FUNCTIONAL EQUATIONS OF \(L\)-FUNCTIONS
FOR SYMMETRIC PRODUCTS
OF THE KLOOSTERMAN SHEAF

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ABSTRACT. We determine the (arithmetic) local monodromy at 0 and at \(\infty\) of the Kloosterman sheaf using local Fourier transformations and Laumon’s stationary phase principle. We then calculate \(\epsilon\)-factors for symmetric products of the Kloosterman sheaf. Using Laumon’s product formula, we get functional equations of \(L\)-functions for these symmetric products and prove a conjecture of Evans on signs of constants of functional equations.

INTRODUCTION

Let \(p \neq 2\) be a prime number and let \(\mathbb{F}_p\) be the finite field with \(p\) elements. Fix an algebraic closure \(\mathbb{F}\) of \(\mathbb{F}_p\). Denote the projective line over \(\mathbb{F}_p\) by \(\mathbb{P}^1\). For any power \(q\) of \(p\), let \(\mathbb{F}_q\) be the finite subfield of \(\mathbb{F}\) with \(q\) elements. Let \(\ell\) be a prime number different from \(p\). Fix a nontrivial additive character \(\psi: \mathbb{F}_p \to \mathbb{Q}_\ell^\times\). For any \(x \in \mathbb{F}_q^\times\), we define the one-variable Kloosterman sum by

\[
\text{Kl}_2(\mathbb{F}_q, x) = \sum_{\lambda \in \mathbb{F}_q^\times} \psi \left( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \left( \lambda + \frac{x}{\lambda} \right) \right).
\]

In [3], Deligne constructs a lisse \(\mathbb{Q}_\ell\)-sheaf \(\text{Kl}_2\) of rank 2 on \(\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}\), which we call the Kloosterman sheaf, such that for any \(x \in \mathbb{G}_m(\mathbb{F}_q) = \mathbb{F}_q^\times\), we have

\[
\text{Tr}(F_x, \text{Kl}_2, x) = -\text{Kl}_2(\mathbb{F}_q, x),
\]

where \(F_x\) is the geometric Frobenius element at the point \(x\). For a positive integer \(k\), the \(L\)-function \(L(\mathbb{G}_m, \text{Sym}^k(\text{Kl}_2), T)\) of the \(k\)-th symmetric product of \(\text{Kl}_2\) was first studied by Robba [16] via Dwork’s \(p\)-adic methods. Motivated by applications in coding theory, by connections with modular forms, \(p\)-adic modular forms and Dwork’s unit root zeta functions, there has been a great deal of recent interest to understand \(L(\mathbb{G}_m, \text{Sym}^k(\text{Kl}_2), T)\) as much as possible for all \(k\) and for all \(p\). This quickly raises a large number of interesting new problems.

Let \(j: \mathbb{G}_m \to \mathbb{P}^1\) be the inclusion. We shall be interested in the \(L\)-function

\[
M_k(p, T) := L(\mathbb{P}^1, j_*(\text{Sym}^k(\text{Kl}_2)), T).
\]

This is the nontrivial factor of \(L(\mathbb{G}_m, \text{Sym}^k(\text{Kl}_2), T)\). The trivial factor of \(L(\mathbb{G}_m, \text{Sym}^k(\text{Kl}_2), T)\) was completely determined in Fu-Wan [7]. By the general
theory of Grothendieck-Deligne, the nontrivial factor $M_k(p, T)$ is a polynomial in $T$ with integer coefficients, pure of weight $k + 1$. Its degree $\delta_k(p)$ can be easily extracted from Fu-Wan [8], Proposition 2.3, Lemmas 4.1 and 4.2:

$$\delta_k(p) = \begin{cases} \frac{k-1}{2} - \left[ \frac{k}{2p} + \frac{1}{2} \right] & \text{if } k \text{ is odd,} \\ 2 \left( \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{2p} \right\rfloor \right) & \text{if } k \text{ is even.} \end{cases}$$

For fixed $k$, the variation of $M_k(p, T)$ as $p$ varies should be explained by an automorphic form; see Choi-Evans [2] and Evans [4] for the precise relations in the cases $k \leq 7$ and Fu-Wan [9] for a motivic interpretation for all $k$. For $k \leq 4$, the degree $\delta_k(p) \leq 1$ and $M_k(p, T)$ can be determined easily. For $k = 5$, the degree $\delta_5(p) = 2$ for $p > 5$. The quadratic polynomial $M_5(p, T)$ is explained by an explicit modular form [15]. For $k = 6$, the degree $\delta_6(p) = 2$ for $p > 6$. The quadratic polynomial $M_6(p, T)$ is again explained by an explicit modular form [10]. For $k = 7$, the degree $\delta_7(p) = 3$ for $p > 7$. The cubic polynomial $M_7(p, T)$ is conjecturally explained in a more subtle way by an explicit modular form in Evans [4]. We will return to this conjecture later in the Introduction.

For fixed $p$, the variation of $M_k(p, T)$ as $k$ varies $p$-adically should be related to $p$-adic automorphic forms and $p$-adic $L$-functions. No progress has been made along this direction. The $p$-adic limit of $M_k(p, T)$ as $k$ varies $p$-adically links to an important example of Dwork’s unit root zeta function; see the introduction in Wan [19]. The polynomial $M_k(p, T)$ can be used to determine the weight distribution of certain codes, see Moisio [13, 14], and this has been studied extensively for small $p$ and small $k$. The $p$-adic Newton polygon (the $p$-adic slopes) of $M_k(p, T)$ remains largely mysterious.

By Katz [11], 4.11.11, we have $(\Kl_2)^\vee = \Kl_2 \otimes \overline{\Q}_\ell(1)$. So for any natural number $k$, we have

$$(\text{Sym}^k(\Kl_2))^\vee = \text{Sym}^k(\Kl_2) \otimes \overline{\Q}_\ell(k).$$

The general theory (confer [12], 3.1.1) shows that $M_k(p, T)$ satisfies the functional equation

$$M_k(p, T) = c T^\delta M_k \left( p, \frac{1}{p^{k+1}T} \right),$$

where

$$c = \prod_{i=0}^{2} \det(-F, H^i(\P^1_{\Q}, j^*(\text{Sym}^k(\Kl_2))))^{(-1)^{i+1}},$$

$$\delta = -\chi(\P^1_{\Q}, j^*(\text{Sym}^k(\Kl_2))) = \delta_k(p),$$

and $F$ denotes the Frobenius correspondence. Applying the functional equation twice, we get

$$c^2 = p^{(k+1)\delta}.$$

Based on numerical computation, Evans (4, [5]) suggests that the sign of $c$ should be $-\left( \frac{p}{115} \right)$ (the Jacobi symbol) for $k = 7$, $-\left( \frac{p}{115} \right)$ for $k = 11$, and $\left( \frac{p}{1051} \right)$ for $k = 13$. In this paper, we determine $c$ for all $k$ and all $p > 2$. The main result of this paper is the following theorem.

**Theorem 0.1.** Let $p > 2$ be an odd prime. If $k$ is even, we have

$$c = p^{(k+1)(\left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{2p} \right\rfloor)}.$$
If $k$ is odd, we have
\[ c = (-1)^{\frac{k-1}{2}} \prod_{j \in \{0,1,\ldots,\frac{k}{2}\}, \ p/2j+1} \left( \frac{-2}{p} \right)^{\frac{1}{2}} \frac{1}{p} \prod_{j \in \{0,1,\ldots,\frac{k}{2}\}, \ p/2j+1} \left( \frac{-1}{p} \right)^{(2j+1)} \frac{1}{p} \]  

**Corollary 0.2.** If $k$ is even and $p > 2$, the sign of $c$ is always 1. If $k$ is odd and $p > k$, the sign of $c$ is
\[ (-1)^{\frac{k-1}{2}} \prod_{j \in \{0,1,\ldots,\frac{k}{2}\}, \ p/2j+1} \left( \frac{-1}{p} \right)^{(2j+1)} \frac{1}{p} \]  

In the above corollary, if we take $k = 7$, we see that the sign of $c$ for $p > 7$ is
\[ -\left( \frac{1 \cdot (-3) \cdot 5 \cdot (-7)}{p} \right) = -\left( \frac{105}{p} \right) = -\left( \frac{105}{p} \right) \]  
if we take $k = 11$, we see that the sign of $c$ for $p > 11$ is
\[ -\left( \frac{1 \cdot (-3) \cdot 5 \cdot (-7) \cdot 9 \cdot (-11)}{p} \right) = -\left( \frac{-1155}{p} \right) = -\left( \frac{-1155}{p} \right) \]  
if we take $k = 13$, we see that the sign of $c$ for $p > 13$ is
\[ \left( \frac{1 \cdot (-3) \cdot 5 \cdot (-7) \cdot 9 \cdot (-11) \cdot 13}{p} \right) = \left( \frac{-15015}{p} \right) = \left( \frac{-15015}{p} \right) \]  
consistent with Evans’ calculation.

In the case $k = 7$, Evans proposed a precise description of $M_T(p, T)$ in terms of modular forms. For $k = 7$ and $p > 7$, the polynomial $M_T(p, T)$ has degree 3. Write
\[ M_T(p, T) = 1 + a_p T + d_p T^2 + e_p T^3. \]

The functional equation and our sign determination show that one of the reciprocal roots for $M_T(p, T)$ is $\left( \frac{a_p}{105} \right) p^4$ and $e_p = -\left( \frac{d_p}{105} \right) p^{12}$. Denote the other two reciprocal roots by $\lambda_p$ and $\mu_p$, which are Weil numbers of weight 8. We deduce that
\[ a_p = -\left( \left( \frac{p}{105} \right)^{p^4} + \lambda_p + \mu_p \right), \ \lambda_p \mu_p = p^8, \ |\lambda_p| = |\mu_p| = p^4. \]

To explain the numerical calculation of Evans, Katz suggests that there exists a two-dimensional representation
\[ \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}(\mathbb{Q}_p^2) \]
unramified for $p > 7$ and a Dirichlet character $\chi$ such that
\[ \alpha_p^2 = \chi(p) \left( \frac{p}{105} \right) \frac{\lambda_p}{p^4}, \]
\[ \beta_p^2 = \chi(p) \left( \frac{p}{105} \right) \frac{\mu_p}{p^4}, \]
\[ \alpha_p \beta_p = \chi(p), \]
where $\alpha_p$ and $\beta_p$ are the eigenvalues of the geometric Frobenius element at $p$ under $\rho$. We then have
\[ 1 - \left( \frac{p}{105} \right) \frac{a_p}{p^4} = 2 + \left( \frac{p}{105} \right) \frac{\lambda_p}{p^4} + \left( \frac{p}{105} \right) \frac{\mu_p}{p^4} \]
\[ = \chi(p)(2\alpha_p + \alpha_p^2 + \beta_p^2) \]
\[ = \chi(p)(\alpha_p + \beta_p)^2. \]
Set \( b(p) = p(\alpha_p + \beta_p) \). Evans \([4]\) conjectured that \( b(p) \) is the \( p \)-th Hecke eigenvalue for a weight 3 newform \( f \) on \( \Gamma_0(525) \). Our \( \alpha_p \) equals \(-c_pp^2\) in \([4]\).

Our proof of Theorem 0.1 naturally splits into two parts, corresponding to the two ramification points at 0 and \( \infty \). Let \( t \) be the coordinate of \( \mathbb{A}^1 = \mathbb{P}^1 - \{\infty\} \). For any closed point \( x \) in \( \mathbb{P}^1 \), let \( \mathbb{P}^1(x) \) be the henselization of \( \mathbb{P}^1 \) at \( x \). By Laumon’s product formula \([12]\), 3.2.1.1, we have

\[
c = p^{k+1} \prod_{x \in [\mathbb{P}^1]} \epsilon(\mathbb{P}^1(x), j_*(\text{Sym}^k(\text{Kl}_2))|_{\mathbb{P}^1(x)}, dt|_{\mathbb{P}^1(x)}),
\]

where \( [\mathbb{P}^1] \) is the set of all closed points of \( \mathbb{P}^1 \). When \( x \neq 0, \infty \), the sheaf \( \text{Sym}^k(\text{Kl}_2)|_{\mathbb{P}^1(x)} \) is lisse and the order of \( dt \) at \( x \) is 0. So by \([12]\), 3.1.5.4 (ii) and (v), we have

\[
\epsilon(\mathbb{P}^1(x), j_*(\text{Sym}^k(\text{Kl}_2))|_{\mathbb{P}^1(x)}, dt|_{\mathbb{P}^1(x)}) = 1
\]

for \( x \neq 0, \infty \). Therefore

\[
c = p^{k+1} \epsilon(\mathbb{P}^1(0), j_*(\text{Sym}^k(\text{Kl}_2))|_{\mathbb{P}^1(0)}, dt|_{\mathbb{P}^1(0)}) \epsilon(\mathbb{P}^1(\infty), j_*(\text{Sym}^k(\text{Kl}_2))|_{\mathbb{P}^1(\infty)}, dt|_{\mathbb{P}^1(\infty)}).
\]

In §1, we prove the following.

**Proposition 0.3.** We have

\[
\epsilon(\mathbb{P}^1(0), j_*(\text{Sym}^k(\text{Kl}_2))|_{\mathbb{P}^1(0)}, dt|_{\mathbb{P}^1(0)}) = (-1)^k p^{\frac{k(k+1)}{2}}.
\]

In §2, we prove the following.

**Proposition 0.4.** \( \epsilon(\mathbb{P}^1(\infty), j_*(\text{Sym}^k(\text{Kl}_2))|_{\mathbb{P}^1(\infty)}, dt|_{\mathbb{P}^1(\infty)}) \) equals

\[
p^{-(k+1)(\frac{k+1}{2}+\frac{1}{p})}
\]

if \( k = 2r \) for an even \( r \),

\[
p^{-(k+1)(\frac{k+6}{2}+\frac{1}{p})}
\]

if \( k = 2r \) for an odd \( r \), and

\[
(-1)^{\frac{k+1}{2}+\frac{1}{p}} p^{-\frac{k+1}{2}+\frac{k+6}{2}+\frac{1}{p}} \left( \frac{-2}{p} \right)^{\left(\frac{k}{p}-\frac{1}{2}\right)} \prod_{j \in \{0, 1, \ldots, \left(\frac{k}{2}\right)\}} \left( \frac{-1}{j} \right) \left( \frac{2j+1}{p} \right)\]

if \( k = 2r + 1 \).

We deduce from the above two propositions the constant \( c \) as stated in Theorem 0.1 using the following facts:

\[
\left\lfloor \frac{k-2}{4} \right\rfloor = \begin{cases} 
\frac{k-4}{2} & \text{if } k = 2r \text{ for an even } r, \\
\frac{k-2}{2} & \text{if } k = 2r \text{ for an odd } r,
\end{cases}
\]

\[
\left\lfloor \frac{k}{p} \rightfloor - \left\lfloor \frac{k}{2p} \rightfloor = \left\lfloor \frac{k}{2p} + \frac{1}{2} \right\rfloor \text{ if } k \text{ is odd.}
\]

To get Proposition 0.4, we first have to determine the local (arithmetic) monodromy of \( \text{Kl}_2 \) at \( \infty \). This is Theorem 2.1 in §2, which is of interest itself, and is proved by using local Fourier transformations and Laumon’s stationary phase principle.
1. Calculation of $\epsilon(\mathbb{P}^1_{(0)}, j_*(\text{Sym}^k(Kl_2)|_{\mathbb{P}^1_{(0)}}), dt|_{\mathbb{P}^1_{(0)}})$

Let $\eta_0$ be the generic point of $\mathbb{P}^1_{(0)}$, let $\bar{\eta}_0$ be a geometric point located at $\eta_0$, and let $V$ be a $\overline{\mathbb{Q}}_\ell$-representation of $\text{Gal}(\bar{\eta}_0/\eta_0)$. Suppose the inertia subgroup $I_0$ of $\text{Gal}(\bar{\eta}_0/\eta_0)$ acts unipotently on $V$. Consider the $\ell$-adic part of the cyclotomic character

\[ t_\ell : I_0 \to \mathbb{Z}_\ell(1), \quad \sigma \mapsto \left( \frac{\sigma(\sqrt[\ell]{t})}{\sqrt[\ell]{t}} \right). \]

Note that for any $\sigma$ in the inertia subgroup, the $\ell^n$-th root of unity $\sigma(\sqrt[\ell^n]{t})$ does not depend on the choice of the $\ell^n$-th root $\sqrt[\ell^n]{t}$ of $t$. Since $I_0$ acts on $V$ unipotently, there exists a nilpotent homomorphism $N : V(1) \to V$ such that the action of $\sigma \in I_0$ on $V$ is given by $\exp(t_\ell(\sigma)N)$. Fix a lifting $F \in \text{Gal}(\bar{\eta}_0/\eta_0)$ of the geometric Frobenius element in $\text{Gal}(\bar{F}/F_p)$.

**Lemma 1.1.** Notation as above. Let $V = Kl_2, \bar{\eta}_0$. There exists a basis $\{e_0, e_1\}$ of $V$ such that

- $F(e_0) = e_0$, $F(e_1) = pe_1$,
- $N(e_0) = 0$, $N(e_1) = e_0$.

**Proof.** This is the $n = 2$ case of Proposition 1.1 in [7]. □

**Lemma 1.2.** Keep the notation in Lemma 1.1. Let $\{f_0, \ldots, f_k\}$ be the basis of $\text{Sym}^k(V) = \text{Sym}^k(Kl_2, \bar{\eta}_0)$ defined by $f_i = \frac{1}{i!}e_0^{k-i}e_1^i$. We have

- $F(f_i) = p^i f_i$, $N(f_i) = f_{i-1}$,

where we regard $f_{i-1}$ as 0 if $i = 0$.

**Proof.** Use the fact that for any $v_1, \ldots, v_k \in V$, we have the following identities in $\text{Sym}^k(V)$:

- $F(v_1 \cdots v_k) = F(v_1) \cdots F(v_k)$,
- $N(v_1 \cdots v_k) = \sum_{i=1}^k v_1 \cdots v_{i-1}N(v_i)v_{i+1} \cdots v_k$.

□

**Corollary 1.3.** The sheaf $\text{Sym}^k(Kl_2)|_{\eta_0}$ has a filtration

\[ 0 = \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \text{Sym}^k(Kl_2)|_{\eta_0} \]

such that

\[ \mathcal{F}_i/\mathcal{F}_{i-1} \cong \overline{\mathbb{Q}}_\ell(-i) \]

for any $i = 0, \ldots, k$.

**Proof.** This follows from Lemma 1.2 by taking $\mathcal{F}_i$ to be the sheaf on $\eta_0$ corresponding to the Galois representation $\text{Span}(f_0, \ldots, f_i)$ of $\text{Gal}(\bar{\eta}_0/\eta_0)$.

□

The following is Proposition 0.3 in the Introduction.
Proposition 1.4. We have
\[ \epsilon(\mathbb{P}^1_{(0)}, j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) = (-1)^k p^{\frac{k(k+1)}{2}}. \]

Proof. Let \( u : \eta_0 \to \mathbb{P}^1_{(0)} \) and \( v : \{0\} \to \mathbb{P}^1_{(0)} \) be the immersions. By [12], 3.1.5.4 (iii) and (v), we have
\[ \epsilon(\mathbb{P}^1_{(0)}, u_*(\mathcal{Q}_\ell(-i)), dt|_{\mathbb{P}^1_{(0)}}) = 1, \]
\[ \epsilon(\mathbb{P}^1_{(0)}, v_*(\mathcal{Q}_\ell(-i)), dt|_{\mathbb{P}^1_{(0)}}) = \det(-F_0, \mathcal{Q}_\ell(-i))^{-1} = -\frac{1}{p^i}. \]

We have an exact sequence
\[ 0 \to u_*(\mathcal{Q}_\ell(-i)) \to u_*(\mathcal{Q}_\ell(-i)) \to v_*(\mathcal{Q}_\ell(-i)) \to 0. \]

It follows from [12], 3.1.5.4 (ii) that we have
\[ \epsilon(\mathbb{P}^1_{(0)}, u_*(\mathcal{Q}_\ell(-i)), dt|_{\mathbb{P}^1_{(0)}}) = \epsilon(\mathbb{P}^1_{(0)}, v_*(\mathcal{Q}_\ell(-i)), dt|_{\mathbb{P}^1_{(0)}}) \]
\[ = -p^i. \]

By Corollary 1.3 and [12], 3.1.5.4 (ii), we have
\[ \epsilon(\mathbb{P}^1_{(0)}, j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) = \prod_{i=0}^{k} \epsilon(\mathbb{P}^1_{(0)}, u_*(\mathcal{F}_i), dt|_{\mathbb{P}^1_{(0)}}) \]
\[ = \prod_{i=0}^{k} \epsilon(\mathbb{P}^1_{(0)}, u_*(\mathcal{Q}_\ell(-i)), dt|_{\mathbb{P}^1_{(0)}}) \]
\[ = \prod_{i=0}^{k} (-p^i). \]

Moreover, by Lemma 1.2, we have
\[ v^*(j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}} \cong \mathcal{Q}_\ell, \]
and hence
\[ \epsilon(\mathbb{P}^1_{(0)}, v_*v^*(j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) = -1. \]

So we have
\[ \epsilon(\mathbb{P}^1_{(0)}, j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) \]
\[ = \epsilon(\mathbb{P}^1_{(0)}, j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) \epsilon(\mathbb{P}^1_{(0)}, v_*v^*(j_*(\text{Sym}^k(K_{L_2})))|_{\mathbb{P}^1_{(0)}}, dt|_{\mathbb{P}^1_{(0)}}) \]
\[ = \prod_{i=1}^{k} (-p^i) \]
\[ = (-1)^k p^{\frac{k(k+1)}{2}}. \]
2. Calculation of $\epsilon(\mathbb{P}^1_{(\infty)}, j_*(\text{Sym}^k(Kl_2))[v^\infty], dt|_{v^\infty})$

We first introduce some notation. Fix a nontrivial additive character $\psi : \mathbb{F}_p \to \mathbb{C}_\ell$ and define $Kl_2$ as in the Introduction. Fix a separable closure $\mathbb{F}_p(t)$ of $\mathbb{F}_p(t)$. Let $x$ be an element in $\mathbb{F}_p(t)$ satisfying $x^p - x = t$. Then $\mathbb{F}_p(t, x)$ is Galois over $\mathbb{F}_p(t)$. We have a canonical isomorphism

$$\mathbb{F}_p \cong \text{Gal}(\mathbb{F}_p(t, x)/\mathbb{F}_p(t))$$

which sends each $a \in \mathbb{F}_p$ to the element in Gal($\mathbb{F}_p(t, x)/\mathbb{F}_p(t)$) defined by $x \mapsto x + a$. Let $L_\psi$ be the Galois representation defined by

$$\text{Gal}(\mathbb{F}_p(t)/\mathbb{F}_p(t)) \to \text{Gal}(\mathbb{F}_p(t, x)/\mathbb{F}_p(t)) \xrightarrow{\cong} \mathbb{F}_p \xrightarrow{\psi^{-1}} \mathbb{Q}_\ell^*.$$

It is unramified outside $\infty$ and totally wild at $\infty$ with Swan conductor 1. This Galois representation defines a lisse $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1$, which we still denote by $L_\psi$. Let $X$ be an $\mathbb{F}_p$-scheme. Any section $f$ in $\mathcal{O}_X(X)$ defines an $\mathbb{F}_p$-algebra homomorphism

$$\mathbb{F}_p[t] \to \mathcal{O}_X(X), \ t \mapsto f,$$

and hence an $\mathbb{F}_p$-morphism of schemes

$$f : X \to \mathbb{A}^1.$$

We denote the lisse $\mathbb{Q}_\ell$-sheaf $f^*L_\psi$ on $X$ by $L_\psi(f)$. For any $f_1, f_2 \in \mathcal{O}_X(X)$, we have

$$L_\psi(f_1) \otimes L_\psi(f_2) \cong L_\psi(f_1 + f_2).$$

Recall that $p \neq 2$. Let $y$ be an element in $\mathbb{F}_p(t)$ satisfying $y^2 = t$. Then $\mathbb{F}_p(t, y)$ is Galois over $\mathbb{F}_p(t)$. We have a canonical isomorphism

$$\{\pm 1\} \cong \text{Gal}(\mathbb{F}_p(t, y)/\mathbb{F}_p(t))$$

which sends $-1$ to the element in Gal($\mathbb{F}_p(t, y)/\mathbb{F}_p(t)$) defined by $y \mapsto -y$. Let

$$\chi : \{\pm 1\} \to \mathbb{Q}_\ell^*$$

be the (unique) nontrivial character. Define $L_\chi$ to be the Galois representation defined by

$$\text{Gal}(\mathbb{F}_p(t)/\mathbb{F}_p(t)) \to \text{Gal}(\mathbb{F}_p(t, y)/\mathbb{F}_p(t)) \xrightarrow{\cong} \{\pm 1\} \xrightarrow{\chi^{-1}} \mathbb{Q}_\ell^*.$$

It is unramified outside 0 and $\infty$ and tamely ramified at 0 and $\infty$. This Galois representation defines a lisse $\mathbb{Q}_\ell$-sheaf on $\mathbb{G}_m$, which we still denote by $L_\chi$.

Let $\theta : \text{Gal}(\mathbb{F}/\mathbb{F}_p) \to \mathbb{Q}_\ell^*$ be a character of the Galois group of the finite field. Denote by $L_\theta$ the Galois representation

$$\text{Gal}(\mathbb{F}_p(t)/\mathbb{F}_p(t)) \to \text{Gal}(\mathbb{F}/\mathbb{F}_p) \xrightarrow{\theta} \mathbb{Q}_\ell^*.$$

It is unramified everywhere and hence defines a lisse $\mathbb{Q}_\ell$-sheaf on $\mathbb{P}^1$, which we still denote by $L_\theta$.

**Theorem 2.1.** Notation as above. Let $\eta_{\infty}$ be the generic point of $\mathbb{P}^1_{(\infty)}$. Then $Kl_2|_{\eta_{\infty}}$ is isomorphic to the restriction to $\eta_{\infty}$ of the sheaf

$$[2]_\psi(L_\psi(2t) \otimes L_\chi) \otimes L_\theta,$$

where $[2] : \mathbb{G}_m \to \mathbb{G}_m$ is the morphism defined by $x \mapsto x^2$, and

$$\theta_0 : \text{Gal}(\mathbb{F}/\mathbb{F}_p) \to \mathbb{Q}_\ell^*.$$
is the character sending the geometric Frobenius element $F$ in $\text{Gal}(\mathbb{F}/\mathbb{F}_p)$ to the Gauss sum 

$$\theta_0(F) = g(\chi, \psi) = - \sum_{x \in \mathbb{F}_p} \left( \frac{x}{p} \right) \psi(x).$$

Proof. By [9], Proposition 1.1, we have

$$Kl_2 = F(j_! L_\psi \left( \frac{1}{t} \right)) \big|_{G_m},$$

where $F$ is the $\ell$-adic Fourier transformation and $j : G_m \to \mathbb{A}^1$ is the inclusion. Let

$$\pi_1, \pi_2 : G_m \times_{\mathbb{F}_p} G_m \to G_m$$

be the projections. Using the proper base change theorem and the projection formula ([1], XVII, 5.2.6 and 5.2.9), one can verify

$$[2]^* \left( F(j_! L_\psi \left( \frac{1}{t} \right)) \big|_{G_m} \right) \cong R\pi_2! \left( L_\psi \left( \frac{1}{t} + tt^2 \right) \right) [1],$$

where

$$\frac{1}{t} + tt^2 : G_m \times_{\mathbb{F}_p} G_m \to A^1$$

is the morphism corresponding to the $\mathbb{F}_p$-algebra homomorphism

$$\mathbb{F}_p[t] \to \mathbb{F}_p[t, 1/t, t', 1/t'], \ t \mapsto \frac{1}{t} + tt^2.$$

Consider the isomorphism

$$\tau : G_m \times_{\mathbb{F}_p} G_m \to G_m \times_{\mathbb{F}_p} G_m, \ (t, t') \mapsto \left( \frac{t}{t'}, t' \right).$$

We have $\pi_2 \tau = \pi_2$. So

$$R\pi_2! \left( L_\psi \left( \frac{1}{t} + tt^2 \right) \right) \cong R(\pi_2 \tau)! \tau^* \left( L_\psi \left( \frac{1}{t} + tt^2 \right) \right) \cong R\pi_2! L_\psi \left( \left( \frac{1}{t} + t \right) t' \right).$$

Consider the morphism

$$g : G_m \to \mathbb{A}^1, \ t \mapsto \frac{1}{t} + t.$$

Again using the proper base change theorem and the projection formula, one can verify

$$F(Rg_! \mathbb{G}_k) \cong R\pi_2! L_\psi \left( \left( \frac{1}{t} + t \right) t' \right) [1].$$

From the isomorphisms (1)-(4), we get

$$[2]^* Kl_2 \cong F(Rg_! \mathbb{G}_k)|_{\eta_{\infty}}.$$

By Lemma 2.2 below, the stationary phase principle of Laumon [12], 2.3.3.1 (iii), and [12], 2.5.3.1, as representations of $\text{Gal}(\overline{\eta}_{\infty}/\eta_{\infty})$, we have

$$\mathcal{H}^0(F(Rg_! \mathbb{G}_k))|_{\eta_{\infty}} \cong F^{(2, \infty)}(\mathcal{L}_\chi) \oplus F^{(-2, \infty)}(\mathcal{L}_\chi) \cong (\mathcal{L}_\psi(2t') \otimes F^{(0, \infty)}(\mathcal{L}_\chi)) \oplus (\mathcal{L}_\psi(-2t) \otimes F^{(0, \infty)}(\mathcal{L}_\chi)) \cong (\mathcal{L}_\psi(2t') \otimes \mathcal{L}_\chi \otimes \mathcal{L}_{\theta_0}) \oplus (\mathcal{L}_\psi(-2t) \otimes \mathcal{L}_\chi \otimes \mathcal{L}_{\theta_0}).$$

Hence

$$([2]^* Kl_2)|_{\eta_{\infty}} \cong (\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi \otimes \mathcal{L}_{\theta_0})|_{\eta_{\infty}} \oplus (\mathcal{L}_\psi(-2t) \otimes \mathcal{L}_\chi \otimes \mathcal{L}_{\theta_0})|_{\eta_{\infty}}.$$
Note that this decomposition of $([2] \ast K_{l2})|_{\eta_{\infty}}$ is nonisotypical. By [17], Proposition 24 on p. 61, and the fact that $K_{l2}|_{\eta_{\infty}}$ is irreducible (since its Swan conductor is 1), we have

$$K_{l2}|_{\eta_{\infty}} \cong ([2] \ast (L_{\psi}(2t) \otimes L_{\chi} \otimes L_{\theta_0}))[\eta_{\infty}].$$

We have

$$[2] \ast (L_{\psi}(2t) \otimes L_{\chi} \otimes L_{\theta_0}) \cong [2] \ast (L_{\psi}(2t) \otimes L_{\chi} \otimes [2] \ast L_{\theta_0}) \cong [2] \ast (L_{\psi}(2t) \otimes L_{\chi}) \otimes L_{\theta_0}.$$

Here we use the fact that $[2] \ast L_{\theta_0} \cong L_{\theta_0}$.

Hence

$$K_{l2}|_{\eta_{\infty}} \cong \left([2] \ast (L_{\psi}(2t) \otimes L_{\chi}) \otimes L_{\theta_0}\right)[\eta_{\infty}].$$

\[\square\]

Lemma 2.2. For the morphism

$$g : G_m \to \mathbb{A}^1, \ t \mapsto \frac{1}{t} + t,$$

the following holds:

(i) $Rg_{\mathbb{Q}_\ell}$ is a $\mathbb{Q}_\ell$-sheaf on $\mathbb{A}^1$ which is lisse outside the rational points 2 and $-2$.

(ii) $Rg_{\mathbb{Q}_\ell}$ is unramified at $\infty$.

(iii) Let $P$ be one of the rational points 2 or $-2$, and let $\tilde{\mathbb{A}}^1_P$ be the henselization of $\mathbb{A}^1$ at $P$. We have

$$(Rg_{\mathbb{Q}_\ell})|_{\tilde{\mathbb{A}}^1_P} \cong \mathbb{Q}_\ell \oplus L_{\chi}!,$$

where $L_{\chi}!$ denotes the extension by 0 of the Kummer sheaf $L_{\chi}$ on the generic point of $\tilde{\mathbb{A}}^1_P$ to $\tilde{\mathbb{A}}^1_P$.

Proof. We have

$$\frac{\partial g}{\partial t} = -\frac{1}{t^2} + 1.$$

So $\frac{\partial g}{\partial t}$ vanishes at the points $t = \pm 1$. We have

$$g(\pm 1) = \pm 2,$$

$$\frac{\partial^2 g}{\partial t^2}(\pm 1) = \pm 2 \neq 0.$$

It follows that $g$ is tamely ramified above $\pm 2$ with ramification index 2, and $g$ is étale elsewhere. Consider the morphism

$$\tilde{g} : \mathbb{P}^1 \to \mathbb{P}^1, \ [t_0 : t_1] \mapsto [t_0 t_1 : t_0^2 + t_1^2].$$

We have $\tilde{g}^{-1}(\infty) = \{0, \infty\}$. Hence

$$\tilde{g}^{-1}(\mathbb{A}^1) = G_m.$$

It is clear that

$$\tilde{g}|_{G_m} = g.$$

So $g : G_m \to \mathbb{A}^1$ is a finite morphism of degree 2. Near 0, the morphism $\tilde{g}$ can be expressed as

$$t \mapsto \frac{t}{1 + t^2}.$$

Hence $\tilde{g}$ is unramified at 0. Similarly $\tilde{g}$ is also unramified at $\infty$. Our lemma follows from these facts. \[\square\]
Remark 2.3. The first attempt to determine the monodromy at $\infty$ of the $(n-1)$-variable Kloosterman sheaf $Kl_n|_{\eta_\infty}$ is done in Fu-Wan \cite{fuwan}, Theorem 1.1, where we deduce from Katz \cite{Katz} that
\[
Kl_n|_{\eta_\infty} \cong \left([n]_*(\mathcal{L}_\psi(nt) \otimes \mathcal{L}_{\chi_n^{-1}}) \otimes \mathcal{L}_\theta \otimes \mathcal{U}_\ell \left(\frac{1-n}{2}\right)\right)|_{\eta_\infty}
\]
for some character $\theta : \text{Gal}(\overline{F}/F_p) \rightarrow \overline{Q}_\ell^\ast$, and an explicit description of $\theta^2$ is given. Using induction on $n$, \cite{fuwan}, Proposition 1.1, and adapting the argument in \cite{fuwan} to a nonalgebraically closed ground field, we can get an explicit description of $\theta$. See \cite{fuwan} where the monodromy of the more general hypergeometric sheaf is treated (over an algebraically closed field).

Lemma 2.4. Keep the notation in Theorem 2.1. Let
\[
\theta_1 : \text{Gal}(\overline{F}/F_p) \rightarrow \overline{Q}_\ell^\ast
\]
be the character defined by
\[
\theta_1(\sigma) = \chi \left(\frac{\sigma(\sqrt{-1})}{\sqrt{-1}}\right)
\]
for any $\sigma \in \text{Gal}(\overline{F}/F_p)$. Note that the above expression is independent of the choice of the square root $\sqrt{-1} \in F$ of $-1$.

(i) If $k = 2r$ is even, $\text{Sym}^k(Kl_2)|_{\eta_\infty}$ is isomorphic to the restriction to $\eta_\infty$ of the sheaf
\[
(\mathcal{L}_{\chi_r} \otimes \mathcal{L}_{\phi_y}) \otimes \left(\bigoplus_{i=0}^{r-1} [2]^* \mathcal{L}_\psi((4i-4r)t) \otimes \mathcal{L}_{\phi_y^*}\right).
\]

(ii) If $k = 2r + 1$ is odd, $\text{Sym}^k(Kl_2)|_{\eta_\infty}$ is isomorphic to the restriction to $\eta_\infty$ of the sheaf
\[
\bigoplus_{i=0}^{r} [2]^* \mathcal{L}_\psi((4i-4r-2)t) \otimes \mathcal{L}_\chi) \otimes \mathcal{L}_{\phi_y^{2r+1}}.
\]

Proof. By Theorem 2.1, it suffices to calculate the restriction to $\eta_\infty$ of $\text{Sym}^k([2]^* \mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)).$ Let $y, z, w$ be elements in $\overline{F}(t)$ satisfying
\[
y^2 = t, z^p - z = y, w^2 = y.
\]
Fix a square root $\sqrt{-1}$ of $-1$ in $F$. Then $\overline{F}(z, w, \sqrt{-1})$ and $\overline{F}(y)$ are Galois extensions of $\overline{F}(t)$. Let $G = \text{Gal}(\overline{F}(z, w, \sqrt{-1})/\overline{F}(t))$ and $H = \text{Gal}(\overline{F}(z, w, \sqrt{-1})/\overline{F}(y))$. Then $H$ is normal in $G$, and we have the canonical isomorphisms
\[
G/H \xrightarrow{\cong} \text{Gal}(\overline{F}(y)/\overline{F}(t)) \xrightarrow{\cong} \{\pm 1\}.
\]
Consider the case where $\sqrt{-1}$ does not lie in $F_p$. We have an isomorphism
\[
\overline{F}_p \times \{\pm 1\} \xrightarrow{\cong} H = \text{Gal}(\overline{F}_p(z, w, \sqrt{-1})/\overline{F}_p(y)),
\]
which maps $(a, \mu', \mu'') \in \overline{F}_p \times \{\pm 1\} \times \{\pm 1\}$ to the element $g(a, \mu', \mu'') \in \text{Gal}(\overline{F}_p(z, w, \sqrt{-1})/\overline{F}_p(y))$ defined by
\[
g(a, \mu', \mu'')(z) = z + a, g(a, \mu', \mu'')(w) = \mu'w, g(a, \mu', \mu'')(\sqrt{-1}) = \mu'' \sqrt{-1}.
\]
(In the case where $\sqrt{-1}$ lies in $F_p$, we have $\overline{F}_p(z, w, \sqrt{-1}) = \overline{F}_p(z, w)$, and we have an isomorphism
\[
\overline{F}_p \times \{\pm 1\} \xrightarrow{\cong} H = \text{Gal}(\overline{F}_p(z, w)/\overline{F}_p(y)),
\]
which maps \((a, \mu) \in \mathbb{F}_p \times \{\pm 1\}\) to the element \(g(a, \mu) \in \text{Gal}(\mathbb{F}_p(z, w)/\mathbb{F}_p(y))\) defined by

\[
g(a, \mu)(z) = z + a, \quad g(a, \mu)(w) = \mu w.
\]

The following argument works for this case with slight modification. We leave it to the reader to treat this case.) Let \(V\) be a one-dimensional \(\mathbb{Q}_\ell\)-vector space with a basis \(e_0\). Define an action of \(H\) on \(V\) by

\[
g(a, \mu', \mu'')(e_0) = \psi(-2a)\chi(\mu'^{-1})e_0.
\]

Then \([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)\) is just the composition of \(\text{Ind}_H^G(V)\) with the canonical homomorphism

\[
\text{Gal}(\mathbb{F}_p(t)/\mathbb{F}_p(t)) \to \text{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(t)) = G.
\]

Let \(g\) be the element in \(G = \text{Gal}(\mathbb{F}_p(z, w, \sqrt{-1})/\mathbb{F}_p(t))\) defined by

\[
g(z) = -z, \quad g(w) = \sqrt{-1}w, \quad g(\sqrt{-1}) = \sqrt{-1}.
\]

Then the image of \(g\) in \(G/H\) is a generator of the cyclic group \(G/H\). So \(G\) is generated by \(g(a, \mu', \mu'') \in H = (\mathbb{F}_p \times \{\pm 1\} \times \{\pm 1\})\) and \(g\). The space \(\text{Ind}_H^G(V)\) has a basis \(\{e_0, e_1\}\) with

\[
g(e_0) = e_1, \quad g(e_1) = g^2(e_0) = g(0, -1, 1)(e_0) = -e_0.
\]

Suppose \(k = 2r\) is even. \(\text{Sym}^k(\text{Ind}_H^G(V))\) has a basis

\[
\{e_1^k, g(e_1^k), e_0 e_1^{k-1}, g(e_0 e_1^{k-1}), \ldots, e_0 e_1^{r-1} e_1^r, g(e_0 e_1^{r-1} e_1^r), e_0 e_1^r\},
\]

and for each \(i = 0, 1, \ldots, r\), we have

\[
g(a, \mu', \mu'')(e_i^k e_1^{k-i}) = \psi(-2ia)\chi(\mu'^{-i})\psi(2(k - i)a)\chi(\mu'^{-(k-i)} \mu'^{-r-(k-i)})e_0 e_1^{k-i} = \psi(2(k - 2i)a)\chi(\mu'^{-k})\chi(\mu'^{-r-(k-i)})e_0 e_1^{k-i}.
\]

Using the fact that \(k\) is even and \(\chi^2 = 1\), we get

\[
g(a, \mu', \mu'')(e_i^k e_1^{k-i}) = \psi(2(k - 2i)a)\chi(\mu'^{r})e_0^r e_1^r.
\]

In particular, we have

\[
g(a, \mu', \mu'')(e_0^r e_1^r) = \chi(\mu'^{r})e_0^r e_1^r.
\]

Moreover, we have

\[
g(e_0^r e_1^r) = e_1^r (g(e_1^r)) = (-1)^r e_0^r e_1^r.
\]

It follows that

\[
\text{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)) \cong (\mathcal{L}_{\chi^r} \otimes \mathcal{L}_{\theta_1^r}) \oplus \left( \bigoplus_{i=0}^{r-1} [2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^r}) \right).
\]

We have

\[
[2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^r}) \cong [2]_*(\mathcal{L}_\psi(2(2i-k)t) \otimes [2]_* \mathcal{L}_{\theta_1^r}) \cong [2]_* \mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^r}.
\]

So we have

\[
\text{Sym}^k([2]_*(\mathcal{L}_\psi(2t) \otimes \mathcal{L}_\chi)) \cong (\mathcal{L}_{\chi^r} \otimes \mathcal{L}_{\theta_1^r}) \oplus \left( \bigoplus_{i=0}^{r-1} \mathcal{L}_\psi(2(2i-k)t) \otimes \mathcal{L}_{\theta_1^r} \right).
\]
Suppose $k = 2r + 1$ is odd. $\text{Sym}^k(\text{Ind}_{H}^{G}(V))$ has a basis
\[ \{ e_1^{k}, g(e_1^{i}), e_0 e_1^{-1}, g(e_0 e_1^{k-1}), \ldots, e_0 e_1^{-r+1}, g(e_0 e_1^{r+1}) \} . \]
Using the same calculation as above, we get
\[ \text{Sym}^k([2]_{\ast}(\mathcal{L}_{\psi}(2t) \otimes \mathcal{L}_{\chi})) \cong \bigoplus_{i=0}^{r} [2]_{\ast}(\mathcal{L}_{\psi}(2(2i-k)t) \otimes \mathcal{L}_{\chi}) \otimes \mathcal{L}_{\mathcal{O}_{\ell}^{i+1}} . \]
Lemma 2.4 follows by twisting the above expressions of $\text{Sym}^k([2]_{\ast}(\mathcal{L}_{\psi}(2t) \otimes \mathcal{L}_{\chi}))$ by $\mathcal{L}_{\mathcal{O}_{\ell}^{i}}$.

**Lemma 2.5.** Assume $a \in \mathbb{F}_{p}$ is nonzero. We have the following identities.

(i) $\epsilon(\mathbb{P}_{1}(\infty), \overline{\mathbb{Q}_{\ell}}, dt|_{\mathbb{P}_{1}\infty}) = \frac{1}{p^r}$.

(ii) $\epsilon(\mathbb{P}_{1}(\infty), \overline{\mathbb{Q}_{\ell}}, dt^2|_{\mathbb{P}_{1}\infty}) = \frac{1}{p^r}$.

(iii) $\epsilon(\mathbb{P}_{1}(\infty), Js \mathcal{L}_{\chi}|_{\mathbb{P}_{1}\infty}, dt|_{\mathbb{P}_{1}\infty}) = -g(x, \psi)$.

(iv) $\epsilon(\mathbb{P}_{1}(\infty), Js \mathcal{L}_{\chi}|_{\mathbb{P}_{1}\infty}, dt^2|_{\mathbb{P}_{1}\infty}) = -g(x, \psi^2)\left(\frac{-2}{p}\right)$.

(v) $\epsilon(\mathbb{P}_{1}(\infty), Js (\mathcal{L}_{\psi}(at) \otimes \mathcal{L}_{\chi})|_{\mathbb{P}_{1}\infty}, dt^2|_{\mathbb{P}_{1}\infty}) = \frac{1}{p^r} \left(\frac{2a}{p}\right)$.

(vi) $\epsilon(\mathbb{P}_{1}(\infty), Js \mathcal{L}_{\chi}|_{\mathbb{P}_{1}\infty}, dt|_{\mathbb{P}_{1}\infty}) = \frac{1}{p^r}$.

(vii) $\epsilon(\mathbb{P}_{1}(\infty), Js [2]_{\ast}(\mathcal{L}_{\psi}(at) \otimes \mathcal{L}_{\chi})|_{\mathbb{P}_{1}\infty}, dt|_{\mathbb{P}_{1}\infty}) = 0$.

(viii) $\epsilon(\mathbb{P}_{1}(\infty), Js [2]_{\ast}(\mathcal{L}_{\psi}(at) \otimes \mathcal{L}_{\chi})|_{\mathbb{P}_{1}\infty}, dt^2|_{\mathbb{P}_{1}\infty}) = 0$.

**Proof.** Let $K_{\infty}$ be the completion of the field $k(\eta_{\infty})$, let $\mathcal{O}_{\infty}$ be the ring of integers in $K_{\infty}$, and let $s = \frac{1}{2}$. Then $s$ is a uniformizer of $K_{\infty}$. Denote the inclusion $\eta_{\infty} \rightarrow \mathbb{P}_{1}(\infty)$ also by $j$. Let $V$ be a $\overline{\mathbb{Q}_{\ell}}$-sheaf of rank 1 on $\eta_{\infty}$, and let $\phi : K_{\infty}^{\ast} \rightarrow \overline{\mathbb{Q}_{\ell}}^{\ast}$ be the character corresponding to $V$ via the reciprocity law. The Artin conductor $a(\phi)$ of $\phi$ is defined to be the smallest integer $m$ such that $\phi|_{1+s=\mathcal{O}_{\infty}} = 1$. For any nonzero meromorphic differential 1-form $\omega = fds$ on $\mathbb{P}_{1}(\infty)$, define the order $v_{\infty}(\omega)$ of $\omega$ to be the valuation $v_{\infty}(f)$ of $f$. By [12], 3.1.5.4 (v), we have
\[ \epsilon(\mathbb{P}_{1}(\infty), Js, V, \omega) = \begin{cases} 0, & \text{if } \phi|_{\mathcal{O}_{\infty}} = 1, \\ \int_{x(\omega(\zeta)+v_{\infty}(\omega))\mathcal{O}_{\infty}} \phi^{-1}(\zeta)\psi(x\omega(z))dz, & \text{if } \phi|_{\mathcal{O}_{\infty}} \neq 1, \end{cases} \]
where $\text{Res}_{\mathcal{O}_{\infty}}$ denotes the residue of a meromorphic 1-form at $\infty$, and the integral is taken with respect to the Haar measure $dz$ on $K_{\infty}$ normalized by $\int_{\mathcal{O}_{\infty}} dz = 1$.

Note that $dt = -\frac{dz}{2\pi}$ has order $-2$ at $\infty$ and $dt^2 = -\frac{2dz}{\pi}$ has order $-3$. Applying the first case of the above formula for the $\epsilon$-factor, we get (i) and (ii).

(iii) Taking $a = t = \frac{1}{2}$ and $b = z$ in the explicit reciprocity law in [13], XIV, §3, Proposition 8, we obtain that the character $\chi' : K_{\infty}^{\ast} \rightarrow \overline{\mathbb{Q}_{\ell}}^{\ast}$ corresponding to $\mathcal{L}_{\chi}$ is given by
\[ \chi'(z) = \chi^{-1} \left( \left( \frac{\zeta}{p} \right) \right), \]
where
\[ c = (-1)^{-v_\infty(z)} z^{-1} \frac{z^{-1}}{s^{-v_\infty(z)}}, \]
which is a unit in \( \mathcal{O}_\infty \), \( \bar{c} \) is the residue class of \( c \) in \( \mathcal{O}_\infty / s\mathcal{O}_\infty \cong \mathbb{F}_p \), and \( \left( \frac{\bar{c}}{p} \right) \) is the Legendre symbol of \( \bar{c} \). Note that our formula for \( c \) is the reciprocal of the formula in [18] because the reciprocity map in [18] maps uniformizers in \( K \) to arithmetic Frobenius elements in \( \text{Gal}(\overline{K}_\infty / K_\infty)_{ab} \), whereas the reciprocity map in [12] maps uniformizers in \( K \) to geometric Frobenius elements. One can verify that \( a(\chi') = 1 \).

For any \( z \in s\mathcal{O}_\infty^* \), write
\[ z = s(r_0 + r_1 s + \cdots) \]
with \( r_i \in \mathbb{F}_p \) and \( r_0 \neq 0 \). We then have
\[ \bar{c} = -r_0^{-1}, \]
\[ \text{Res}_\infty(zdt) = -r_0. \]

So we have
\[
\epsilon(\mathbb{P}^1_\infty, \mathcal{J}_s \mathcal{L}_\chi|_{\mathbb{P}^1_\infty}, dt|_{\mathbb{P}^1_\infty}) = \int_{s\mathcal{O}_\infty^*} \chi^{-1}(z) \psi(\text{Res}_\infty(zdt)) dz
\]
\[
= \int_{s\mathcal{O}_\infty^*} \chi \left( \left( \frac{-r_0}{p} \right) \right) \psi(-r_0) dz
\]
\[
= \sum_{r_0 \in \mathbb{F}_p^*} \int_{s\mathcal{O}_\infty^*} \left( \frac{-r_0}{p} \right) \psi(-r_0) dz
\]
\[
= \sum_{r_0 \in \mathbb{F}_p^*} \left( \frac{-r_0}{p} \right) \psi(-r_0) \int_{s\mathcal{O}_\infty^*} dz
\]
\[
= \frac{1}{p^2} \sum_{r_0 \in \mathbb{F}_p^*} \left( \frac{-r_0}{p} \right) \psi(-r_0)
\]
\[
= -g(\chi, \psi) \frac{1}{p^2}.
\]

(iv) We can use the same method as in (iii), or use the formula [12], 3.1.5.5 to get
\[
\epsilon(\mathbb{P}^1_\infty, \mathcal{J}_s \mathcal{L}_\chi|_{\mathbb{P}^1_\infty}, dt^2|_{\mathbb{P}^1_\infty}) = \epsilon(\mathbb{P}^1_\infty, \mathcal{J}_s \mathcal{L}_\chi|_{\mathbb{P}^1_\infty}, 2tdt|_{\mathbb{P}^1_\infty})
\]
\[
= \chi' \left( \frac{2}{s} \right) p^{v_\infty(\frac{2}{s})} \epsilon(\mathbb{P}^1_\infty, \mathcal{J}_s \mathcal{L}_\chi|_{\mathbb{P}^1_\infty}, dt|_{\mathbb{P}^1_\infty})
\]
\[
= \left( \frac{-2}{p} \right) \frac{1}{p} \epsilon(\mathbb{P}^1_\infty, \mathcal{J}_s \mathcal{L}_\chi|_{\mathbb{P}^1_\infty}, dt|_{\mathbb{P}^1_\infty})
\]
\[
= -g(\chi, \psi) \frac{-2}{p^3}.
\]

(v) Taking \( a \) to be \( a = \frac{a}{s} \) and \( b = z \) in the explicit reciprocity law in [18], XIV, \$5$, Proposition 15, we see that the character
\[ K^*_\infty \rightarrow \mathbb{Q}_l^* \]
corresponding to $L_\chi(at)$ is
\[ z \mapsto \psi^{-1}\left( -\text{Res}_\infty \left( \frac{a \cdot dz}{s \cdot z} \right) \right). \]
(We add the negative sign to the formula in [18] since the reciprocity map in [18] is different from the one used in [12].) So the character
\[
\phi : K^*_\infty \to \mathbb{Q}_l
\]
corresponding to $L_\chi(at) \otimes L_\chi$ is given by
\[
\phi(z) = \psi^{-1}\left( -\text{Res}_\infty \left( \frac{a \cdot dz}{s \cdot z} \right) \chi^{-1} \left( \left( \frac{c}{p} \right) \right) \right),
\]
where $c = (-1)^{-v_\infty(z)} \frac{\bar{c}}{s^{-1} v_\infty(z)}$. One can verify that $a(\phi) = 2$. For any $z \in sO^*_\infty$, write
\[ z = s(r_0 + r_1 s + \cdots) \]
with $r_i \in \mathbb{F}_p$ and $r_0 \neq 0$. We then have
\[
\text{Res}_\infty \left( \frac{a \cdot dz}{s \cdot z} \right) = \frac{ar_1}{r_0},
\]
\[ \bar{c} = -r_0^{-1}, \]
\[
\text{Res}_\infty(z dt^2) = -2r_1.
\]
So we have
\[
\epsilon(P_{(\infty)}^1, j_\ast L_\chi|_{P_{(\infty)}^1}, dt^2|_{P_{(\infty)}^1})
\]
\[ = \int_{sO^*_\infty} \phi^{-1}(z) \psi(\text{Res}_\infty(z dt^2)) dz
\]
\[ = \int_{sO^*_\infty} \psi \left( -\frac{ar_1}{r_0} \right) \chi \left( \left( -\frac{-r_0^{-1}}{p} \right) \right) \psi(-2r_1) dz
\]
\[ = \int_{sO^*_\infty} \left( -\frac{r_0}{p} \right) \psi \left( -r_1 \left( \frac{a}{r_0} + 2 \right) \right) dz
\]
\[ = \sum_{r_0, r_1 \in \mathbb{F}_p, r_0 \neq 0} \int_{s(r_0 + r_1 s)(1 + s^2 O_\infty)} \left( -\frac{r_0}{p} \right) \psi \left( -r_1 \left( \frac{a}{r_0} + 2 \right) \right) dz
\]
\[ = \frac{1}{p^3} \sum_{r_0, r_1 \in \mathbb{F}_p, r_0 \neq 0} \left( -\frac{r_0}{p} \right) \psi \left( -r_1 \left( \frac{a}{r_0} + 2 \right) \right)
\]
\[ = \frac{1}{p^3} \sum_{r_0 \in \mathbb{F}_p} \left( -\frac{r_0}{p} \right) \sum_{r_1 \in \mathbb{F}_p} \psi \left( -r_1 \left( \frac{a}{r_0} + 2 \right) \right)
\]
\[ = \frac{1}{p^3} \cdot \left( -\frac{2a}{p} \right) \cdot p
\]
\[ = \frac{1}{p^2} \left( \frac{2a}{p} \right).
\]
We omit the proof of (vi), which is similar to the proof of (v).
(vii) We have $[2]_* \overline{\epsilon}_t \cong \overline{\epsilon}_t \oplus j_* \mathcal{L}_\chi$. So
\[
\epsilon(\ell)_1, \left[ 2 \right]_* \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} = \epsilon(\ell)_1, \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)}.
\]
We then use (i) and (iii).

(viii) We can define $\epsilon$-factors for virtual sheaves on $\mathcal{P}_1^{\left( s \right)}$. By [12], 3.1.5.4 (iv), we have
\[
\epsilon(\ell)_1, \left[ 2 \right]_* \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} = \epsilon(\ell)_1, \left( j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right) \big|_{\mathcal{P}_1}^{\left( s \right)}.
\]
Hence
\[
\epsilon(\ell)_1, \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} = \epsilon(\ell)_1, \left[ 2 \right]_* \overline{\epsilon}_t \big|_{\mathcal{P}_1}^{\left( s \right)}.
\]
We then apply the formulas (ii), (v), and (vii).

We omit the proofs of (ix) and (x), which are similar to the proof of (viii).

Lemma 2.6. We have
\[
\epsilon(\ell)_1, \left[ j_* \left( \mathcal{L}_\chi \otimes \mathcal{L}_\theta \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} = \begin{cases} \frac{g(\chi, \psi)^2}{p^2} \frac{\left( -1 \right)^r}{p^2} & \text{if } r \text{ is even}, \\ \frac{g(\chi, \psi)^2}{p^2} \frac{\left( -1 \right)^{r+1}}{p^2} & \text{if } r \text{ is odd}, \end{cases}
\]

Proof. Let $F_\infty$ be the geometric Frobenius element at $\infty$. We have
\[
\theta_0(F_\infty) = g(\chi, \psi), \quad \theta_1(F_\infty) = \left( -\frac{1}{p} \right).
\]
Using the notation in [12], 3.1.5.1, we have
\[
\begin{align*}
\epsilon(\ell)_1, \left[ 2 \right]_* \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} &= -2, \\
\epsilon(\ell)_1, \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} &= -1, \\
\epsilon(\ell)_1, \left[ 2 \right]_* \epsilon(\ell)_1, \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} &= -3, \\
\epsilon(\ell)_1, \left[ 2 \right]_* \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} &= -1 (a \in \mathbb{F}_p), \\
\epsilon(\ell)_1, \left[ 2 \right]_* \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} &= -2, \\
\epsilon(\ell)_1, \left[ 2 \right]_* \left[ j_* \left( \mathcal{L}_\psi (at) \otimes \mathcal{L}_\chi \right) \right] \big|_{\mathcal{P}_1}^{\left( s \right)} &= -1 (a \in \mathbb{F}_p).
\end{align*}
\]
So by [12, 3.1.5.6, we have

\[
\epsilon(P_{(\infty)}, j_*(\mathcal{L}_\chi \otimes \mathcal{L}_{\theta_0^i \theta_1^j})|_{P_{(\infty)}^i}, dt|_{P_{(\infty)}^i})
= \begin{cases} 
\epsilon(P_{(\infty)}, j_*(\mathcal{L}_\chi))^{-2} \epsilon(P_{(\infty)}, \mathcal{O}_\mathcal{E}, dt|_{P_{(\infty)}^i}) & \text{if } r \text{ is even,} \\
\epsilon(P_{(\infty)}, j_*(\mathcal{L}_\chi))^{-1} \epsilon(P_{(\infty)}, j_*(\mathcal{L}_\chi)|_{P_{(\infty)}^i}, dt|_{P_{(\infty)}^i}) & \text{if } r \text{ is odd,}
\end{cases}
\]

\[
\epsilon(P_{(\infty)}, j_*(\mathcal{L}_\psi((4i-4r)t) \otimes \mathcal{L}_{\theta_0^i \theta_1^j})|_{P_{(\infty)}^i}, dt|_{P_{(\infty)}^i})
= \begin{cases} 
\epsilon(P_{(\infty)}, j_*(\mathcal{L}_\psi))^{-2} \epsilon(P_{(\infty)}, [2], \mathcal{O}_\mathcal{E}, dt|_{P_{(\infty)}^i}) & \text{if } p|j - r, \\
\epsilon(P_{(\infty)}, j_*(\mathcal{L}_\psi((4i-4r-2t) \otimes \mathcal{L}_\chi)) \otimes \mathcal{L}_{g_0^r g_1^r + 1})|_{P_{(\infty)}^i}, dt|_{P_{(\infty)}^i}) & \text{if } p|j - r - 1,
\end{cases}
\]

We then apply the formulas in Lemma 2.5.

The following is Proposition 0.4 in the Introduction.

**Proposition 2.7.** \(\epsilon(P_{(\infty)}, j_*(\text{Sym}^k(Kl_2))|_{P_{(\infty)}^i}, dt|_{P_{(\infty)}^i})\) equals

\[
p^{-(k+1)(\frac{k+8}{2}+\frac{k}{p})}
\]

if \(k = 2r\) for an even \(r\),

\[
p^{-(k+1)(\frac{k+8}{2}+\frac{k}{p})}
\]

if \(k = 2r\) for an odd \(r\), and

\[
(-1)^{\frac{k+8}{2}+\frac{k}{p}} \prod_{j \in \{0, \ldots, \frac{k}{p}\}} \left(\frac{-2}{p}\right)^{\frac{k}{p}} \prod_{p|j+j+1} \left(\frac{-1}{p}\right)^{\frac{k}{p}}
\]

if \(k = 2r + 1\).

**Proof.** By Lemmas 2.4 and 2.6, \(\epsilon(P_{(\infty)}, j_*(\text{Sym}^k(Kl_2))|_{P_{(\infty)}^i}, dt|_{P_{(\infty)}^i})\) equals

\[
g(\chi, \psi)\frac{-4r}{p^2} \prod_{i \in \{0, \ldots, r-1\}, \ p|j-i}\left(\frac{-1}{p}\right)^{i}\left(\frac{-1}{p}\right)^{i}
\]

\[
\times \prod_{i \in \{0, \ldots, r-1\}, \ p|j-i} \left(\frac{-1}{p}\right)^{4i}
\]

if \(k = 2r\) for an even \(r\),

\[
\frac{g(\chi, \psi)}{p^2} \prod_{i \in \{0, \ldots, r-1\}, \ p|j-i}\left(\frac{-1}{p}\right)^{i}\left(\frac{-1}{p}\right)^{i}
\]

\[
\times \prod_{i \in \{0, \ldots, r-1\}, \ p|j-i} \left(\frac{-1}{p}\right)^{4i}
\]

if \(k = 2r + 1\).
if \( k = 2r \) for an odd \( r \), and
\[
\prod_{i \in \{0, \ldots, r\}, \, p \not| 2i - 2r - 1} \left( \frac{g(\chi, \psi)^{-4r}}{p^4} \left( -\frac{2}{p} \right)^i \right) \\
\times \prod_{i \in \{0, \ldots, r\}, \, p \not| 2i - 2r - 1} \left( -\frac{g(\chi, \psi)^{-2r}}{p^3} \left( -\frac{1}{p} \right)^i \right)
\]
if \( k = 2r + 1 \). Let’s simplify the above expressions. Recall that \( g(\chi, \psi)^2 = p \left( \frac{-1}{p} \right) \).

If \( k = 2r \) with \( r \) even, we have
\[
g(\chi, \psi)^{-4r} \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i-r} \left( -\frac{g(\chi, \psi)^{-6r+1}}{p^4} \left( -\frac{1}{p} \right)^i \right) \\
\times \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i-r} \left( -\frac{g(\chi, \psi)^{-2r+1}}{p^3} \left( -\frac{1}{p} \right)^i \right)
\]
\[
g(\chi, \psi)^{-4r} \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i} \left( -\frac{g(\chi, \psi)^{-6r+1}}{p^4} \left( -\frac{1}{p} \right)^i \right) \\
\times \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i} \left( -\frac{g(\chi, \psi)^{-2r+1}}{p^3} \left( -\frac{1}{p} \right)^i \right)
\]
\[
p^{-2r} \left( \frac{-2}{p} \right)^r \frac{\left( \frac{-1}{p} \right)^{r(\frac{r}{2}+1)}}{p^{3r}} \left( -\frac{1}{p} \right)^{\frac{r(r-1)}{2}}
\]
\[
p^{-k+1}(\frac{k+1}{2}+1)(\frac{k}{2})
\]

If \( k = 2r \) with \( r \) odd, we have
\[
g(\chi, \psi)^{-2r+1} \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i-r} \left( -\frac{1}{p} \right)^i \left( -\frac{g(\chi, \psi)^{-6r+1}}{p^4} \left( -\frac{1}{p} \right)^i \right) \\
\times \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i-r} \left( -\frac{g(\chi, \psi)^{-2r+1}}{p^3} \left( -\frac{1}{p} \right)^i \right)
\]
\[
g(\chi, \psi)^{-2r+1} \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i} \left( -\frac{1}{p} \right)^i \left( -\frac{g(\chi, \psi)^{-6r+1}}{p^4} \left( -\frac{1}{p} \right)^i \right) \\
\times \prod_{i \in \{0, \ldots, r-1\}, \, p \not| i} \left( -\frac{g(\chi, \psi)^{-2r+1}}{p^3} \left( -\frac{1}{p} \right)^i \right)
\]
\[
- g(\chi, \psi)^{-2r+1} \left( \frac{-1}{p} \right) \left( \frac{-g(\chi, \psi)^{-4r}}{p^3} \right)^{\frac{r}{2}} \left( \frac{\left( \frac{-g(\chi, \psi)^{-2r+1}}{p^3} \right)^r}{p^{\frac{r(r-1)}{2}}} \right)
\]
If \( k \) is odd, we have

\[
\epsilon(P^1_{\infty}, j_*(\text{Sym}^k(\mathcal{K}_L)))|_{P^1_{\infty}}, dt|_{P^1_{\infty}}
\]

\[
= \prod_{i \in \{0, \ldots, r\}, p \mid 2i - 2r - 1} \left( \frac{g(\chi, \psi)^{-2r}}{p^2} \left( \frac{-2}{p} \right) \right) \times \prod_{i \in \{0, \ldots, r\}, p \nmid 2i - 2r - 1} \left( -\frac{g(\chi, \psi)^{-2r}}{p^2} \left( \frac{-1}{p} \right) \right) \times \prod_{j \in \{0, \ldots, r\}, p \mid 2j + 1} \left( \frac{-2}{p} \right) (-1)^{r-j}(2j + 1)
\]

\[
= (-1)^{r+1} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{k+3}{2})} p^{-r(1+r)(\frac{k+5}{2}) - r(1+r)(\frac{k+3}{2})} \frac{-2}{p} \frac{-2}{p} \frac{-2}{p} (-1)^{\frac{k+1}{2} + \frac{1}{2} \frac{k+3}{2}} (-1)^{\frac{1}{2} \frac{k+3}{2}} (-1)^{\frac{k+1}{2} + \frac{1}{2} \frac{k+5}{2}} \frac{-2}{p} \frac{-2}{p} \frac{-2}{p}
\]

\[
\times \prod_{j \in \{0, \ldots, r\}, p \nmid 2j + 1} \left( \frac{-1}{p} \right) (-1)^{r-j}(2j + 1)
\]

\[
= (-1)^{\frac{k+1}{2} + \frac{1}{2} \frac{k+3}{2}} \frac{-2}{p} \frac{-2}{p} \frac{-2}{p} \frac{-2}{p} \frac{-2}{p} \frac{-2}{p}
\]

\[
\times \prod_{j \in \{0, \ldots, r\}, p \nmid 2j + 1} \left( \frac{-1}{p} \right) (-1)^{j}(2j + 1)
\]

\[
\square
\]

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