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ERRATUM AND ADDENDUM TO "CENTRAL EXTENSIONS OF CURRENT ALGEBRAS"

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Regretfully, the author has spotted, with almost two decades delay, errors in [Z1].

In [Z1, Proposition 3.3], the dimension of the first order cyclic cohomology $HC^1(O_n)$ of the reduced polynomial algebra $O_n = K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ over a field K of characteristic $p > 0$, is specified incorrectly.

The correct value can be easily obtained as follows. For $n = 1$, we have $\dim HC^1(O_1) = 1$ (see, for example, [L, Corollary 5.4.17]), the basic cocycle being

$$\alpha(x^i, x^j) = \begin{cases} i, & \text{if } i + j = p \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Now, from the fact that

$$O_n \simeq O_{n-1} \otimes O_1, \tag{2}$$

and the Künneth formula for the cyclic cohomology (see formula (3) below), we get by induction $\dim HC^1(O_n) = np^{n-1}$.

[Z1, Proposition 3.4] is wrong. The correct statement is given below. Recall that for an associative commutative algebra A, its derivation algebra $Der(A)$ acts on $HC^1(A)$ by the formula

$$(D\varphi)(a, b) = \varphi(D(a), b) + \varphi(a, D(b))$$

for any $D \in Der(A)$, $\varphi \in HC^1(A)$, $a, b \in A$.

Proposition 1.

$$HC^1(O_n)^{Der(O_n)} = \begin{cases} HC^1(O_1), & n = 1 \\ 0, & n > 1. \end{cases}$$

Proof. In the case $n = 1$, it is easy to check that the cocycle (1) is invariant under the action of $x^k \frac{d}{dx}$ for any $0 \le k < p$.

Lemma. *Let A, B be two associative commutative algebras with unit. Then*

$$HC^1(A \otimes B)^{Der(A \otimes B)} \simeq \Big(HC^1(A)^{Der(A)} \otimes (B^*)^{Der(B)}\Big) \oplus \Big((A^*)^{Der(A)} \otimes HC^1(B)^{Der(B)}\Big).$$

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Proof. By the Künneth exact sequence for the cyclic cohomology (see [L, §4.4.10]),

$$HC^1(A\otimes B)\simeq\Big(HC^1(A)\otimes B^*\Big)\oplus\Big(A^*\otimes HC^1(B)\Big), \tag{3}$$

each cyclic 1-cocycle on $A\otimes B$ being represented as the linear span of cocycles of the form

$$(a\otimes b)\wedge(a'\otimes b')\mapsto\varphi(a,a')\beta(bb')+\alpha(aa')\psi(b,b'), \tag{4}$$

where $\varphi\in HC^1(A)$, $\psi\in HC^1(B)$, $\alpha\in A^*$, $\beta\in B^*$, and $a,a'\in A$, $b,b'\in B$.

Similarly, by the Künneth formula for the Hochschild cohomology (derivations are just Hochschild cocycles of order 1),

$$Der(A\otimes B)\simeq\Big(Der(A)\otimes B\Big)\oplus\Big(A\otimes Der(B)\Big),$$

each derivation of $A\otimes B$ being represented as the linear span of derivations of the form $D\otimes R_b+R_a\otimes F$, where $D\in Der(A)$, $F\in Der(B)$, $a\in A$, $b\in B$, and R_u denotes the multiplication by an element u in the respective algebra.

Using these explicit isomorphisms, we get that the result of the action of the derivation $D\otimes 1_B$, where $D\in Der(A)$ and 1_B is the unit of B, on a cocycle of the form (4), is equal to

$$(D\varphi)(a,a')\otimes\beta(bb')+\alpha(D(aa'))\otimes\psi(b,b'),$$

and hence

$$HC^1(A\otimes B)^{Der(A)\otimes 1_B}\simeq\Big(HC^1(A)^{Der(A)}\otimes B^*\Big)\oplus\Big((A^*)^{Der(A)}\otimes HC^1(B)\Big).$$

Similarly,

$$HC^1(A\otimes B)^{1_A\otimes Der(B)}\simeq\Big(HC^1(A)\otimes(B^*)^{Der(B)}\Big)\oplus\Big(A^*\otimes HC^1(B)^{Der(B)}\Big).$$

Taking intersection of the right-hand sides in the last two formulas, we arrive at the right-hand side of the isomorphism claimed. It is straightforward to check that each cocycle of the form (4), where $\varphi\in HC^1(A)^{Der(A)}$, $\psi\in HC^1(B)^{Der(B)}$, $\alpha\in(A^*)^{Der(A)}$, $\beta\in(B^*)^{Der(B)}$, is invariant under the action of $Der(A\otimes B)$. □

Remark. Similar, but much more involved arguments could be used to prove the analogous statements for not necessarily commutative algebras, for algebras without unit, and for the cyclic cohomology of higher degrees. As we are interested here solely in application to the central extensions of modular semisimple Lie algebras, as specified below, we will not go into detail.

Continuation of the proof of Proposition 1. The case $n>1$ follows from (2), the lemma just proved, and the obvious fact that $(Der(O_n))(O_n)=O_n$ for any n. □

These errors do not affect other statements in [Z1].

A more general, then those presented in [Z1], formula for the second cohomology with trivial coefficients of the current Lie algebra, allows us to obtain results more general than [Z1, Proposition 3.2]. Namely, for modular semisimple Lie algebras which are represented as the semidirect sum $(S\otimes O_n)\Subset\mathscr{D}$, where S is a simple Lie algebra, and $\mathscr{D}$ is a Lie subalgebra of $Der(O_n)$, we have:

Proposition 2.

$$H^2\Big((S\otimes O_n)\Subset\mathscr{D}\Big)\simeq\Big(H^2(S)\otimes(O_n^*)^{\mathscr{D}}\Big)\oplus\Big(\mathcal{B}(S)\otimes HC^1(O_n)^{\mathscr{D}}\Big)\oplus H^2(\mathscr{D}).$$

Here $\mathcal{B}(S)$ denotes the space of symmetric invariant bilinear forms on S.

Proof. This is an immediate application of [Z1, Lemma 3.1] (which, in its turn, is a simple consequence of the Hochschild–Serre spectral sequence) and a cohomological version of [Z2, Theorem 0.1]. □

It is interesting to compare this situation with a somewhat dual one in [DZ, §6]. There, we have a similar formula for the space of so-called commutative 2-cocycles on semisimple modular Lie algebras of the form $(S \otimes O_n) \in \mathscr{D}$, but we do not bother with the cyclic cohomology, as the corresponding direct summand vanishes due to vanishing of its first tensor factor – a certain skew-symmetric analog of $\mathcal{B}$.

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