SPLITTING OF GAUGE GROUPS

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Abstract. Let $G$ be a topological group and let $P$ be a principal $G$-bundle over a based space $B$. We denote the gauge group of $P$ by $\mathcal{G}(P)$ and the based gauge group of $P$ by $\mathcal{G}_0(P)$. Then the inclusion of the basepoint of $B$ induces the exact sequence of topological groups $1 \to \mathcal{G}_0(P) \to \mathcal{G}(P) \to G \to 1$. We study the splitting of this exact sequence in the category of $A_n$-spaces and $A_n$-maps in connection with the triviality of the adjoint bundle of $P$ and with the higher homotopy commutativity of $G$.

1. Introduction

We will always assume each space has the homotopy type of a CW-complex. Let $G$ be a topological group and let $P$ be a principal $G$-bundle over a space $B$. The gauge group of $P$, denoted $\mathcal{G}(P)$, is the group of automorphisms of $P$ covering the identity of $B$.

Fix a basepoint $b_0$ of $B$. Then the basepoint inclusion $b_0 \hookrightarrow B$ induces a homomorphism of topological groups

$$\mathcal{G}(P) \to \mathcal{G}(P|_{b_0}) \cong G.$$ 

Since we work with CW-complexes which are normal, this homomorphism is easily seen to be a surjection. We call the kernel of this homomorphism the based gauge group of $P$ and denote it by $\mathcal{G}_0(P)$. Namely, $\mathcal{G}_0(P)$ consists of automorphisms of $P$ covering $1_B$ which restrict to the identity on the fibre at the basepoint $b_0$. Now we have an extension of topological groups:

$$1 \to \mathcal{G}_0(P) \xrightarrow{\iota} \mathcal{G}(P) \xrightarrow{\pi} G \to 1. \tag{1.1}$$

The second author [14] classified the homotopy types of $\mathcal{G}(P)$ as spaces, not as topological groups, when $P$ runs all over principal SU(2)-bundles over $S^4$. Later, Crabb and Sutherland [4] studied the homotopy type of $\mathcal{G}(P)$ as $H$-spaces for a general $P$. Moreover, when $B$ is a simply connected 4-manifold and $G = \text{SU}(2)$, Tsukuda and the second author [15], [24] classified the homotopy types of the classifying spaces $B\mathcal{G}(P)$, equivalently, the homotopy types of $\mathcal{G}(P)$ as loop spaces. These results suggest to us to study the homotopy theory of gauge groups as spaces with intermediate higher homotopy associativity in the sense of Stasheff [20], that is, as $A_n$-spaces. In particular, we may study the group extension $\mathcal{G}_0(P) \to \mathcal{G}(P) \to G \to 1$ in the

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category of $A_n$-spaces and $A_n$-maps. The aim of this article is to study a splitting of (1.1) in the category of $A_n$-spaces and $A_n$-maps which we call an $A_n$-splitting. More precisely, we will consider:

**Question 1.1.** (1) Formulate an $A_n$-splitting.

(2) What is the geometric meaning of an $A_n$-splitting of (1.1)?

(3) Give a criterion for an $A_n$-splitting of (1.1).

Regarding the first question, we consider a relation between an $A_n$-splitting of (1.1) and the bundle $P$. Let $\text{ad}: G \to \text{aut}G$ be the adjoint action of $G$ on itself and let $\text{ad}P = P \times_{\text{ad}} G$, the adjoint bundle of $P$. Introducing the fibrewise analogue of $A_n$-maps between topological monoids, we obtain:

**Theorem 1.2.** There is an $A_n$-splitting of (1.1) if and only if $\text{ad}P$ is fibrewise $A_n$-equivalent to the trivial bundle $B \times G$.

Let $\text{map}(X,Y; f)$ be the path component of the space of maps from $X$ to $Y$ containing $f$, where we will always take $f$ to be basepoint preserving. Denote the universal $G$-bundle by $EG \to BG$. Regarding the second question, we will be concerned with the classical result of Atiyah and Bott [2]:

$$ (1.2) \quad BG(P) \simeq \text{map}(B, BG; \alpha), $$

where $\alpha$ is the classifying map of $P$. Naturality of this homotopy equivalence allows us to identify the map $B\pi : B\gamma(P) \to BG$ with the evaluation fibration $\text{map}(B, BG; \alpha) \to BG$. This leads us to the definition of $H(k,l)$-spaces having the following property.

**Theorem 1.3.** There is an $A_1$-splitting of (1.1) if $BG$ is an $H(k,l)$-space and $\text{cat} B \leq k$.

As above, an $H(k,l)$-space is motivated by the evaluation fibration $\text{map}(B, BG; \alpha) \to BG$ and, in particular, an $H(1,n)$-space can be described by the connecting map $\delta : G \to \text{map}_0(B, BG; \alpha)$ in the fibre sequence $G \to \text{map}_0(B, BG; \alpha) \to \text{map}(B, BG; \alpha) \to BG$, where $\text{map}_0(X,Y; f)$ is the subspace of $\text{map}(X,Y; f)$ consisting of based maps. Note that the adjoint action $\text{ad} : G \to \text{aut}(G)$ induces a map $\text{Bad} : G \to \text{map}_0(BG, BG; 1)$ which assigns each $g \in G$ to the map $\text{Bad}(g) : BG \to BG$. Here we must notice that $\text{Bad}$ does not mean the map $BG \to \text{Baut}(G)$ induced from the adjoint action $\text{ad} : G \to \text{aut}(G)$. Then we obtain:

**Theorem 1.4.** The connecting map $\delta : G \to \text{map}_0(B, BG; \alpha)$ is given by $\delta(g) = \text{Bad}(g) \circ \alpha$ for $g \in G$.

Let $E_nG \to B_nG$ be the $n$-th stage of Milnor's construction of the universal bundle $EG \to BG$ [18]. By definition, $BG$ is an $H(1,n)$-space if and only if the connecting map $\delta$ in Theorem 1.4 is trivial for the inclusion $i_n : B_nG \to BG$. Then we have:

**Corollary 1.5.** $BG$ is an $H(1,n)$-space if and only if $\text{Bad} \circ i_n : G \to \text{map}_0(B_nG, BG; i_n)$ is null-homotopic.

We will investigate an $H(k,l)$-space further in view of higher homotopy commutativity as follows. By definition, the loop space of an $H(1,1)$-space is homotopy
commutative and an $H(\infty, \infty)$-space is an $H$-space. On the other hand, Sugawara [23] constructed a class of spaces between homotopy commutative topological monoids and the loop spaces of $H$-spaces, called higher homotopy commutativity. Then we expect that the loop spaces of $H(k,l)$-spaces form a new class of higher homotopy commutativity. Kawamoto and Hemmi [12] introduced $H_k(n)$-spaces to unify Aguadé’s $T_k$-spaces [1] and Félix and Tanré’s $H(n)$-spaces [6]. They also introduced higher homotopy commutativity called $C_k(n)$-spaces to describe $H_k(n)$-spaces by higher homotopy. Moreover, in describing an $H_k(n)$-space by a $C_k(n)$-space, they worked at the level of $H(i,j)$-spaces. This leads us to define a new class of higher homotopy commutativity, $C(k,l)$-spaces, by cutting $C_k(n)$-spaces into pieces and obtain:

**Theorem 1.6.** A connected topological monoid is a $C(k,l)$-space if and only if its classifying space is an $H(k,l)$-space.

Combining Theorem 1.2, Theorem 1.3 and Theorem 1.6, we can conclude:

**Corollary 1.7.** Let $G$ be a connected topological group and let $P$ be a principal $G$-bundle over $B$. If $G$ is a $C(k,l)$-space, then there is an $A_n$-splitting of $G$; equivalently, $adP$ is fibrewise $A_n$-homotopy equivalent to the trivial bundle $B \times G$.

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2. $A_n$-SPLITTING

In this section, we formulate a splitting of an extension of topological groups in the category of $A_n$-spaces and $A_n$-maps which we call an $A_n$-splitting. An $A_n$-space was introduced by Stasheff [20] to be a space with a multiplication which enjoys a certain higher homotopy associativity. Then an $A_n$-map should be a map between $A_n$-spaces preserving their $A_n$-space structures. Stasheff [21] defined an $A_n$-map between $A_\infty$-spaces. Later, he [22] defined an $A_n$-map from an $A_n$-space to an $A_\infty$-space and implied an $A_n$-map between $A_n$-spaces. Finally, Iwase and Mimura [13] described an $A_n$-map between $A_n$-spaces completely. Of course, these definitions of $A_n$-maps are consistent, and we will use them conveniently case by case.

An $A_n$-splitting of an extension of topological groups should be analogous to a splitting in the category of topological groups and their homomorphisms. However, the existence of a section is not enough for an $A_n$-splitting since it does not imply directly the “splitting” of an $A_n$-space. Then we define an $A_n$-splitting of an extension of topological groups as follows.

**Definition 2.1.** An $A_n$-splitting of an extension of topological groups $1 \to K \to \tilde{H} \to H \to 1$ consists of the following:

1. There is an $A_n$-structure on $H \times K$, the direct product as spaces, not as topological groups, which restricts to the canonical group structures on $H \times \{1\}$ and $\{1\} \times K$. 

(2) There is an $A_n$-map $\theta : H \times K \to \tilde{H}$ with respect to the above $A_n$-structure on $H \times K$ satisfying the homotopy commutative diagram:

\[
\begin{array}{cccccc}
1 & \to & K & \to & \tilde{H} & \to & H & \to & 1 \\
1 & \downarrow & \downarrow & \downarrow & \theta & \downarrow & \downarrow & \downarrow & 1 \\
1 & \to & K & \to & H \times K & \to & H & \to & 1,
\end{array}
\]

where $\pi$ is the second projection.

Let $1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1$ be an extension of topological groups. A splitting of this extension as groups can be completely described by a section of $\pi$ which is a group homomorphism. We shall show that there is an analogy for an $A_n$-splitting. Namely, a homotopy section of $\pi$ which is an $A_n$-map, called an $A_n$-section, implies an $A_n$-splitting of the extension, where the homotopy section of $\pi$ is a map $s : H \to \tilde{H}$ such that $\pi \circ s \simeq 1_H$.

Let us first recall Stasheff’s polytope, the associahedron, which was used to define an $A_n$-space and an $A_n$-map from an $A_n$-space to an $A_\infty$-space (see [20] and [22]). The $i$-th associahedron $K_i$ is an $(i-2)$-dimensional convex polytope having the face maps

$$\partial_k(r,s) : K_r \times K_s \to K_i$$

for $r + s = i + 1$ and $1 \leq k \leq i - s + 1$ and the degeneracy maps

$$s_j : K_i \to K_{i-1}$$

for $1 \leq j \leq i$. In particular, we have the relations:

$$s_j \circ \partial_k(r,s) = \begin{cases} 
\partial_k(r,s-1) \circ (1 \times s_{j-k+1}) & k \leq j < k + s, \\
\partial_k(r-1,s) \circ (s_{j-s+1} \times 1) & j \geq k + s.
\end{cases} \quad (2.1)$$

There is a one-to-one correspondence between vertices of $K_i$ and connected binary trees with $n$ leaves. In order to define an $A_n$-space structure from an $A_n$-section, we consider the following operations of binary trees. Let $T_n$ be the set of connected binary trees with $n$ leaves and let $\hat{T}_n$ be the set of ordered binary trees, not necessarily connected, with $n$ leaves. Then we can label each leaf of an element of $\hat{T}_n$ from 1 to $n$ in the obvious way. Define a map $\delta : T_{n+1} \to \hat{T}_n$ by deleting the branches from the root to the $n$-th leaf. For example, $\delta : T_7 \to \hat{T}_6$ is:

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Then $\delta$ is a bijection. Analogously we define a map $\hat{\delta} : \hat{T}_n \to \hat{T}_{n-1}$ by applying the above map $\delta$ to the connected binary tree having the leaf labelled by $n$. Then $\delta : T_n \to \hat{T}_{n-1}$ is the restriction of $\hat{\delta} : \hat{T}_n \to \hat{T}_{n-1}$.
Let \( X \) be an \( H \)-space. For \( x_1, \ldots, x_n \in X \) and \( t \in T_n \), we define \( t(x_1, \ldots, x_n) \) as in \([22]\), which is consistent with the definition of \( A_n \)-spaces. For example, if \( t \in T_4 \) is

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]

then \( t(x_1, x_2, x_3, x_4) = x_1((x_2x_3)x_4) \). Let \( G \) be a topological group. Using the above map \( t \), for a map \( f : X \to G \), we define a map \( \hat{f} : \hat{T}_n \times X^n \to G \) by

\[
\hat{f}(\hat{t}, x_1, \ldots, x_n) = f(t_1(x_1, \ldots, x_{n_1}))f(t_2(x_{n_1+1}, \ldots, t_{n_1+n_2})) \cdots f(t_k(x_{n_1+\cdots+n_{k-1}+1}, \ldots, x_n)),
\]

where \( \hat{t} = t_1 \sqcup \cdots \sqcup t_k \in \hat{T}_n \) such that \( t_1 < \cdots < t_k \) and \( t_i \in T_n \).

Now we consider an extension of topological groups \( 1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1 \). Suppose that \( \pi \) admits an \( A_n \)-section \( s \) whose \( A_n \)-form is \( \{m_i : K_{i+1} \times H^i \to \tilde{H}\}_{i \leq n} \). As noted above, for a vertex \( v \in K_{i+1} \) corresponding to \( \hat{t} \in \hat{T}_i \), we have

\[
h_i(v, x_1, \ldots, x_i) = \hat{s}(\hat{t}, x_1, \ldots, x_i).
\]

We write \( \gamma_j(\tau, x) = h_j(s_{j+1}s_{j+2} \cdots s_i(\tau), \pi_{j+1}\pi_{j+2} \cdots \pi_i(x)) \) for \( \tau \in K_n, x \in K^i \)
and the projection \( \pi_j : K^i \to K^{i-1} \) omitting the \( i \)-th entry. Then, for a vertex \( v \in K_i \) corresponding to \( t \in T_i \) and \( x = (x_1, \ldots, x_i) \in K^i \), it follows from \([2.1]\) that

\[
\gamma_j(v, x) = s(\hat{d}^{-j}t, x_1, \ldots, x_i).
\]

Define \( M_i : K_i \times (H \times K)^i \to H \times K \)

\[
M_i(\tau, (h_1, k_1), \ldots, (h_i, k_i)) = (h_1h_2(\gamma_1(\tau, k_1))h_3(\gamma_2(\tau, k_2)) \cdots h_i(\gamma_i(\tau, k_i), k_1 \cdots k_i)
\]

for \( \tau \in K_i \) and \( k = (k_1, \ldots, k_i) \in K^i \), where \( g^h = hgh^{-1} \) for \( g, h \in H \). Then, by \([22]\), it is straightforward to check that \( \{M_i : K_i \times (H \times K)^i \to H \times K\}_{2 \leq i \leq n+1} \)

is an \( A_{n+1} \)-form on \( H \times K \) such that, for a vertex \( v \in K_i \) corresponding to \( \hat{t} \in T_i \), \( M_i(v, (h_1, k_1), \ldots, (h_i, k_i)) = t((h_1, k_1), \ldots, (h_i, k_i)) \). In particular, the multiplication of \( H \times K \) is defined by

\[
(h_1, k_1) \cdot (h_2, k_2) = (h_1(h_2^{(k_1)}), k_1 k_2),
\]

which is analogous to semidirect products of groups.

By a quite analogous observation, we can see that the map \( \theta : H \times K \to \tilde{H} \)

defined by

\[
\theta(h, k) = h \cdot s(k)
\]

for \( h \in H \) and \( k \in K \) admits an \( A_n \)-form. Summarizing, we have established:

**Lemma 2.2.** An extension of topological groups \( 1 \to K \to \tilde{H} \xrightarrow{\pi} H \to 1 \) has an \( A_n \)-splitting if and only if \( \pi \) admits an \( A_n \)-section.
In this section, we introduce the fibrewise analogue of $A_n$-maps between topological monoids and characterize them by using the fibrewise analogue of projective spaces. Let us first recall from [3] some notation and terminologies of fibrewise homotopy theory. Fix a space $B$. A fibrewise space over $B$ is an arrow $X \overset{\pi}{\to} B$. $\pi_X$ is called the projection and $\pi_X^{-1}(b)$ for $b \in B$ is called a fibre at $b$. Then the direct product $A \times B$ is a fibrewise space over $B$ and, in particular, so is $B$ itself. A fibrewise map from a fibrewise space $X \overset{\pi}{\to} B$ to $Y \overset{\pi_Y}{\to} B$ is a commutative diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow \pi_X & & \downarrow \pi_Y \\
B & \longrightarrow & B.
\end{array}
$$

Then fibrewise spaces over $B$ and fibrewise maps between them form a category which is nothing but the comma category $\text{Top} \downarrow B$, where $\text{Top} \downarrow B$ is the category of topological spaces and continuous maps. Fibrewise homotopy theory is not homotopy theory by the canonical model category structure on $\text{Top} \downarrow B$ induced from $\text{Top}$, but it respects fibre homotopy equivalence in the classical sense. With this in mind, we recall basic constructions in fibrewise homotopy theory. The fibrewise product $X \times_B Y$ of $X \overset{\pi}{\to} B$ and $Y \overset{\pi_Y}{\to} B$ is the pullback of the triad $X \overset{\pi}{\to} B \overset{\pi_Y}{\longrightarrow} Y$, that is,

$$X \times_B Y = \{(x, y) \in X \times Y \mid \pi_X(x) = \pi_Y(y)\}.$$

Then the diagonal map restricts to the fibrewise diagonal map $X \to X \times_B X$, denoted by $\Delta_B$. We often abbreviate the fibrewise product of $n$ copies of a fibrewise space $X \to B$ by $X^n$ by abuse of notation. We denote the fibrewise space $[0, 1] \times B \to B$ by $I_B$ and call it the fibrewise interval; here the projection is the second projection. A fibrewise homotopy is a fibrewise map $X \times_B I_B \to Y$ and we have a fibrewise homotopy equivalence in the obvious sense, which are the classical fibre homotopy and fibre homotopy equivalence, respectively. With this notion of fibrewise homotopies, we have a fibrewise fibration and a fibrewise cofibration, which are characterized by a fibrewise homotopy lifting property and a fibrewise homotopy extension property, respectively (see [3]).

The fibrewise unreduced cone of $X \overset{\pi}{\to} B$, denoted $C_BX$, is a pushout of the cotriad $I_B \times_B X \leftarrow \{0\} \times X \overset{\pi}{\to} B$. Similarly, the fibrewise unreduced suspension of $X \overset{\pi}{\to} B$, denoted $\Sigma_BX$, is a pushout of the cotriad $I_B \times_B X \leftarrow \{0, 1\} \times X \overset{\pi_1 \times \pi_X}{\to} \{0, 1\} \times B$. A fibrewise pointed space is a fibrewise space $X \to B$ with a distinguished section, and then we assume $B \subset X$. We have a fibrewise pointed map in the obvious sense. The fibrewise reduced cone $C^n_BX$ and the fibrewise reduced suspension $\Sigma^n_BX$ are the fibrewise collapses $C_BX/\Sigma_BX$ and $\Sigma_BX/\Sigma_BX$, respectively (see [3] p.55). A fibrewise pointed space is said to be well-pointed if the section is a fibrewise cofibration. Then, as in the usual case, if a fibrewise pointed space $X$ over $B$ is well-pointed, then $C_BX$ is fibrewise homotopy equivalent to $C^n_BX$ relative to $X$. In particular, $\Sigma_BX$ is fibrewise homotopy equivalent to $\Sigma^n_BX$.

In order to introduce a fibrewise analogue of $A_n$-maps between topological monoids, we need to have a fibrewise analogue of topological monoids which is
Given by replacing spaces and structure maps with fibrewise spaces and fibrewise maps of topological monoids as follows. A fibrewise topological monoid over $B$ is a fibrewise space $X \rightarrow B$ with fibrewise maps $\epsilon : B \rightarrow X$ and $\mu : X \times_B X \rightarrow X$ satisfying two conditions:

$$\mu \circ (\mu \times 1) = \mu \circ (1 \times \mu), \quad \mu \circ (1 \times \epsilon \pi_X) \circ \Delta_B = 1 = \mu \circ (\epsilon \circ \pi_X \times 1).$$

In particular, a fibrewise topological monoid is a fibrewise pointed space, and each of its fibres is a topological monoid. We usually abbreviate $\mu(x, y)$ by $xy$. A fibrewise topological monoid $X \rightarrow B$ is a fibrewise topological group if it has a fibrewise map $\iota : X \rightarrow X$ satisfying

$$\mu \circ (1 \times \iota) \circ \Delta_B = \epsilon \circ \pi_X = \mu \circ (\iota \times 1) \circ \Delta_B.$$

Let us look at examples of fibrewise topological monoids.

**Example 3.1.** Let $X \rightarrow B$ be a fibrewise pointed space with a distinguished section $s$. The fibrewise Moore path space of $X$ is

$$\Omega'_B X = \coprod_{b \in B} \Omega'(\pi^{-1}(b))$$

equipped with an appropriate topology (see [3]), where $\Omega'Y$ is the Moore path space of a space $Y$. Then the loop multiplication of $\Omega'(\pi^{-1}(b))$ makes $\Omega'_B X$ into a fibrewise topological monoid.

**Example 3.2.** Let $G$ be a topological group and let $\pi : P \rightarrow B$ be a principal $G$-bundle. Then the adjoint bundle $\text{ad}P$ is a fibrewise topological group with the structure maps:

$$\epsilon : B \rightarrow \text{ad}P, \quad \epsilon(b) = [\pi^{-1}(b), 1],$$

$$\mu : \text{ad}P \times_B \text{ad}P \rightarrow \text{ad}P, \quad \mu([x, g], [x, h]) = [x, gh],$$

$$\iota : \text{ad}P \rightarrow \text{ad}P, \quad \iota([x, g]) = [x, g^{-1}],$$

where $[(x, g)]$ is an equivalence class of $(x, g) \in P \times G$ in $\text{ad}P$.

Now we define a fibrewise $A_n$-map between fibrewise topological monoids just by replacing objects and arrows with fibrewise ones and the interval $[0, 1]$ with the fibrewise interval $I_B$ (see [21] for the definition of the usual $A_n$-maps between topological monoids).

**Definition 3.3.** Let $X$ and $Y$ be fibrewise topological monoids over $B$. A fibrewise map $f : X \rightarrow Y$ is called a fibrewise $A_n$-map if there exists a sequence of fibrewise maps $\{h_i : I_B^{-1} \times_B X^i \rightarrow Y\}_{1 \leq i \leq n}$ such that $h_1 = f$ and

$$h_i(t_1, \ldots, t_{i-1}, x_1, \ldots, x_i) = \begin{cases} h_{i-1}(t_1, \ldots, t_{i-1}, x_1, \ldots, x_jx_{j+1}, \ldots, x_i) & t_j = 0, \\ h_j(t_1, \ldots, t_{j-1}, x_1, \ldots, x_j)h_{i-j}(t_{j+1}, \ldots, t_{i-1}, x_{j+1}, \ldots, x_i) & t_j = 1. \end{cases}$$

By a quite analogous proof to [22] and [7], we can see the following properties of fibrewise $A_n$-maps.

**Proposition 3.4.**

1. If a fibrewise map $f$ is fibrewise homotopic to a fibrewise $A_n$-map, then so is $f$.
2. The composition of fibrewise $A_n$-maps is a fibrewise $A_n$-map.
(3) A homotopy inverse of a fibrewise homotopy equivalence which is a fibrewise $A_n$-map is a fibrewise $A_n$-map.

It follows from the above proposition that fibrewise homotopy equivalences which are fibrewise $A_n$-maps give an equivalence relation among fibrewise topological monoids. We call this equivalence a fibrewise $A_n$-equivalence.

Let us characterize fibrewise $A_n$-maps using the fibrewise analogue of projective spaces as in [21]. Note that we do not have appropriate quasi-fibrations in our fibrewise category. That is, we do not have weak equivalences nor quasi-fibrations, which can be replaced with fibrewise fibrations by weak equivalences, in our fibrewise category. Then it seems impossible to mimic the proof of [21, Theorem 4.5] directly. However, we only need to deal with fibrewise topological groups and we can overcome the above difficulty by restricting ourselves to fibrewise topological groups.

Let $G$ be a fibrewise topological group over $B$. Then, by [8 p.37], we have a fibrewise analogue of the Milnor construction for classifying spaces. Denote the $n$-th stage of the fibrewise Milnor construction for $G$ by $E^n_B G \rightarrow B^n_B G$, which is a finite numerable fibrewise bundle. Thus, by a quite analogous observation of [19] Corollary 14, we have:

**Lemma 3.5.** The fibrewise map $E^n_B G \rightarrow B^n_B G$ is a fibrewise fibration.

It will be convenient for later use to state a characterization of fibrewise $A_n$-maps by using a fibrewise analogue of the Dold-Lashof construction which coincides with the Milnor construction in the usual case (see, for example, [8]). Then we define the fibrewise Dold-Lashof construction only by replacing everything in the Dold-Lashof construction with a fibrewise one as follows. Let $H$ be a fibrewise topological monoid having a fibrewise action on $E$, denoted $m : H \times_B E \rightarrow E$ (see [3 p.15]). Start with a fibrewise map $q : E \rightarrow X$ enjoying $q(m(h, x)) = p(x)$ for $(h, x) \in H \times_B X$. Let $DL_B(E)$ be the fibrewise quotient of $(H \times_B C_B E) \cup E$ by the relation $(h, (1, x)) \sim \mu(h, x)$ for $(h, (1, x)) \in H \times_B C_B E$ and let $DL_B(X)$ be the fibrewise quotient of $C_B E \sqcup X$ by $(1, x) \sim q(x)$ for $(1, x) \in C_B E$. Then the Dold-Lashof construction for $q$ is the fibrewise map

$$DL_B(q) : DL_B(E) \rightarrow DL_B(X), \ (h, (t, x)) \mapsto (t, x).$$

Note that we do not have to take much care for topologies of $DL_B(E)$ and $DL_B(X)$ since we work in the category of spaces having the homotopy types of CW-complexes. Since $H$ is fibrewise associative, we can apply the Dold-Lashof construction iteratively. We denote the iterated Dold-Lashof construction $DL^n_B(H) \rightarrow DL^n_B(B)$ for the projection $\pi_H : H \rightarrow B$ by $\pi^n_B : E^n_B H \rightarrow P^n_B H$. As in the usual case, we can easily verify that if $H$ is a fibrewise topological group, $\pi^n_B : E^n_B H \rightarrow P^n_B H$ coincides with the $n$-th stage of the Milnor construction $E^n_B H \rightarrow B^n_B H$ [8].

We follow [13] to characterize fibrewise $A_n$-maps. Then we first define a fibrewise $A_n$-structure of a fibrewise $A_n$-map. Let $D^n_B X = C_B E^n_B X$ for a fibrewise topological monoid $X$.

**Definition 3.6.** Let $X$ and $Y$ be fibrewise topological monoids over $B$. A fibrewise $A_n$-structure of a fibrewise map $f : X \rightarrow Y$ consists of:

(1) $f$ respects fibrewise units of $X$ and $Y$. 
(2) There are sequences of commutative squares of fibrewise maps

\[
\begin{array}{ccc}
(D_{i+1}^i B, E_i^i X) & \xrightarrow{f_{i+1}} & (D_{i+1}^i Y, E_i^i Y) \\
\pi_{i+1} & & \pi_{i+1} \\
(P_{i+1}^i B, P_i^i X) & \xrightarrow{f_i^i} & (P_{i+1}^i Y, P_i^i Y)
\end{array}
\]

for \(1 \leq i \leq n - 1\) such that \(f_1^1|X = f, \quad f_{i+1}^{i+1}|D_b^b X = f_{i+1}^{i+1}, \quad f_i^i|P_b^b X = f_{i}^{i-1}\).

Now we give a characterization of a fibrewise \(A_n\)-map.

**Theorem 3.7.** Let \(X\) be a fibrewise topological monoid over \(B\) and let \(Y\) be a fibrewise well-pointed topological group over \(B\). A fibrewise map \(f : X \to Y\) is a fibrewise \(A_n\)-map if and only if \(f\) possesses a fibrewise \(A_n\)-structure.

**Proof.** The if part is done by Sugawara’s construction [23]. In order to prove the only if part, we can mimic the proof of [21, Theorem 4.5] without replacement of quasi-fibrations with fibrations by Lemma 3.5. \(\square\)

### 4. Set of sections

In this section, we consider the set of sections of a fibrewise space and prove Theorem 1.2. Let \(X\) be a fibrewise space over \(B\). We denote the set of sections of \(X\) by \(\Gamma(X)\). Then it is obvious that \(\Gamma\) is a functor from \(\text{Top} \downarrow B\) to \(\text{Top}\). Note that, by the pointwise multiplication, \(\Gamma(X)\) is a topological monoid and a topological group according as \(X\) is a fibrewise topological monoid and a fibrewise topological group. In particular, for a principal bundle \(P\), \(\Gamma(\text{ad}P)\) is a topological group by which we have an isomorphism of topological groups

\[
\mathcal{G}(P) \cong \Gamma(\text{ad}P)
\]

(see [2]).

Let \(C : \text{Top} \to \text{Top}\) be the unreduced cone functor. We define a natural transformation \(\lambda : C \Gamma \to \Gamma C_B\) by

\[
\lambda : C \Gamma(X) \to \Gamma(C_B X), \quad \lambda(t, s)(b) = (t(b))
\]

for \(b \in B\). Let \(H\) be a fibrewise topological monoid with a fibrewise action \(\mu : H \times_B E \to E\) and let \(q : E \to X\) be a fibrewise map such that \(q(\mu(h, x)) = x\) for \((h, x) \in H \times_B E\). Then, by definition, the natural transformation \(\lambda\) induces a commutative diagram

\[
\begin{array}{ccc}
\text{DL}(\Gamma(E)) & \xrightarrow{\lambda} & \Gamma(\text{DL}_B(E)) \\
\text{DL}(\Gamma(q)) & & \Gamma(\text{DL}_B(q)) \\
\text{DL}(\Gamma(X)) & \xrightarrow{\lambda} & \Gamma(\text{DL}_B(X))
\end{array}
\]
fibrewise map evaluation at the basepoint $\Gamma(ad_{Lashof})$. Then it follows that we have a commutative square

$$
\begin{array}{ccc}
(D^{n+1}\Gamma(H), E^n\Gamma(H)) & \xrightarrow{\tilde{\lambda}_n} & (\Gamma(D^{n+1}_B H), \Gamma(E^n_B H)) \\
\pi^{n+1} & \downarrow & \pi^{n+1}_B \\
(P^{n+1}\Gamma(H), P^n\Gamma(H)) & \xrightarrow{\lambda_n} & (\Gamma(P^{n+1}_B H), \Gamma(P^n_B H))
\end{array}
$$

for all $n$ such that

$$\tilde{\lambda}_n|_{D^n\Gamma(H)} = \tilde{\lambda}_{n-1}, \quad \lambda_n|_{P^n\Gamma(H)} = \lambda_{n-1},$$

where, for a topological monoid $Y$, $\pi^n : E^n \times Y \to P^n \times Y$ is DL$^n(*) : DL(Y) \to DL(*)$ and $D^{n+1}Y = CE^nY$.

**Proof of Theorem 1.2** Suppose that we have an $A_n$-splitting of (1.1). Then, by Lemma 2.2, we have an $A_n$-section $\sigma$ of $\pi : G(P) \to G$ which is identified with the evaluation at the basepoint $\Gamma(\text{ad}P) \to G$ through the isomorphism 4.1. Define a fibrewise map

$$\theta : B \times G \to \text{ad}P, \quad \theta(b, g) = \sigma(g)(b).$$

Then we have $\theta|_{(b_0) \times G} \simeq 1_G$ since $\sigma$ is a section of the evaluation at the basepoint $\Gamma(\text{ad}P) \to G$, where $b_0$ is the basepoint of $B$. Thus, by Dold’s theorem 5.1, $\theta$ is a fibrewise homotopy equivalence.

Since $\sigma$ is an $A_n$-map, it possesses an $A_n$-structure in the sense of [13]; that is, there is a sequence of homotopy commutative squares

$$
\begin{array}{ccc}
(D^{i+1}_G, E^iG) & \xrightarrow{\sigma^i_E} & (D^{i+1}\Gamma(\text{ad}P), E^i\Gamma(\text{ad}P)) \\
\downarrow & & \downarrow \\
(P^{i+1}_G, P^iG) & \xrightarrow{\sigma^i_P} & (P^{i+1}\Gamma(\text{ad}P), P^i\Gamma(\text{ad}P))
\end{array}
$$

for $i = 1, \ldots, n - 1$ in which $\sigma^i_E = \sigma, \sigma^i_E|_{D^n} = f^i_E, \sigma^i_P|_{P^n G} = f^i_P$. Note that

$$D^{i+1}_B(B \times G) = B \times G, \quad E^i(B \times G) = B \times E^iG, \quad P^i(B \times G) = B \times P^iG$$

and then we shall make these identifications. Define fibrewise maps

$$\theta^i_E : (D^{i+1}_B(B \times G), E^i_B(B \times G)) \to (D^{i+1}\text{ad}P, E^i_B\text{ad}P)$$

and

$$\theta^i_P : (P^{i+1}_B(B \times G), P^i_B(B \times G)) \to (P^{i+1}\text{ad}P, P^i_B\text{ad}P)$$

by

$$\theta^i_E(b, x) = \tilde{\lambda}_i(\sigma^i_E(x))(b), \quad \theta^i_P(b, y) = \lambda_i(\sigma^i_P(y))(b)$$

for $b \in B, x \in D^{i+1}_B, y \in P^{i+1}G$. Then these fibrewise maps give a fibrewise $A_n$-structure of $\theta$ and therefore, by Theorem 5.1, $\theta$ is a fibrewise $A_n$-equivalence.

Let $X$ be a fibrewise space over $B$. As in [22], we have a map

$$\rho : [0, 1] \times \Gamma(V) \to \Gamma(I_B \times_B X), \quad \rho(t, s)(b) = (t, s(b))$$

for $(t, s) \in [0, 1] \times \Gamma(V)$ and $b \in B$. Then a fibrewise $A_n$-map $f : X \to Y$ for fibrewise topological monoids $X, Y$ induces an $A_n$-map $\Gamma(f) : \Gamma(X) \to \Gamma(Y)$ in the sense of [21].
Suppose that we have a fibrewise $A_n$-equivalence $\theta : B \times G \to \text{ad}P$. Then it follows that we have an $A_n$-equivalence $\Gamma(\theta) : \Gamma(B \times G) \to \Gamma(\text{ad}P)$. Now we have an isomorphism of topological groups $\Gamma(B \times G) \cong \text{map}(B, G)$ which is natural with respect to $B$. Then the evaluation at the basepoint $\Gamma(B \times G) \to G$ is nothing but the evaluation at the basepoint map$(B, G) \to G$ which admits a section as topological groups. Then we obtain an $A_n$-section of $\pi : \Gamma(\text{ad}P) \to G$ and thus, by Lemma 2.2 we have established an $A_n$-splitting of (1.1). □

5. $H(k,l)$-SPACE

In this section, we consider the second question, that is, a criterion for an $A_n$-splitting of (1.1). Our major tool is the homotopy equivalence (1.2). Then let us first recall the construction of the homotopy equivalence (1.2). Let $G$ be a topological group. We denote by map$^G(X,Y)$ the space of all $G$-equivariant maps from $X$ to $Y$ for $G$-spaces $X, Y$. Let $P$ and $Q$ be principal $G$-bundles. Then $G(P)$ acts on map$^G(X,Y)$ by composition. Now we consider the case $Q = EG$. Then we have:

**Lemma 5.1** ([11, Theorem 5.2], [2, Proposition 2.4]).

1. map$^G(P,EG)$ is contractible.
2. The action of $G(P)$ on map$^G(P,EG)$ is free.

Then we have the universal $G(P)$-bundle:

\[
G(P) \to \text{map}^G(P,EG) \to \text{map}^G(P,EG)/G(P).
\]

Let us denote by $\theta$ the map $\text{map}^G(P,EG) \to \text{map}(B,BG;\alpha)$ induced from the projections $P \to B$ and $EG \to BG$, where $B$ is the base space of $P$ and $\alpha$ is the classifying map of $P$. Then one can easily see that the map $\theta$ induces a homeomorphism

\[
\bar{\theta} : \text{map}^G(P,EG)/G(P) \cong \text{map}(B,BG;\alpha),
\]

which is natural with respect to $P$. Thus we obtain a homotopy equivalence

\[
\hat{\theta} : BG(P) \cong \text{map}(B,BG;\alpha),
\]

which is natural with respect to $P$.

Consider the topological group $G$ as the principal $G$-bundle over a point and identify $G(G)$ with $G$. Then the basepoint inclusion $i : b_0 \to B$ induces a homotopy commutative diagram:

\[
\begin{array}{ccc}
BG(P) & \xrightarrow{\beta\pi} & BG \\
\downarrow{\hat{\theta}} & & \downarrow{\theta} \\
\text{map}(B,BG;\alpha) & \xrightarrow{i^*} & \text{map}(b_0,BG;0),
\end{array}
\]

where 0 stands for the constant map. Then the evaluation at the basepoint $e : \text{map}(B,BG;\alpha) \to BG$ is a model for $B\pi :BG(P) \to BG$, and this leads us to the following definition of $H(k,l)$-spaces. Let $i_k : P^k\Omega X \to \Omega P^\infty \Omega X \simeq X$ denote the canonical inclusion.
Definition 5.2. A space $X$ is called an $H(k,l)$-space if there is a map $m : P^k\Omega X \times P^l\Omega X \to X$ satisfying a homotopy commutative diagram:

$$
\begin{array}{c}
P^k\Omega X \vee P^l\Omega X \xrightarrow{i_k \vee i_l} X \\
\downarrow j \\
P^k\Omega X \times P^l\Omega X \xrightarrow{m} X,
\end{array}
$$

where $j$ is the inclusion.

It is obvious that an $H(k,l)$-space is an $H(k',l')$-space if $k \geq k'$ or $l \geq l'$. The loop space of an $H(1,1)$-space is homotopy commutative and an $H(\infty,\infty)$-space is an $H$-space. The loop spaces of $H(k,l)$-spaces give intermediate states between $H$-spaces and the loop spaces of $H$-spaces which will be discussed in section 7. On the other hand, an $H(\infty,k)$-space is Aguadé’s $T_\infty$-space \[1\]. In particular, an $H(1,\infty)$-space is Aguadé’s $T$-space, and this can also be seen by the fibrewise homotopy equivalence $adEG \simeq LBG$ over $BG$, where $LX$ is the free loop space of $X$.

An $H(k,l)$-space is defined to satisfy the following lemma:

Lemma 5.3. If a classifying space of a topological group $G$ is an $H(k,l)$-space, then there is an $A_\infty$-splitting of the exact sequence $1 \to \mathcal{G}_0(E^kG) \to \mathcal{G}(E^kG) \to G \to 1$.

Proof. Recall first from \[22\] that, for $A_{\infty}$-spaces $X,Y$, a map $f : X \to Y$ is an $A_\infty$-map if and only if its adjoint $\tilde{f} : \Sigma X \to P^\infty Y$ extends to $P^n X \to P^\infty Y$ up to homotopy.

Suppose that $X$ is an $H(k,l)$-space by $m : P^k\Omega X \times P^l\Omega X \to X$. Then, by the exponential law, the adjoint of $m$ restricts to a map $\tilde{m} : \Sigma \Omega X \to \Omega \map(P^k\Omega X, X ; i_k)$ such that $e \circ \tilde{m} \simeq i_1$, where $e : \Omega \map(P^k\Omega X, X ; i_k) \to X$ is the evaluation at the basepoint. Then the adjoint of $\tilde{m}$, say $\bar{m} : \Omega X \to \Omega \map(P^k\Omega X, X ; i_k)$, is a homotopy section of $\Omega e$ and thus $\bar{m}$ is an $A_\infty$-map. Therefore, by Lemma \[22\] and \[5.3\], Lemma 5.3 is established.

Proof of Theorem \[1.3\]. It is well known that $\text{cat} B \leq k$ if and only if $i_k : P^k\Omega B \to B$ admits a homotopy section. Then, by naturality of $i_k$, if $\text{cat} B \leq k$, each map $f : B \to BG$ admits a map $\tilde{f} : B \to P^kG$ such that $i_k \circ \tilde{f} \simeq f$. This implies that $\tilde{f}^{-1} E^kG \cong P$ and then an $A_\infty$-section for $\pi : \mathcal{G}(E^kG) \to G$ induces that of $\pi : \mathcal{G}(P) \to G$. Thus, by Lemma 5.3 the proof is completed.

6. Investigating $H(1,n)$-spaces

In the previous section, we have obtained the universal $\mathcal{G}(P)$-bundle \[5.1\]. Then it follows from \[5.2\] that there is a homotopy equivalence $\varphi : \map^G(P,EG;\alpha)/\mathcal{G}_0(P) \to \map_0(B,BG;\alpha)$ and $\tilde{\varphi} : \mathcal{G}_0(P) \to \map^G(P,EG;\alpha)/\mathcal{G}_0(P)$ such that the following diagram of fibre sequences is homotopy commutative:

$$
\begin{array}{cccccccc}
G & \to & BG_0(P) & \xrightarrow{B_1} & BG(P) & \xrightarrow{B_\pi} & BG \\
\downarrow & & \downarrow \varphi \simeq & & \downarrow \tilde{\varphi} \simeq & & \downarrow \\
\mathcal{G}(P)/\mathcal{G}_0(P) & \xrightarrow{\delta_\lambda} & \map^G(P,EG;f)/\mathcal{G}_0(P) & \xrightarrow{\map^G(P,EG;\alpha)/\mathcal{G}(P)} & BG \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G & \xrightarrow{\delta_\lambda} & \map_0(B,BG;\alpha) & \to & \map(B,BG;\alpha) & \xrightarrow{e} & BG.
\end{array}
$$
The aim of this section is to study the connecting map $\delta_\alpha$ and characterize $H(1, n)$-spaces by it. Consider the following commutative diagram:

$$
\begin{array}{c}
G \xrightarrow{\delta} \text{map}_0(B, BG; \alpha) \\
\downarrow \alpha^* \\
G \xrightarrow{\delta_\alpha} \text{map}_0(B, BG; \alpha) \\
\end{array}
\xrightarrow{\alpha^*} 
\begin{array}{c}
\text{map}_0(BG, BG; 1) \\
\xrightarrow{e} BG \\
\end{array}
\xrightarrow{\alpha^*} 
\begin{array}{c}
\text{map}_0(BG, BG; \alpha) \\
\xrightarrow{e} BG \\
\end{array}
$$

Then it is sufficient to consider the universal connecting map $\delta : G \to \text{map}_0(B, BG; \alpha)$.

Put $E = \text{map}_G(EG, EG)$, $G = G(EG)$, and $G_0 = G_0(EG)$. Let $E_0$ be the subspace of $E$ consisting of $G$-equivariant maps $EG \to EG$ restricting to the identity on the fibre at the basepoint. Then we have a fibre sequence $E_0 \to E \to \text{map}_G(EG, EG)$ induced from the basepoint inclusion of $BG$. Then it follows from Lemma 5.1 that $E_0$ is contractible and $G_0$ acts freely on $E_0$ by composition. Then we have the universal $G_0$-bundle

$$
G_0 \to E_0 \to \text{map}_0(BG, BG; \alpha).
$$

On the other hand, the projection $\theta_0 : E_0 \to \text{map}_0(B, BG; \alpha)$ induces a homeomorphism

$$
\bar{\theta}_0 : E_0/G_0 \cong \text{map}_0(BG, BG; \alpha).
$$

Note that the inclusion $\kappa : E_0 \to E$ induces a map $\bar{\kappa} : E_0/G_0 \to E/G_0$ by which the diagram

$$
\begin{array}{c}
E_0/G_0 \\
\downarrow \bar{\theta}_0 \\
\text{map}_0(BG, BG; \alpha) \\
\end{array}
\xrightarrow{\bar{\kappa}} 
\begin{array}{c}
E/G_0 \\
\downarrow \bar{\phi} \\
\text{map}_0(BG, BG; \alpha) \\
\end{array}
$$

commutes up to homotopy.

Let us construct an alternative universal $G$-bundle to describe the connecting map $\delta$. Following Milnor [18], we denote an element of $EG$ by $t_0g_0 \oplus t_1g_1 \oplus \cdots$ for $\sum_i t_i = 1$, $t_i \geq 0$ and $g_i \in G$ such that finite $t_i$'s are positive. The basepoint of $EG$ is $e \oplus 0 \oplus 0 \oplus \cdots$, where $e$ is the unity of $G$. For $g \in G$, we denote by $\xi_g$ the principal bundle map

$$
EG \to EG, ~ t_0g_0 \oplus t_1g_1 \oplus \cdots \mapsto t_0g^{-1}g_0 \oplus t_1g^{-1}g_1 \oplus \cdots.
$$

Then we have a commutative diagram:

$$
\begin{array}{c}
EG \\
\downarrow \xi_g \\
BG \xrightarrow{\text{Bad}(g)} BG \\
\end{array}
\xrightarrow{\text{Bad}(g)} 
\begin{array}{c}
EG \\
\end{array}
$$

Now we let $G$ act on $E_0 \times EG$ from the right by

$$(f, x) \cdot g = (\xi_g \cdot (f \circ g, x \cdot \pi(g)))$$

for $g \in G$ and $(f, x) \in E_0 \times EG$. One can easily check that this action is free, and then we have established the universal $G$-bundle

$$
G \to E_0 \times EG \to (E_0 \times EG)/G.
$$
Thus there exist a homotopy equivalence $\mathcal{E}/\mathcal{G} \to (\mathcal{E}_0 \times EG)/\mathcal{G}$ and a $\mathcal{G}$-equivariant homotopy equivalence $\nu : \mathcal{E} \to \mathcal{E}_0 \times EG$ by which the diagram

$$\begin{array}{cccccc}
\mathcal{G} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/\mathcal{G} & \longrightarrow & BG \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq & & \\
\mathcal{G} & \longrightarrow & \mathcal{E}_0 \times EG & \longrightarrow & (\mathcal{E}_0 \times EG)/\mathcal{G} & \longrightarrow & BG
\end{array}$$

commutes up to homotopy. Since the above diagram is that of $\mathcal{G}_0$-spaces and $\mathcal{G}_0$-equivariant maps, we obtain a homotopy commutative diagram:

$$\begin{array}{cccccc}
G & \longrightarrow & \mathcal{G}/\mathcal{G}_0 & \longrightarrow & \mathcal{E}/\mathcal{G}_0 & \longrightarrow & \mathcal{E}/\mathcal{G} & \longrightarrow & BG \\
\downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
\mathcal{G}/\mathcal{G}_0 & \longrightarrow & (\mathcal{E}_0 \times EG)/\mathcal{G}_0 & \longrightarrow & (\mathcal{E}_0 \times EG)/\mathcal{G} & \longrightarrow & BG.
\end{array}$$

Note that the above action of $\mathcal{G}$ on $\mathcal{E}_0 \times EG$ restricts to the product of the usual action of $\mathcal{G}_0$ on $\mathcal{E}_0$ and the trivial action of $\mathcal{G}_0$ on $EG$. Then we have $(\mathcal{E}_0 \times EG)/\mathcal{G}_0 = \mathcal{E}_0/\mathcal{G}_0 \times EG$ and thus the first projection $\pi_1 : \mathcal{E}_0 \times EG \to \mathcal{E}_0$ induces a homotopy equivalence $\bar{\pi}_0 : (\mathcal{E}_0 \times EG)/\mathcal{G}_0 \simeq \mathcal{E}_0/\mathcal{G}_0$. Since the map $\delta^0(\mathcal{E}_0, \mathcal{E}_0)$ is contractible, in particular, path connected, the $\mathcal{G}_0$-equivariant map $\bar{\pi}_0 \circ \nu \circ \kappa$ is homotopic to the identity of $\mathcal{E}_0$ as $\mathcal{G}_0$-equivariant maps. Then, by (6.2), we have established a homotopy commutative diagram:

$$\begin{array}{cccccc}
\mathcal{E}_0/\mathcal{G}_0 & \longrightarrow & \mathcal{E}/\mathcal{G}_0 & \longrightarrow & \mathcal{E}/\mathcal{G} & \longrightarrow & BG \\
\uparrow \bar{\pi}_0 & & \uparrow \simeq & & \uparrow \simeq & & \\
\mathcal{G} & \longrightarrow & (\mathcal{E}_0 \times EG)/\mathcal{G}_0 & \longrightarrow & (\mathcal{E}_0 \times EG)/\mathcal{G} & \longrightarrow & BG \\
\downarrow \bar{\delta}_0 & & \downarrow \simeq & & \downarrow \simeq & & \\
\mathcal{G} & \longrightarrow & \text{map}_0(BG, BG; 1) & \longrightarrow & \text{map}(BG, BG; 1).
\end{array}$$

Therefore we have obtained:

**Lemma 6.1.** There is a homotopy commutative diagram:

$$\begin{array}{cccccc}
\mathcal{G} & \longrightarrow & \mathcal{E}_0/\mathcal{G}_0 & \longrightarrow & (\mathcal{E}_0 \times EG)/\mathcal{G} & \longrightarrow & BG \\
\downarrow \simeq & & \downarrow \delta_0 & & \downarrow \simeq & & \\
\mathcal{G} & \longrightarrow & \text{map}_0(BG, BG; 1) & \longrightarrow & \text{map}(BG, BG; 1).
\end{array}$$

In particular, the connecting map $\delta$ is $\text{Bad}$.

Theorem 1.4 follows from (6.1).
Remark 6.2. It is an old problem to describe the connecting map of the evaluation fibration:
\[ \text{map}_0(X, Y; f) \to \text{map}(X, Y; f) \to Y. \]

Whitehead [25] first studied the case when \( X \) is a sphere and, later, Lang [10] studied the case when \( X \) is a suspension. They described the connecting map by Whitehead products. By [17], \( Y \) has the homotopy type of the classifying space of some topological group when \( Y \) is a connected countable simplicial complex. Then Theorem 1.4 is not only a characterization of \( H(1, n) \)-spaces but also an answer to the above old problem. One can easily deduce the results of Whitehead [25] and Lang [10] by using the adjointness of Whitehead products and Samelson products.

7. \( C(k, l) \)-SPACE

In this section, we discuss a relation between \( H(k, l) \)-spaces and higher homotopy commutativity as promised in section 5. Higher homotopy commutativity was first introduced by Sugawara [23] as intermediate states between loop spaces and loop spaces of \( H \)-spaces. Later, Williams [26] introduced another kind of higher homotopy commutativity using associahedra in section 2. Recently, Hemmi and Kawamoto [12] studied a relation between higher homotopy commutativity, Aguadé’s \( T_k \)-spaces [1] and Félix and Tanré’s \( H(n) \)-spaces [6]. In order to relate them, they introduced \( H_k(n) \)-spaces and \( C_k(n) \)-spaces. \( H_k(n) \)-spaces collect Aguadé’s \( T_k \)-spaces and Félix and Tanré’s \( H(n) \)-spaces whose definition is given by a sequence of \( H(k, l) \)-spaces for \( k + l = n \) (see [12]). On the other hand, \( C_k(n) \)-spaces are defined as follows by using Gel’fand, Kapranov and Zelevinsky’s polytopes called resultohedra (see [9], [10] for the definition of resultohedra).

Let \( R_+ = \{ x \in R | x \geq 0 \} \). The resultohedron \( N_{m,n} \) is an \((m+n-1)\)-dimensional polytope in \( R^{m+n+2} \) which consists of all points \((p_0, \ldots, p_m, q_0, \ldots, q_n) \in R^{m+n+2} \) satisfying:
\[
\sum_{i=0}^{m} p_i = n, \quad \sum_{i=0}^{n} q_i = m, \quad h_{i,j} \geq 0, \quad h_{m,n} = 0,
\]
where
\[
h_{i,j} = \sum_{k=0}^{i} (i-k)p_k + \sum_{l=0}^{j} (j-l)q_l - ij
\]
for \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). Then, in particular, \( N_{0,0} \) is the one-point set and \( N_{k,1} \) and \( N_{1,k} \) are affinely homeomorphic to the \( k \)-simplex \( \Delta^k \). Vertices of \( N_{m,n} \) are labelled by integer lattice paths from \((0, 0)\) to \((m, n)\).

For \( x = p_i, q_j \) and \( h_{i,j} \) in (7.1), we put
\[
N(x) = \{ (p_0, \ldots, p_m, q_0, \ldots, q_n) \in N_{m,n} | x = 0 \}.
\]

Gel’fand, Kapranov and Zelevinsky [10] described the face maps
\[
\epsilon^{(p_i)} : N_{m-1,n} \to N(p_i), \quad \epsilon^{(q_j)} : N_{m,n-1} \to N(q_j), \quad \epsilon^{(h_{i,j})} : N_{i,j} \times N_{m-1,n-1-j} \to N(h_{i,j}).
\]

On the other hand, Hemmi and Kawamoto [12] described the degeneracy maps
\[
\delta_i : N_{m,n} \to N_{m-1,n}, \quad \delta'_j : N_{m,n} \to N_{m,n-1}.
\]
Now a $C_k(n)$-space is defined by a coherent sequence of maps $Q_{r,s} : N_{r,s} \times X^{r+s} \to X$ for a topological monoid $X$, $r + s \leq n$ and $s \leq k$ (see [12] for a precise definition). The main result of [12] is:

**Theorem 7.1** ([12] Theorem A). A connected topological monoid is a $C_k(n)$-space if and only if its classifying space is an $H_k(n)$-space.

As noted above, the definition of an $H_k(n)$-space is a collection of $H(k,l)$-spaces for $k + l \leq n$ and, actually, the proof of Theorem 7.1 is done by collecting constructions on $H(k,l)$-spaces. Then, by defining $C(k,l)$-spaces as follows, which is a modification of that of $C_k(n)$-spaces, we obtain Theorem 1.6.

**Definition 7.2.** A topological monoid $X$ is a $C(k,l)$-space if there exists a sequence of maps $Q_{r,s} : N_{r,s} \times X^{r+s} \to X$ for $0 \leq r \leq k$ and $0 \leq s \leq l$ satisfying:

\[Q_{r,0}(\ast, x_1, \ldots, x_r) = x_1 \cdots x_r, \quad Q_{0,s}(\ast, y_1, \ldots, y_s) = y_1 \cdots y_s,\]

\[Q_{r,s}(\epsilon^{(p_i)}(\sigma), x_1, \ldots, x_r, y_1, \ldots, y_s) = \begin{cases} x_1 \cdot Q_{r-1,s}(\sigma, x_2, \ldots, y_s) & i = 0 \\ Q_{r-1,s}(\sigma, x_1, \ldots, x_i x_{i+1}, \ldots, y_s) & 0 < i < r \\ Q_{r-1,s}(\sigma, x_1, \ldots, x_r-1, y_1, \ldots, y_s) & i = r, \end{cases}\]

\[Q_{r,s}(\epsilon^{(q_j)}(\sigma), x_1, \ldots, x_r, y_1, \ldots, y_s) = \begin{cases} y_1 \cdot Q_{r,s-1}(\sigma, x_1, \ldots, x_r, y_2, \ldots, y_s) & j = 0 \\ Q_{r,s-1}(\sigma, x_1, \ldots, y_j y_{j+1}, \ldots, y_s) & 0 < j < s \\ Q_{r,s-1}(\sigma, x_1, \ldots, y_{s-1}) & j = s, \end{cases}\]

\[Q_{r,s}(\epsilon^{(h_{i,j})}(\sigma_1, \sigma_2), x_1, \ldots, x_r, y_1, \ldots, y_s) = Q_{i,j}(\sigma_1, x_1, \ldots, x_i, y_1, \ldots, y_j) \cdot Q_{r-i,s-j}(\sigma_2, x_{i+1}, \ldots, x_y, y_j y_{j+1}, \ldots, y_s),\]

\[Q_{r,s}(\sigma, x_1, \ldots, x_i-1, \ast, x_{i+1}, \ldots, y_j, \ldots, y_s) = Q_{r-1,s}(\delta_i(\sigma), x_1, \ldots, x_i-1, x_{i+1}, \ldots, y_s),\]

\[Q_{r,s}(\sigma, x_1, \ldots, y_j-1, \ast, y_{j+1}, \ldots, y_s) = Q_{r,s-1}(\delta_j(\sigma), x_1, \ldots, y_j-1, y_{j+1}, \ldots, y_s).\]

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