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HKR CHARACTERS, p-DIVISIBLE GROUPS AND THE GENERALIZED CHERN CHARACTER

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Abstract. In this paper we describe the generalized Chern character of classifying spaces of finite groups in terms of Hopkins-Kuhn-Ravenel generalized group characters. For this purpose we study the p-divisible group and its level structures associated with the $K(n)$-localization of the $(n+1)$st Morava E-theory.

1. Introduction

For a finite group G, we let BG be its classifying space. A generalized cohomology theory $h^*(-)$ defines a contravariant functor from the category of finite groups to the category of graded modules by assigning to a finite group G a graded module $h^*(BG)$. When $h^*(-)$ has a graded commutative multiplicative structure, this functor takes its values in the category of graded commutative h^*-algebras. If we have a description of this functor in terms of finite groups G, then it is considered that we have some intrinsic information for the cohomology theory $h^*(-)$. For example, when $h^*(-)$ is the complex K-theory $K^*(-)$, there is a natural ring homomorphism from the complex representation ring $R(G)$ to $K^*(BG)$, and Atiyah [4] has shown that this induces a natural isomorphism after completion at the augmentation ideal I of $R(G)$,

$$K^*(BG) \cong R(G)^\wedge_I,$$

where we regard $K^*(-)$ as a $\mathbb{Z}/2\mathbb{Z}$-graded cohomology theory. This isomorphism is considered to be intimately related to the definition of K-theory by vector bundles.

It is known that representations of finite groups are studied by their characters. Let $\mathbb{Q}^{\mathrm{ab}}$ be the maximal abelian extension of the field $\mathbb{Q}$, the rational numbers. We denote by $\mathrm{Rep}(\mathbb{Z}, G)$ the set of all conjugacy classes of G. A class function on G is a function on $\mathrm{Rep}(\mathbb{Z}, G)$. The character of a finite dimensional complex representation is obtained by taking the trace of representation matrices, which is a class function on G with values in $\mathbb{Q}^{\mathrm{ab}}$. We denote by $Ch(G)$ the ring of class functions with values in $\mathbb{Q}^{\mathrm{ab}}$, and call it the character ring of G,

$$Ch(G) = \mathrm{Map}(\mathrm{Rep}(\mathbb{Z}, G), \mathbb{Q}^{\mathrm{ab}}).$$

Then there is an embedding of the representation ring $R(G)$ into the character ring $Ch(G)$. For $t \in \mathbb{Z}$, the t-th power map $g \mapsto g^t$ on G gives a map from $\mathrm{Rep}(\mathbb{Z}, G)$ to

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itself. Hence we obtain a ring homomorphism $\Psi^t : Ch(G) \to Ch(G)$ by $\Psi^t f(g) = f(g^t)$ for $f \in Ch(G)$. This restricts to a ring homomorphism $\Psi^t : R(G) \to R(G)$, which corresponds to the Adams operation Ψ^t in K-theory under the map $R(G) \to K(BG)$. Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$, and let $\widehat{\mathbb{Z}}^\times$ be the group of units in $\widehat{\mathbb{Z}}$. Since the order of G is finite, the above construction extends from $\mathbb{Z}$ to $\widehat{\mathbb{Z}}$. In particular, $\widehat{\mathbb{Z}}^\times$ acts on $\mathrm{Rep}(\mathbb{Z}, G)$. On the other hand, $\widehat{\mathbb{Z}}^\times$ can be identified with the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Hence the two actions of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ on $\mathrm{Rep}(\mathbb{Z}, G)$ and $\mathbb{Q}^{\mathrm{ab}}$ define an action on the character ring $Ch(G) = \mathrm{Map}(\mathrm{Rep}(G), \mathbb{Q}^{\mathrm{ab}})$ by conjugation. Let f be a character associated with a representation ρ. For $g \in G$, we may take a representation matrix $M(g)$ of $\rho(g)$ as an upper triangular matrix. Since $M(g^t) = M(g)^t$ for $t \in \widehat{\mathbb{Z}}^\times$, we see that $\Psi^t f(g) = f(g^t) = \sigma_t(f(g))$, where $\sigma_t \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ corresponds to t under the identification $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times$. Hence the embedding $R(G) \hookrightarrow Ch(G)$ factors through the invariant subring of $Ch(G)$ under the action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. Furthermore, this induces an isomorphism of commutative rings after tensoring with $\mathbb{Q}$ (cf. [26, Chapters 12–13]):

$$R(G) \otimes \mathbb{Q} \cong H^0(\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}); Ch(G)).$$

Note that the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ is obtained by adjoining all roots of unity. The degree 0 formal group associated with the complex K-theory $K^*(-)$ is the multiplicative formal group $\widehat{\mathbf{G}}_m$ over $\mathbb{Z}$, which is the formal completion of the multiplicative group scheme $\mathbf{G}_m$. So $\mathbb{Q}^{\mathrm{ab}}$ is obtained from $\mathbb{Q}$ by adjoining all torsion points of $\mathbf{G}_m$ in the algebraic closure $\overline{\mathbb{Q}}$.

Let E_n be the nth Morava E-theory spectrum at a prime number p. The cohomology theory $E_n^*(-)$ associated with E_n is a generalization of the complex K-theory $K^*(-)$ to a higher chromatic level from the point of view of chromatic stable homotopy theory. The Morava E-theory E_n is important since it governs the $K(n)$-local category, that is, the Bousfield localization of the stable homotopy category with respect to the Morava K-theory $K(n)$, which is a building block of the stable homotopy category (cf. [24, 17, 18]). Furthermore, E_n is an E_∞-ring spectrum by the Goerss-Hopkins-Miller theorem [25, 8]. This fact has important applications in stable homotopy theory (cf. [13, 14, 7]). We let $\mathrm{Rep}(\mathbb{Z}_p^n, G)$ be the set of all conjugacy classes of continuous homomorphisms from $\mathbb{Z}_p^n$ to a finite group G, where $\mathbb{Z}_p$ is the ring of p-adic integers. Then $\mathrm{Rep}(\mathbb{Z}_p^n, G)$ is interpreted as the set of all conjugacy classes of n-tuples of commuting group elements with p-power order. Hopkins, Kuhn, and Ravenel [16, 15] have defined generalized group characters as functions on $\mathrm{Rep}(\mathbb{Z}_p^n, G)$, and they have described the E_n-cohomology ring of classifying spaces of finite groups in terms of generalized group characters. We have a formal group law $\mathbf{F}_n$ associated with the complex oriented commutative ring spectrum E_n, which is a universal deformation of the height n Honda formal group law $\mathbf{H}_n$. For $r \geq 0$, we let $D_n(r)$ be the representing ring of the functor classifying Drinfeld $(\mathbb{Z}/p^r\mathbb{Z})^n$-level structures on a deformation of $\mathbf{H}_n$ (cf. [6, 1, 27, 16]). The ring $D_n(r)$ is obtained from E_n^0 by adjoining all p^r-torsion points of $\mathbf{F}_n$ in the algebraic closure of the field of fractions of E_n^0. The ring $D_n(r)$ is a finite flat extension of E_n^0, and there is an action of $\mathrm{Aut}((\mathbb{Z}/p^r\mathbb{Z})^n)$ on the ring $D_n(r)$. Then the $\mathrm{Aut}((\mathbb{Z}/p^r\mathbb{Z})^n)$-invariant subring of $D_n(r)$ is E_n^0, and the extension $E_n^0 \to D_n(r)$ induces an $\mathrm{Aut}((\mathbb{Z}/p^r\mathbb{Z})^n)$-Galois extension on the fraction fields. Note that $D_n(r)$ is not an étale extension over E_n^0 since there occurs a ramification on the special fiber. We set $D_n = \mathrm{colim}_{r\geq 0}\, D_n(r)$. Then $\mathrm{GL}_n(\mathbb{Z}_p) = \lim_{r\geq 0}\, \mathrm{Aut}((\mathbb{Z}/p^r\mathbb{Z})^n)$

acts on the ring D_n. In particular, when $n = 1$, the 1st Morava E-theory E_1 is the p-adic complex K-theory $K_p^\wedge$. Then the associated formal group law $\mathbf{F}_1$ is the p-typification of the multiplicative formal group law $\widehat{\mathbf{G}}_m$ over $\mathbb{Z}_p$. Then the ring D_1 is obtained from $\mathbb{Z}_p$ by adjoining all roots of unity of p-power order. Then $D_1 \otimes \mathbb{Q}$ is a maximal totally ramified abelian extension of the field $\mathbb{Q}_p$, the p-adic number field with Galois group $\mathrm{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p^\times$ (cf. [19]).

A generalized group character is defined to be a function on $\mathrm{Rep}(\mathbb{Z}_p^n, G)$ with its values in the ring D_n. We denote by $\mathrm{Ch}_{n,p}(G)$ the ring of all generalized group characters

$$\mathrm{Ch}_{n,p}(G) = \mathrm{Map}(\mathrm{Rep}(\mathbb{Z}_p^n, G), D_n).$$

There is an action of $\mathrm{GL}_n(\mathbb{Z}_p)$ on both $\mathrm{Rep}(\mathbb{Z}_p^n, G)$ and D_n, and hence $\mathrm{GL}_n(\mathbb{Z}_p)$ acts on $\mathrm{Ch}_{n,p}(G)$ by conjugation. We denote by $\mathrm{HKR}(G; E_n)$ the invariant subring of $\mathrm{Ch}_{n,p}(G)$ under the action of $\mathrm{GL}_n(\mathbb{Z}_p)$:

$$\mathrm{HKR}(G; E_n) = H^0(\mathrm{GL}_n(\mathbb{Z}_p); \mathrm{Ch}_{n,p}(G)).$$

Then there is a natural E_n^0-algebra homomorphism $E_n^0(BG) \to \mathrm{HKR}(G; E_n)$, and Hopkins-Kuhn-Ravenel [16, 15] (see also [12]) have shown that this ring homomorphism is rationally an isomorphism, that is, an isomorphism after tensoring with $\mathbb{Q}$:

$$E_n^0(BG) \otimes \mathbb{Q} \cong \mathrm{HKR}(G; E_n) \otimes \mathbb{Q}.$$

In [3], Ando, Morava and Sadofsky have considered a generalization of the Chern character which is a multiplicative natural transformation from a height $(n+1)$ cohomology theory to a height n cohomology theory. In [32], we have refined this construction. We have constructed a commutative ring spectrum $\mathbb{B}_n$ and a ring spectrum map $i : E_n \to \mathbb{B}_n$, where $\mathbb{B}_n$ is even-periodic and Landweber exact of height n. The ring spectrum map i induces a faithfully flat ring homomorphism $E_n^0 \to \mathbb{B}_n^0$ on the degree 0 homotopy rings, and hence we can consider that $\mathbb{B}_n$ is an extension of E_n. We can define a commutative $\mathbb{B}_n^0$-algebra $\mathrm{HKR}(G; \mathbb{B}_n)$ in the same way as the construction of the commutative E_n^0-algebra $\mathrm{HKR}(G; E_n)$. Then there is a natural $\mathbb{B}_n^0$-algebra homomorphism $\mathbb{B}_n^0(BG) \to \mathrm{HKR}(G; \mathbb{B}_n)$ which induces an isomorphism after tensoring with $\mathbb{Q}$. The generalized Chern character constructed in [32] has the following form:

$$ch : E_{n+1}^*(-) \longrightarrow \mathbb{B}_n^*(-),$$

which is a multiplicative natural transformation from the height $(n+1)$ cohomology theory $E_{n+1}^*(-)$ to the height n cohomology theory $\mathbb{B}_n^*(-)$. The map ch factors through $\mathbb{A}_n = L_{K(n)}E_{n+1}$, the Bousfield localization of E_{n+1} with respect to $K(n)$, and the induced map from $\mathrm{Spf}\,\mathbb{B}_n^0 \to \mathrm{Spf}\,\mathbb{A}_n^0$ on the reduced schemes is the limit of the tower of Igusa varieties (cf. [11, Chap. II.1]; see also [31, §2]). In particular, when $n = 1$ and $p > 3$, the completion of an (even-periodic) elliptic cohomology at a supersingular prime is isomorphic to the 2nd Morava E-theory E_2, and the induced map $\mathrm{Spec}\,\mathbb{B}_1^0/(p) \to \mathrm{Spec}\,\mathbb{A}_1^0/(p)$ is the limit of the tower of Igusa curves on the moduli space of elliptic curves around the supersingular point (cf. [20, Chap.4] and [21, Chap.12]). We would like to have a model of the natural transformation $ch(G) : E_{n+1}^0(BG) \to \mathbb{B}_n^0(BG)$ in terms of generalized group characters, that is, we would like to have a natural ring homomorphism $ch\mathrm{HKR}(G) : \mathrm{HKR}(G; E_{n+1}) \to \mathrm{HKR}(G; \mathbb{B}_n)$ which is identified with $ch(G)$ after tensoring with $\mathbb{Q}$.

For this purpose, we study the p-divisible groups $\mathbf{F}_n(\infty)$ and $\mathbf{F}_{n+1}(\infty)$ associated with the formal group laws $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$, respectively. By the base changes along the map $E_n^0 \to \mathbb{B}_n^0$ induced by i, and the map $E_{n+1}^0 \to \mathbb{B}_n^0$ induced by ch, we have two p-divisible groups $\mathbf{F}_n(\infty)_{\mathbb{B}}$ and $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ over $\operatorname{Spec}\mathbb{B}_n^0$. In general, to a p-divisible group G over a locally Noetherian scheme Y such that p is locally nilpotent, we can associate a natural filtration of reduced closed subschemes $Y^{[h]}$ of Y such that the p-divisible group G over $Y^{(h)} = Y^{[h]} - Y^{[h-1]}$ is an extension of an étale p-divisible group with constant height h by its identity component (cf. [11, Chap.II.1]). In particular, for the p-divisible group $\mathbf{F}_{n+1}(\infty)$ over $X = \operatorname{colim}_r \operatorname{Spec} E_{n+1}^0/(p^r)$, it is connected of height $(n+1)$ over the stratum $X^{(0)}$, and is an extension of an étale p-divisible group of height 1 by a p-divisible formal group of height n over the stratum $X^{(1)}$. Then the attaching map of $X^{(1)}$ to $X^{(0)}$ is important to understand the relationship between the $K(n)$-local category and the $K(n+1)$-local category. The connection between extensions of an étale p-divisible group of height 1 by a p-divisible formal group and stable homotopy theory has been discussed in [2, §5]; see also [10, 30, 31]. The p-divisible group $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ over $\operatorname{Spf}\mathbb{B}_n^0$ can be written as an extension of an étale p-divisible group of height 1 by its identity component. We show that there exists an exact sequence of p-divisible groups

$$0 \to \mathbf{F}_n(\infty)_{\mathbb{B}} \longrightarrow \mathbf{F}_{n+1}(\infty)_{\mathbb{B}} \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)_{\mathbb{B}} \to 0$$

over $\operatorname{Spec}\mathbb{B}_n^0$, where $\mathbf{F}_n(\infty)_{\mathbb{B}}$ is identified with the identity component of $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ over $\operatorname{Spf}\mathbb{B}_n^0$ (cf. Theorem 5.3).

Let $\mathbb{K}$ be the field of fractions of $\mathbb{B}_n^0$. To compare the level structures on $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ and that on $\mathbf{F}_n(\infty)_{\mathbb{B}}$, we consider $W = \mathbf{F}_{n+1}(\infty)(\overline{\mathbb{K}})$ and $V = \mathbf{F}_n(\infty)(\overline{\mathbb{K}})$, the abelian groups of all torsion points of $\mathbf{F}_{n+1}(\infty)$ and $\mathbf{F}_n(\infty)$ in the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, respectively. By the above exact sequence, we can regard V as a subgroup of W. Since $\mathbf{F}_{n+1}(\infty)$ is étale over $\mathbb{K}$, $\mathbf{F}_{n+1}(\infty)_{\mathbb{K}}$ is determined by the action of the absolute Galois group $\pi_1(\mathbb{K}) = \operatorname{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ on W. We define $B(W,V)$ to be the subgroup of $\operatorname{Aut}(W)$ consisting of automorphisms of W which preserve the subgroup V and induce the identity on the quotient group W/V. Then the monodromy representation $\pi_1(\mathbb{K}) \to \operatorname{Aut}(W)$ factors through $B(W,V)$. We show that $B(W,V)$ is identified with the Galois group of the extension $\mathbb{K}[W]$ over $\mathbb{K}$, and the monodromy representation is surjective on $B(W,V)$ (cf. Theorem 6.2).

Let $\operatorname{Level}(W;\mathbf{F}_{n+1}(\infty))$ (resp. $\operatorname{Level}(V;\mathbf{F}_n(\infty))$) be the functor which assigns to a complete local $\mathbb{B}_n^0$-algebra S the set of all level W-structures on $\mathbf{F}_{n+1}(\infty)$ (resp. level V-structures on $\mathbf{F}_n(\infty)$) over S. We consider a subfunctor $\operatorname{Level}(W,V;\mathbf{F}_{n+1}(\infty))$ of $\operatorname{Level}(W;\mathbf{F}_{n+1}(\infty))$ consisting of level W-structures which send V into $\mathbf{F}_n(\infty)(S)$. We show that $\operatorname{Level}(W,V;\mathbf{F}_{n+1}(\infty))$ is representable by a complete regular semi-local ring $\mathbb{I}_n$ (cf. Proposition 7.1). Let $A(W,V)$ be the subgroup of $\operatorname{Aut}(W)$ consisting of automorphisms of W which preserve the subgroup V:

$$A(W,V) = \{g \in \operatorname{Aut}(W) \mid g(V) = V\}.$$

The group $A(W,V)$ naturally acts on the functor $\operatorname{Level}(W,V;\mathbf{F}_{n+1}(\infty))$, and so it acts on the representing ring $\mathbb{I}_n$. Then it is shown that $\mathbb{B}_n^0$ is the invariant subring of $\mathbb{I}_n$ under the action of $A(W,V)$ (Corollary 7.3). Note that $B(W,V)$ is a subgroup of $A(W,V)$ and the quotient group $A(W,V)/B(W,V)$ is identified with the set of connected components of $\operatorname{Level}(W,V;\mathbf{F}_{n+1}(\infty))$. There are natural transformations of functors

$$\mathrm{Level}(W;\mathbf{F}_{n+1}(\infty)) \longleftarrow \mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty)) \longrightarrow \mathrm{Level}(V;\mathbf{F}_n(\infty)),$$

which are $A(W,V)$-equivariant. Hence there are $A(W,V)$-equivariant ring homomorphisms of representing rings

$$D_{n+1} \longrightarrow \mathbb{I}_n \longleftarrow D_n.$$

Note that $\mathrm{Aut}(W) \cong \mathrm{GL}_{n+1}(\mathbb{Z}_p)$ since $W \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{n+1}$. Hence $A(W,V)$ can be regarded as a subgroup of $\mathrm{GL}_{n+1}(\mathbb{Z}_p)$, and it acts on $\mathrm{Rep}(\mathbb{Z}_p^{n+1},G)$. Also, $\mathrm{Aut}(V) \cong \mathrm{GL}_n(\mathbb{Z}_p)$ since $V \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$. Then $A(W,V)$ acts on $\mathrm{Rep}(\mathbb{Z}_p^n,G)$ through the projection $A(W,V) \to \mathrm{Aut}(V) \cong \mathrm{GL}_n(\mathbb{Z}_p)$. The ring $\mathrm{HKR}(G;\mathbb{B}_n)$ of generalized group characters in $\mathbb{B}_n$ is identified with the set of all $A(W,V)$-equivariant functions on $\mathrm{Rep}(\mathbb{Z}_p^n,G)$ with their values in $\mathbb{I}_n$:

$$\mathrm{HKR}(G;\mathbb{B}_n) \cong H^0(A(W,V);\mathrm{Map}(\mathrm{Rep}(\mathbb{Z}_p^n,G),\mathbb{I}_n)).$$

Then the $A(W,V)$-equivariant ring homomorphism $D_{n+1} \to \mathbb{I}_n$ induces a ring homomorphism $ch\mathrm{HKR}(G) : \mathrm{HKR}(G;E_{n+1}) \to \mathrm{HKR}(G;\mathbb{B}_n)$, and the following is the main theorem of this paper.

Theorem 1.1 (cf. Theorem 8.1)**.** *For any finite group G, there is a natural ring homomorphism*

$$ch\mathrm{HKR}(G) : \mathrm{HKR}(G;E_{n+1}) \longrightarrow \mathrm{HKR}(G;\mathbb{B}_n),$$

which covers the ring homomorphism $E^0_{n+1} \to \mathbb{B}^0_n$ induced by ch. Then the following diagram commutes:

$$\begin{array}{ccc} E^0_{n+1}(BG) & \xrightarrow{ch(BG)} & \mathbb{B}^0_n(BG) \\ \big\downarrow & & \big\downarrow \\ \mathrm{HKR}(G;E_{n+1}) & \xrightarrow{ch\mathrm{HKR}(G)} & \mathrm{HKR}(G;\mathbb{B}_n), \end{array}$$

where the vertical arrows are isomorphisms after tensoring with $\mathbb{Q}$.

Note that there is a profinite group $\mathbb{G}$ which is some amalgamation of the nth extended Morava stabilizer group $\mathbb{G}_n$ and the $(n+1)$st extended Morava stabilizer group $\mathbb{G}_{n+1}$. There are natural actions of $\mathbb{G}$ on the commutative rings $E^0_{n+1}(BG)$, $\mathbb{B}^0_n(BG)$, $\mathrm{HKR}(G;E_{n+1})$ and $\mathrm{HKR}(G;\mathbb{B}_n)$. Then the above commutative diagram is compatible with all the actions of $\mathbb{G}$.

The organization of this paper is as follows: In §2 we study the p-divisible group $\mathbf{G}(\infty)$ over a ring of formal power series which is obtained by some base change from the p-divisible group $\mathbf{F}_{n+1}(\infty)$ associated with E_{n+1}. Then we show that the identity component and the étale quotient are trivialized over the residue field L of $\mathbb{B}^0_n$. In §3 we review the generalized Chern character map $ch : E_{n+1} \to \mathbb{B}_n$ and the inclusion map $i : E_n \to \mathbb{B}_n$. We recall the action of a profinite group $\mathbb{G}$ on $\mathbb{B}_n$ and the fact that ch and i are $\mathbb{G}$-equivariant. In §4 we review the theory of generalized group characters due to Hopkins-Kuhn-Ravenel. We recall that the theory of level structures defines a tower of extension rings over E^0_n in which the generalized group characters take their values. We have two p-divisible groups over $\mathbb{B}^0_n$. The one $\mathbf{F}_n(\infty)_{\mathbb{B}}$ is obtained from the p-divisible group associated with E_n by the base change along the map induced by i, and the other $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ is obtained from the p-divisible group associated with E_{n+1} by the base change along the map induced by ch. In §5 we study the relationship between these two p-divisible groups. In §6 we study the Galois module structure on the geometric points of $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ over the generic fiber and the ring extensions obtained by adjoining torsion points. In §7 we study level structures on the p-divisible group $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ and the representing ring of level structures. In §8 we construct a model of generalized Chern characters for classifying spaces of finite groups up to torsion in terms of generalized group characters. Furthermore, we show that this ring homomorphism respects the actions of the profinite group $\mathbb{G}$. As an application we describe the generalized Chern character of the extended power spectrum DS^0 in §9.

2. The p-divisible group $\mathbf{G}(\infty)$

In this section we define a p-divisible group $\mathbf{G}(\infty)$ over a complete discrete valuation ring of equicharacteristic $p > 0$. Then we study Galois representations associated with the identity component and the étale quotient of $\mathbf{G}(\infty)$ over the generic point.

We recall the definition of p-divisible groups or Barsotti-Tate groups (cf. [29, 5, 22]). Let p be a prime number, and let h be a positive integer. A p-divisible group of height h over a commutative ring S is a system $G(\infty) = (G(r), i(r))_{r\geq 0}$ of group schemes over S such that

(1) $G(r)$ is a finite flat commutative group scheme of rank p^{hr} which is of finite presentation over S, and
(2) $i(r) : G(r) \to G(r+1)$ is a morphism of group schemes which induces an isomorphism $G(r) \overset{\cong}{\to} \ker(p^r : G(r+1) \to G(r+1))$.

If we have a homomorphism $S \to T$ of commutative rings, then we can define a p-divisible group $G(\infty)_T = (G(r)_T, i(r)_T)_{r\geq 0}$ over T, where $G(r)_T$ and $i(r)_T$ are the base changes of $G(r)$ and $i(r)$ along the map $S \to T$, respectively.

Let n be a positive integer. We fix an algebraic extension $\mathbb{F}$ of the prime field $\mathbb{F}_p$ which contains the finite fields $\mathbb{F}_{p^n}$ and $\mathbb{F}_{p^{n+1}}$. Let $R = \mathbb{F}[\![t]\!]$ be the ring of formal power series over $\mathbb{F}$ with variable t. Then R is a complete discrete valuation ring with maximal ideal generated by t and residue field $\mathbb{F}$. Let $\mathbf{G}$ be the p-typical formal group law over R satisfying the following relation on the p-series:

$$[p]^{\mathbf{G}}(X) = tX^{p^n} +_{\mathbf{G}} X^{p^{n+1}},$$

where $+_{\mathbf{G}}$ means the formal sum defined by $\mathbf{G}$. Note that $\mathbf{G}$ is obtained from the formal group law associated with the $(n+1)$st Morava E-theory E_{n+1} by the base change along the reduction map $E^0_{n+1} \to E^0_{n+1}/I_n \cong R$ (see §3 below).

For $r \geq 0$, the p^r-times map $p^r : \mathbf{G} \to \mathbf{G}$ is an isogeny with a finite flat group scheme $\mathbf{G}(r)$ as the kernel. By assembling these finite group schemes, we obtain a p-divisible group $\mathbf{G}(\infty)$ over R. By the base change along the reduction map $R \to \mathbb{F}$ to the residue field, we obtain a p-divisible group $\mathbf{G}(\infty)_{\mathbb{F}}$ over $\mathbb{F}$. Then there is an isomorphism

$$\mathbf{G}(\infty)_{\mathbb{F}} \cong \mathbf{H}_{n+1}(\infty),$$

where $\mathbf{H}_{n+1}$ is the Honda formal group over $\mathbb{F}$ of height $(n+1)$, and $\mathbf{H}_{n+1}(\infty)$ is the associated p-divisible group.

Let $K = \mathbb{F}((t))$ be the field of fractions of R. We denote by $\mathbf{G}(\infty)_K$ the p-divisible group over K obtained by the base change along the map $R \to K$. There is an exact sequence of p-divisible groups over K:

$$0 \longrightarrow \mathbf{G}(\infty)_K^0 \longrightarrow \mathbf{G}(\infty)_K \longrightarrow \mathbf{G}(\infty)_K^{\text{ét}} \longrightarrow 0.$$

The p-divisible group $\mathbf{G}(\infty)_K^0$ is connected of dimension 1 and of height n. The p-divisible group $\mathbf{G}(\infty)_K^{\text{ét}}$ is étale of height 1. Furthermore, on the separable closure K^{sep} of K, there are isomorphisms of p-divisible groups

$$\mathbf{G}(\infty)_{K^{\text{sep}}}^0 \cong \mathbf{H}_n(\infty)_{K^{\text{sep}}}, \tag{2.1}$$

$$\mathbf{G}(\infty)_{K^{\text{sep}}}^{\text{ét}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)_{K^{\text{sep}}}, \tag{2.2}$$

where $\mathbf{H}_n(\infty)$ is the p-divisible group associated with the Honda formal group $\mathbf{H}_n$ of height n, and $\mathbb{Q}_p/\mathbb{Z}_p$ is the p-divisible group consisting of the constant group schemes $\mathbb{Z}/p^r\mathbb{Z}$.

The monodromy representation

$$\pi_1(K) = \mathrm{Gal}(K^{\text{sep}}/K) \longrightarrow \mathrm{Aut}(\mathbf{G}_K(\infty))$$

gives two p-adic Galois representations

$$\begin{aligned} \rho_{1/n} : \mathrm{Gal}(K^{\text{sep}}/K) &\longrightarrow \mathrm{Aut}(\mathbf{G}(\infty)_{K^{\text{sep}}}^0), \\ \rho_{0/1} : \mathrm{Gal}(K^{\text{sep}}/K) &\longrightarrow \mathrm{Aut}(\mathbf{G}(\infty)_{K^{\text{sep}}}^{\text{ét}}). \end{aligned}$$

Let $\mathbb{S}_n$ be the nth Morava stabilizer group, which is the group of automorphisms of the formal group $\mathbf{H}_n$ over $\mathbb{F}$. The isomorphism (2.1) implies an isomorphism $\mathrm{Aut}(\mathbf{G}(\infty)_{K^{\text{sep}}}^0) \cong \mathbb{S}_n$. Hence we obtain a homomorphism $\rho_{1/n} : \mathrm{Gal}(K^{sep}/K) \to \mathbb{S}_n$.

Theorem 2.1 (Gross [9, Theorem 3.5])**.** *The Galois representation $\rho_{1/n}$ is surjective.*

We define L to be the subfield of K^{sep} corresponding to the kernel of $\rho_{1/n}$. Then L is a Galois extension of K with Galois group $\mathrm{Gal}(L/K) \cong \mathbb{S}_n$. The field L is obtained by trivializing the connected group $\mathbf{G}(\infty)_K^0$ to $\mathbf{H}_n(\infty)$.

Corollary 2.2. *There is an isomorphism $\mathbf{G}(\infty)_L^0 \cong \mathbf{H}_n(\infty)_L$ of p-divisible groups over L.*

The isomorphism (2.2) implies an isomorphism $\mathrm{Aut}(\mathbf{G}(\infty)_{K^{\text{sep}}}^{\text{ét}}) \cong \mathbb{Z}_p^\times$. Hence we obtain a homomorphism $\rho_{0/1} : \mathrm{Gal}(K^{\text{sep}}/K) \to \mathbb{Z}_p^\times$. The representations $\rho_{1/n}$ and $\rho_{0/1}$ are not independent. Let $\mathrm{Nm} : \mathbb{S}_n \to \mathbb{Z}_p^\times$ be the reduced norm map. Then the relationship between $\rho_{1/n}$ and $\rho_{0/1}$ is given by the following theorem.

Theorem 2.3 (Gross [9, Theorem 2.7])**.** *For any $g \in \mathrm{Gal}(K^{\text{sep}}/K)$, we have* $(\mathrm{Nm} \circ \rho_{1/n})(g) = \rho_{0/1}(g)^{-1}$.

By Theorems 2.1 and 2.3, we see that $\rho_{0/1}$ is surjective. We define NL to be the subfield of K^{sep} corresponding to the kernel of $\rho_{0/1}$. Then NL is a Galois extension of K with Galois group $\mathrm{Gal}(NL/K) \cong \mathbb{Z}_p^\times$. The field NL is obtained by trivializing the étale group $\mathbf{G}(\infty)_K^{\text{ét}}$ to $\mathbb{Q}_p/\mathbb{Z}_p$.

Corollary 2.4. *There is an isomorphism $\mathbf{G}(\infty)_{NL}^{\text{ét}} \cong (\mathbb{Q}_p/\mathbb{Z}_p)_{NL}$ of p-divisible groups over NL.*

By Theorem 2.3, we have $\ker \rho_{1/n} \subset \ker \rho_{0/1}$. Then the Galois theory implies that the field L contains NL.

Corollary 2.5. *There is an exact sequence of p-divisible groups over L:*

$$0 \to \mathbf{H}_n(\infty)_L \longrightarrow \mathbf{G}(\infty)_L \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)_L \to 0.$$

By the Weierstrass preparation theorem, there is a decomposition of the p^r-series $[p^r]^{\mathbf{G}}(X)$ of the formal group law $\mathbf{G}$ as

$$[p^r]^{\mathbf{G}}(X) = \phi_r(X^{p^{nr}}) \cdot u_r(X),$$

where $\phi_r(Y)$ is a monic polynomial in $R[Y]$ of degree p^r, and $u_r(X)$ is a unit in $R[\![X]\!]$. The group scheme $\mathbf{G}(r)_L$ is represented by the ring $L[X]/(\phi_r(X^{p^{nr}}))$ and $\mathbf{G}(r)_L^{\text{ét}}$ is represented by $L[Y]/(\phi_r(Y))$. The epimorphism $\mathbf{G}(r)_L \to \mathbf{G}(r)_L^{\text{ét}}$ is represented by the ring homomorphism $L[Y]/(\phi_r(Y)) \to L[X]/(\phi_r(X^{p^{nr}}))$ given by $Y \mapsto X^{p^{nr}}$.

Let $\overline{K}$ be the algebraic closure of K. We define $U(r)$ to be the set of $\overline{K}$-valued points of $\mathbf{G}(r)$:

$$U(r) = \mathbf{G}(r)(\overline{K}).$$

Note that $U(r)$ is identified with the set of all roots of $\phi_r(Y)$ in $\overline{K}$. Since the étale quotient $\mathbf{G}(\infty)_K^{\text{ét}}$ is of height 1, there is an isomorphism of groups $U(r) \cong \mathbb{Z}/p^r\mathbb{Z}$. The morphism $\mathbf{G}(r) \to \mathbf{G}(r+1)$ induces an inclusion $U(r) \hookrightarrow U(r+1)$, and we set $U = U(\infty) = \bigcup_r U(r) \cong \mathbb{Q}_p/\mathbb{Z}_p$. Let α_r be a root of $\phi_r(Y)$ which corresponds to a generator of $U(r)$. Then $K(\alpha_r)$ is a subfield of NL corresponding to the kernel of the reduction map $\mathrm{Gal}(NL/K) \cong \mathbb{Z}_p^\times \to (\mathbb{Z}/p^r\mathbb{Z})^\times$.

Lemma 2.6. *The field $K(\alpha_r)$ is a totally ramified extension of K, and $K(\alpha_r) = \mathbb{F}((\alpha_r))$.*

Proof. We set $\phi_0(X) = X$. Then $\phi_r(X)^{p^n}$ is a factor of $\phi_{r+1}(X^{p^n})$ for $r \geq 0$. Set $\psi_{r+1}(X^{p^n}) = \phi_{r+1}(X^{p^n})/\phi_r(X)^{p^n}$. Then $\psi_{r+1}(X) \in R[X]$ is a monic polynomial of degree $p^r(p-1)$. Since $\psi_{r+1}(X)$ satisfies $\psi_{r+1}(0) = t$ and $\psi_{r+1}(X) \equiv X^{p^r(p-1)}$ mod (t), $\psi_{r+1}(X)$ is an Eisenstein polynomial over R. Then the lemma follows from the fact that α_r is a root of $\psi_r(X)$. $\square$

Lemma 2.7. *For $r > 0$, $\sqrt[p]{\alpha_r} \notin L$.*

Proof. Suppose that $\sqrt[p]{\alpha_r} \in L$. Since L is separable over $K(\alpha_r)$, $\sqrt[p]{\alpha_r} \in K(\alpha_r)$. This contradicts the fact that $K(\alpha_r) = \mathbb{F}((\alpha_r))$ by Lemma 2.6. $\square$

For a generator $\alpha_r \in U(r)$, there is a corresponding map $\mathrm{spec}(L) \to \mathbf{G}(r)_L^{\text{ét}}$. If there exists a splitting $\mathbf{G}(r)_L^{\text{ét}} \to \mathbf{G}(r)_L$, we obtain a map $\mathrm{spec}(L) \to \mathbf{G}(r)_L$ such that the composition with $\mathbf{G}(r)_L \to \mathbf{G}(r)_L^{\text{ét}}$ corresponds to α_r. This implies that $\sqrt[p^{nr}]{\alpha_r} \in L$, which contradicts Lemma 2.7. Hence we see that the exact sequence in Corollary 2.5 does not split over L.

Remark 2.8. The exact sequence in Corollary 2.5 canonically splits over the algebraic closure $\overline{K}$ of K.

3. The generalized Chern character

In this section we review the generalized Chern character constructed in [32]. We recall the construction of a commutative ring spectrum $\mathbb{B}_n$ and two ring spectrum maps $ch : E_{n+1} \to \mathbb{B}_n$ and $i : E_n \to \mathbb{B}_n$. Furthermore, we recall the action of a profinite group $\mathbb{G}$ on $\mathbb{B}_n$ and the fact that the ring spectrum maps ch and i are $\mathbb{G}$-equivariant.

Let W be the ring of Witt vectors with coefficients in $\mathbb{F}$. We consider variants of the nth Morava E-theory spectrum E_n and the $(n+1)$st Morava E-theory spectrum E_{n+1} such that the coefficient rings are given by

$$\begin{aligned} E_n^* &= W[\![w_1,\ldots,w_{n-1}]\!][w^{\pm 1}], \\ E_{n+1}^* &= W[\![u_1,\ldots,u_n]\!][u^{\pm 1}]. \end{aligned}$$

Since the Morava E-theories E_n and E_{n+1} are complex oriented, there are associated formal group laws. We denote by $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$ the formal group laws associated with E_n and E_{n+1}, and we also denote by $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$ the associated formal groups, respectively. We abbreviate to $[m]^n(X)$ the m-series $[m]^{\mathbf{F}_n}(X)$ of $\mathbf{F}_n$, and to $[m]^{n+1}(X)$ the m-series $[m]^{\mathbf{F}_{n+1}}(X)$ of $\mathbf{F}_{n+1}$. The formal group laws $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$ are p-typical and they are characterized by their p-series

$$\begin{aligned} [p]^n(X) &= pX \underset{\mathbf{F}_n}{+} w_1X^p \underset{\mathbf{F}_n}{+} \cdots \underset{\mathbf{F}_n}{+} w_{n-1}X^{p^{n-1}} \underset{\mathbf{F}_n}{+} X^{p^n}, \\ [p]^{n+1}(X) &= pX \underset{\mathbf{F}_{n+1}}{+} u_1X^p \underset{\mathbf{F}_{n+1}}{+} \cdots \underset{\mathbf{F}_{n+1}}{+} u_nX^{p^n} \underset{\mathbf{F}_{n+1}}{+} X^{p^{n+1}}, \end{aligned}$$

where $\underset{\mathbf{F}_n}{+}$ and $\underset{\mathbf{F}_{n+1}}{+}$ are formal sums defined by $\mathbf{F}_n$ and $\mathbf{F}_{n+1}$, respectively.

Let $K(n)$ be the nth Morava K-theory spectrum at p. We define a spectrum $\mathbb{A}_n$ to be the Bousfield localization of E_{n+1} with respect to $K(n)$,

$$\mathbb{A}_n = L_{K(n)}E_{n+1}.$$

Then $\mathbb{A}_n$ is a $K(n)$-local Landweber exact commutative ring spectrum. The coefficient ring of $\mathbb{A}_n$ is given by

$$\mathbb{A}_n^* = W((u_n))_p^{\wedge}[\![u_1,\ldots,u_{n-1}]\!][u^{\pm 1}],$$

where $W((u_n))_p^{\wedge}$ is the p-adic completion of the ring of Laurent series with variable u_n and coefficients in W. Hence the degree 0 subring $\mathbb{A}_n^0$ is a complete Noetherian regular local ring with maximal ideal generated by $p, u_1, \ldots, u_{n-1}$ and residue field $\mathbb{F}((u_n))$.

Let I_n be the ideal of E_{n+1}^0 generated by $p, u_1, \ldots, u_{n-1}$. We identify the ring E_{n+1}^0/I_n with $R = \mathbb{F}[\![t]\!]$ by $u_n = t$, and the residue field $\mathbb{A}_n^0/I_n\mathbb{A}_n^0$ with $K = \mathbb{F}((t))$. Then the formal group law $\mathbf{G}$ in §2 is obtained from $\mathbf{F}_{n+1}$ by the base change along the reduction map $E_{n+1}^0 \to E_{n+1}^0/I_n = R$. By Theorem 2.1, there is a Galois extension L of K with Galois group $\mathrm{Gal}(L/K) \cong \mathbb{S}_n$. In [32], we constructed a Landweber exact $K(n)$-local commutative ring spectrum $\mathbb{B}_n$. The coefficient ring $\mathbb{B}_n^*$ is even-periodic, and the degree 0 subring $\mathbb{B}_n^0$ is a complete Noetherian regular local ring with maximal ideal generated by $p, u_1, \ldots, u_{n-1}$ and residue field L. There is a ring spectrum map $\mathbb{A}_n \to \mathbb{B}_n$ which covers the inclusion $K \hookrightarrow L$ on the degree 0 coefficient rings. By the composition $E_{n+1} \longrightarrow L_{K(n)}E_{n+1} = \mathbb{A}_n \longrightarrow \mathbb{B}_n$, we obtain a morphism of commutative ring spectra

$$ch : E_{n+1} \longrightarrow \mathbb{B}_n.$$

We say that ch is the generalized Chern character since ch induces a multiplicative natural transformation of the height $(n+1)$ cohomology theory $E_{n+1}^*(-)$ to the height n cohomology theory $\mathbb{B}_n^*(-)$. Let $(\mathbf{F}_{n+1})_{\mathbb{B}}$ be the formal group law over $\mathbb{B}_n^0$ obtained from the formal group law $\mathbf{F}_{n+1}$ over E_{n+1}^0 by the base change along the map $E_{n+1}^0 \to \mathbb{B}_n^0$ induced by ch. Then $(\mathbf{F}_{n+1})_{\mathbb{B}}$ gives a height n formal group law $\mathbf{G}_L$ over the residue field L. Note that the p-divisible group associated with $\mathbf{G}_L$

is the identity component $\mathbf{G}(\infty)^0_L$. By Corollary 2.2, there is an isomorphism of formal group laws between $\mathbf{G}_L$ and the height n Honda formal group law $(\mathbf{H}_n)_L$,

$$\Phi : \mathbf{G}_L \xrightarrow{\cong} (\mathbf{H}_n)_L.$$

Since the formal group law $\mathbf{F}_n$ associated with the nth Morava E-theory spectrum E_n is a universal deformation of $\mathbf{H}_n$, this isomorphism gives a continuous ring homomorphism from E_n^0 to $\mathbb{B}_n^0$, and this ring homomorphism extends to a morphism of commutative ring spectra

$$i : E_n \longrightarrow \mathbb{B}_n.$$

Let $\mathbb{S}_n$ (resp. $\mathbb{S}_{n+1}$) be the nth (resp. the $(n+1)$st) Morava stabilizer group. The group $\mathbb{S}_n$ (resp. $\mathbb{S}_{n+1}$) is the group of automorphisms of the Honda formal group laws $\mathbf{H}_n$ of height n (resp. $\mathbf{H}_{n+1}$ of height $(n+1)$) over $\mathbb{F}$. Let Γ be the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$. The extended Morava stabilizer group $\mathbb{G}_n$ (resp. $\mathbb{G}_{n+1}$) is defined to be the semi-direct product $\Gamma \ltimes \mathbb{S}_n$ (resp. $\Gamma \ltimes \mathbb{S}_{n+1}$). There are projections $\mathbb{G}_n \to \Gamma$ and $\mathbb{G}_{n+1} \to \Gamma$. We define a group $\mathbb{G}$ to be the fiber product of $\mathbb{G}_n$ and $\mathbb{G}_{n+1}$ over Γ:

$$\mathbb{G} = \mathbb{G}_n \times_\Gamma \mathbb{G}_{n+1}.$$

Note that $\mathbb{G}$ is a profinite group as well as $\mathbb{G}_n$ and $\mathbb{G}_{n+1}$. We set $\mathbb{S} = \mathbb{S}_n \times \mathbb{S}_{n+1}$. Then the Galois group Γ acts on $\mathbb{S}$ diagonally and there is an isomorphism $\mathbb{G} \cong \Gamma \ltimes \mathbb{S}$. In [31, §2.4], we showed that there is an action of $\mathbb{G}$ on L, which is an extension of the action of $\mathbb{G}_{n+1}$ on K and the action of $\mathbb{S}_n$ on L as a Galois group. This action of $\mathbb{G}$ on L extends to an action of $\mathbb{G}$ on the ring spectrum $\mathbb{B}_n$ in the stable homotopy category. By the projection $\mathbb{G} \to \mathbb{G}_n$, we can consider that $\mathbb{G}$ acts on the commutative ring spectrum E_n. Also, by the projection $\mathbb{G} \to \mathbb{G}_{n+1}$, we can consider that $\mathbb{G}$ acts on the commutative ring spectrum E_{n+1}. Then the ring spectrum maps $ch : E_{n+1} \to \mathbb{B}_n$ and $i : E_n \to \mathbb{B}_n$ are $\mathbb{G}$-equivariant.

4. HKR characters

In [16, 15], Hopkins, Kuhn and Ravenel defined a generalization of group characters, and they showed that the cohomology ring of classifying spaces of finite groups tensored with the field $\mathbb{Q}$ of rational numbers can be described in terms of generalized group characters for some class of complex oriented cohomology theories. In this section we review the theory of generalized group characters (HKR characters) in the case of Morava E-theory $E_n^*(-)$.

Let $C(r) = \mathbb{Z}/p^r\mathbb{Z}$ be the finite cyclic group of order p^r, and let $BC(r)$ be its classifying space. The E_n-cohomology ring of $BC(r)$ is concentrated in even degrees, and the degree 0 subring is given by

$$E_n^0(BC(r)) = E_n[\![X]\!]/([p^r]^n(X)).$$

In particular, $E_n^0(BC(r))$ is a complete local E_n^0-algebra with residue field $\mathbb{F}$. By the Weierstrass preparation theorem, we can decompose $[p^r]^n(X)$ as

$$[p^r]^n(X) = f_r(X) \cdot v_r(X),$$

where $f_r(X) \in E_n^0[X]$ is a monic polynomial of degree p^{nr}, and $v_r(X) \in E_n^0[\![X]\!]$ is a unit in the power series ring. Then there is an isomorphism of rings

$$E_n^0(BC(r)) \cong E_n^0[X]/(f_r(X)).$$

In particular, $E_n^0(BC(r))$ is a free E_n^0-module of rank p^{nr}. Let Ω be an algebraically closed field which contains E_n^0. We define $V_n(r)$ to be the set of all roots of $f_r(X)$

in Ω. Note that $V_n(r)$ is identified with the set of all E_n^0-algebra homomorphisms from $E_n^0(BC(r))$ to Ω:

$$V_n(r) \cong E_n^0\text{-Alg}\,(E_n^0(BC(r)), \Omega).$$

Since $E_n^0(BC(r))$ is a co-commutative Hopf algebra over E_n^0, there is an abelian group structure on $V_n(r)$, and there is an isomorphism $V_n(r) \cong (\mathbb{Z}/p^r\mathbb{Z})^n$. The canonical inclusion $C(r) \hookrightarrow C(r+1)$ induces a surjective ring homomorphism $E_n^0(BC(r+1)) \to E_n^0(BC(r))$. This ring homomorphism induces an injection $V_n(r) \hookrightarrow V_n(r+1)$ of abelian groups. We define an abelian group $V_n = V_n(\infty)$ to be the colimit of $V_n(r)$:

$$V_n = V_n(\infty) = \underset{\longrightarrow r}{\operatorname{colim}}\, V_n(r).$$

Then we have $V_n \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$ and $\mathrm{Aut}(V_n) \cong \mathrm{GL}_n(\mathbb{Z}_p)$.

For a complete local E_n^0-algebra $(R, \mathfrak{m})$, we denote by $(\mathfrak{m}, \mathbf{F}_n)$, the abelian group $\mathfrak{m}$ with respect to the formal sum $+_{\mathbf{F}_n}$. There is a functor $\mathrm{Hom}(V_n, \mathbf{F}_n)(r)$, which assigns to a complete local E_n^0-algebra $(R, \mathfrak{m})$ the set of all homomorphisms from $V_n(r)$ to $(\mathfrak{m}, \mathbf{F}_n)$. Then the functor $\mathrm{Hom}(V_n, F_n)(r)$ is represented by $E_n^0(BV_n(r)^*)$, where $V_n(r)^* = \mathrm{Hom}(V_n(r), \mathbb{Q}_p/\mathbb{Z}_p)$ is the dual abelian group. A homomorphism $\phi \in \mathrm{Hom}(V_n, \mathbf{F}_n)(r)(R)$ is said to be a level $V_n(r)$-structure on $(\mathbf{F}_n)_R$ if the polynomial $\prod_{a \in V_n(1)}(X - \phi(a))$ divides the p-series $[p]^n(X)$ in $R[\![X]\!]$. Then we have a subfunctor $\mathrm{Level}(V_n, \mathbf{F}_n)(r)$ of $\mathrm{Hom}(V_n, \mathbf{F}_n)(r)$, which assigns to a complete local E_n^0-algebra R the set of all level $V_n(r)$-structures on $(\mathbf{F}_n)_R$. Then the functor $\mathrm{Level}(V_n, \mathbf{F}_n)(r)$ is representable (cf. [6, 1, 27, 16]). We define $D_n(r)$ to be the commutative E_n^0-algebra generated by $V_n(r)$:

$$D_n(r) = E_n^0[V_n(r)].$$

The ring $D_n(r)$ is a complete Noetherian regular local ring with residue field $\mathbb{F}$ and represents the functor $\mathrm{Level}(V_n, \mathbf{F}_n)(r)$. The canonical inclusion $V_n(r) \hookrightarrow D_n(r)$ factors through the maximal ideal of $D_n(r)$, and this gives a homomorphism in $\mathrm{Hom}(V_n, \mathbf{F}_n)(r)(D_n(r))$ which is a universal level $V_n(r)$-structure on $\mathbf{F}_n$ over $D_n(r)$. Hence we have a continuous E_n^0-algebra homomorphism $E_n^0(BV_n(r)^*) \to D_n(r)$, which corresponds to the natural transformation $\mathrm{Level}(V_n, \mathbf{F}_n)(r) \to \mathrm{Hom}(V_n, \mathbf{F}_n)(r)$. The action of $\mathrm{Aut}(V_n(r)) \cong \mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})$ on $V_n(r)$ extends to an action on the ring $D_n(r)$, and the invariant subring under this action is E_n^0. We define a ring $D_n = D_n(\infty)$ to be the colimit of $D_n(r)$:

$$D_n = D_n(\infty) = \underset{\longrightarrow r}{\operatorname{colim}}\, D_n(r).$$

Then D_n is a commutative E_n^0-algebra generated by V_n. The action of $\mathrm{Aut}(V_n)$ on V_n extends to an action on the ring D_n, and the invariant subring under this action is E_n^0:

$$H^0(\mathrm{Aut}(V_n); D_n) = E_n^0.$$

Let $V_n^* = \mathrm{Hom}(V_n, \mathbb{Q}_p/\mathbb{Z}_p)$ be the dual abelian group. Note that V_n^* is a profinite abelian group, and there exists an isomorphism $V_n^* \cong \mathbb{Z}_p^n$. For a finite group G, we let $\mathrm{Hom}^c(V_n^*, G)$ be the set of all continuous homomorphisms from V_n^* to G. Then there exists an action of G on $\mathrm{Hom}^c(V_n^*, G)$ by conjugation at the target. We define $\mathrm{Rep}(V_n^*, G)$ to be the quotient set of $\mathrm{Hom}^c(V_n^*, G)$ under this conjugation action of G:

$$\mathrm{Rep}(V_n^*, G) = \mathrm{Hom}^c(V_n^*, G)/G.$$

If we fix an isomorphism $V_n^* \cong \mathbb{Z}_p^n$, then $\mathrm{Rep}(V_n^*, G)$ is identified with the set of all conjugacy classes of n-tuples $(g_1, \dots, g_n)$ such that $g_i^{p^m} = e$ for some m and $[g_i, g_j] = e$. We consider that $\mathrm{Rep}(V_n^*, G)$ is a generalization of the set of conjugacy classes. A representative of an element in $\mathrm{Rep}(V_n^*, G)$ factors through $V_n(r)^*$ for some r. This induces a ring homomorphism $E_n^0(BG) \to E_n^0(BV_n(r)^*) \to D_n(r) \to D_n$. Then we can verify that this ring homomorphism is independent of the choice of representative and r. Hence we obtain a map $E_n^0(BG) \times \mathrm{Rep}(V_n^*, G) \to D_n$. As its adjoint, we obtain an E_n^0-algebra homomorphism

$$E_n^0(BG) \longrightarrow \mathrm{Map}(\mathrm{Rep}(V_n^*, G), D_n), \tag{4.1}$$

which is natural for finite groups G. The target $\mathrm{Map}(\mathrm{Rep}(V_n^*, G), D_n)$ is the ring of n-dimensional generalized group characters due to Hopkins, Kuhn and Ravenel. The group $\mathrm{Aut}(V_n)$ acts on $\mathrm{Rep}(V_n^*, G)$ in the obvious way, and there is an action of $\mathrm{Aut}(V_n)$ on D_n. These two actions induce an action of $\mathrm{Aut}(V_n)$ on $\mathrm{Map}(\mathrm{Rep}(V_n^*, G), D_n)$ by conjugation. We define $\mathrm{HKR}(G, E_n)$ to be the $\mathrm{Aut}(V_n)$-invariant subring

$$\mathrm{HKR}(G, E_n) = H^0(\mathrm{Aut}(V_n); \mathrm{Map}(\mathrm{Rep}(V_n^*, G), D_n)).$$

Then the E_n^0-ring homomorphism (4.1) factors through $\mathrm{HKR}(G, E_n)$.

Theorem 4.1 (Hopkins–Kuhn–Ravenel [16, 15])**.** *The E_n^0-algebra homomorphism*

$$E_n^0(BG) \longrightarrow \mathrm{HKR}(G, E_n)$$

is an isomorphism after tensoring with $\mathbb{Q}$.

The nth extended Morava stabilizer group $\mathbb{G}_n$ acts on the nth Morava E-theory $E_n^*(-)$ by multiplicative cohomology operations. Since the functor $\mathrm{Hom}(V_n, \mathbf{F}_n)(r)$ is represented by $E_n^0(BV_n(r)^*)$, there is an action of $\mathbb{G}_n$ on $\mathrm{Hom}(V_n, \mathbf{F}_n)(r)$. Then this action restricts to an action on the subfunctor $\mathrm{Level}(V_n, \mathbf{F}_n)(r)$. Since $\mathrm{Level}(V_n, \mathbf{F}_n)(r)$ is represented by the ring $D_n(r)$, we obtain an action of $\mathbb{G}_n$ on $D_n(r)$, which commutes with the action of $\mathrm{Aut}(V_n(r))$. By taking the colimit on r, we obtain an action of $\mathbb{G}_n$ on the ring D_n, which commutes with the action of $\mathrm{Aut}(V_n)$. This induces a natural action of $\mathbb{G}_n$ on the ring of generalized group characters $\mathrm{Map}(\mathrm{Rep}(V_n^*, G), D_n)$, which commutes with the action of $\mathrm{Aut}(V_n)$. Hence we obtain an action of $\mathbb{G}_n$ on the invariant subring $\mathrm{HKR}(G, E_n)$. We can verify that the ring homomorphism (4.1) is $\mathbb{G}_n$-equivariant. By Theorem 4.1, there is an isomorphism of $\mathbb{G}_n$-modules

$$E_n^0(BG) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{HKR}(G, E_n) \otimes \mathbb{Q},$$

which is natural for finite groups G.

5. The p-divisible group $\mathbf{F}_{n+1}(\infty)$ over $\mathbb{B}_n$

For the formal group $\mathbf{F}_n$, we can consider the corresponding p-divisible group $\mathbf{F}_n(\infty)$ over E_n^0. We denote by $\mathbf{F}_n(\infty)_{\mathbb{B}}$ and $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ the p-divisible groups over $\mathbb{B}_n^0$ obtained by the base change along the map $E_n^0 \to \mathbb{B}_n^0$ induced by i, and the map $E_{n+1}^0 \to \mathbb{B}_n^0$ induced by ch, respectively. In this section we study the relationship between the p-divisible groups $\mathbf{F}_n(\infty)_{\mathbb{B}}$ and $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$.

The E_n-cohomology ring of the classifying space $BC(r)$ of the cyclic group $C(r)$ is a co-commutative Hopf algebra over E_n^0 which is free of rank p^{nr}. Hence $E_n^0(BC(r))$ gives a finite flat commutative group scheme $\mathbf{F}_n(r)$ of finite presentation over E_n^0.

The canonical inclusion $C(r) \hookrightarrow C(r+1)$ induces a morphism $i_n(r) : \mathbf{F}_n(r) \to \mathbf{F}_n(r+1)$ of group schemes which induces an isomorphism

$$\mathbf{F}_n(r) \cong \ker(p^r : \mathbf{F}_n(r+1) \to \mathbf{F}_n(r+1)).$$

Hence we obtain a p-divisible group $\mathbf{F}_n(\infty)$ of height n over E_n^0. In the same way, we obtain a p-divisible group $\mathbf{F}_{n+1}(\infty)$ of height $(n+1)$ over E_{n+1}^0. By the base change along the map $E_{n+1}^0 \to \mathbb{B}_n^0$ induced by ch, there is a p-divisible group $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ over $\mathbb{B}_n^0$. Since $\mathbb{B}_n^0$ is a Henselian local ring, there is a canonical exact sequence of p-divisible groups

$$0 \to \mathbf{F}_{n+1}(\infty)_{\mathbb{B}}^0 \longrightarrow \mathbf{F}_{n+1}(\infty)_{\mathbb{B}} \longrightarrow \mathbf{F}_{n+1}(\infty)_{\mathbb{B}}^{\text{ét}} \to 0,$$

where $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}^0$ is connected over $\mathrm{Spf}\,\mathbb{B}_n^0$, and $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}^{\text{ét}}$ is étale.

For the abelian group $U(r)$ of $\overline{K}$-valued points of the finite flat group scheme $\mathbf{G}(r)$, we let $U(r)_{\mathbb{B}}$ be the corresponding constant group scheme over $\mathbb{B}_n^0$. Then the system $\{U(r)_{\mathbb{B}}\}_{r\geq 0}$ defines a p-divisible group $U(\infty)_{\mathbb{B}}$ that is constant of height 1.

Lemma 5.1. $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}^{\text{ét}} \cong U(\infty)_{\mathbb{B}}$.

Proof. Since $(\mathbf{F}_{n+1})_R \cong \mathbf{G}$, there is an isomorphism $\mathbf{F}_{n+1}(\infty)_R \cong \mathbf{G}(\infty)$. Corollary 2.4 implies an isomorphism $\mathbf{F}_{n+1}(\infty)_L \cong U(\infty)_L$. Hence $\mathbf{F}_{n+1}(\infty)(\overline{K}) \cong U(\infty)$ is a trivial $\mathrm{Gal}(\overline{K}/L)$-module. Then the lemma follows from the fact that the category of finite étale abelian group schemes over $\mathbb{B}_n^0$ is equivalent to the category of finite $\mathrm{Gal}(\overline{K}/L)$-modules. □

By the Weierstrass preparation theorem, there is a decomposition of $[p^r]^{n+1}(X)$ in $E_{n+1}^0[\![X]\!]$ as

$$[p^r]^{n+1}(X) = g_r(X) \cdot w_r(X),$$

where $g_r(X) \in E_{n+1}^0[X]$ is a monic polynomial of degree $p^{(n+1)r}$, and $w_r(X)$ is a unit in $E_{n+1}^0[\![X]\!]$. Then the finite group scheme $\mathbf{F}_{n+1}(r)_{\mathbb{B}}$ is represented by $\mathbb{B}_n^0[X]/(g_r(X))$. Note that $g_r(X) \equiv \phi_r(X^{p^{nr}}) \mod I_n$ in $L[X]$, and there is a decomposition $\phi_r(Y) = \prod_{\alpha\in U(r)}(Y-\alpha)$ in $L[Y]$. By Hensel's lemma, there exists a decomposition of $g_r(X)$ as

$$g_r(X) = \prod_{\alpha\in U(r)} g_{r,\alpha}(X)$$

in $\mathbb{B}_n^0[X]$, where $g_{r,\alpha}(X)$ is a monic polynomial of degree p^{nr} such that $g_{r,\alpha}(X) \equiv X^{p^{nr}} - \alpha \mod I_n$ in $L[X]$. This decomposition gives an isomorphism of rings

$$\mathbb{B}_n^0[X]/(g_r(X)) \xrightarrow{\cong} \prod_{\alpha\in U(r)} \mathbb{B}_n^0[X]/(g_{r,\alpha}(X)),$$

where $\mathbb{B}_n^0[X]/(g_{r,\alpha}(X))$ represents the connected component of $\mathbf{F}_{n+1}(r)_{\mathbb{B}}$ corresponding to $\alpha \in U(r)$.

By the Weierstrass preparation theorem, there is also a decomposition of $[p^r]^{n+1}(X)$ in $\mathbb{B}_n^0[\![X]\!]$ as

$$[p^r]^{n+1}(X) = h_r(X) \cdot s_r(X),$$

where $h_r(X) \in \mathbb{B}_n^0[X]$ is a monic polynomial of degree p^{nr}, and $s_r(X)$ is a unit in $\mathbb{B}_n^0[\![X]\!]$. Then $g_{r,0}(X) = h_r(X)$ since $h_r(X)$ is a factor of $g_r(X)$ and $h_r(X) \equiv X^{p^{nr}}$

mod I_n in $L[X]$. This implies that the identity component $\mathbf{F}_{n+1}(r)^0_{\mathbb{B}}$ is isomorphic to $(\mathbf{F}_{n+1})_{\mathbb{B}}(r)$. By assembling this isomorphism for all $r \geq 0$, we obtain an isomorphism of p-divisible groups

$$\mathbf{F}_{n+1}(\infty)^0_{\mathbb{B}} \cong (\mathbf{F}_{n+1})_{\mathbb{B}}(\infty).$$

Lemma 5.2. $\mathbf{F}_{n+1}(\infty)^0_{\mathbb{B}} \cong \mathbf{F}_n(\infty)_{\mathbb{B}}$.

Proof. By Corollary 2.2, there is an isomorphism $\mathbf{G}_L \cong (\mathbf{H}_n)_L$ of formal groups over L. By [32, §4], this lifts to an isomorphism $(\mathbf{F}_{n+1})_{\mathbb{B}} \cong (\mathbf{F}_n)_{\mathbb{B}}$ of formal groups over $\mathbb{B}^0_n$. Hence we obtain an isomorphism $(\mathbf{F}_{n+1})_{\mathbb{B}}(\infty) \cong (\mathbf{F}_n)_{\mathbb{B}}(\infty)$ of p-divisible groups. Then the lemma follows from the fact that $(\mathbf{F}_n)_{\mathbb{B}}(\infty) \cong \mathbf{F}_n(\infty)_{\mathbb{B}}$. □

By Lemmas 5.1 and 5.2, we obtain the following theorem.

Theorem 5.3. *There exists an exact sequence of p-divisible groups over $\mathbb{B}^0_n$:*

$$0 \to \mathbf{F}_n(\infty)_{\mathbb{B}} \longrightarrow \mathbf{F}_{n+1}(\infty)_{\mathbb{B}} \longrightarrow U(\infty)_{\mathbb{B}} \to 0.$$

6. The Galois module $\mathbf{F}_{n+1}(\infty)(\overline{\mathbb{K}})$

Let $\mathbb{K}$ be the field of fractions of $\mathbb{B}^0_n$, and let $\overline{\mathbb{K}}$ be its algebraic closure. In this section we study the Galois module structure on $\mathbf{F}_{n+1}(\infty)(\overline{\mathbb{K}})$ and the ring extensions obtained by adjoining torsion points of $\mathbf{F}_{n+1}(\infty)(\overline{\mathbb{K}})$.

By the base change along the map $\mathbb{B}^0_n \to \mathbb{K}$, the exact sequence in Theorem 5.3 induces an exact sequence of p-divisible groups over $\mathbb{K}$:

$$0 \to \mathbf{F}_n(\infty)_{\mathbb{K}} \longrightarrow \mathbf{F}_{n+1}(\infty)_{\mathbb{K}} \longrightarrow U(\infty)_{\mathbb{K}} \to 0.$$

Let $\pi_1(\mathbb{K})$ be the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Since $\mathbf{F}_{n+1}(\infty)_{\mathbb{K}}$ is étale, it is determined by the continuous $\pi_1(\mathbb{K})$-module structure on the discrete module

$$\mathbf{F}_{n+1}(\infty)(\overline{\mathbb{K}}) = \underset{\longrightarrow}{\mathrm{colim}}\, \mathbf{F}_{n+1}(\infty)(L),$$

where the colimit is taken over all finite extensions L over K. For $0 \leq r \leq \infty$, we define abelian groups $W(r)$ and $V(r)$ by

$$\begin{array}{rcccl} W(r) & = & V_{n+1}(r) & = & \mathbf{F}_{n+1}(r)(\overline{\mathbb{K}}), \\ V(r) & = & V_n(r) & = & \mathbf{F}_n(r)(\overline{\mathbb{K}}), \end{array}$$

and we abbreviate $W(\infty)$ and $V(\infty)$ to W and V, respectively. Then there is an exact sequence of abelian groups

$$0 \to V(r) \longrightarrow W(r) \longrightarrow U(r) \to 0 \tag{6.1}$$

for all $0 \leq r \leq \infty$. Note that there are isomorphisms $V \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$ and $W \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{n+1}$. Since $V(r)$ and $W(r)$ are identified with the set of all roots of $f_r(X)$ and $g_r(X)$ in $\overline{\mathbb{K}}$, we regard V and W as subsets of $\overline{\mathbb{K}}$.

For $0 \leq r \leq \infty$, we define commutative $\mathbb{B}^0_n$-algebras $\mathbb{D}_n(r)$ and $\mathbb{J}_n(r)$ to be the $\mathbb{B}^0_n$-algebras generated by $V(r)$ and $W(r)$, respectively,

$$\begin{array}{rcl} \mathbb{D}_n(r) & = & \mathbb{B}_n[V(r)], \\ \mathbb{J}_n(r) & = & \mathbb{B}_n[W(r)], \end{array}$$

and we abbreviate $\mathbb{D}_n(\infty) = \mathbb{B}_n[V]$ and $\mathbb{J}_n(\infty) = \mathbb{B}_n[W]$ to $\mathbb{D}_n$ and $\mathbb{J}_n$, respectively.

Lemma 6.1. *For $0 \leq r < \infty$, $\mathbb{D}_n(r)$ is a complete regular local ring of Krull dimension n with residue field L. The field of fractions of $\mathbb{D}_n(r)$ is a Galois extension of $\mathbb{K}$ with Galois group $\mathrm{Aut}(V(r))$.*

Proof. The lemma follows from the fact that $\mathbb{D}_n(r) \cong \mathbb{B}_n^0 \otimes_{E_n^0} D_n(r)$. □

For $g \in \pi_1(\mathbb{K})$, the homomorphism $\pi_1(\mathbb{K}) \to \mathrm{Aut}(W)$ gives a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & V & \longrightarrow & W & \longrightarrow & U & \to & 0 \\ & & \downarrow{\scriptstyle g_V} & & \downarrow{\scriptstyle g_W} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \to & V & \longrightarrow & W & \longrightarrow & U & \to & 0, \end{array}$$

where the right vertical map is the identity since $U(\infty)_{\mathbb{K}}$ is a constant p-divisible group.

For $0 \leq r \leq \infty$, we define a subgroup $B(W,V)(r)$ of $\mathrm{Aut}(W(r))$ by

$$B(W,V)(r) = \{g \in \mathrm{Aut}(W(r)) |\ g(V(r)) = V(r),\ \overline{g} = id_{U(r)}\},$$

where $\overline{g}$ is an automorphism on $U(r)$ induced by g. We abbreviate $B(W,V)(\infty)$ to $B(W,V)$. Then the homomorphism $\pi_1(\mathbb{K}) \to \mathrm{Aut}(W)$ factors through $B(W,V)$.

Theorem 6.2. $\pi_1(\mathbb{K}) \to B(W,V)$ *is surjective.*

Proof. Set $M(r) = \mathbb{K}[W(r)]$ and $N(r) = \mathbb{K}[V(r)]$ for $0 \leq r < \infty$. Then $M(r)$ and $N(r)$ are the fields of fractions of $\mathbb{D}_n(r)$ and $\mathbb{J}_n(r)$, respectively. By Lemma 6.1, $N(r)$ is a Galois extension of $\mathbb{K}$ with Galois group $\mathrm{Aut}(V(r))$. Note that $W(r)/V(r)$ can be identified with $U(r)$. Let $\alpha \in U(r)$ be a generator. We consider that $g_{r,\alpha}(X)$ is a polynomial in $\mathbb{D}_n[X]$. By Lemma 6.1, the residue field of $\mathbb{D}_n$ is L, and $g_{r,\alpha}(X) \equiv X^{p^{nr}} - \alpha$ in $L[X]$. Then we see that $g_{r,\alpha}(X)$ is irreducible in $\mathbb{D}_n[X]$ by Lemma 2.7. Since $\mathbb{D}_n$ is an integrally closed domain by Lemma 6.1, $g_{r,\alpha}(X)$ is irreducible in the fraction field $N(r)$. Since $M(r)$ is the minimum decomposition field of $g_{r,\alpha}(X)$, $M(r)$ is a Galois extension over $N(r)$ of degree $\geq p^{nr}$. There is an obvious injection $\mathrm{Gal}(M(r)/\mathbb{K}) \hookrightarrow B(W,V)(r)$. By comparing the orders of groups, we see that this is an isomorphism $\mathrm{Gal}(M(r)/\mathbb{K}) \cong B(W,V)(r)$. Then we obtain that $\mathrm{Gal}(\mathbb{K}[W]/\mathbb{K}) \cong B(W,V)$, and we can identify the homomorphism $\pi_1(\mathbb{K}) \to B(W,V)$ with $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \to \mathrm{Gal}(\mathbb{K}[W]/\mathbb{K})$. This completes the proof. □

Proposition 6.3. *For $0 \leq r < \infty$, $\mathbb{J}_n(r)$ is a complete regular local ring of Krull dimension n. The field of fractions of $\mathbb{J}_n(r)$ is a Galois extension of $\mathbb{K}$ with Galois group $B(W,V)(r)$.*

Proof. Take a generator $\alpha \in U(r)$. There is an isomorphism of rings $\mathbb{J}_n(r) \cong \mathbb{D}_n(r)[X]/(g_{r,\alpha}(X))$. Then $L \otimes_{\mathbb{D}_n(r)} \mathbb{J}_n(r) \cong L[X]/(X^{p^{nr}} - \alpha)$ is a field by Lemma 2.7. Hence we see that the maximal ideal of $\mathbb{D}_n(r)$ generates a maximal ideal of $\mathbb{J}_n(r)$. Since $\mathbb{J}_n(r)$ is a finitely generated free $\mathbb{D}_n(r)$-module, we see that $\mathbb{J}_n(r)$ is a complete regular local ring. This completes the proof. □

There is a canonical surjection $B(W,V) \to \mathrm{Aut}(V)$. We denote by $Q(W,V)$ its kernel. Then there exists an exact sequence

$$1 \to Q(W,V) \longrightarrow B(W,V) \longrightarrow \mathrm{Aut}(V) \to 1.$$

Proposition 6.4. *$H^0(Q(W,V);\mathbb{J}_n)=\mathbb{D}_n$ and $H^0(B(W,V);\mathbb{J}_n)=H^0(\mathrm{Aut}(V);\mathbb{D}_n)=\mathbb{B}_n^0$.*

Proof. In general, we let S be an integrally closed domain, and let T be an integral closure in some finite Galois extension of the field of fractions of S with Galois group G. We denote by $Q(S)$ and $Q(T)$ their fields of fractions, respectively. Then G acts on T, and the invariant subring $H^0(G;T)$ is a subring of $H^0(G;Q(T))=Q(S)$ consisting of integral elements over S. Hence we see that $H^0(G;T)=S$. Note that a regular local ring is an integrally closed domain. Then the proposition follows from Lemma 6.1 and Proposition 6.3. □

7. Level structures on $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$

Let $\mathcal{C}(\mathbb{B})$ be the category of complete local commutative $\mathbb{B}_n^0$-algebras and continuous $\mathbb{B}_n^0$-algebra homomorphisms. Suppose we have a p-divisible group $G(\infty)=\{G(r),i(r)\}_{r\geq 0}$ of height h over $\mathbb{B}_n^0$. For $S\in\mathcal{C}(\mathbb{B})$, there is a p-divisible group $G(\infty)_S$ over S obtained by base change. Let A be an abelian group isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^h$. For $r\geq 0$, we set $A(r)=\ker(p^r:A\to A)$, the kernel of the p^r-times map on A. A level $A(r)$-structure on $G(\infty)_S$ is a homomorphism $\varphi: A(r)\to G(r)(S)$ such that the set of $\varphi(a)$ for $a\in A(r)$ forms a full set of sections of $G(r)_S$ in the sense of [21, 1.8]. Note that this definition of level structures on p-divisible groups is consistent with the definition of level structures on the formal group law in §4 by [11, Corollary II.2.3]. We let $\mathrm{Level}(A;G(\infty))(r)(S)$ be the set of all level $A(r)$-structures on $G(\infty)_S$. Then $\mathrm{Level}(A;G(\infty))(r)$ defines a functor from $\mathcal{C}(\mathbb{B})$ to the category of sets

$$\mathrm{Level}(A;G(\infty))(r):\mathcal{C}(\mathbb{B})\longrightarrow \mathbf{Sets}.$$

In this section we study level structures on the p-divisible group $\mathbf{F}_{n+1}(\infty)_{\mathbb{B}}$ and the representing ring of level structures.

By (6.1), we have an exact sequence of abelian groups $0\to V(r)\to W(r)\to U(r)\to 0$. For $S\in\mathcal{C}(\mathbb{B})$, by taking the sets of S-valued points on the exact sequence of p-divisible groups in Theorem 5.3, we obtain an exact sequence of abelian groups

$$0\longrightarrow \mathbf{F}_n(r)(S)\longrightarrow \mathbf{F}_{n+1}(r)(S)\longrightarrow U(r).$$

Let $\varphi: W(r)\to \mathbf{F}_{n+1}(r)(S)$ be a homomorphism of abelian groups such that $\varphi(V(r))$ lies in the image of $\mathbf{F}_n(r)(S)$. Then we obtain a homomorphism $\varphi_U: U(r)\to U(r)$. If φ is a level $W(r)$-structure on $\mathbf{F}_{n+1}(\infty)_S$, then φ_U is an isomorphism by [21, Proposition 1.11.2]. We define a set $\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)(S)$ to be the subset of $\mathrm{Level}(W;\mathbf{F}_{n+1}(\infty))(r)(S)$ consisting of φ such that $\varphi(V(r))$ lies in the image of $\mathbf{F}_n(r)(S)$:

$$\begin{aligned}&\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)(S)\\ &\quad=\{\varphi\in\mathrm{Level}(W;\mathbf{F}_{n+1}(\infty))(r)(S)\mid \varphi(V(r))\subset\mathbf{F}_n(r)(S)\}.\end{aligned}$$

Then $\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)$ defines a subfunctor of $\mathrm{Level}(W,\mathbf{F}_{n+1}(\infty))(r)$ from $\mathcal{C}(\mathbb{B})$ to the category of sets

$$\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r):\mathcal{C}(\mathbb{B})\longrightarrow\mathbf{Sets}.$$

Then there is a natural transformation of functors on $\mathcal{C}(\mathbb{B})$:

$$\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)\longrightarrow\mathrm{Level}(W,\mathbf{F}_{n+1}(\infty))(r). \tag{7.1}$$

For $\varphi \in \mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)(S)$, by restriction to $V(r)$, we obtain a homomorphism $\varphi_V : V(r) \to \mathbf{F}_n(r)(S)$ of abelian groups. Then we see that $\varphi_V \in \mathrm{Level}(V; \mathbf{F}_n(\infty))(r)(S)$ by [21, Proposition 1.11.2]. Hence we obtain a natural transformation of functors on $\mathcal{C}(\mathbb{B})$:

$$\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r) \longrightarrow \mathrm{Level}(V; \mathbf{F}_n(\infty))(r). \tag{7.2}$$

We let $A(W, V)(r)$ be the subgroup of $\mathrm{Aut}(W(r))$ consisting of g such that $g(V(r)) = V(r)$:

$$A(W, V)(r) = \{g \in \mathrm{Aut}(W(r)) |\ g(V(r)) = V(r)\}.$$

We abbreviate $A(W, V)(\infty)$ to $A(W, V)$. Note that $B(W, V)(r)$ is a normal subgroup of $A(W, V)(r)$ and $A(W, V)(r)/B(W, V)(r) \cong \mathrm{Aut}(U(r))$. The group $A(W, V)(r)$ acts on the exact sequence (6.1). This action induces an action of $A(W, V)(r)$ on the functor $\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)$. Then the natural transformation (7.1) is $A(W, V)(r)$-equivariant, where $A(W, V)(r)$ acts on the functor $\mathrm{Level}(W, \mathbf{F}_{n+1}(\infty))(r)$ through the inclusion $A(W, V)(r) \hookrightarrow \mathrm{Aut}(W(r))$. By restriction to $V(r)$, we have a homomorphism $A(W, V)(r) \to \mathrm{Aut}(V(r))$. Through this homomorphism, we can consider that $A(W, V)(r)$ acts on the functor $\mathrm{Level}(V; \mathbf{F}_n(\infty))(r)$. Then the natural transformation (7.2) is also $A(W, V)(r)$-equivariant.

We define $\mathrm{Level}(W, V, \mathbf{F}_{n+1}(\infty))(r)_\chi$ to be the subfunctor of $\mathrm{Level}(W, V, \mathbf{F}_{n+1}(\infty))(r)$ which consists of $\varphi \in \mathrm{Level}(W, V, \mathbf{F}_{n+1}(\infty))(r)(S)$ such that $\varphi_U = \chi$ for $\chi \in \mathrm{Aut}(U(r))$. Then we have a decomposition of $\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)$ as

$$\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r) = \coprod_{\chi \in \mathrm{Aut}(U(r))} \mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)_\chi.$$

There is a canonical homomorphism $A(W, V)(r) \to \mathrm{Aut}(U(r))$ and the action $A(W, V)(r)$ permutes the components of $\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)$ through $A(W, V)(r) \to \mathrm{Aut}(U(r))$. In particular, all components $\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)_\chi$ are equivalent to each other.

Let $U(r)^\times$ be the subset of $U(r)$ consisting of generators of $U(r)$. We define a commutative $\mathbb{B}_n^0$-algebra $\mathbb{I}_n(r)$ by

$$\mathbb{I}_n(r) = \prod_{\alpha \in U(r)^\times} \mathbb{D}_n(r)[X]/(g_{r,\alpha}(X)).$$

Then $\mathbb{I}_n(r)$ is a complete Noetherian regular semi-local ring of equidimension n.

Proposition 7.1. *The functor* $\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)$ *is represented by* $\mathbb{I}_n(r)$.

Proof. Let Z be the component of $\mathrm{Level}(W, V; \mathbf{F}_{n+1}(\infty))(r)$ corresponding to the identity map in $\mathrm{Aut}(U(r))$. We fix $\alpha \in U(r)^\times$ and put $P = \mathbb{B}_n(r)[X]/(g_r(X))$ and $Q = \mathbb{B}_n(r)[X]/(g_{r,\alpha}(X))$. Then it is sufficient to show that Z is represented by the commutative $\mathbb{B}_n^0$-algebra $\mathbb{D}_n(r) \otimes_{\mathbb{B}_n^0} Q$. We fix a lifting $\alpha' \in W(r)$ of $\alpha \in U(r)$. For $S \in \mathcal{C}(\mathbb{B})$, a morphism $\mathbb{D}_n(r) \otimes_{\mathbb{B}_n^0} Q \to S$ in $\mathcal{C}(\mathbb{B})$ induces morphisms $\mathbb{D}_n(r) \to S$ and $Q \to S$ in $\mathcal{C}(\mathbb{B})$. A homomorphism $\mathbb{D}_n(r) \to S$ gives a level $V(r)$-structure on $\mathbf{F}_n(r)_S$, which induces a homomorphism $\psi : V(r) \to \mathbf{F}_n(r)(S) \to \mathbf{F}_{n+1}(r)(S)$. A homomorphism $Q \to S$ gives a homomorphism $P \to Q \to S$, which corresponds to an S-valued point $a \in \mathbf{F}_{n+1}(r)(S)$. Then we can extend a homomorphism ψ to a homomorphism $\varphi : W(r) \to \mathbf{F}_{n+1}(r)(S)$ by sending α' to $a \in \mathbf{F}_{n+1}(r)(S)$. Then φ is a level $W(r)$-structure on $\mathbf{F}_{n+1}(r)_S$ by [21, Proposition 1.11.2]. Since φ_U is the identity, we obtain $\varphi \in Z(S)$. This gives a natural bijection between the set of morphisms $\mathbb{D}_n(r) \otimes_{\mathbb{B}_n} Q \to S$ in $\mathcal{C}(\mathbb{B})$ and $Z(S)$. This completes the proof. $\square$

By Proposition 7.1, we see that the natural transformation (7.1) is represented by a continuous $\mathbb{B}_n^0$-algebra homomorphism $\mathbb{B}_n^0 \otimes_{E_{n+1}^0} D_{n+1}(r) \to \mathbb{I}_n(r)$. Hence we obtain a ring homomorphism $\widehat{ch}(r) : D_{n+1}(r) \longrightarrow \mathbb{I}_n(r)$, which covers the ring homomorphism $E_{n+1}^0 \to \mathbb{B}_n^0$ induced by ch. Since the natural transformation (7.1) is $A(W,V)(r)$-equivariant, the ring homomorphism $\widehat{ch}(r)$ is $A(W,V)(r)$-equivariant. There is a natural transformation $\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r+1) \to \mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)$ of functors by restriction. This induces a continuous $\mathbb{B}_n^0$-algebra homomorphism $\mathbb{I}_n(r) \to \mathbb{I}_n(r+1)$. We define a commutative $\mathbb{B}_n^0$-algebra $\mathbb{I}_n = \mathbb{I}_n(\infty)$ to be the direct limit of $\mathbb{I}_n(r)$. Then we obtain an $A(W,V)$-equivariant ring homomorphism

$$\widehat{ch} : D_{n+1} \longrightarrow \mathbb{I}_n,$$

which covers the ring homomorphism $E_{n+1}^0 \to \mathbb{B}_n^0$ induced by ch.

The natural transformation (7.2) corresponds to the canonical $\mathbb{B}_n^0$-algebra homomorphism $\widehat{i}(r) : \mathbb{D}_n(r) \longrightarrow \mathbb{I}_n(r)$. Since $\widehat{i}(r)$ is compatible with r, we obtain a $\mathbb{B}_n^0$-algebra homomorphism

$$\widehat{i} : \mathbb{D}_n \to \mathbb{I}_n$$

by taking the direct limit on r. Then we can regard $\mathbb{D}_n$ as a subring of $\mathbb{I}_n$ through $\widehat{i}$. We define $P(W,V)(r)$ to be the kernel of the canonical homomorphism $A(W,V)(r) \to \mathrm{Aut}(V(r))$, and we abbreviate $P(W,V)(\infty)$ to $P(W,V)$. Then $Q(W,V)(r)$ is a normal subgroup of $P(W,V)(r)$, and $P(W,V)(r)/Q(W,V)(r)$ is identified with $\mathrm{Aut}(U(r))$.

Lemma 7.2. $H^0(P(W,V);\mathbb{I}_n) = \mathbb{D}_n$.

Proof. It is sufficient to show that $H^0(P(W,V)(r);\mathbb{I}_n(r)) = \mathbb{D}_n(r)$ for all r. For $\alpha \in U(r)^\times$, we let $\mathbb{I}_n(r)_\alpha$ be the commutative $\mathbb{B}_n^0$-algebra $\mathbb{D}_n[X]/(g_{r,\alpha}(X))$. Then there is an isomorphism $\mathbb{I}_n(r)_\alpha \cong \mathbb{J}_n(r)$ for any $\alpha \in U(r)^\times$. We have $H^0(Q(W,V)(r);\mathbb{I}_n(r)_\alpha) = \mathbb{D}_n(r)$ by Proposition 6.4. Hence $H^0(Q(W,V)(r);\mathbb{I}_n(r)) = \prod_\alpha \mathbb{D}_n(r)$, where the product is taken over $\alpha \in \mathrm{Aut}(U(r))$. Then the lemma follows from the fact that the action of $\mathrm{Aut}(U(r))$ permutes the components freely and transitively. $\square$

Corollary 7.3. $H^0(A(W,V);\mathbb{I}_n) = \mathbb{B}_n^0$.

There is an action of $\mathbb{G}$ on the exact sequence of p-divisible groups in Theorem 5.3 which covers the action on $\mathbb{B}_n^0$. Hence we have an isomorphism of exact sequences for $g \in \mathbb{G}$:

$$\begin{array}{ccccccccc}
0 & \to & \mathbf{F}_n(\infty)_{\mathbb{B}} & \longrightarrow & \mathbf{F}_{n+1}(\infty)_{\mathbb{B}} & \longrightarrow & U(\infty)_{\mathbb{B}} & \to & 0 \\
 & & \downarrow{\scriptstyle \rho_V(g)} & & \downarrow{\scriptstyle \rho_W(g)} & & \downarrow{\scriptstyle \rho_U(g)} & & \\
0 & \to & \mathbf{F}_n(\infty)_{\mathbb{B}} & \longrightarrow & \mathbf{F}_{n+1}(\infty)_{\mathbb{B}} & \longrightarrow & U(\infty)_{\mathbb{B}} & \to & 0.
\end{array}$$

For $\varphi \in \mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)(S)$ and $h \in A(W,V)(r)$, we see that

$$\rho_W(g)(S) \circ \varphi \circ h \in \mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)(S).$$

This gives an action of $\mathbb{G} \times A(W,V)(r)$ on the functor $\mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r)$. Then the following diagram commutes for all $(g,h) \in \mathbb{G} \times A(W,V)(r)$:

$$\begin{array}{ccccc}
\mathrm{Level}(W;\mathbf{F}_{n+1}(\infty))(r) & \leftarrow & \mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r) & \rightarrow & \mathrm{Level}(V;\mathbf{F}_{n}(\infty))(r) \\
\big\downarrow{\scriptstyle (g,h)} & & \big\downarrow{\scriptstyle (g,h)} & & \big\downarrow{\scriptstyle (g,h)} \\
\mathrm{Level}(W;\mathbf{F}_{n+1}(\infty))(r) & \leftarrow & \mathrm{Level}(W,V;\mathbf{F}_{n+1}(\infty))(r) & \rightarrow & \mathrm{Level}(V;\mathbf{F}_{n}(\infty))(r).
\end{array}$$

This implies the following commutative diagram of representing rings for all $(g,h) \in \mathbb{G} \times A(W,V)(r)$:

$$\begin{array}{ccccc}
D_{n+1}(r) & \xrightarrow{\widehat{ch}(r)} & \mathbb{I}_n(r) & \xleftarrow{\widehat{i}(r)} & \mathbb{D}_n(r) \\
\big\downarrow{\scriptstyle (g,h)} & & \big\downarrow{\scriptstyle (g,h)} & & \big\downarrow{\scriptstyle (g,h)} \\
D_{n+1}(r) & \xrightarrow{\widehat{ch}(r)} & \mathbb{I}_n(r) & \xleftarrow{\widehat{i}(r)} & \mathbb{D}_n(r).
\end{array}$$

Hence we obtain the following theorem.

Theorem 7.4. *The group $\mathbb{G} \times A(W,V)$ acts on the ring $\mathbb{I}_n$. The ring homomorphisms $\widehat{ch}$ and $\widehat{i}$ are $\mathbb{G} \times A(W,V)$-equivariant.*

8. Morphism between generalized group characters

In this section we construct a ring homomorphism from the ring of invariant n-dimensional generalized group characters to the ring of invariant $(n+1)$-dimensional generalized group characters. Then we show that this ring homomorphism is a model of the generalized Chern character for classifying spaces of finite groups up to torsion. Furthermore, we show that the ring homomorphism is equivariant under the action of $\mathbb{G}$. This corresponds to the fact that the generalized Chern character respects the action of $\mathbb{G}$.

By Theorem 7.4, we have the $A(W,V)$-equivariant ring homomorphism $\widehat{ch}$: $D_{n+1} \longrightarrow \mathbb{I}_n$. Since $A(W,V)$ is a subgroup of $\mathrm{Aut}(W)$ consisting of $g \in \mathrm{Aut}(W)$ that preserves the subgroup V, $A(W,V)$ acts on $\mathrm{Rep}(V^*,G)$ through the projection $A(W,V) \to \mathrm{Aut}(V)$. Then we obtain a ring homomorphism

(8.1)
$$H^0(\mathrm{Aut}(W);\mathrm{Map}(\mathrm{Rep}(W^*,G),D_{n+1})) \longrightarrow H^0(A(W,V);\mathrm{Map}(\mathrm{Rep}(V^*,G),\mathbb{I}_n)).$$

Since $P(W,V)$ is the kernel of the projection $A(W,V) \to \mathrm{Aut}(V)$, the action of $P(W,V)$ on $\mathrm{Rep}(V^*,G)$ is trivial. By Corollary 7.3, we have $H^0(P(W,V);\mathbb{I}_n) = \mathbb{D}_n$. Hence we obtain an isomorphism of commutative rings:

$$H^0(A(W,V);\mathrm{Map}(\mathrm{Rep}(V^*,G),\mathbb{I}_n)) \cong H^0(\mathrm{Aut}(V);\mathrm{Map}(\mathrm{Rep}(V^*,G),\mathbb{D}_n)). \tag{8.2}$$

We define commutative rings $\mathrm{HKR}(G;E_{n+1})$ and $\mathrm{HKR}(W,G;\mathbb{B}_n)$ by

$$\begin{aligned}
\mathrm{HKR}(G;\mathbb{B}_n) &= H^0(\mathrm{Aut}(V);\mathrm{Map}(\mathrm{Rep}(V^*,G),\mathbb{D}_n)), \\
\mathrm{HKR}(G;E_{n+1}) &= H^0(\mathrm{Aut}(W);\mathrm{Map}(\mathrm{Rep}(W^*,G),D_{n+1})).
\end{aligned}$$

By the ring homomorphism (8.1) and the isomorphism (8.2), we obtain a ring homomorphism from the ring of invariant n-dimensional generalized group characters to the ring of invariant $(n+1)$-dimensional generalized group characters

$$ch\mathrm{HKR}(G) : \mathrm{HKR}(G, E_{n+1}) \longrightarrow \mathrm{HKR}(G, \mathbb{B}_n).$$

By Theorem 7.4, the ring homomorphism $\widehat{ch}$ is $\mathbb{G} \times A(W,V)$-equivariant. This implies that $ch\mathrm{HKR}(G)$ is $\mathbb{G}$-equivariant. Hence we obtain the following theorem.

Theorem 8.1. *For any finite group G, the following diagram commutes:*

$$\begin{tikzcd} E^0_{n+1}(BG) \arrow[r, "ch(BG)"] \arrow[d] & \mathbb{B}^0_n(BG) \arrow[d] \\ \mathrm{HKR}(G; E_{n+1}) \arrow[r, "ch\mathrm{HKR}(G)"] & \mathrm{HKR}(G; \mathbb{B}_n), \end{tikzcd}$$

where the vertical arrows are isomorphisms after tensoring with $\mathbb{Q}$. Furthermore, the profinite group $\mathbb{G}$ acts on this commutative diagram.

We fix an isomorphism $W \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{n+1}$ and regard V as the direct summand consisting of elements of the form $(x_1, \ldots, x_n, 0)$. Then $\mathrm{Aut}(W) = \mathrm{GL}_{n+1}(\mathbb{Z}_p)$. Since $W^* = (\mathbb{Z}_p)^{n+1}$, we can consider that $\mathrm{Rep}(W^*, G) = \mathrm{Rep}((\mathbb{Z}_p)^{n+1}, G)$ is the set of all conjugacy classes of $(n+1)$-tuples $(g_1, \ldots, g_{n+1})$ of elements in G such that g_i has a p-power order and $[g_i, g_j] = e$. Then $\mathrm{Rep}(V^*, G) = \mathrm{Rep}((\mathbb{Z}_p)^n, G)$ is the subset of $\mathrm{Rep}((\mathbb{Z}_p)^{n+1}, G)$ consisting of conjugacy classes of the form $(g_1, \ldots, g_n, e)$. Let $f \in \mathrm{HKR}(G; E_{n+1})$. Then f is a $\mathrm{GL}_{n+1}(\mathbb{Z}_p)$-equivariant function on $\mathrm{Rep}((\mathbb{Z}_p)^{n+1}, G)$ with values in D_{n+1}. The composition

$$\mathrm{Rep}((\mathbb{Z}_p)^n, G) \hookrightarrow \mathrm{Rep}((\mathbb{Z}_p)^{n+1}, G) \xrightarrow{f} D_{n+1} \xrightarrow{\widehat{ch}} \mathbb{I}_n$$

factors through the subring $\mathbb{D}_n$ of $\mathbb{I}_n$, which is $\mathrm{GL}_n(\mathbb{Z}_p)$-equivariant. This gives a description of $ch\mathrm{HKR}(G)(f)$ in $\mathrm{HKR}(G; \mathbb{D}_n)$.

9. The extended power spectrum DS^0

In [28], Strickland and Turner gave a description of the rational E_n-cohomology of the extended power spectrum DS^0. In this section, as an application of Theorem 8.1, we describe the rational generalized Chern character $ch\widehat{\otimes}\mathbb{Q}$ of DS^0 in terms of generalized group characters.

For a group G, we denote by $\mathbf{A}(G)$ the set of all isomorphism classes of finite G-sets. There are two binary operations on $\mathbf{A}(G)$: a sum by the disjoint union $[X] * [Y] = [X \coprod Y]$, and a product by the Cartesian product $[X] \circ [Y] = [X \times Y]$. These operations define a commutative semiring structure on $\mathbf{A}(G)$. Hence we obtain a contravariant functor $\mathbf{A}(-)$ from the category of groups to the category of semirings. For a commutative ring S, we let $S[\mathbf{A}(G)]$ be the free S-module generated by $\mathbf{A}(G)$. Then $S[\mathbf{A}(G)]$ has a natural structure of semiring-ring. Furthermore, we have two coproducts $\psi_\times$ and $\psi_\bullet$ on $S[\mathbf{A}(G)]$ defined by $\psi_\times[X] = \sum_{X = Y \coprod Z} [Y] \otimes [Z]$ and $\psi_\bullet[X] = [X] \otimes [X]$. Note that the unit η_* of $*$ is given by $[\emptyset]$, and the unit $\eta_\circ$ of $\circ$ is given by the one point set $[1]$. The counit $\varepsilon_\times$ of $\psi_\times$ is given by $\varepsilon_\times[X] = 1$ if $[X] = [\emptyset]$, and 0 otherwise. The counit $\varepsilon_\bullet$ of $\psi_\bullet$ is given by $\varepsilon_\bullet[X] = 1$ for all $[X]$.

There is a dual object of $S[\mathbf{A}(G)]$. We let $\mathrm{Map}(\mathbf{A}(G), S)$ be the set of all functions from $\mathbf{A}(G)$ to S. We regard S as a discrete topological ring and give a topology on $\mathrm{Map}(\mathbf{A}(G), S)$ by the product topology. Then there are duality isomorphisms $\mathrm{Hom}_S(S[\mathbf{A}(G)], S) \cong \mathrm{Map}(\mathbf{A}(G), S)$ and $\mathrm{Hom}^c_S(\mathrm{Map}(\mathbf{A}(G), S), S) \cong S[\mathbf{A}(G)]$, where $\mathrm{Hom}^c_S(-,-)$ stands for the set of all continuous S-module homomorphisms.

By these duality isomorphisms, we have two products $\times$, $\bullet$ and two coproducts ψ_*, $\psi_\circ$ on $\mathrm{Map}(\mathbf{A}(G), S)$. In summary, for $f, g \in \mathrm{Map}(\mathbf{A}(G), S)$ we have

$$\begin{aligned}(f \times g)(X) &= \textstyle\sum_{X=Y\coprod Z} f(Y)g(Z),\\ (f \bullet g)(X) &= f(X)g(X),\\ (\psi_* f)(X,Y) &= f(X \coprod Y),\\ (\psi_\circ f)(X,Y) &= f(X \times Y).\end{aligned}$$

Let $\mathcal{S}_m$ be the symmetric group of m letters, and let $B\mathcal{S}_m$ be its classifying space. We define a spectrum DS^0 by

$$DS^0 = \bigvee_{m\geq 0} \Sigma^\infty B\mathcal{S}_{m+},$$

where X_+ stands for a space X with disjoint base point $+$. Then $E_n^0(DS^0) = \prod_{m\geq 0} E_n^0(B\mathcal{S}_m)$. The extended power spectrum DS^0 has a rich structure, which in particular induces two products and two coproducts on $E_n^0(DS^0)$. We set

$$\mathrm{HKR}(DS^0; E_n \otimes \mathbb{Q}) = \prod_{m\geq 0} (\mathrm{HKR}(\mathcal{S}_m; E_n) \otimes \mathbb{Q}).$$

There is a subfunctor $\mathbf{A}(-)_m$ of $\mathbf{A}(-)$ which consists of $[X] \in \mathbf{A}(G)$ with $|X| = m$. Then we see that $\mathrm{Rep}(G, \mathcal{S}_m) \cong \mathbf{A}(G)_m$. Hence we obtain an isomorphism

$$\mathrm{HKR}(DS^0; E_n \otimes \mathbb{Q}) \cong H^0(\mathrm{Aut}(V); \mathrm{Map}(\mathbf{A}(V^*), D_n \otimes \mathbb{Q}).$$

Then the operations $\times$, $\bullet$, ψ_* and $\psi_\circ$ on $\mathrm{Map}(\mathbf{A}(V)^*, D_n \otimes \mathbb{Q})$ descend to operations on the invariant subring $\mathrm{HKR}(DS^0; E_n \otimes \mathbb{Q})$, and there is an isomorphism between $E_n^0(DS^0)\widehat{\otimes}\mathbb{Q} = \prod_{m\geq 0}(E_n^0(B\mathcal{S}_m) \otimes \mathbb{Q})$ and $\mathrm{HKR}(DS^0; E_n \otimes \mathbb{Q})$ which respects operations by [28, Theorem 4.2].

The inclusion $V \hookrightarrow W$ induces a homomorphism of semirings $\mathbf{A}(V^*) \to \mathbf{A}(W^*)$. By the ring homomorphism $\widehat{ch} : D_{n+1} \longrightarrow \mathbb{I}_n$, we obtain a map

$$ch\mathrm{HKR}(DS^0) : \mathrm{HKR}(DS^0; E_{n+1} \otimes \mathbb{Q}) \longrightarrow \mathrm{HKR}(DS^0; \mathbb{B}_n \otimes \mathbb{Q}),$$

which covers the ring homomorphism $E_{n+1}^0 \otimes \mathbb{Q} \to \mathbb{B}_n \otimes \mathbb{Q}$ induced by ch and respects two products $\times$, $\bullet$ and two coproducts ψ_*, $\psi_\circ$. By Theorem 8.1, we obtain the following theorem.

Theorem 9.1. *The following diagram commutes:*

$$\begin{array}{ccc} E_{n+1}(DS^0) & \xrightarrow{ch(DS^0)} & \mathbb{B}_n(DS^0) \\ \downarrow & & \downarrow \\ \mathrm{HKR}(DS^0; E_{n+1} \otimes \mathbb{Q}) & \xrightarrow{ch\mathrm{HKR}(DS^0)} & \mathrm{HKR}(DS^0; \mathbb{B}_n \otimes \mathbb{Q}). \end{array}$$

The vertical arrows are isomorphisms after we take the complete tensor product with $\mathbb{Q}$. Furthermore, the profinite group $\mathbb{G}$ acts on this commutative diagram.

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