhood, there exists a local basis \(\{I, J, K\}\) of \(V\) such that

\[
I^2 = J^2 = K^2 = -1,
IJ = -JI = K,
JK = -KJ = I,
KI = -IK = J
\]

and

\[
g(X, Y) = g(I X, I Y) = g(J X, J Y) = g(K X, K Y),
\]

for any \(X, Y \in TM\).

2. If \(\phi \in \Gamma(V)\), then \(\nabla_X \phi \in \Gamma(V)\) for any \(X \in TM\).
We define the following two-forms:
\[
\omega_1(X,Y) = g(X, Y),
\omega_2(X,Y) = g(X, JY),
\omega_3(X,Y) = g(X, KY).
\]
The parallel 4-form of \( M \) is then given by
\[
\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.
\]
Let \( \{e_i\}_{i=1}^n \) be a local orthonormal frame and let
\[
\{\omega_i\}_{i=1}^n = \{\theta^1, ..., \theta^n, I\theta^1, ..., I\theta^n, J\theta^1, ..., J\theta^n, K\theta^1, ..., K\theta^n\}
\]
be the orthonormal coframe. Let \( \omega = \sum_{i=1}^n a_i \omega^i \in H^1(L^2(M)) \). Using the above formula of \( \Omega \) and calculating as in Theorem 4.2 (or see [11]), we have
\[
a_{ii} + a_{i+n,i+n} + a_{i+2n,i+2n} + a_{i+3n,i+3n} = 0,
\]
for any \( 1 \leq i \leq n \). We now proceed as in Theorem 4.2. Let \( h = |\omega| \). It is not difficult to see that \( h \) satisfies the Bochner formula of the following form:
\[
\Delta h \geq \frac{\text{Ric}_M(\omega, \omega)}{h} + \frac{1}{3} \left| \nabla h \right|^2 h.
\]
Applying Corollary 3.3 with \( b = \frac{1}{3} \), the result follows. Indeed, let \( \{e_i\}_{i=1}^n \) be as described above with \( e_1 \) such that \( \omega(e_1) = |\omega| \) and \( \omega(e_j) = 0 \) for any \( j \neq 1 \) at a point \( p \). We compute
\[
\left| \nabla \theta \right|^2 = \sum_{i,j=1} a_{ij}^2 \\
\geq a_{11}^2 + a_{1+n,1+n}^2 + a_{1+2n,1+2n}^2 + a_{1+3n,1+3n}^2 + 2 \sum_{j=2}^{2n} a_{ij}^2 \\
\geq a_{11}^2 + \frac{1}{3} \left( a_{1+n,1+n} + a_{1+2n,1+2n} + a_{1+3n,1+3n} \right)^2 + 2 \sum_{j=2}^{2n} a_{ij}^2 \\
= \frac{4}{3} a_{11}^2 + 2 \left( \sum_{j=2}^{2n} a_{ij}^2 \right) \\
= \frac{4}{3} \left( a_{11}^2 + \sum_{j=2}^{2n} a_{ij}^2 \right) \\
= \frac{4}{3} |\nabla h|^2
\]
at \( p \), where the third inequality and the fourth equality follow from Schwarz’s inequality and from (4.7), respectively. Combining the above inequality with the
Bochner formula gives us

\[ \frac{1}{2} \Delta (h^2) = \text{Ric}(\omega, \omega) + |\nabla \theta|^2 \]

\[ \geq \frac{4}{3} |\nabla h|^2 + \text{Ric}(\omega, \omega). \]

Hence

\[ \Delta h \geq \frac{\text{Ric}_M(\omega, \omega)}{h} + \frac{1}{3} \frac{|\nabla h|^2}{h}. \]

\( \square \)

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