

GEOMETRIC IDEALIZER RINGS

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ABSTRACT. Let $B = B(X, \mathcal{L}, \sigma)$ be the twisted homogeneous coordinate ring of a projective variety X over an algebraically closed field \mathbb{k} . We construct and investigate a large class of interesting and highly noncommutative noetherian subrings of B . Specifically, let Z be a closed subscheme of X and let $I \subseteq B$ be the corresponding right ideal of B . We study the subalgebra

$$R = k + I$$

of B . Under mild conditions on Z and $\sigma \in \text{Aut}_{\mathbb{k}}(X)$, R is the *idealizer* of I in B : the maximal subring of B in which I is a two-sided ideal.

Our main result gives geometric conditions on Z and σ that determine the algebraic properties of R . We say that

$$\{\sigma^n(Z)\}$$

is *critically transverse* if for any closed subscheme Y of Z , for $|n| \gg 0$ the subschemes Y and $\sigma^n(Z)$ are homologically transverse. We show that if $\{\sigma^n(Z)\}$ is critically transverse, then R is left and right noetherian, has finite left and right cohomological dimension, is strongly right noetherian but not strongly left noetherian, and satisfies right χ_d (where $d = \text{codim } Z$) but fails left χ_1 . This generalizes results of Rogalski in the case that Z is a point in \mathbb{P}^d . We also give an example of a right noetherian ring with infinite right cohomological dimension, partially answering a question of Stafford and Van den Bergh.

Further, we study the geometry of critical transversality and show that it is often generic behavior, in a sense that we make precise.

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1. INTRODUCTION

One of the contributions of noncommutative algebraic geometry to noncommutative algebra is a gallery of interesting counterexamples: rings with previously unobserved, unusual, and counter-intuitive properties. Many of these examples are even noetherian \mathbb{N} -graded domains, but their subtler properties, such as whether or not they are strongly noetherian or satisfy the Artin-Zhang χ conditions, can be surprisingly pathological. Examples of such rings include the naïve blowup algebras first analyzed by Rogalski in [20] and later studied in [17] and the idealizers constructed by Rogalski in [21].

In this paper we use noncommutative algebraic geometry to construct and analyze a large class of rings with interesting algebraic behavior. Our rings, like those mentioned above, will be constructed as subrings of twisted homogeneous coordinate rings. We recall Artin and Van den Bergh's construction [3].

Let X be a projective scheme, let \mathcal{L} be an invertible sheaf on X , and let σ be an automorphism of X . We denote the pullback of \mathcal{L} along σ by

$$\sigma^*\mathcal{L} = \mathcal{L}^\sigma.$$

For any $n \geq 0$, we define

$$\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}.$$

Then the *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma)$ is defined to be

$$\bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n).$$

If \mathcal{L} is appropriately ample (the technical term is σ -ample), then $B(X, \mathcal{L}, \sigma)$ is noetherian [3, Theorem 1.4].

Recall that if I is a right ideal of a ring B , the *idealizer* $\mathbb{I}_B(I)$ of I in B is the maximal subring of B in which I becomes a two-sided ideal. That is,

$$\mathbb{I}_B(I) = \{x \in B \mid xI \subseteq I\}.$$

In this paper, we study idealizer subrings of twisted homogeneous coordinate rings. Our basic construction is the following:

Construction 1.1. Let X be a projective variety, let σ be an automorphism of X , and let \mathcal{L} be a σ -ample invertible sheaf on X . Let Z be a closed subscheme of X . Let $B = B(X, \mathcal{L}, \sigma)$, and let I be the right ideal of B generated by sections that vanish on Z .

Our object of study is the idealizer ring

$$R = R(X, \mathcal{L}, \sigma, Z) = \mathbb{I}_B(I) = \{x \in B \mid xI \subseteq I\}.$$

We refer to R as a *geometric idealizer*, or more specifically, as the (*right*) *idealizer in B at Z* .

Our main result gives geometric criteria that determine the algebraic properties of geometric idealizers. In particular, we characterize when such rings are noetherian. We also analyze when idealizers are strongly noetherian, satisfy various χ conditions, and have finite cohomological dimension. Here we define the properties that we will investigate.

Throughout, we fix an algebraically closed ground field \mathbb{k} . All rings R we consider will be *connected* \mathbb{N} -graded \mathbb{k} -algebras; that is, R is \mathbb{N} -graded, with $R_0 = \mathbb{k}$.

Definition 1.2. Let R be a finitely generated, connected \mathbb{N} -graded \mathbb{k} -algebra, and let $j \in \mathbb{N}$. We say that R *satisfies right* χ_j if, for all $i \leq j$ and for all finitely generated graded right R -modules M , we have

$$\dim_{\mathbb{k}} \underline{\text{Ext}}_R^i(\mathbb{k}, M) < \infty.$$

We say that R *satisfies right* χ if R satisfies right χ_j for all $j \in \mathbb{N}$. We similarly define *left* χ_j and *left* χ ; we say R *satisfies* χ if it satisfies left and right χ .

Definition 1.3. A \mathbb{k} -algebra Rq is *strongly right (left) noetherian* if, for any commutative noetherian \mathbb{k} -algebra C , the ring $R \otimes_{\mathbb{k}} C$ is right (left) noetherian.

Finitely generated commutative graded rings are always strongly noetherian by the Hilbert basis theorem and satisfy χ by [4, Corollary 8.12]. For noncommutative rings, these two conditions are important because their presence allows one to use powerful techniques inspired by commutative algebraic geometry. The strong noetherian property, in particular, is needed in order for modules over R to be parameterized by a (commutative) projective scheme [5]. If R satisfies χ_1 , then one may reconstruct R from the noncommutative projective scheme associated to R [4]. The higher conditions χ_j for $j > 1$ are less well understood. However, if a ring satisfies right or left χ , then it is well-behaved in some important ways. For example, the χ conditions are needed in order to have a version of Serre duality for a noncommutative ring R [29, Theorem 6.3], [32, Theorem 4.2]. This is known as the existence of a *balanced dualizing complex*.

Examples are known of noetherian rings that are not strongly noetherian on one or both sides, or which satisfy χ_d on one side and fail χ_1 on the other side. However, in general, the ways in which it is possible for these properties to fail, and to fail asymmetrically, are still poorly understood. Thus another goal of the work in this paper is to gain more insight into situations where some of these properties do not hold. We study these properties for geometric idealizers and show that, roughly speaking, if σ and Z are in general position (in a sense we make precise), then $R = R(X, \mathcal{L}, \sigma, Z)$ is noetherian, strongly right noetherian, and satisfies right χ_d , where $d = \text{codim } Z$. On the other hand, R is not strongly left noetherian and fails left χ_1 .

We also study the *cohomological dimension* of geometric idealizers, since there are many open questions about this invariant. Recall that in noncommutative algebraic geometry, one often works with the category $\text{qgr-}R$, defined roughly as

$$\{\text{graded right } R\text{-modules}\} / \{\text{finite-dimensional modules}\}.$$

The category $\text{qgr-}R$ is the appropriate analogue of $\text{Proj } R$ for a commutative graded ring R . It has a global section functor, defined as

$$\Gamma(\mathcal{M}) = \text{Hom}_{\text{qgr-}R}([R], \mathcal{M}),$$

where $[R]$ denotes the image of R in $\text{qgr-}R$. The cohomological dimension of the functor Γ is referred to as the *right cohomological dimension of R* ; of course, one may also define *left cohomological dimension*. Stafford and Van den Bergh [26, page 194] have asked if a noetherian graded ring must have finite left and right cohomological dimension. (This is true for commutative graded rings by a well-known result of Grothendieck [12].) We answer Stafford and Van den Bergh's question in the affirmative for geometric idealizers. We also give the first known example of a right noetherian graded ring with infinite right cohomological dimension.

Our work generalizes work of Rogalski [21], who used algebraic techniques to investigate idealizers of maximal nonirrelevant graded right ideals in Zhang twists of polynomial rings. In our language, Rogalski studied geometric idealizers in the special setting that $X = \mathbb{P}^d$, $\mathcal{L} = \mathcal{O}(1)$, and $Z = \{p\}$ is a point. (His work generalized earlier work of Stafford and Zhang [27], who studied idealizers on \mathbb{P}^1 .) Rogalski discovered that the algebraic behavior of $R(X, \mathcal{O}(1), \sigma, \{p\})$ is controlled by the geometry of the orbit of p , as follows:

Definition 1.4. Let X be a variety, let $p \in X$ and let $\sigma \in \text{Aut}_{\mathbb{k}}(X)$. The orbit $\{\sigma^n(p)\}_{n \in \mathbb{Z}}$ is *critically dense* if it is infinite, and any infinite subset is Zariski dense in X .

Theorem 1.5 (Rogalski). *Let $\sigma \in \text{PGL}_{d+1}$ and let $p \in \mathbb{P}^d$. Assume that p is of infinite order under σ , and let $R = R(\mathbb{P}^d, \mathcal{O}(1), \sigma, \{p\})$. Then*

- (1) R is strongly right noetherian.
- (2) R fails left χ_1 .

Further, if the set $\{\sigma^n(p)\}$ is critically dense, then:

- (3) R is left noetherian, but not strongly left noetherian if $d \geq 2$.
- (4) R satisfies right χ_{d-1} , but fails right χ_d .
- (5) $R \otimes_{\mathbb{k}} R$ is right but not left noetherian. □

Rogalski's work shows that the algebraic conclusions of (3), (4), and (5) are controlled by rather subtle geometry. In particular, right idealizers at points of infinite order are automatically right noetherian, but in order for them to be left noetherian, σ must move p significantly and in some sense uniformly around \mathbb{P}^d . One naturally asks if there is a higher-dimensional analogue of critical density that controls the behavior of more general idealizers than those studied in Theorem 1.5. One of the main results of this paper is that the answer is "yes." We define:

Definition 1.6. Let X be a variety and let $Z, Y \subseteq X$ be closed subschemes. We say that Z and Y are *homologically transverse* if

$$\text{Tor}_j^X(\mathcal{O}_Z, \mathcal{O}_Y) = 0$$

for all $j \geq 1$.

Definition 1.7. Let X be a variety and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. Let $Z \subseteq X$ be a closed subscheme. The set $\{\sigma^n Z\}_{n \in \mathbb{Z}}$ is *critically transverse* if for all closed subschemes $Y \subseteq X$, the subschemes $\sigma^n(Z)$ and Y are homologically transverse for all but finitely many n .

In this paper, we generalize Theorem 1.5 to arbitrary idealizers in twisted homogeneous coordinate rings. We show that critical transversality controls the behavior of these rings, and we prove:

Theorem 1.8 (Theorem 10.2). *Let X be a projective variety, let σ be an automorphism of X , and let \mathcal{L} be a σ -ample invertible sheaf on X . Form the ring $R = R(X, \mathcal{L}, \sigma, Z)$ as above. For simplicity, assume that Z is irreducible and of infinite order under σ . (We treat the general case in the body of the paper.)*

If for all $p \in Z$, the set $\{n \geq 0 \mid \sigma^n(p) \in Z\}$ is finite, then:

- (1) *R is strongly right noetherian.*
- (2) *R fails left χ_1 .*

If the set $\{\sigma^n Z\}_{n \in \mathbb{Z}}$ is critically transverse, then:

- (3) *R is left noetherian, but R is strongly left noetherian if and only if all components of Z have codimension 1.*
- (4) *Let $d = \text{codim } Z$. Then R fails right χ_d . If X and Z are smooth, then R satisfies right χ_{d-1} .*
- (5) *If R is noetherian, then R has finite left and right cohomological dimension.*
- (6) *If Z is not of pure codimension 1, then $R \otimes_{\mathbb{k}} R$ is not left noetherian.*

One motivation for undertaking the investigations in this paper was to make progress on the classification of noncommutative projective surfaces: graded noetherian domains of GK-dimension 3. Even for the nicest possible such surfaces, those that are birational to a commutative surface in the sense of [1], current classification results such as [22] depend on technical conditions such as being generated in degree 1. Since ideally a classification effort in noncommutative geometry would be *sui generis*, without requiring such conditions, understanding idealizers in twisted homogeneous coordinate rings has important applications to classifying noncommutative surfaces. In a future paper, we will give a full classification of birationally commutative projective surfaces, using many of the results in the current work.

In the remainder of the introduction, we discuss the geometry underlying the technical-looking definition of critical transversality. We first explain the use of the term “transverse.” Let Y and Z be closed subschemes of X . Recall [14, p. 427] Serre’s definition of the intersection multiplicity of Y and Z along the proper component P of their intersection:

$$i(Y, Z; P) = \sum_{i \geq 0} (-1)^i \text{len}_P(\text{Tor}_i^X(\mathcal{O}_Y, \mathcal{O}_Z)),$$

where $\text{len}_P(\mathcal{F})$ is the length of \mathcal{F}_P over the local ring $\mathcal{O}_{X,P}$. Thus if Y and Z are homologically transverse, their intersection multiplicity is given by the naïve formula that

$$i(Y, Z; P) = \text{len}_P(\mathcal{O}_Y \otimes_X \mathcal{O}_Z).$$

That is, $i(Y, Z; P)$ is, as we might hope, the length of the structure sheaf of the scheme-theoretic intersection of Y and Z over the local ring at P .

Another way of phrasing the critical transversality of $\{\sigma^n(Z)\}$ is that for any Y , the general translate of Z by a power of σ is homologically transverse to Y . This sort of statement is clearly reminiscent of the Kleiman-Bertini theorem, and in fact the investigations in this chapter have led to a new, purely algebro-geometric, generalization of this classical result; see [24]. In this paper, we apply the results in

[24] to obtain a simple criterion for the critical transversality of $\{\sigma^n(Z)\}$ in many cases.

Theorem 1.9 (Theorem 5.11). *Let \mathbb{k} be an algebraically closed field of characteristic 0, X be a variety over \mathbb{k} , let Z be a closed subscheme of X , and let σ be an element of an algebraic group G that acts on X . Then $\{\sigma^n Z\}$ is critically transverse if and only if Z is homologically transverse to all reduced σ -invariant subschemes of X .*

It is reasonable to say that σ and Z are in *general position* if Z is homologically transverse to σ -invariant subschemes of X . Thus, in characteristic 0 and if σ is an element of an algebraic group, Theorems 1.9 and 1.8 together imply that if σ and Z are in general position, then $R(X, \mathcal{L}, \sigma, Z)$ has the properties enumerated in Theorem 1.8.

2. DEFINITIONS

Throughout, we let \mathbb{k} be a fixed algebraically closed field; all schemes are of finite type over \mathbb{k} .

The starting point of this paper is Rogalski's work in [21], and specifically Theorem 1.5. One goal of the current work is to place the results in [21] in a more geometric context. In that sense, this paper may be viewed as analogous to [17], which defines naïve blowup algebras and gives a geometric construction of the algebras investigated by Rogalski in [20]. As in [17], one of our main techniques will be to work, not with the ring $R(X, \mathcal{L}, \sigma, Z)$, but with an associated quasicoherent sheaf on X . This object is known as a *bimodule algebra*, and is, roughly speaking, a sheaf with multiplicative structure.

In this section, we give the definitions and notation to allow us to work with bimodule algebras. Most of the material in this section was developed in [28] and [3], and we refer the reader there for references. We will not work in full generality, however, and our presentation will follow that in [17, Section 2].

A bimodule algebra on a variety X is, roughly speaking, a quasicoherent sheaf with a multiplicative structure.

Definition 2.1. Let X be a variety; that is, a projective integral separated scheme (of finite type over \mathbb{k}). An \mathcal{O}_X -bimodule is a quasicoherent $\mathcal{O}_{X \times X}$ -module \mathcal{F} , such that for every coherent $\mathcal{F}' \subseteq \mathcal{F}$, the projection maps $p_1, p_2 : \text{Supp } \mathcal{F}' \rightarrow X$ are both finite morphisms. The left and right \mathcal{O}_X -module structures associated to an \mathcal{O}_X -bimodule \mathcal{F} are defined respectively as $(p_1)_* \mathcal{F}$ and $(p_2)_* \mathcal{F}$. We make the notational convention that when we refer to an \mathcal{O}_X -bimodule simply as an \mathcal{O}_X -module, we are using the left-handed structure (for example, when we refer to the global sections or higher cohomology of an \mathcal{O}_X -bimodule).

There is a tensor product operation on the category of bimodules that has the expected properties; see [28, Section 2].

All the bimodules that we consider will be constructed from bimodules of the following form:

Definition 2.2. Let X be a variety and let $\sigma, \tau \in \text{Aut}_{\mathbb{k}}(X)$. Let (σ, τ) denote the map

$$\begin{aligned} X &\rightarrow X \times X, \\ x &\mapsto (\sigma(x), \tau(x)). \end{aligned}$$

If \mathcal{F} is a quasicoherent sheaf on X , we define the \mathcal{O}_X -bimodule ${}_\sigma\mathcal{F}_\tau$ to be

$${}_\sigma\mathcal{F}_\tau = (\sigma, \tau)_*\mathcal{F}.$$

If $\sigma = 1$ is the identity, we will often omit it; thus we write \mathcal{F}_τ for ${}_1\mathcal{F}_\tau$ and \mathcal{F} for the \mathcal{O}_X -bimodule ${}_1\mathcal{F}_1 = \Delta_*\mathcal{F}$, where $\Delta : X \rightarrow X \times X$ is the diagonal.

We quote a lemma that shows how to work with bimodules of the form ${}_\sigma\mathcal{F}_\tau$ and, in particular, how to form their tensor product. If σ is an automorphism of X and \mathcal{F} is a sheaf on X , recall the notation that $\mathcal{F}^\sigma = \sigma^*\mathcal{F}$. Thus σ acts on functions by sending f to $f^\sigma = f \circ \sigma$.

Lemma 2.3 ([17, Lemma 2.3]). *Let X be a variety, let \mathcal{F}, \mathcal{G} be coherent \mathcal{O}_X -modules, and let $\sigma, \tau \in \text{Aut}_{\mathbb{k}} X$. Then:*

- (1) ${}_\tau\mathcal{F}_\sigma \cong (\mathcal{F}^{\tau^{-1}})_{\sigma\tau^{-1}}$.
- (2) $\mathcal{F}_\sigma \otimes \mathcal{G}_\tau \cong (\mathcal{F} \otimes \mathcal{G}^\sigma)_{\tau\sigma}$.
- (3) In particular, if \mathcal{L} is an invertible sheaf on X , then $\mathcal{L}_\sigma^{\otimes n} = (\mathcal{L}_n)_{\sigma^n}$. □

Definition 2.4. Let X be a variety and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. An \mathcal{O}_X -bimodule algebra, or simply a *bimodule algebra*, \mathcal{B} is an algebra object in the category of bimodules. That is, there are a unit map $1 : \mathcal{O}_X \rightarrow \mathcal{B}$ and a product map $\mu : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$ that have the usual properties.

We follow [17] and define

Definition 2.5. Let X be a variety and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. A bimodule algebra \mathcal{B} is a *graded* (\mathcal{O}_X, σ) -bimodule algebra if:

- (1) there are coherent sheaves \mathcal{B}_n on X such that

$$\mathcal{B} = \bigoplus_{n \in \mathbb{Z}} {}_1(\mathcal{B}_n)_{\sigma^n};$$

- (2) $\mathcal{B}_0 = \mathcal{O}_X$;

- (3) the multiplication map μ is given by \mathcal{O}_X -module maps $\mathcal{B}_n \otimes \mathcal{B}_m^{\sigma^n} \rightarrow \mathcal{B}_{n+m}$, satisfying the obvious associativity conditions.

Definition 2.6. Let X be a variety and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. Let \mathcal{R} be a graded (\mathcal{O}_X, σ) -bimodule algebra. A *right \mathcal{R} -module* \mathcal{M} is a quasicoherent \mathcal{O}_X -module \mathcal{M} together with a right \mathcal{O}_X -module map $\mu : \mathcal{M} \otimes \mathcal{R} \rightarrow \mathcal{M}$ satisfying the usual axioms. We say that \mathcal{M} is *graded* if there is a direct sum decomposition

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n)_{\sigma^n}$$

with multiplication giving a family of \mathcal{O}_X -module maps $\mathcal{M}_n \otimes \mathcal{R}_m^{\sigma^n} \rightarrow \mathcal{M}_{n+m}$, obeying the appropriate axioms.

We say that \mathcal{M} is *coherent* if there are a coherent \mathcal{O}_X -module \mathcal{M}' and a surjective map $\mathcal{M}' \otimes \mathcal{R} \rightarrow \mathcal{M}$ of ungraded \mathcal{O}_X -modules. We make similar definitions for left \mathcal{R} -modules. The bimodule algebra \mathcal{R} is *right (left) noetherian* if every right (left) ideal of \mathcal{R} is coherent. By standard arguments, a graded (\mathcal{O}_X, σ) -bimodule algebra is right (left) noetherian if and only if every graded right (left) ideal is coherent.

We alert the reader to a technicality of notation. Suppose that \mathcal{R} is a bimodule algebra on X , that \mathcal{M} is a right \mathcal{R} -module, and that \mathcal{F} is a subsheaf of \mathcal{M} . We will denote the image of $\mathcal{F} \otimes \mathcal{R}$ under the multiplication map

$$\mathcal{F} \otimes \mathcal{R} \longrightarrow \mathcal{M} \otimes \mathcal{R} \xrightarrow{\mu} \mathcal{M}$$

by $\mathcal{F} \cdot \mathcal{R}$. In the case that $\mathcal{F} \subseteq \mathcal{O}_X$ is also an ideal sheaf on X , to avoid ambiguity we will denote the image of $\mathcal{F} \otimes \mathcal{R}$ under the canonical map

$$\mathcal{F} \otimes \mathcal{R} \longrightarrow \mathcal{O}_X \otimes \mathcal{R} = \mathcal{R}$$

by \mathcal{FR} .

Coherence for \mathcal{R} -modules should be viewed as analogous to finite generation, but it is unknown whether, for a general noetherian bimodule algebra, every submodule of a coherent module is coherent! Fortunately, in our situation the usual intuitions do hold. We restate [17, Proposition 2.10] as:

Lemma 2.7. *Let X be a variety and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. Let $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{R}_n)_{\sigma^n}$ be a graded (\mathcal{O}_X, σ) -bimodule algebra such that each \mathcal{R}_n is a subsheaf of an invertible sheaf. Then \mathcal{R} is right (left) noetherian if and only if all submodules of coherent right (left) \mathcal{R} -modules are coherent. \square*

We recall here some standard notation on module categories over rings and bimodule algebras. Let R be an \mathbb{N} -graded ring. We define $\text{Gr-}R$ to be the category of \mathbb{Z} -graded right R -modules; morphisms in $\text{Gr-}R$ preserve degree. Let $\text{Tors-}R$ be the full subcategory of modules that are direct limits of finite-dimensional modules. This is a Serre subcategory of $\text{Gr-}R$, so we may form the *quotient category*

$$\text{Qgr-}R := \text{Gr-}R / \text{Tors-}R.$$

(We refer the reader to [11] as a reference for the category theory used here.) There is a canonical *quotient functor* $\pi : \text{Gr-}R \rightarrow \text{Qgr-}R$.

We make similar definitions on the left. Further, throughout this paper, we adopt the convention that if Xyz is a category, then xyz is the full subcategory of noetherian objects. Thus we have $\text{gr-}R$ and $\text{qgr-}R = \pi(\text{gr-}R)$, $R\text{-qgr}$, etc. If X is a variety, we will denote the category of quasicohherent (respectively coherent) sheaves on X by $\mathcal{O}_X\text{-Mod}$ (respectively $\mathcal{O}_X\text{-mod}$).

Given a module $M \in \text{gr-}R$, we define $M[n] = \bigoplus_{i \in \mathbb{Z}} M[n]_i$, where

$$M[n]_i = M_{n+i}.$$

If $M, N \in \text{gr-}R$, let

$$\underline{\text{Hom}}_{\text{gr-}R}(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{gr-}R}(M, N[n]).$$

Similarly, if $\mathcal{M}, \mathcal{N} \in \text{qgr-}R$, we define

$$\underline{\text{Hom}}_{\text{qgr-}R}(\mathcal{M}, \mathcal{N}) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{qgr-}R}(\mathcal{M}, \mathcal{N}[n]).$$

The $\underline{\text{Hom}}$ functors have derived functors $\underline{\text{Ext}}_{\text{gr-}R}$ and $\underline{\text{Ext}}_{\text{qgr-}R}$.

For a graded (\mathcal{O}_X, σ) -bimodule algebra \mathcal{R} , we likewise define $\text{Gr-}\mathcal{R}$ and $\text{gr-}\mathcal{R}$. The full subcategory $\text{Tors-}\mathcal{R}$ of $\text{Gr-}\mathcal{R}$ consists of direct limits of modules that are coherent as \mathcal{O}_X -modules, and we similarly define

$$\text{Qgr-}\mathcal{R} := \text{Gr-}\mathcal{R} / \text{Tors-}\mathcal{R}.$$

We define $\text{qgr-}\mathcal{R}$ in the obvious way.

If \mathcal{R} is a graded (\mathcal{O}_X, σ) -bimodule algebra, we may form its *section algebra*

$$H^0(X, \mathcal{R}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{R}_n).$$

Multiplication on $H^0(X, \mathcal{R})$ is induced from the multiplication map μ on \mathcal{R} ; that is, from the maps

$$H^0(X, \mathcal{R}_n) \otimes H^0(X, \mathcal{R}_m) \xrightarrow{1 \otimes \sigma^n} H^0(X, \mathcal{R}_n) \otimes H^0(X, \mathcal{R}_m^{\sigma^n}) \xrightarrow{\mu} H^0(X, \mathcal{R}_{n+m}).$$

If \mathcal{M} is a graded right \mathcal{R} -module, then

$$H^0(X, \mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{M}_n)$$

is a right $H^0(X, \mathcal{R})$ -module in the obvious way; thus $H^0(X, _)$ is a functor from $\text{Gr-}\mathcal{R}$ to $\text{Gr-}H^0(X, \mathcal{R})$.

If $R = H^0(X, \mathcal{R})$ and M is a graded right R -module, define $M \otimes_R \mathcal{R}$ to be the sheaf associated to the presheaf $V \mapsto M \otimes_R \mathcal{R}(V)$. This is a graded right \mathcal{R} -module, and the functor $_ \otimes_R \mathcal{R} : \text{Gr-}R \rightarrow \text{Gr-}\mathcal{R}$ is a right adjoint to $H^0(X, _)$.

The fundamental result on when one can more closely relate $\text{Gr-}\mathcal{R}$ and $\text{Gr-}R$ is due to Van den Bergh. We first give a definition:

Definition 2.8. Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let $\{\mathcal{R}_n\}_{n \in \mathbb{N}}$ be a sequence of coherent sheaves on X . The sequence of bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}_{n \in \mathbb{N}}$ is *right ample* if for any coherent \mathcal{O}_X -module \mathcal{F} , the following properties hold:

- (i) $\mathcal{F} \otimes \mathcal{R}_n$ is globally generated for $n \gg 0$;
- (ii) $H^q(X, \mathcal{F} \otimes \mathcal{R}_n) = 0$ for $n \gg 0$ and all $q \geq 1$.

The sequence $\{(\mathcal{R}_n)_{\sigma^n}\}_{n \in \mathbb{N}}$ is *left ample* if for any coherent \mathcal{O}_X -module \mathcal{F} , the following properties hold:

- (i) $\mathcal{R}_n \otimes \mathcal{F}^{\sigma^n}$ is globally generated for $n \gg 0$;
- (ii) $H^q(X, \mathcal{R}_n \otimes \mathcal{F}^{\sigma^n}) = 0$ for $n \gg 0$ and all $q \geq 1$.

We say that an invertible sheaf \mathcal{L} is σ -*ample* if the \mathcal{O}_X -bimodules

$$\{(\mathcal{L}_n)_{\sigma^n}\}_{n \in \mathbb{N}} = \{\mathcal{L}_{\sigma^n}^{\otimes n}\}_{n \in \mathbb{N}}$$

form a right ample sequence. By [15, Theorem 1.2], this is true if and only if the \mathcal{O}_X -bimodules $\{(\mathcal{L}_n)_{\sigma^n}\}_{n \in \mathbb{N}}$ form a left ample sequence.

The following result is a special case of a result due to Van den Bergh [28, Theorem 5.2], although we follow the presentation of [17, Theorem 2.12]:

Theorem 2.9 (Van den Bergh). *Let X be a projective scheme and let σ be an automorphism of X . Let $\mathcal{R} = \bigoplus (\mathcal{R}_n)_{\sigma^n}$ be a right noetherian graded (\mathcal{O}_X, σ) -bimodule algebra, such that the bimodules $\{(\mathcal{R}_n)_{\sigma^n}\}$ form a right ample sequence. Then $R = H^0(X, \mathcal{R})$ is also right noetherian, and the functors $H^0(X, _)$ and $_ \otimes_R \mathcal{R}$ induce an equivalence of categories*

$$\text{qgr-}\mathcal{R} \simeq \text{qgr-}R.$$

□

Example 2.10 (Twisted bimodule algebras). Let X be a projective scheme, let $\sigma \in \text{Aut}_{\mathbb{k}}(X)$, and let \mathcal{L} be an invertible sheaf on X . We define the *twisted bimodule algebra of \mathcal{L}* to be

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} \mathcal{L}_n.$$

Then \mathcal{B} is an (\mathcal{O}_X, σ) -graded bimodule algebra. Taking global sections of $\mathcal{B}(X, \mathcal{L}, \sigma)$ gives the twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$.

Assume now that \mathcal{L} is σ -ample. Here we record some important properties of B and \mathcal{B} . By Theorem 2.9, the categories $\text{qgr-}\mathcal{B}$ and $\text{qgr-}B$ are equivalent. This was originally proved by Artin and Van den Bergh [3], who also observed that both categories are equivalent to $\mathcal{O}_X\text{-mod}$. The equivalence between $\text{qgr-}B$ and $\mathcal{O}_X\text{-mod}$ is given as follows. Define a functor

$$\Gamma_* : \mathcal{O}_X\text{-mod} \rightarrow \text{gr-}B,$$

$$\mathcal{F} \mapsto \bigoplus_{n \geq 0} H^0(X, \mathcal{F} \otimes \mathcal{L}_n).$$

The composition $\pi\Gamma_*$ has a quasi-inverse, induced by the functor

$$\widetilde{} : \text{gr-}B \rightarrow \mathcal{O}_X\text{-mod}.$$

To define this functor, let $M \in \text{gr-}B$. There is a unique coherent sheaf \mathcal{F} such that $\mathcal{F} \otimes \mathcal{L}_n = (M \otimes_B \mathcal{B})_n$ for all $n \gg 0$. Define $\widetilde{M} = \mathcal{F}$.

Since σ -ampleness is left-right symmetric, there is also an equivalence $B\text{-qgr} \simeq \mathcal{O}_X\text{-mod}$. The quasi-inverses between these two categories are defined by letting

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \geq 0} H^0(\mathcal{L}_n \otimes \mathcal{F}^{\sigma^n}) \in B\text{-gr}$$

and letting \widetilde{M} be the unique \mathcal{F} such that $\mathcal{L}_n \otimes \mathcal{F}^{\sigma^n} = (\mathcal{B} \otimes_B M)_n$ for all $n \gg 0$.

We note that the shift functors $(_)[n]$ on $\text{gr-}B$ induce autoequivalences of $\mathcal{O}_X\text{-mod}$. By [26, (3.1)], if $N \in \text{gr-}B$, then

$$(2.1) \quad \widetilde{N[n]} \cong (\widetilde{N} \otimes \mathcal{L}_n)^{\sigma^{-n}}$$

for all $n \geq 0$.

Throughout this paper, we will work with sub-bimodule algebras of the twisted bimodule algebra $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$. Here we note that the invertible sheaf \mathcal{L} makes only a formal difference.

Lemma 2.11. *Let X be a projective scheme with automorphism σ , and let \mathcal{L} be an invertible sheaf on X . Let*

$$\mathcal{R} = \bigoplus_{n \geq 0} (\mathcal{R}_n)_{\sigma^n}$$

be a graded (\mathcal{O}_X, σ) -sub-bimodule algebra of the twisted bimodule algebra $\mathcal{B}(X, \mathcal{L}, \sigma)$. Let $\mathcal{S}_n = \mathcal{R}_n \otimes \mathcal{L}_n^{-1}$ for $n \geq 0$.

(1) *Let \mathcal{S} be the graded (\mathcal{O}_X, σ) -bimodule algebra defined by*

$$\mathcal{S} = \bigoplus_{n \geq 0} (\mathcal{S}_n)_{\sigma^n}.$$

Then the categories $\text{gr-}\mathcal{R}$ and $\text{gr-}\mathcal{S}$ are equivalent, and the categories $\mathcal{S}\text{-gr}$ and $\mathcal{R}\text{-gr}$ are equivalent.

(2) *Let \mathcal{H} be an invertible sheaf on X and let $k \in \mathbb{Z}$. Then the functor $\mathcal{H}_{\sigma^k} \otimes _$ that maps*

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n)_{\sigma^n} \mapsto \bigoplus_{n \in \mathbb{Z}} (\mathcal{H} \otimes \mathcal{M}_n^{\sigma^k})_{\sigma^{k+n}} = \mathcal{H}_{\sigma^k} \otimes \mathcal{M}$$

is an autoequivalence of $\text{gr-}\mathcal{R}$.

Proof. (1) By symmetry, it suffices to prove that $\text{gr-}\mathcal{R} \simeq \text{gr-}\mathcal{S}$. For $n < 0$, define

$$\mathcal{L}_n = (\mathcal{L}^{-1})^{\sigma^n} \otimes (\mathcal{L}^{-1})^{\sigma^{n+1}} \otimes \cdots \otimes (\mathcal{L}^{-1})^{\sigma^{-1}}.$$

Define a functor $F : \text{gr-}\mathcal{R} \rightarrow \text{gr-}\mathcal{S}$ as follows: if

$$\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n)_{\sigma^n}$$

is a graded right \mathcal{R} -module, define

$$F(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} (\mathcal{M}_n \otimes (\mathcal{L}_n)^{-1})_{\sigma^n}.$$

The inverse functor $G : \text{gr-}\mathcal{S} \rightarrow \text{gr-}\mathcal{R}$ is defined as follows: if

$$\mathcal{N} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{N}_n)_{\sigma^n}$$

is a graded right \mathcal{S} -module, let

$$G(\mathcal{N}) = \bigoplus_{n \in \mathbb{Z}} (\mathcal{N}_n \otimes \mathcal{L}_n)_{\sigma^n}.$$

It is trivial that $GF \cong \text{Id}_{\text{gr-}\mathcal{R}}$ and that $FG \cong \text{Id}_{\text{gr-}\mathcal{S}}$.

(2) By Lemma 2.3(2), we have that

$$((\mathcal{H}^{\sigma^{-k}})^{-1})_{\sigma^{-k}} \otimes \mathcal{H}_{\sigma^k} \cong_1 ((\mathcal{H}^{\sigma^{-k}})^{-1} \otimes \mathcal{H}^{\sigma^{-k}})_1 \cong \mathcal{O}_X.$$

Thus the functor $((\mathcal{H}^{\sigma^{-k}})^{-1})_{\sigma^{-k}} \otimes _$ is a quasi-inverse to $\mathcal{H}_{\sigma^k} \otimes _$. □

To end this section, we give the sheaf-theoretic versions of some standard results on primary decomposition of ideals in a commutative ring. Let X be a locally noetherian scheme and let \mathcal{I} be a proper ideal sheaf on X . We will say that \mathcal{I} is *prime* if it defines a reduced and irreducible subscheme of X . We say that \mathcal{I} is *\mathcal{P} -primary* if there is a prime ideal sheaf \mathcal{P} such that some $\mathcal{P}^n \subseteq \mathcal{I}$, and for all ideal sheaves \mathcal{J} and \mathcal{K} on X , if $\mathcal{J}\mathcal{K} \subseteq \mathcal{I}$ but $\mathcal{J} \not\subseteq \mathcal{P}$, then $\mathcal{K} \subseteq \mathcal{I}$.

Since primary decompositions localize, the theory of primary decomposition of ideals in a commutative ring translates straightforwardly to ideal sheaves on X . In particular, any ideal sheaf \mathcal{I} has a *minimal primary decomposition*

$$\mathcal{I} = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_c,$$

where each \mathcal{I}_i is \mathcal{P}_i -primary for some prime ideal sheaf \mathcal{P}_i , the \mathcal{P}_i are all distinct, and \mathcal{I} may not be written as an intersection with fewer terms. If \mathcal{P}_i is a minimal prime over \mathcal{I} , then we will refer to \mathcal{I}_i as a *minimal primary component* of \mathcal{I} . If \mathcal{P}_i is not minimal over \mathcal{I} , we will refer to \mathcal{I}_i as an *embedded primary component*. As is well known, the primes \mathcal{P}_i and the minimal primary components of \mathcal{I} are uniquely determined by \mathcal{I} , while the embedded primary components are not necessarily unique.

Now let Z be a closed subscheme of X and let \mathcal{I} be the ideal sheaf of Z . Let $\mathcal{I} = \mathcal{I}_1 \cap \cdots \cap \mathcal{I}_c$ be a minimal primary decomposition of \mathcal{I} . We will refer to the closed subschemes Z_i defined by the minimal primary components \mathcal{I}_i of \mathcal{I} as the *irreducible components* of Z . We will refer to the subschemes defined by embedded primary components as *embedded components* of Z . Together, the irreducible and embedded components make up the *primary components* of Z .

For any two ideal sheaves \mathcal{K} and \mathcal{J} on X , we define the *ideal quotient*

$$(\mathcal{J} : \mathcal{K})$$

to be the maximal coherent subsheaf \mathcal{F} of \mathcal{O}_X such that $\mathcal{K}\mathcal{F} \subseteq \mathcal{J}$. We record the following elementary lemmas for future use.

Lemma 2.12. *Let $\mathcal{I} = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_c$ be a primary decomposition of the ideal sheaf \mathcal{I} on the locally noetherian scheme X , where each \mathcal{I}_i is \mathcal{Q}_i -primary for some prime ideal sheaf \mathcal{Q}_i .*

(1) *If \mathcal{K} and \mathcal{J} are ideal sheaves so that $\mathcal{K} \not\subseteq \mathcal{Q}_i$ for some i , then*

$$(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})) \subseteq (\mathcal{I}_i : \mathcal{J}).$$

(2) *If \mathcal{K} is not contained in any \mathcal{Q}_i , then $(\mathcal{I} : \mathcal{K}) = \mathcal{I}$.*

Proof. (1) We have

$$(\mathcal{I} : (\mathcal{K} \cap \mathcal{J}))\mathcal{J}\mathcal{K} \subseteq (\mathcal{I} : (\mathcal{K} \cap \mathcal{J}))(\mathcal{K} \cap \mathcal{J}) \subseteq \mathcal{I} \subseteq \mathcal{I}_i.$$

As $\mathcal{K} \not\subseteq \mathcal{Q}_i$ and \mathcal{I}_i is \mathcal{Q}_i -primary, $(\mathcal{I} : (\mathcal{K} \cap \mathcal{J}))\mathcal{J} \subseteq \mathcal{I}_i$, and so by definition

$$(\mathcal{I} : (\mathcal{K} \cap \mathcal{J})) \subseteq (\mathcal{I}_i : \mathcal{J}).$$

(2) Applying (1) with $\mathcal{J} = \mathcal{O}_X$, we see that

$$(\mathcal{I} : \mathcal{K}) \subseteq \bigcap_{i=1}^c (\mathcal{I}_i : \mathcal{O}_X) = \bigcap_{i=1}^c \mathcal{I}_i = \mathcal{I}.$$

The other containment is automatic. □

Lemma 2.13. *Let \mathcal{P} and \mathcal{I} be ideal sheaves on the locally noetherian scheme X , where \mathcal{P} is prime and \mathcal{I} is \mathcal{P} -primary. If \mathcal{J} is an ideal sheaf on X that is not contained in \mathcal{I} , then $(\mathcal{I} : \mathcal{J})$ is also \mathcal{P} -primary.*

Proof. Since $\mathcal{J} \not\subseteq \mathcal{I}$, we have that $(\mathcal{I} : \mathcal{J}) \neq \mathcal{O}_X$. Suppose that \mathcal{F} and \mathcal{G} are ideal sheaves with $\mathcal{F} \not\subseteq \mathcal{P}$ and $\mathcal{F}\mathcal{G} \subseteq (\mathcal{I} : \mathcal{J})$. Thus $\mathcal{F}\mathcal{G}\mathcal{J} \subseteq \mathcal{I}$, and since \mathcal{I} is \mathcal{P} -primary, $\mathcal{G}\mathcal{J} \subseteq \mathcal{I}$. This precisely says that $\mathcal{G} \subseteq (\mathcal{I} : \mathcal{J})$. Since for some m we have $\mathcal{P}^m \subseteq \mathcal{I} \subseteq (\mathcal{I} : \mathcal{J})$, we see that $(\mathcal{I} : \mathcal{J})$ is \mathcal{P} -primary. □

3. RIGHT NOETHERIAN BIMODULE ALGEBRAS

Let X, \mathcal{L}, σ , and Z be as in Construction 1.1, and let R be the geometric idealizer ring

$$R = R(X, \mathcal{L}, \sigma, Z).$$

In this section, we begin the study of R by working with the corresponding bimodule algebra. Here we introduce the notation we will use.

Notation 3.1. Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let \mathcal{L} be an invertible sheaf on X . Let Z be a closed subscheme of X and let $\mathcal{I} = \mathcal{I}_Z$ be its defining ideal. Let

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} (\mathcal{L}_n)_{\sigma^n},$$

and let $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ be the graded (\mathcal{O}_X, σ) -sub-bimodule algebra of \mathcal{B} defined by

$$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} ((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n)_{\sigma^n}.$$

It is easy to see that \mathcal{R} is the maximal sub-bimodule algebra of \mathcal{B} such that $\mathcal{I}\mathcal{B}_+$ is a two-sided ideal of \mathcal{R} , and we will write

$$\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I}\mathcal{B}_+)$$

and speak of \mathcal{R} as an *idealizer bimodule algebra* inside \mathcal{B} . As usual we write

$$\mathcal{R} = \bigoplus_{n \geq 0} (\mathcal{R}_n)_{\sigma^n},$$

so

$$\mathcal{R}_n = (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n.$$

In the next two sections, we give geometric conditions on the defining data $(X, \mathcal{L}, \sigma, Z)$ that determine when $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is left or right noetherian. In this section, we consider when $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is right noetherian; we will show this is controlled by a straightforward geometric property of the intersection of Z with σ -orbits.

Definition 3.2. Let $x \in X$ and let σ be an automorphism of X . The *forward σ -orbit* or *forward orbit* of x is the set

$$\{\sigma^n(x) \mid n \geq 0\}.$$

If $Z \subset X$ is such that for any $x \in X$ the set

$$\{n \geq 0 \mid \sigma^n(x) \in Z\}$$

is finite, we say that Z has *finite intersection with forward orbits*. In particular, if Z has finite intersection with forward orbits, it contains no points of finite order under σ .

Recall that if R is a \mathbb{Z} -graded or \mathbb{N} -graded ring, and $1 \leq n \in \mathbb{N}$, then the *n th Veronese of R* is the graded ring $R^{(n)}$ defined by

$$R_i^{(n)} = R_{ni}.$$

We note here that

$$\mathcal{B}(X, \mathcal{L}, \sigma)^{(n)} \cong \mathcal{B}(X, \mathcal{L}_n, \sigma^n)$$

and that

$$\mathcal{R}(X, \mathcal{L}, \sigma, Z)^{(n)} \cong \mathcal{R}(X, \mathcal{L}_n, \sigma^n, Z).$$

We will show:

Theorem 3.3. *Assume Notation 3.1. Let*

$$(3.1) \quad \mathcal{I} = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_c \cap \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_e$$

be a minimal primary decomposition of \mathcal{I} , where each \mathcal{J}_i is \mathcal{P}_i -primary for some prime ideal sheaf \mathcal{P}_i of finite order under σ , and each \mathcal{K}_j is \mathcal{Q}_j -primary for some prime ideal sheaf \mathcal{Q}_j of infinite order under σ . Let W be the closed subscheme of Z defined by the ideal sheaf $\mathcal{K} = \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_e$, and let $\mathcal{J} = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_c$. Then the following are equivalent:

- (1) $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is right noetherian;
- (2) there is some n so that $\mathcal{J}^{\sigma^n} = \mathcal{J}$, and either $W = X$ or W has finite intersection with forward σ -orbits.

Furthermore, if (2) and (1) hold, then \mathcal{R} is a finite right module over $\mathcal{R}^{(n)}$, and there are an integer n_0 and a closed subscheme W' of W , with ideal sheaf \mathcal{K}' , so that $(W')^{\text{red}} = W^{\text{red}}$, and such that

$$\mathcal{R}(X, \mathcal{L}_n, \sigma^n, Z)_{\geq n_0} = \mathcal{R}(X, \mathcal{L}_n, \sigma^n, W')_{\geq n_0} = \mathcal{K}'(\mathcal{B}^{(n)})_{\geq n_0}.$$

That is, any right noetherian right idealizer is a finite right module over a right idealizer at a subscheme without fixed components.

Before proving Theorem 3.3, we give some preliminary lemmas.

Lemma 3.4. *Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let \mathcal{L} be an invertible sheaf on X . Let \mathcal{B} be a graded (\mathcal{O}_X, σ) -sub-bimodule algebra of $\mathcal{B}(X, \mathcal{L}, \sigma)$, and let \mathcal{R} and \mathcal{R}' be graded (\mathcal{O}_X, σ) -sub-bimodule algebras of \mathcal{B} . Suppose that \mathcal{R} is right noetherian and contains a nonzero graded right ideal of \mathcal{B} and that there is some n_0 so that*

$$\mathcal{R}_{\geq n_0} \subseteq \mathcal{R}'_{\geq n_0}.$$

Then \mathcal{R}' is right noetherian. If $\mathcal{R} \subseteq \mathcal{R}'$, then \mathcal{R}' is a coherent right \mathcal{R} -module.

Proof. By Lemma 2.11, without loss of generality we may assume that $\mathcal{L} = \mathcal{O}_X$.

We note that $\mathcal{R}_{\geq n_0}$ also contains a nonzero graded right ideal of \mathcal{B} . Further, $\mathcal{R} \cap \mathcal{R}'$ is also right noetherian, as $(\mathcal{R} \cap \mathcal{R}')_{\geq n_0} = \mathcal{R}_{\geq n_0}$. Thus without loss of generality we may assume that $\mathcal{R} \subseteq \mathcal{R}'$.

Let \mathcal{J} be a nonzero graded right ideal of \mathcal{B} that is contained in \mathcal{R} , and let m be such that $\mathcal{J}_m \neq 0$. Since X is a variety, there is an invertible ideal sheaf \mathcal{H} contained in \mathcal{J}_m . As \mathcal{R} is right noetherian and $\mathcal{H} \cdot \mathcal{R}' \subseteq \mathcal{J}_m \cdot \mathcal{B} \subseteq \mathcal{R}$, we see that $\mathcal{H} \cdot \mathcal{R}'$ is a coherent right \mathcal{R} -module. Lemma 2.11 now implies that \mathcal{R}' is a coherent right \mathcal{R} -module.

Any right ideal of \mathcal{R}' is also a right \mathcal{R} -submodule, and so it is coherent as an \mathcal{R} -module. It is thus also coherent as an \mathcal{R}' -module. Thus \mathcal{R}' is right noetherian. \square

The next few lemmas involve translating general results on idealizers to the context of bimodule algebras. The proofs are all easy generalizations of the original proofs.

The following result is originally due to Robson [19, Proposition 2.3(i)], although we will follow Stafford’s restatement of it.

Lemma 3.5. [25, Lemma 1.1] *Let I be a right ideal of a right noetherian ring B , and let $R = \mathbb{I}_B(I)$. If B/I is a right noetherian R -module, then R is right noetherian. \square*

Our version of this is the following lemma.

Lemma 3.6. *Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let \mathcal{L} be an invertible sheaf on X . Let $\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$, and let $\mathcal{I} = \bigoplus (\mathcal{I}_n)_{\sigma^n}$ be a nonzero graded right ideal of \mathcal{B} . Let $\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I})$. Then \mathcal{B}/\mathcal{I} is a noetherian right \mathcal{R} -module if and only if \mathcal{R} is right noetherian.*

Proof. The proof is a straightforward translation of [19, Proposition 2.3(i)] into sheaf terminology. By Lemma 2.11, without loss of generality we may let $\mathcal{L} = \mathcal{O}_X$. Thus all \mathcal{R}_n and all \mathcal{I}_n are ideal sheaves on X .

By Lemma 3.4, if \mathcal{R} is right noetherian, certainly $\mathcal{B}_{\mathcal{R}}$ and thus $(\mathcal{B}/\mathcal{I})_{\mathcal{R}}$ are also right noetherian. So suppose that \mathcal{B}/\mathcal{I} is a noetherian right \mathcal{R} -module. Let \mathcal{J} be a right ideal of \mathcal{R} ; we will show that \mathcal{J} is coherent. Because \mathcal{B} is right noetherian,

we may choose a coherent sheaf $\mathcal{J}' \subseteq \mathcal{J}$ such that $\mathcal{J}' \cdot \mathcal{B} = \mathcal{J} \cdot \mathcal{B}$. It suffices to show that $\mathcal{J}/(\mathcal{J}' \cdot \mathcal{R})$ is a coherent right \mathcal{R} -module.

Now, $\mathcal{J}/(\mathcal{J}' \cdot \mathcal{R})$ is a submodule of $((\mathcal{J}' \cdot \mathcal{B}) \cap \mathcal{R})/(\mathcal{J}' \cdot \mathcal{R})$. Further, it is killed by \mathcal{I} and so is a subfactor of $\mathcal{J}' \otimes (\mathcal{B}/\mathcal{I})$. Since \mathcal{B}/\mathcal{I} is a noetherian right \mathcal{R} -module, so is $\mathcal{J}' \otimes (\mathcal{B}/\mathcal{I})$. Thus the subfactor $\mathcal{J}/(\mathcal{J}' \cdot \mathcal{R})$ is coherent. \square

The criterion in Lemma 3.5 can be hard to test. Stafford [25, Lemma 1.2] gave a different criterion for an idealizer to be noetherian; it was later slightly strengthened by Rogalski [21, Proposition 2.1]. We give the following version, which is adequate for our needs.

Lemma 3.7. *Let B be a right noetherian domain, let I be a right ideal of B , and let $R = \mathbb{I}_B(I)$. Then the following are equivalent:*

- (1) *R is right noetherian;*
- (2) *B_R is finitely generated, and for all right ideals J of B such that $J \supseteq I$, the right R -module (or R/I -module) $\text{Hom}_B(B/I, B/J)$ is noetherian.*

Proof. (2) \Rightarrow (1) is [25, Lemma 1.2]. For (1) \Rightarrow (2), note that if R is noetherian, as B is a domain we have $B_R \hookrightarrow R_R$, and so B_R is finitely generated. The rest of the argument is [21, Proposition 2.1]. \square

Our version of this is the following lemma:

Lemma 3.8. *Let X be a projective variety, and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. Let \mathcal{B} be a right noetherian graded (\mathcal{O}_X, σ) -sub-bimodule algebra of the twisted bimodule algebra $\mathcal{B}(X, \mathcal{O}_X, \sigma)$, and let $\mathcal{I} = \bigoplus (\mathcal{I}_n)_{\sigma^n}$ be a nonzero graded right ideal of \mathcal{B} . Let $\mathcal{R} = \mathbb{I}_{\mathcal{B}}(\mathcal{I})$. Suppose that for all graded right ideals $\mathcal{J} \supseteq \mathcal{I}$ of \mathcal{B} , for $n \gg 0$,*

$$\mathcal{B}_n \cap \bigcap_{m \geq 0} (\mathcal{J}_{n+m} : \mathcal{I}_m^{\sigma^n}) = \mathcal{J}_n.$$

Then \mathcal{R} is right noetherian.

Proof. We follow Stafford's proof of [25, Lemma 1.2]. Assume that the hypotheses of the lemma hold; we claim that \mathcal{B}/\mathcal{I} is a noetherian right \mathcal{R} -module.

Let \mathcal{G} be a graded right \mathcal{R} -module with $\mathcal{I} \subseteq \mathcal{G} \subseteq \mathcal{B}$. We seek to prove that \mathcal{G}/\mathcal{I} is coherent. Let \mathcal{J} be the largest graded right ideal of \mathcal{B} of the form $\mathcal{G}' \cdot \mathcal{I}$ for some coherent graded \mathcal{O}_X -submodule \mathcal{G}' of \mathcal{G} . (\mathcal{J} exists because \mathcal{B} is right noetherian.) By maximality of \mathcal{J} , we have $\mathcal{I} \subseteq \mathcal{J}$.

Using Zorn's lemma, let \mathcal{C} be the maximal quasicohherent subsheaf of \mathcal{B} such that $\mathcal{C} \cdot \mathcal{I} \subseteq \mathcal{J}$. Obviously, \mathcal{C} is graded. Note that

$$\mathcal{C}_n = \mathcal{B}_n \cap \bigcap_{m \geq 0} (\mathcal{J}_{n+m} : \mathcal{I}_m^{\sigma^n}).$$

Since $\mathcal{C} \cdot \mathcal{R} \cdot \mathcal{I} \subseteq \mathcal{C} \cdot \mathcal{I} \subseteq \mathcal{J}$, we have that $\mathcal{C} \cdot \mathcal{R} \subseteq \mathcal{C}$ and \mathcal{C} is a right \mathcal{R} -submodule of \mathcal{B} . Since by assumption $\mathcal{C}_n = \mathcal{J}_n$ for $n \gg 0$, the right \mathcal{R} -module \mathcal{C}/\mathcal{J} is in fact a coherent \mathcal{O}_X -module.

We claim that $\mathcal{G} \subseteq \mathcal{C}$. Suppose not. We may choose a coherent graded \mathcal{O}_X -submodule \mathcal{G}'' of \mathcal{G} such that $\mathcal{G}'' \not\subseteq \mathcal{C}$, and so $\mathcal{G}'' \cdot \mathcal{I} \not\subseteq \mathcal{J}$. Then $(\mathcal{G}' + \mathcal{G}'') \cdot \mathcal{I} \not\subseteq \mathcal{J}$ by choice of \mathcal{G}' , contradicting the maximality of \mathcal{J} . Thus $\mathcal{G} \subseteq \mathcal{C}$.

Since \mathcal{C}/\mathcal{J} is a coherent \mathcal{O}_X -module, so is the submodule \mathcal{G}/\mathcal{J} . Since $\mathcal{J}_{\mathcal{R}}$ is coherent and \mathcal{G}/\mathcal{J} is a coherent \mathcal{O}_X -module, $\mathcal{G}_{\mathcal{R}}$ is coherent. Thus $(\mathcal{G}/\mathcal{I})_{\mathcal{R}}$ is also

coherent. Since \mathcal{G} was arbitrary, we have shown that \mathcal{B}/\mathcal{I} is a right noetherian \mathcal{R} -module, as claimed.

Applying Lemma 3.6, we obtain that \mathcal{R} is a right noetherian bimodule algebra. \square

We are almost ready to give the proof of Theorem 3.3. One technical difficulty in the proof is that if Z has multiple components, it may be difficult to compute $(\mathcal{I} : \mathcal{I}^{\sigma^n})$ and thus \mathcal{R} . However, if Z is irreducible, then computing \mathcal{R} is straightforward; we record this in the next lemma.

Lemma 3.9. *Assume Notation 3.1. Suppose in addition that Z is irreducible and without embedded components. If Z^{red} has infinite order under σ , then $\mathcal{R} = \mathcal{O}_X \oplus \mathcal{I}\mathcal{B}_+$.*

Proof. Let \mathcal{P} be the ideal sheaf of Z^{red} . For $n \geq 1$, clearly $\mathcal{I}^{\sigma^n} \not\subseteq \mathcal{P}$, since \mathcal{I}^{σ^n} is \mathcal{P}^{σ^n} -primary and $\mathcal{P}^{\sigma^n} \neq \mathcal{P}$. The result follows from Lemma 2.12(2) and the identification $\mathcal{R}_n = (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n$. \square

We can now show that Theorem 3.3 holds under some additional assumptions on Z .

Lemma 3.10. *Assume Notation 3.1. Let*

$$\mathcal{I} = \mathcal{K}_1 \cap \cdots \cap \mathcal{K}_c$$

be a minimal primary decomposition of $\mathcal{I} = \mathcal{I}_Z$, where each \mathcal{K}_i is \mathcal{Q}_i -primary for some prime ideal sheaf \mathcal{Q}_i . For $i = 1, \dots, c$, let Z_i be the primary component of Z corresponding to \mathcal{K}_i , and let

$$\mathcal{R}^i = \mathbb{I}_{\mathcal{B}}(\mathcal{K}_i\mathcal{B}_+) = \mathcal{R}(X, \mathcal{L}, \sigma, Z_i).$$

Suppose that for all $1 \leq i, j \leq c$ the set

$$\{m \geq 0 \mid \mathcal{K}_j^{\sigma^m} \subseteq \mathcal{Q}_i\}$$

is finite. (In particular, we assume that the \mathcal{Q}_i are of infinite order under σ .) Then $\mathcal{R}_m = \mathcal{I}\mathcal{L}_m$ for $m \gg 0$. Further, the following are equivalent:

- (1) \mathcal{R} is right noetherian;
- (2) \mathcal{R}^i is right noetherian for $i = 1, \dots, c$;
- (3) Z has finite intersection with forward orbits;
- (4) if \mathcal{J} is an ideal sheaf on X such that $\mathcal{J} \supseteq \mathcal{I}$, then $(\mathcal{J} : \mathcal{I}^{\sigma^m}) = \mathcal{J}$ for $m \gg 0$;
- (5) the bimodule algebra $\mathcal{O}_X \oplus \mathcal{I}\mathcal{B}_+$ is right noetherian.

We note that the assumptions of the lemma are satisfied if Z consists of one primary component such that Z^{red} is of infinite order under σ .

Proof. By Lemma 2.11, we may without loss of generality assume that $\mathcal{L} = \mathcal{O}_X$. By Lemma 2.12(2),

$$(\mathcal{I} : \mathcal{I}^{\sigma^m}) = \mathcal{I}$$

for $m \gg 0$. Thus $\mathcal{R}_m = \mathcal{I}$ for $m \gg 0$, as claimed. Note that this implies that (1) \iff (5).

(1) \implies (2). Fix $1 \leq i \leq c$. By Lemma 3.9, $(\mathcal{R}^i)_m = \mathcal{K}_i$ for all $m \geq 1$. As $\mathcal{R}_m = \mathcal{I}$ for $m \gg 0$, there is some m_0 so that for $m \geq m_0$,

$$\mathcal{R}(X, \mathcal{O}_X, \sigma, Z)_m = \mathcal{I} \subseteq \mathcal{K}_i = (\mathcal{R}^i)_m.$$

By Lemma 3.4, \mathcal{R}^i is right noetherian.

(2) \Rightarrow (3) Since Z is the set-theoretic union of finitely many irreducible components, it is enough to prove (3) in the case that \mathcal{I} is itself primary; that is, in the case that $i = 1$. In this case, since $\mathcal{R} = \mathcal{R}^1$ is noetherian by assumption, by Lemma 3.4 $\mathcal{B}_{\mathcal{R}}$ is coherent. By Lemma 3.9, $\mathcal{R} = \mathcal{O}_X \oplus \mathcal{I}\mathcal{B}_+$.

Fix $x \in X$, and let \mathcal{I}_x be its ideal sheaf. Let

$$\mathcal{M} = \bigoplus_{n \geq 0} (\mathcal{I}_x : \mathcal{I}^{\sigma^n})_{\sigma^n} \subseteq \mathcal{B} = \bigoplus_{n \geq 0} (\mathcal{O}_X)_{\sigma^n}.$$

Let $m \geq 1$ and $n \geq 0$. By Lemma 3.9, $\mathcal{R}_m = \mathcal{I}$. Therefore,

$$(\mathcal{M}_n \cdot \mathcal{R})_{m+n} = (\mathcal{I}_x : \mathcal{I}^{\sigma^n})_{\sigma^n} \subseteq \mathcal{I}_x \subseteq \mathcal{M}_{m+n},$$

so \mathcal{M} is a right \mathcal{R} -submodule of \mathcal{B} . It is therefore coherent, and the quotient $\mathcal{M}/\mathcal{I}_x\mathcal{B}$ is as well. Since $\mathcal{M} \cdot \mathcal{I}\mathcal{B}_+ \subseteq \mathcal{I}_x\mathcal{B}$, the \mathcal{R} -action on $\mathcal{M}/\mathcal{I}_x\mathcal{B}$ factors through $\mathcal{R}/\mathcal{I}\mathcal{B}_+ = \mathcal{O}_X$. In other words, $\mathcal{M}/\mathcal{I}_x\mathcal{B}$ is a noetherian and therefore coherent \mathcal{O}_X -module, and so the ideal sheaves $(\mathcal{I}_x : \mathcal{I}^{\sigma^n})$ and \mathcal{I}_x are equal for $n \gg 0$. For fixed n , this is true if and only if $x \notin \sigma^{-n}Z$ or $\sigma^n(x) \notin Z$. Thus $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is finite.

(3) \Rightarrow (4). Let \mathcal{P} be a nonzero prime ideal sheaf, defining a reduced and irreducible subscheme $W \subset X$. Since for any $m \in \mathbb{Z}$ we have that $\mathcal{I}^{\sigma^m} \subseteq \mathcal{P}$ if and only if $\sigma^m(W)$ is (set-theoretically) contained in Z , we see that the set

$$\{m \geq 0 \mid \mathcal{I}^{\sigma^m} \subseteq \mathcal{P}\}$$

is finite.

Now let $\mathcal{J} \supseteq \mathcal{I}$ be an ideal sheaf on X , and let $\mathcal{J} = \mathcal{J}_1 \cap \dots \cap \mathcal{J}_e$ be a primary decomposition of \mathcal{J} , where \mathcal{J}_i is \mathcal{P}_i -primary for a suitable prime ideal sheaf \mathcal{P}_i . For $m \gg 0$ and for $i = 1, \dots, e$, we have $\mathcal{I}^{\sigma^m} \not\subseteq \mathcal{P}_i$. Therefore by Lemma 2.12(2), $(\mathcal{J} : \mathcal{I}^{\sigma^m}) = \mathcal{J}$ for $m \gg 0$.

(4) \Rightarrow (1). Suppose that for all $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{O}_X$, we have $(\mathcal{J} : \mathcal{I}^{\sigma^n}) = \mathcal{J}$ for $n \gg 0$. Let $\mathcal{F} \supseteq \mathcal{I}\mathcal{B}$ be a graded right ideal of \mathcal{B} , and for all $m \geq 0$ let

$$\mathcal{C}_m = \bigcap_{n \geq 0} (\mathcal{F}_{n+m} : \mathcal{I}^{\sigma^m}).$$

We saw in Example 2.10 that the categories $\text{qgr-}\mathcal{B}$ and $\mathcal{O}_X\text{-mod}$ are equivalent, and that there is an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ such that, for some k , we have

$$\mathcal{F}_{\geq k} = \bigoplus_{m \geq k} (\mathcal{J})_{\sigma^m}.$$

By construction, $\mathcal{J} \supseteq \mathcal{I}$. For $m \geq k$, we have $\mathcal{C}_m = (\mathcal{J} : \mathcal{I}^{\sigma^m})$. By assumption, this is equal to $\mathcal{J} = \mathcal{F}_m$ for $m \gg k$, and so the hypotheses of Lemma 3.8 hold. By Lemma 3.8, \mathcal{R} is right noetherian. \square

We now prove Theorem 3.3.

Proof of Theorem 3.3. We recall our notation:

$$\mathcal{I} = \mathcal{J}_1 \cap \dots \cap \mathcal{J}_c \cap \mathcal{K}_1 \cap \dots \cap \mathcal{K}_e$$

is a minimal primary decomposition of \mathcal{I} , where each \mathcal{J}_i is \mathcal{P}_i -primary for some prime ideal sheaf \mathcal{P}_i of finite order under σ , and each \mathcal{K}_j is \mathcal{Q}_j -primary for some prime ideal sheaf \mathcal{Q}_j of infinite order under σ . The ideal sheaf $\mathcal{K} = \mathcal{K}_1 \cap \dots \cap \mathcal{K}_e$ defines the closed subscheme W of X , and $\mathcal{J} = \mathcal{J}_1 \cap \dots \cap \mathcal{J}_c$.

By Lemma 2.11, we may without loss of generality assume that $\mathcal{L} = \mathcal{O}_X$.

(1) \Rightarrow (2). Suppose that \mathcal{R} is right noetherian. We first show this implies that there is some n so that all \mathcal{J}_i are fixed by σ^n . Suppose, in contrast, that for some i there is no n with $\mathcal{J}_i^{\sigma^n} = \mathcal{J}_i$. Since Veronese rings of \mathcal{R} are also right noetherian and \mathcal{P}_i has finite order under σ , we may assume without loss of generality that \mathcal{P}_i is fixed by σ .

Let $m \geq 1$. Since $(\mathcal{J}_i)^{\sigma^m} \neq \mathcal{J}_i$, by minimality of the primary decomposition (3.1), it is clear that $\mathcal{I}^{\sigma^m} \not\subseteq \mathcal{J}_i$. By Lemma 2.13 $(\mathcal{J}_i : \mathcal{I}^{\sigma^m}) \neq \mathcal{O}_X$ is \mathcal{P}_i -primary. Therefore

$$\mathcal{R}_m = (\mathcal{I} : \mathcal{I}^{\sigma^m}) \subseteq (\mathcal{J}_i : \mathcal{I}^{\sigma^m}) \subseteq \mathcal{P}_i$$

for all $m \geq 1$.

Let $\mathcal{B} = \mathcal{B}(X, \mathcal{O}_X, \sigma)$. For any $k \geq 0$, we have

$$(\mathcal{B}_{\leq k} \cdot \mathcal{R})_{k+1} = \sum_{j=0}^k (\mathcal{R}_{k+1-j})^{\sigma^j} \subseteq \mathcal{P}_i \neq \mathcal{O}_X = \mathcal{B}_{k+1}.$$

We see that $\mathcal{B}_{\mathcal{R}}$ is not finitely generated; by Lemma 3.4, this contradicts the assumption that \mathcal{R} is right noetherian. Thus \mathcal{J}_i is of finite order under σ .

As this holds for all i , there is some n so that $\mathcal{J}^{\sigma^n} = \mathcal{J}$. Suppose that $W \neq X$. Since W has finite intersection with forward σ -orbits if and only if W has finite intersection with forward σ^n -orbits, without loss of generality we may replace R by the Veronese $\mathcal{R}^{(n)}$ and assume that \mathcal{J} is σ -invariant. Suppose that W has infinite intersection with some forward σ -orbit. We will derive a contradiction.

For $i = 1, \dots, e$, let W_i be the primary component of Z defined by \mathcal{K}_i , and let $Y_i = W_i^{\text{red}}$ be the subvariety defined by the prime ideal sheaf \mathcal{Q}_i . We claim that there is some i so that

- (i) $Y_i \not\subseteq \sigma^{-m}(W)$ for $m \geq 1$;
- (ii) for some $x \in X$, the set $\{m \geq 0 \mid \sigma^m(x) \in Y_i\}$ is infinite.

To see this, note that we may define a strict partial order \prec on the set of the Y_i by defining

$$Y_i \prec Y_j \text{ if } Y_i \subseteq \sigma^{-m}(Y_j) \text{ for some } m \geq 1.$$

The order \prec is strict because each Y_i has infinite order under σ . Now (ii) holds for some Y_i by assumption. Thus (ii) holds for some Y_i that is maximal under \prec . But (i) holds for any such maximal Y_i , as the ideal sheaf of Y_i is prime.

Let Y_i satisfy (i) and (ii). We thus have $\mathcal{K}^{\sigma^m} \not\subseteq \mathcal{Q}_i$ for any $m \geq 1$. As

$$\mathcal{I}^{\sigma^m} = \mathcal{K}^{\sigma^m} \cap \mathcal{J}^{\sigma^m} = \mathcal{K}^{\sigma^m} \cap \mathcal{J},$$

by Lemma 2.12(1) we have

$$\mathcal{R}_m = (\mathcal{I} : \mathcal{I}^{\sigma^m}) \subseteq (\mathcal{K}_i : \mathcal{J})$$

for all $m \geq 1$. By minimality of the primary decomposition (3.1) and Lemma 2.13, the ideal sheaf $(\mathcal{K}_i : \mathcal{J})$ is \mathcal{Q}_i -primary.

Let V be the closed subscheme of X defined by $(\mathcal{K}_i : \mathcal{J})$. By Lemma 3.9,

$$\mathcal{R}(X, \mathcal{O}_X, \sigma, V) = \mathcal{O}_X \oplus (\mathcal{K}_i : \mathcal{J})\mathcal{B}_+,$$

so

$$\mathcal{R}(X, \mathcal{O}_X, \sigma, Z) \subseteq \mathcal{R}(X, \mathcal{O}_X, \sigma, V).$$

Thus by Lemma 3.4, $\mathcal{R}(X, \mathcal{O}_X, \sigma, V)$ is right noetherian. But V also has infinite intersection with some forward σ -orbit. This is impossible, by Lemma 3.10.

Thus W has finite intersection with forward σ -orbits.

(2) \Rightarrow (1). Suppose that (2) holds. We claim that

$$(3.2) \quad (\mathcal{I} : \mathcal{I}^{\sigma^m}) = \mathcal{R}_m = (\mathcal{I} : \mathcal{J}^{\sigma^m}) \quad \text{for } m \gg 0.$$

If $W = X$, then $\mathcal{I} = \mathcal{J}$ and (3.2) holds for all m . If $W \neq X$ has finite intersection with forward σ -orbits, then for $m \gg 0$, \mathcal{K}^{σ^m} is not contained in any minimal prime over \mathcal{I} . Thus by Lemma 2.12(1) we have that

$$(\mathcal{I} : \mathcal{I}^{\sigma^m}) \subseteq (\mathcal{I} : \mathcal{J}^{\sigma^m})$$

for $m \gg 0$. As the other containment is automatic, we see that (3.2) holds.

Now, if $n|m$, then

$$(\mathcal{I} : \mathcal{J}^{\sigma^m}) = (\mathcal{I} : \mathcal{J}) = (\mathcal{K} : \mathcal{J}),$$

and so (3.2) implies in particular that $\mathcal{R}^{(n)}$ and $\mathcal{O}_X \oplus (\mathcal{K} : \mathcal{J})(\mathcal{B}^{(n)})_+$ are equal in large degree.

If $W = X$, then $(\mathcal{K} : \mathcal{J}) = \mathcal{O}_X$ and $\mathcal{R}^{(n)} = \mathcal{B}^{(n)}$. If W has finite intersection with forward σ -orbits, then note that $(\mathcal{K} : \mathcal{J})$ is the intersection of the \mathcal{Q}_i -primary ideal sheaves $(\mathcal{K}_i : \mathcal{J})$. Let W' be the closed subscheme of X defined by $\mathcal{K}' = (\mathcal{K} : \mathcal{J})$. Then W' also has finite intersection with forward σ -orbits, and $(W')^{\text{red}} = W^{\text{red}}$. Applying Lemma 3.10 to $\mathcal{R}^{(n)}$, we obtain that $\mathcal{R}^{(n)}$ is right noetherian. By Lemma 3.4, $\mathcal{B}^{(n)}$ is a coherent right $\mathcal{R}^{(n)}$ -module.

Thus in either case, $\mathcal{B}^{(n)}$ is a coherent right $\mathcal{R}^{(n)}$ -module. Therefore, for any m the right ideal

$$(\mathcal{I} : \mathcal{J}^{\sigma^m})(\mathcal{B}^{(n)})$$

of $\mathcal{B}^{(n)}$ is a coherent right $\mathcal{R}^{(n)}$ -module. Applying (3.2) for $m = 0, \dots, n - 1$, we obtain that \mathcal{R} is a finitely generated right $\mathcal{R}^{(n)}$ -module, and so \mathcal{R} is right noetherian. \square

Example 3.11. We give an example illustrating what goes wrong when \mathcal{J} is of infinite order under σ . Let $X = \mathbb{P}^2$ and let

$$\sigma = \begin{bmatrix} 1 & & \\ & p & \\ & & q \end{bmatrix}$$

for some $p, q \in \mathbb{k}^*$ that are not roots of unity. Let $B = B(X, \mathcal{O}(1), \sigma)$. It is easy to see that B may be presented by generators and relations as

$$B \cong \mathbb{k}\langle X, Y, Z \rangle / (XY - pYX, XZ - qZX, YZ - (qp^{-1})ZY).$$

Let $a = [0 : 0 : 1]$ and let $\mathcal{O} = \mathcal{O}_{X,a}$. Let \mathfrak{m} be the maximal ideal of \mathcal{O} . As $\sigma(a) = a$, the automorphism σ acts on \mathcal{O} via

$$\begin{aligned} \sigma(x) &= x, \\ \sigma(y) &= py, \end{aligned}$$

where (x, y) is an appropriate system of parameters for \mathcal{O} .

Let \mathcal{I} be the ideal sheaf cosupported at a so that $\mathcal{I}_a = (x + y, \mathfrak{m}^2)$. Then for any n we have

$$(\mathcal{I}^{\sigma^n})_a = (x + p^n y, \mathfrak{m}^2).$$

Let \mathcal{M} be the ideal sheaf of a . We leave to the reader the computation that

$$(\mathcal{I} : \mathcal{I}^{\sigma^n}) = \mathcal{M}.$$

Thus, if Z is the subscheme defined by \mathcal{I} , we have that

$$R(X, \mathcal{O}(1), \sigma, Z) = \mathbb{k} + xB + yB.$$

This ring is not finitely generated, and so not right or left noetherian. Neither is the bimodule algebra $\mathcal{R}(X, \mathcal{O}(1), \sigma, Z) = \mathcal{O}_X \oplus \mathcal{MB}_+$.

4. LEFT NOETHERIAN BIMODULE ALGEBRAS

We now turn to considering when $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is left noetherian. Since one of our main goals is to understand when idealizers are noetherian, from now on we will assume the condition

Assumption-Notation 4.1. *Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let \mathcal{L} be an invertible sheaf on X . Let Z be a closed subscheme of X and let $\mathcal{I} = \mathcal{I}_Z$ be its defining ideal. Let*

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$$

and let

$$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} ((\mathcal{I} : \mathcal{I}^{\sigma^n}) \mathcal{L}_n)_{\sigma^n}.$$

We assume that $(\mathcal{I} : \mathcal{I}^{\sigma^n}) = \mathcal{I}$ for $n \gg 0$; that is, that $\mathcal{R}_n = \mathcal{I} \mathcal{L}_n$ for $n \gg 0$.

Assumption-Notation 4.1 is satisfied in the situations of most interest to us. In particular, by Theorem 3.3, any right noetherian idealizer bimodule algebra is, up to a finite extension, one whose defining data satisfies Assumption-Notation 4.1. Furthermore, if Z is irreducible and Z^{red} is of infinite order under σ , then Lemma 3.9 implies that Assumption-Notation 4.1 is satisfied. More generally, we have:

Lemma 4.2. *Assume Notation 3.1. Let Z_1, \dots, Z_c be the primary components of Z . For $i = 1, \dots, c$, let $Y_i = Z_i^{\text{red}}$. If $\{n \geq 0 \mid \sigma^n(Y_i) \subseteq Z\}$ is finite for all i , then $\mathcal{R}_n = \mathcal{I} \mathcal{L}_n$ for $n \gg 0$, and so Assumption-Notation 4.1 is satisfied.*

Proof. This follows directly from Lemma 2.12(2). □

As in the previous section, we will translate results on idealizer subrings to the context of bimodule algebras. We quote a result of Rogalski; we note that the original result was stated for left ideals of noetherian rings.

Proposition 4.3 ([21, Proposition 2.2]). *If $R = \mathbb{I}_B(I)$ for some right ideal I of a noetherian ring B , then R is left noetherian if and only if R/I is a left noetherian ring and for all left ideals J of B , the left R -module $\text{Tor}_1^B(B/I, B/J)$ is noetherian.* □

We note that if R/I is finite-dimensional, this result reduces to saying that R is left noetherian if and only if $\text{Tor}_1^B(B/I, B/J)$ is a finite-dimensional vector space for all left ideals J of B .

We now prove a version of Proposition 4.3 for the bimodule algebra \mathcal{R} .

Proposition 4.4. *Assume Assumption-Notation 4.1. Then \mathcal{R} is left noetherian if and only if for all graded left ideals \mathcal{H} of \mathcal{B} , we have*

$$\mathcal{I} \mathcal{L}_n \cap \mathcal{H}_n = (\mathcal{I} \mathcal{H})_n$$

for $n \gg 0$.

Proof. We follow Rogalski's proof of Proposition 4.3.

Since $(\mathcal{I}\mathcal{B} \cap \mathcal{H})/\mathcal{I}\mathcal{H}$ is a subfactor of ${}_{\mathcal{R}}\mathcal{R}$ that is killed on the left by $\mathcal{I}\mathcal{B}$, if \mathcal{R} is left noetherian, then this is a coherent module over $\mathcal{R}/\mathcal{I}\mathcal{B}$ and so is certainly a coherent \mathcal{O}_X -module.

For the other direction, suppose that for all graded left ideals \mathcal{H} of \mathcal{B} we have

$$\mathcal{I}\mathcal{L}_n \cap \mathcal{H}_n = (\mathcal{I}\mathcal{H})_n$$

for $n \gg 0$. Let \mathcal{K} be a graded left ideal of \mathcal{R} . Since \mathcal{B} is noetherian, we may choose a graded coherent \mathcal{O}_X -submodule \mathcal{K}' of \mathcal{K} such that $\mathcal{B} \cdot \mathcal{K} = \mathcal{B} \cdot \mathcal{K}'$. Since $\mathcal{K}/(\mathcal{R} \cdot \mathcal{K}')$ is a submodule of $((\mathcal{B} \cdot \mathcal{K}') \cap \mathcal{R})/(\mathcal{R} \cdot \mathcal{K}')$, it is enough to show that $((\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R})/\mathcal{K}$ is a noetherian left \mathcal{R} -module for any *coherent* graded left ideal \mathcal{K} of \mathcal{R} .

But now consider the exact sequences of left \mathcal{R} -modules

$$(4.1) \quad 0 \rightarrow \frac{\mathcal{K}}{\mathcal{I}\mathcal{B} \cdot \mathcal{K}} \rightarrow \frac{(\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R}}{\mathcal{I}\mathcal{B} \cdot \mathcal{K}} \rightarrow \frac{(\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R}}{\mathcal{K}} \rightarrow 0$$

and

$$(4.2) \quad 0 \rightarrow \frac{(\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{I}\mathcal{B}}{\mathcal{I}\mathcal{B} \cdot \mathcal{K}} \rightarrow \frac{(\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R}}{\mathcal{I}\mathcal{B} \cdot \mathcal{K}} \rightarrow \frac{(\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R}}{(\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{I}\mathcal{B}} \rightarrow 0.$$

Since $((\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R})/((\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{I}\mathcal{B})$ is a coherent \mathcal{O}_X -module, we see that $((\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{R})/\mathcal{K}$ is noetherian if $((\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{I}\mathcal{B})/\mathcal{I}\mathcal{B} \cdot \mathcal{K}$ is noetherian. Since $\mathcal{B} \cdot \mathcal{K}$ is a left ideal of \mathcal{B} , by assumption $((\mathcal{B} \cdot \mathcal{K}) \cap \mathcal{I}\mathcal{B})/\mathcal{I}\mathcal{B} \cdot \mathcal{K}$ is a coherent \mathcal{O}_X -module. In particular, it is noetherian. Thus \mathcal{R} is left noetherian. \square

This allows us to give a necessary and sufficient condition for \mathcal{R} to be left noetherian:

Proposition 4.5. *Assume Assumption-Notation 4.1. Then $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is left noetherian if and only if for all closed subschemes $Y \subseteq X$ the set*

$$\{n \geq 0 \mid \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

Proof. Let \mathcal{J} be an ideal sheaf defining a closed subscheme Y of X . There are identifications of \mathcal{O}_X -modules,

$$\begin{aligned} \frac{\mathcal{I}\mathcal{B} \cap (\mathcal{B} \cdot \mathcal{J})}{\mathcal{I}\mathcal{B} \cdot \mathcal{J}} &\cong \bigoplus_{n \geq 0} \frac{\mathcal{I} \cap \mathcal{J}^{\sigma^n}}{\mathcal{I}\mathcal{J}^{\sigma^n}} \otimes \mathcal{L}_n \\ &\cong \bigoplus_{n \geq 0} \mathcal{T}or_1^X(\mathcal{O}_Z, \mathcal{O}_{\sigma^{-n}Y}) \otimes \mathcal{L}_n \cong \bigoplus_{n \geq 0} \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \otimes \mathcal{L}_n, \end{aligned}$$

using [30, Exercise 3.1.3] and the local property of $\mathcal{T}or$. As $\mathcal{R}/\mathcal{I}\mathcal{B}$ is a coherent \mathcal{O}_X -module,

$$\frac{\mathcal{I}\mathcal{B} \cap (\mathcal{B} \cdot \mathcal{J})}{\mathcal{I}\mathcal{B} \cdot \mathcal{J}}$$

is a coherent left \mathcal{R} -module if and only if it is a coherent \mathcal{O}_X -module. This is true if and only if the set $\{n \geq 0 \mid \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$ is finite. \square

We will explore the geometric meaning of the condition from Proposition 4.5 in the next section. For now, we give it a name for easy reference.

Definition 4.6. Let X be a variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let Z be a closed subscheme of X . We will (provisionally) say that the pair (Z, σ) has *property T* if for all closed subschemes $Y \subseteq X$ the set

$$\{n \geq 0 \mid \mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

5. CRITICAL TRANSVERSALITY

Assume Assumption-Notation 4.1. In the last two sections, we found necessary and sufficient conditions for $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ to be left or right noetherian. We remark that there is a significant contrast between the two sides. By Theorem 3.3, for \mathcal{R} to be right noetherian depends only on a mild condition on the orbits of points in Z . In contrast, property T that by Proposition 4.5 determines when \mathcal{R} is left noetherian is, a priori, much less transparent. In this section, we show that property T does have a natural geometric interpretation. As remarked in the Introduction, it is an analogue of critical density and can be viewed as a transversality property. Further, in many settings property T may be interpreted as saying that σ and Z are in general position. We will see later that in the presence of property T , it is possible to deduce many further nice properties of \mathcal{R} and of Z .

To begin, we define an algebraic generalization of classical transversality.

Definition 5.1. Let X be a variety, and let Y and Z be closed subschemes of X . We say that Y and Z are *homologically transverse* if

$$\mathcal{T}or_i^X(\mathcal{O}_Z, \mathcal{O}_Y) = 0$$

for all $i \geq 1$.

While this appears as an arcane algebraic condition, it does in fact have a geometric basis. As discussed in the Introduction, Serre [23] defines the intersection multiplicity of two closed subschemes Y and Z of X along the proper component P of their intersection by

$$(5.1) \quad i(Y, Z; P) = \sum_{i \geq 0} (-1)^i \text{len}_P(\mathcal{T}or_i^X(\mathcal{O}_Y, \mathcal{O}_Z)).$$

The higher $\mathcal{T}or$ sheaves are needed to correct for a possible mis-counting from the naïve attempt to define $i(Y, Z; P)$ as $\text{len}_P(\mathcal{O}_Y \otimes \mathcal{O}_Z)$. We may think of the nonvanishing of $\mathcal{T}or_{\geq 1}^X(\mathcal{O}_Y, \mathcal{O}_Z)$ as indicating that Y and Z have an extremely non-transverse intersection (for example, the codimension of the intersection is smaller than $\text{codim } Y + \text{codim } Z$).

Definition 5.2. Let X be a variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let Z be a closed subscheme of X . Let $A \subseteq \mathbb{Z}$ be infinite. We say that the set $\{\sigma^n(Z)\}_{n \in A}$ is *critically transverse* if for all closed subschemes Y of X , $\sigma^n(Z)$ and Y are homologically transverse for all but finitely many $n \in A$.

Critical transversality of $\{\sigma^n Z\}_{n \in A}$ is a generic transversality property: for any closed subscheme Y , it implies that the general translate of Z is homologically transverse to Y . In this section, we investigate critical transversality. We will see in particular that critical transversality and property T are equivalent.

Recall that if \mathcal{F} is a coherent sheaf on a projective variety X , we write $\text{hd}_X(\mathcal{F})$ for the minimal length of a locally free resolution of \mathcal{F} . By [14, Ex. III.6.3], we have

$$\text{hd}_X(\mathcal{F}) = \sup_{x \in X} \{\text{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x\}.$$

The following lemma is due to Mel Hochster, and we thank him for allowing us to include it here.

Lemma 5.3 (Hochster). *Suppose that Z is homologically transverse to all parts of the singular stratification of X . Then*

$$\text{hd}_X(\mathcal{O}_Z) \leq \dim X.$$

Proof. Let $X = X^{(0)} \supset X^{(1)} \dots \supset X^{(k)}$ be the singular stratification of X . By assumption, Z is homologically transverse to all $X^{(i)}$. By [9, Corollary 19.5],

$$(5.2) \quad \text{hd}_X(\mathcal{O}_Z) = \sup\{j \mid \text{for some closed point } x \in X, \text{Tor}_j^X(\mathcal{O}_Z, \mathbb{k}_x) \neq 0\}.$$

So fix $x \in X$, and let $\mathcal{O} = \mathcal{O}_{X,x}$. Let $F = \mathcal{O}_{Z,x}$, considered as an \mathcal{O} -module. Let i be such that $x \in X^{(i)} \setminus X^{(i+1)}$. Let J be the ideal of $X^{(i)}$ in \mathcal{O} . By assumption on i , \mathcal{O}/J is a regular local ring; in particular, $\text{pd}_{\mathcal{O}/J} \mathbb{k}_x = \dim X^{(i)} \leq \dim X$.

The change of rings theorem for Tor [30, Theorem 5.6.6] gives a spectral sequence

$$(5.3) \quad \text{Tor}_p^{\mathcal{O}/J}(\text{Tor}_q^{\mathcal{O}}(F, \mathcal{O}/J), \mathbb{k}_x) \Rightarrow \text{Tor}_{p+q}^{\mathcal{O}}(F, \mathbb{k}_x).$$

Now by assumption, Z is homologically transverse to $X^{(i)}$, and so (5.3) collapses for $q \neq 0$. We obtain

$$\text{Tor}_p^{\mathcal{O}/J}(F \otimes_{\mathcal{O}} (\mathcal{O}/J), \mathbb{k}_x) \cong \text{Tor}_p^{\mathcal{O}}(F, \mathbb{k}_x).$$

As \mathcal{O}/J is a regular local ring of dimension no greater than $\dim X$, we have that $\text{pd}_{\mathcal{O}/J} \mathbb{k}_x \leq \dim X$, and so $\text{Tor}_p^{\mathcal{O}}(F, \mathbb{k}_x) = 0$ if $p > \dim X$. By (5.2), $\text{hd}_X(\mathcal{O}_Z) \leq \dim X$. \square

We now show that critical transversality and property T are equivalent.

Lemma 5.4. *Let $A \subseteq \mathbb{Z}$. The following are equivalent:*

(1) *For all closed subschemes Y of X , the set*

$$\{n \in A \mid \text{Tor}_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

(2) *For all reduced and irreducible closed subschemes Y of X , the set*

$$\{n \in A \mid \text{Tor}_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0\}$$

is finite.

(3) *For all closed subschemes Y of X , the set*

$$A'(Y) = \{n \in A \mid \sigma^n Z \text{ is not homologically transverse to } Y\}$$

is finite.

In particular, $\{\sigma^n(Z)\}_{n \geq 0}$ is critically transverse if and only if (Z, σ) has property T .

Proof. The implications (3) \Rightarrow (1) \Rightarrow (2) are trivial. We prove (2) \Rightarrow (3).

Assume (2). We may assume that A is infinite. We first claim that for any coherent sheaf \mathcal{F} and for any $j \geq 1$, the set

$$\{n \in A \mid \text{Tor}_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{F}) \neq 0\}$$

is finite. We induct on j . As any coherent sheaf on a projective variety has a finite filtration by products of invertible sheaves with structure sheaves of reduced and irreducible closed subvarieties, the claim is true for $j = 1$. Let $j > 1$ and fix a coherent sheaf \mathcal{F} . Because X is projective, it has enough locally frees, and there is an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{L} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{L} is locally free and \mathcal{K} is also coherent. The long exact sequence in Tor implies that

$$\text{Tor}_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{F}) \cong \text{Tor}_{j-1}^X(\mathcal{O}_{\sigma^n Z}, \mathcal{K})$$

for any n . By induction, the right-hand side vanishes for all but finitely many $n \in A$.

The claim implies that Z is homologically transverse to any σ -invariant closed subscheme of X , and, in particular, that Z is homologically transverse to the singular stratification of X . By Lemma 5.3, we have $\text{hd}_X(\mathcal{O}_Z) \leq \dim X$. Thus for a fixed Y ,

$$A'(Y) = \{n \in A \mid \text{Tor}_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0 \text{ for some } 1 \leq j \leq \dim X\}.$$

By the claim, this is finite. □

Corollary 5.5. *Assume Assumption-Notation 4.1. Then the bimodule algebra $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is left noetherian if and only if $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse.*

Proof. Combine Lemma 5.4 with Proposition 4.5. □

Next we verify that critical transversality is, as claimed, a generalization of critical density of the orbits of points. We formally define:

Definition 5.6. Let X be a variety, let $x \in X$, and let $\sigma \in \text{Aut}_{\mathbb{k}} X$. Let $A \subseteq \mathbb{Z}$. The set $\{\sigma^n(x) \mid n \in A\}$ is *critically dense* if it is infinite and any infinite subset is dense in X .

We first prove:

Lemma 5.7. *Let $W \subseteq V$ be closed subschemes of a scheme X . Then*

$$\text{Tor}_1^X(\mathcal{O}_V, \mathcal{O}_W) \neq 0.$$

Proof. We work locally; let W' be an irreducible component of W , and let $P = (W')^{\text{red}}$. Let \mathfrak{m} be the maximal ideal of the local ring $\mathcal{O} = \mathcal{O}_{X,P}$. Let J be the ideal of \mathcal{O} defining V and let I be the \mathfrak{m} -primary ideal defining W locally at P . Then we have

$$\text{Tor}_1^X(\mathcal{O}_V, \mathcal{O}_W)_P = \text{Tor}_1^{\mathcal{O}}(\mathcal{O}/J, \mathcal{O}/I) \cong (J \cap I)/JI = J/JI,$$

as $J \subseteq I$. By Nakayama's Lemma, this is nonzero. □

Corollary 5.8. *Let X be a variety, let $\sigma \in \text{Aut}_{\mathbb{k}}(X)$, let Z be a 0-dimensional closed subscheme of X , and let $A \subseteq \mathbb{Z}$ be infinite. The following are equivalent:*

- (1) $\{\sigma^n(Z)\}_{n \in A}$ is critically transverse;
- (2) $\{\sigma^n(x)\}_{n \in A}$ is critically dense for all points $x \in \text{Supp } Z$.

Proof. Because $\text{Tor}_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y)$ is supported on $\sigma^n Z \cap Y$ for any j , (2) \Rightarrow (1). We prove that (1) \Rightarrow (2). By working locally, we may assume that Z is supported on a single point x . Suppose that (2) fails, so there is some infinite $A' \subseteq A$ and some reduced $W \subset X$ such that $\sigma^n(x) \in W$ for all $n \in A'$. Then there is some, not necessarily reduced, W' supported on W such that $\sigma^n(Z) \subseteq W'$ for all $n \in A'$. By Lemma 5.7, $\text{Tor}_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_{W'}) \neq 0$ for any $n \in A'$. Thus (1) also fails. \square

We now turn to investigating when critical transversality occurs. As mentioned, critical transversality is a generic transversality property of the translates of Z , reminiscent of the Kleiman-Bertini theorem. One naturally asks if there are simple conditions on Z and σ sufficient for $\{\sigma^n Z\}$ to be critically transverse. In particular, it is not immediately obvious, even working on nice varieties such as \mathbb{P}^n , that critical transversality *ever* occurs, except for points with critically dense orbits.

If instead of considering a single automorphism σ , we consider instead the action of an algebraic group G on X , then there is a simple condition for generic transversality to hold for translates of Z . This is

Theorem 5.9 ([24, Theorem 1.2]). *Let X be a variety (defined over \mathbb{k}) with a left action of a smooth algebraic group G , and let Z be a closed subscheme of X . Then the following are equivalent:*

- (1) Z is homologically transverse to all G -orbit closures in X .
- (2) For all closed subschemes Y of X , there is a Zariski open and dense subset U of G such that for all closed points $g \in U$, the subscheme gZ is homologically transverse to Y . \square

(We note that if G acts transitively, then (1) and therefore (2) are automatically satisfied; this case of Theorem 5.9 was proved by Miller and Speyer [18].)

In the remainder of this section, we apply Theorem 5.9 to obtain a simple criterion for critical transversality, at least in characteristic 0. It turns out that in many situations, critical transversality is, in a suitable sense, generic behavior.

We will use the following result of Cutkosky and Srinivas.

Theorem 5.10 ([8, Theorem 7]). *Let G be a connected abelian algebraic group defined over an algebraically closed field \mathbb{k} of characteristic 0. Suppose that $g \in G$ is such that the cyclic subgroup $\langle g \rangle$ is dense in G . Then any infinite subset of $\langle g \rangle$ is dense in G . \square*

Here is our simple condition for critical transversality:

Theorem 5.11. *Let \mathbb{k} be an algebraically closed field of characteristic 0, let X be a variety over \mathbb{k} , let Z be a closed subscheme of X , and let σ be an element of an algebraic group G that acts on X . Let $A \subseteq \mathbb{Z}$ be infinite. Then $\{\sigma^n Z\}_{n \in A}$ is critically transverse if and only if Z is homologically transverse to all reduced σ -invariant subschemes of X .*

Proof. If $\{\sigma^n Z\}_{n \in A}$ is critically transverse, then Z is obviously homologically transverse to σ -invariant subschemes. We prove the converse. Assume that Z is homologically transverse to reduced σ -invariant subschemes of X . We consider the abelian subgroup

$$H = \overline{\langle \sigma^n \rangle} \subseteq G.$$

Now the closures of H -orbits in X are σ -invariant and reduced. Thus, by assumption, Z is homologically transverse to all H -orbit closures, and we may apply

Theorem 5.9. Fix a closed subscheme Y of X . By Theorem 5.9, there is a dense open $U \subseteq H$ such that if $g \in U$, then gZ and Y are homologically transverse.

Let H^o be the connected component of the identity in H , so the components of H are $H^o = \sigma^c H^o, \sigma H^o, \dots, \sigma^{c-1} H^o$ for some $c \geq 1$. As $\langle \sigma^c \rangle$ is dense in H^o , it is critically dense by Theorem 5.10.

Fix $0 \leq j \leq c - 1$. The set

$$U_j = \sigma^{-j}(U \cap \sigma^j H^o)$$

is an open dense subset of H^o . By critical density, the set

$$\{m \mid \sigma^{mc} \notin U_j\}$$

is finite. Thus

$$\{n \mid \sigma^n \notin U\} = \bigcup_{j=0}^{c-1} \{n \mid n \equiv j \pmod{c} \text{ and } \sigma^{n-j} \notin U_j\}$$

is also finite. That is to say, for all but finitely many n we have $\sigma^n \in U$, and so $\sigma^n Z$ is homologically transverse to Y . As Y was arbitrary, $\{\sigma^n Z\}_{n \in \mathbb{Z}}$ is critically transverse. Thus $\{\sigma^n Z\}_{n \in A}$ is critically transverse. \square

We note that the case of Theorem 5.11 where Z is a point is proved in [17, Theorem 11.2].

It is reasonable to say that σ and Z are in *general position* if Z is homologically transverse to all σ -fixed subschemes of X . Theorem 5.11 suggests the following conjecture:

Conjecture 5.12. *Let \mathbb{k} be an algebraically closed field of characteristic 0, and let X/\mathbb{k} be a projective variety. Let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let $Z \subseteq X$ be a closed subvariety. Then $\{\sigma^n Z\}$ is critically transverse if and only if σ and Z are in general position.*

If Z is 0-dimensional, then this conjecture reduces to Bell, Ghioca, and Tucker’s recent result [6, Theorem 5.1] that in characteristic 0, the orbit of a point under an automorphism is dense exactly when it is critically dense. If σ is an element of an algebraic group that acts on X , the conjecture is Theorem 5.11. In positive characteristic, the conjecture is known to be false; see [20, Example 12.9] for an example of an automorphism $\sigma \in \mathbb{P}GL_n$ in positive characteristic with a dense but not critically dense orbit.

Suppose now that \mathbb{k} is uncountable (and algebraically closed) and that X is a variety over \mathbb{k} . We say that $x \in X$ is *very general* if there are proper subvarieties $\{Y_i \mid i \in \mathbb{Z}\}$ so that

$$x \notin \bigcup_i Y_i.$$

We can now show that critically transverse subschemes abound. To make the statement easier, the following result is phrased in terms of \mathbb{P}^d ; a similar result holds for any variety with a transitive action by a reductive algebraic group.

Corollary 5.13. *Assume that \mathbb{k} is uncountable and $\text{char } \mathbb{k} = 0$. Let Z be a subscheme of \mathbb{P}^d , and let \mathcal{X} be the $\mathbb{P}GL_{d+1}$ -orbit of Z in the Hilbert scheme of \mathbb{P}^d . Let $\mathcal{Y} = \mathbb{P}GL_{d+1} \times \mathcal{X}$. If (σ, Z') is a very general element of \mathcal{Y} , then σ and Z' are in general position and the set $\{\sigma^n Z'\}$ is critically transverse.*

Proof. By avoiding a countable union of proper subvarieties of $\mathbb{P}GL_{d+1}$, we may ensure that the eigenvalues of σ are algebraically independent over \mathbb{Q} . This implies that the Zariski closure of $\{\sigma^n\}$ in $\mathbb{P}GL_{d+1}$ is a torus \mathbb{T}^d , and that the only reduced subschemes fixed by σ are unions of coordinate linear subspaces with respect to an appropriate choice of coordinates. There are finitely many of these; by repeated applications of Theorem 5.9 with $G = \mathbb{P}GL_{d+1}$ (or by [18]) we see that there is a dense open $U \subseteq \mathbb{P}GL_{d+1}$ such that for all $\tau \in U$, the subscheme $Z' = \tau Z$ is homologically transverse to all unions of coordinate linear subspaces; that is, σ and Z' are in general position. By Theorem 5.11, the set $\{\sigma^n Z'\}$ is critically transverse. \square

6. AMPLENESS

We now return to noncommutative algebra. Our ultimate goal is to study, not the bimodule algebra $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$, but the ring $R = R(X, \mathcal{L}, \sigma, Z)$. In order to apply our knowledge of \mathcal{R} to the ring R , we will need to control the ampleness, in the sense of Definition 2.8, of the sequence $\{(\mathcal{R}_n)_{\sigma^n}\}$. In this section we show that critical transversality of $\{\sigma^n Z\}$, together with σ -ampleness of \mathcal{L} , is enough to show sufficient ampleness of the graded pieces of \mathcal{R} .

Throughout this section we assume Assumption-Notation 4.1. Thus to prove that the sequence $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left or right ample, it suffices to prove that $\{(\mathcal{I} \otimes \mathcal{L}_n)_{\sigma^n}\}$ is left or right ample.

Given σ -ampleness of \mathcal{L} , right ampleness of $\{(\mathcal{R}_n)_{\sigma^n}\}$ is almost trivial; we record this in the next lemma.

Lemma 6.1. *Assume Assumption-Notation 4.1. Assume in addition that \mathcal{L} is σ -ample. Then $\{(\mathcal{R}_n)_{\sigma^n}\}$ is right ample.*

Proof. From Assumption-Notation 4.1, we know that $\mathcal{R}_n = \mathcal{I}\mathcal{L}_n = \mathcal{I} \otimes \mathcal{L}_n$ for $n \gg 0$. Fix a coherent sheaf \mathcal{F} . Then for $n \gg 0$, we have $\mathcal{F} \otimes \mathcal{R}_n = \mathcal{F} \otimes \mathcal{I} \otimes \mathcal{L}_n$. By σ -ampleness of \mathcal{L} , for $n \gg 0$ this is globally generated and has no higher cohomology. \square

Left ampleness, however, is more subtle. In fact, we do not know when, in general, $\{(\mathcal{R}_n)_{\sigma^n}\}$ is left ample. However, we will see that this does hold when \mathcal{R} is left noetherian.

Proposition 6.2. *If \mathcal{L} is σ -ample and $\{\sigma^n(Z)\}_{n \geq 0}$ is critically transverse, then $\{(\mathcal{I} \otimes \mathcal{L}_n)_{\sigma^n}\}$ is a left ample sequence.*

We first prove:

Lemma 6.3. *Let \mathcal{L} be a σ -ample invertible sheaf.*

(1) *If \mathcal{M} and \mathcal{N} are coherent sheaves on X , then there is an integer n_0 so that $\mathcal{M} \otimes \mathcal{L}_n \otimes \mathcal{N}^{\sigma^n}$ is globally generated for all $n \geq n_0$.*

(2) *If \mathcal{E} and \mathcal{F} are invertible sheaves on X , there is an integer m_0 so that $\mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m}$ is ample for all $m \geq m_0$.*

Proof. (1) Using the σ -ampleness of \mathcal{L} , take $i, j \gg 0$ so that $\mathcal{M} \otimes \mathcal{L}_i$ and $\mathcal{L}_j \otimes \mathcal{N}^{\sigma^j}$ are globally generated. Then $\mathcal{L}_j^{\sigma^i} \otimes \mathcal{N}^{\sigma^{i+j}}$ is also globally generated. Since the tensor product of globally generated sheaves is globally generated, $\mathcal{M} \otimes \mathcal{L}_{i+j} \otimes \mathcal{N}^{\sigma^{i+j}}$ is globally generated.

(2) In fact, we will show that $\mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m}$ is very ample for $m \gg 0$. Let \mathcal{C} be an arbitrary very ample invertible sheaf. By (1) we may choose m_0 so that if $m \geq m_0$, the sheaf $\mathcal{K} = \mathcal{C}^{-1} \otimes \mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m}$ is globally generated. By [14, Exercise II.7.5(d)] the tensor product of a very ample invertible sheaf and a globally generated invertible sheaf is very ample. Thus $\mathcal{E} \otimes \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m} \cong \mathcal{C} \otimes \mathcal{K}$ is very ample. \square

Proof of Proposition 6.2. Let \mathcal{M} be an arbitrary coherent sheaf. By Lemma 6.3, we know that $\mathcal{I} \otimes \mathcal{L}_n \otimes \mathcal{M}^{\sigma^n}$ is globally generated for $n \gg 0$. We must establish that $H^j(X, \mathcal{I} \otimes \mathcal{L}_n \otimes \mathcal{M}^{\sigma^n}) = 0$ for all $j \geq 1$ and $n \gg 0$.

We know that $\mathcal{T}or_j^X(\mathcal{O}_{\sigma^n Z}, \mathcal{M}) = 0$ for all $n \gg 0$ and $j \geq 1$. Thus

$$\mathcal{T}or_j^X(\mathcal{I}, \mathcal{M}^{\sigma^n}) \cong \mathcal{T}or_j^X(\mathcal{I}^{\sigma^{-n}}, \mathcal{M})^{\sigma^n} \cong \mathcal{T}or_{j+1}^X(\mathcal{O}_{\sigma^n Z}, \mathcal{M})^{\sigma^n} = 0$$

for all $n \gg 0$ and $j \geq 1$.

First suppose that \mathcal{M} is invertible. By Fujita’s vanishing theorem [10, Theorem 11] choose an invertible sheaf \mathcal{H} such that $H^i(X, \mathcal{I} \otimes \mathcal{H} \otimes \mathcal{F}) = 0$ for all $i \geq 1$ and any ample invertible sheaf \mathcal{F} . By Lemma 6.3(2), we may choose m_0 such that $\mathcal{H}^{-1} \otimes \mathcal{L}_m \otimes \mathcal{M}^{\sigma^m}$ is ample for all $m \geq m_0$. Then $\mathcal{I} \otimes \mathcal{L}_m \otimes \mathcal{M}^{\sigma^m} = \mathcal{I} \otimes \mathcal{H} \otimes \mathcal{H}^{-1} \otimes \mathcal{L}_m \otimes \mathcal{M}^{\sigma^m}$, and so its higher cohomology vanishes.

Now for general \mathcal{M} let the cochain complex

$$\dots \rightarrow \mathcal{P}^{-2} \rightarrow \mathcal{P}^{-1} \rightarrow \mathcal{P}^0 \rightarrow \mathcal{M} \rightarrow 0$$

be a (not necessarily finite!) locally free resolution of \mathcal{M} , where each \mathcal{P}^i is a direct sum of invertible sheaves. By tensoring on the left with $(\mathcal{I} \otimes \mathcal{L}_n)_{\sigma^n}$, we obtain a complex \mathcal{Q}^\bullet , where $\mathcal{Q}^i = \mathcal{I} \otimes \mathcal{L}_n \otimes (\mathcal{P}^i)^{\sigma^n}$. The q th cohomology of \mathcal{Q}^\bullet is isomorphic to $\mathcal{T}or_{-q}^X(\mathcal{I}, \mathcal{M}^{\sigma^n}) \otimes \mathcal{L}_n$. By [30, 5.7.9], using a Cartan-Eilenberg resolution of \mathcal{Q}^\bullet we obtain two spectral sequences

$$(6.1) \quad {}^I E_1^{pq} = H^q(X, \mathcal{Q}^p)$$

and

$$(6.2) \quad {}^{II} E_2^{pq} = H^p(X, \mathcal{T}or_{-q}(\mathcal{I}, \mathcal{M}^{\sigma^n}) \otimes \mathcal{L}_n).$$

Since X has finite cohomological dimension $d = \dim X$, these both converge to the hypercohomology groups $\mathbb{H}^{p+q}(\mathcal{Q}^\bullet)$.

Now, given $p + q = j \geq 1$, by critical transversality we may take $n \gg 0$ so that $\mathcal{T}or_{-q}(\mathcal{I}, \mathcal{M}^{\sigma^n}) = 0$ for all $q \leq -1$; thus (6.2) collapses and we obtain

$$\mathbb{H}^j(\mathcal{Q}^\bullet) = H^j(X, \mathcal{I} \otimes \mathcal{M}^{\sigma^n} \otimes \mathcal{L}_n).$$

On the other hand, since the sheaves \mathcal{P}^i are direct sums of invertible sheaves, applying the invertible case to each summand of \mathcal{P}^i we may further increase n if necessary to obtain that

$$H^q(X, \mathcal{Q}^p) = H^q(X, \mathcal{I} \otimes \mathcal{L}_n \otimes (\mathcal{P}^p)^{\sigma^n}) = 0$$

for $d \geq q \geq 1$ and $1 - d \leq p \leq 0$. Thus if $j \geq 1$, (6.1) collapses to 0. Thus

$$H^j(X, \mathcal{I} \otimes \mathcal{L}_n \otimes \mathcal{M}^{\sigma^n}) = \mathbb{H}^j(\mathcal{Q}^\bullet) = 0$$

for all $n \gg 0$ and $j \geq 1$. \square

Lemma 6.1 and Proposition 6.2, together with Theorem 2.9, will allow us to relate $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$ and its section ring. To conclude this section, we show that, given σ -ampleness of \mathcal{L} , the ring $R(X, \mathcal{L}, \sigma, Z)$ is equal to the section ring of the bimodule algebra $\mathcal{R}(X, \mathcal{L}, \sigma, Z)$.

Lemma 6.4. *Assume Notation 3.1, and let $R = R(X, \mathcal{L}, \sigma, Z)$ as in Construction 1.1. If \mathcal{L} is σ -ample, then*

$$R = R(X, \mathcal{L}, \sigma, Z) = H^0(X, \mathcal{R}(X, \mathcal{L}, \sigma, Z)).$$

Proof. Let $I = \Gamma_*(\mathcal{I})$ be the right ideal of $B(X, \mathcal{L}, \sigma)$ generated by sections vanishing along Z ; thus $R = \mathbb{I}_B(I)$. Suppose that $x \in R_n$, so $xI \subseteq I$. Since \mathcal{L} is σ -ample, $\mathcal{I}\mathcal{L}_m$ is globally generated by $I_m = H^0(X, \mathcal{I}\mathcal{L}_m)$ for $m \gg 0$, and so for $m \gg 0$

$$x\mathcal{O}_X(\mathcal{I}\mathcal{L}^{\sigma^m})^{\sigma^n} = x\mathcal{O}_X(I_m\mathcal{O}_X)^{\sigma^n} \subseteq I_{m+n}\mathcal{O}_X = \mathcal{I}\mathcal{L}_{m+n}$$

for any n . Thus $x\mathcal{O}_X \subseteq (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n$ and $x \in H^0(X, (\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n) = H^0(X, \mathcal{R}_n)$.

For the other containment, suppose that $x \in H^0(X, \mathcal{R}_n)$. Then for any $m \geq 0$ we have

$$xI_m \subseteq H^0((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n) \cdot H^0(X, (\mathcal{I}\mathcal{L}_m)^{\sigma^n}) \subseteq H^0(X, \mathcal{I}\mathcal{L}_{m+n}) = I_{n+m}.$$

Thus $x \in R_n$, and we have established the equality we seek. □

7. NOETHERIAN IDEALIZER RINGS

We are now ready to begin translating our results on bimodule algebras to results on geometric idealizer rings. We will work in the following setting:

Assumption-Notation 7.1. *Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let \mathcal{L} be an invertible sheaf on X , which we now assume to be σ -ample. Let Z be a closed subscheme of X and let $\mathcal{I} = \mathcal{I}_Z$ be its ideal sheaf. We continue to assume that $(\mathcal{I} : \mathcal{I}^{\sigma^n}) = \mathcal{I}$ for $n \gg 0$. Let*

$$\mathcal{B} = \mathcal{B}(X, \mathcal{L}, \sigma)$$

and let

$$B = B(X, \mathcal{L}, \sigma).$$

Let

$$\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z) = \bigoplus_{n \geq 0} ((\mathcal{I} : \mathcal{I}^{\sigma^n})\mathcal{L}_n)_{\sigma^n}.$$

Let

$$I = \bigoplus_{n \geq 0} H^0(X, \mathcal{I}\mathcal{L}_n) = \Gamma_*(\mathcal{I})$$

and let

$$R = R(X, \mathcal{L}, \sigma, Z) = \mathbb{I}_B(I)$$

as in Construction 1.1. By Lemma 6.4,

$$R = \bigoplus_{n \geq 0} H^0(X, \mathcal{R}_n).$$

Our assumptions imply that $\mathcal{R}_n = \mathcal{I}\mathcal{L}_n$ and $R_n = I_n$ for $n \gg 0$.

Assume Assumption-Notation 7.1. In this section, we determine when R is left and right noetherian. We also consider when R is strongly noetherian and when $R \otimes_{\mathbb{k}} R$ is noetherian. We first show that the right noetherian property for R , and in fact the strong right noetherian property, are equivalent to the simple geometric criterion from Theorem 3.3.

Proposition 7.2. *Assume Assumption-Notation 7.1. Then the following are equivalent:*

- (1) Z has finite intersection with forward σ -orbits;
- (2) R is right noetherian;
- (3) R is strongly right noetherian.

Proof. (1) \Rightarrow (3). By Theorem 3.3, if (1) holds, then the bimodule algebra

$$\mathcal{R}(X, \mathcal{L}, \sigma, Z)$$

is right noetherian. Now let C be any commutative noetherian ring, and let

$$X_C = X \times \text{Spec } C$$

and

$$Z_C = Z \times \text{Spec } C \subseteq X_C.$$

Also define

$$\begin{aligned} B_C &= B \otimes_{\mathbb{k}} C, \\ R_C &= R \otimes_{\mathbb{k}} C, \end{aligned}$$

and

$$I_C = I \otimes_{\mathbb{k}} C \cong I \otimes_B B_C.$$

It is clear that

$$R_C = \mathbb{I}_{B_C}(I_C)$$

and that R_C/I_C is a finitely generated C -module. Let $p : X_C \rightarrow X$ be projection onto the first factor.

The idea behind our proof is very simple: if Z has finite intersection with forward σ -orbits, then Z_C has finite intersection with forward $(\sigma \times 1)$ -orbits, and so R_C should be noetherian by Theorem 3.3 and Theorem 2.9. However, neither of these were proved over an arbitrary base ring C ; to work scheme-theoretically we instead follow the proof of [2, Proposition 4.13].

By [2, Proposition 4.13], B_C is noetherian. The proof of this proposition uses the fact that the shift functor in $\text{qgr-}B_C$ satisfies the hypotheses of [4, Theorem 4.5]. By [4, Theorem 4.5], B_C satisfies right χ_1 . In particular, for any graded right ideal J of B_C , the natural map

$$(7.1) \quad \underline{\text{Hom}}_{B_C}(B_C/I_C, B_C/J) \rightarrow \underline{\text{Hom}}_{\text{qgr-}B_C}(\pi(B_C/I_C), \pi(B_C/J))$$

is an isomorphism in large degree, by [4, Proposition 3.5].

As $\text{qgr-}B \simeq \mathcal{O}_X\text{-mod}$, it is clear that

$$(7.2) \quad \text{qgr-}B_C \simeq \mathcal{O}_{X_C}\text{-mod}.$$

We note that B_C/I_C corresponds to \mathcal{O}_{Z_C} under this equivalence.

Let J be a graded right ideal of B_C containing I_C . We claim that

$$\underline{\text{Hom}}_{B_C}(B_C/I_C, B_C/J)$$

is a finitely generated C -module. To see this, let $Y \subseteq Z_C$ be the closed subscheme of X_C such that B_C/J corresponds to \mathcal{O}_Y under the equivalence (7.2). By (2.1), $(B_C/J)[n]$ corresponds to

$$(\mathcal{O}_Y \otimes p^* \mathcal{L}_n)^{(\sigma^{-n} \times 1)} \cong \mathcal{O}_{(\sigma^n \times 1)Y} \otimes p^*(\mathcal{L}_n^{\sigma^{-n}})$$

under (7.2). Thus

$$\underline{\text{Hom}}_{\text{qgr-}B_C}(\pi(B_C/I_C), \pi(B_C/J))_{\geq 0} \cong \bigoplus_{n \geq 0} \text{Hom}_{X_C}(\mathcal{O}_{Z_C}, \mathcal{O}_{(\sigma^n \times 1)Y} \otimes p^*(\mathcal{L}_n^{\sigma^{-n}})).$$

Now, Z_C has finite intersection with forward $(\sigma \times 1)$ -orbits, and so for $n \gg 0$, no component of $(\sigma^n \times 1)Y$ is contained in Z_C . Thus

$$\text{Hom}_{X_C}(\mathcal{O}_{Z_C}, \mathcal{O}_{(\sigma^n \times 1)Y} \otimes p^*(\mathcal{L}_n^{\sigma^{-n}})) = 0$$

for $n \gg 0$. As the map (7.1) is an isomorphism in large degree, we see that

$$\underline{\text{Hom}}_{B_C}(B_C/I_C, B_C/J)_n = 0$$

for $n \gg 0$, and so

$$\underline{\text{Hom}}_{B_C}(B_C/I_C, B_C/J)$$

is a finitely generated C -module, as claimed. As this is true for any graded $J \supseteq I_C$, by Lemma 3.7, R_C is right noetherian.

(3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Let $x \in X$ and let J be the right ideal $\Gamma_*(\mathcal{I}_x)$ of B . As B and R are right noetherian, by Lemma 3.7,

$$\text{Hom}_B(B/I, B/J) \cong \{r \in B \mid rI \subseteq J\}/J$$

is a noetherian right R/I -module. It is thus finite-dimensional, as R/I is finite-dimensional by assumption.

As \mathcal{L} is σ -ample, \mathcal{L}_n is globally generated for $n \gg 0$; in particular, $J_n \subsetneq B_n$ for $n \gg 0$. Now, suppose that

$$\{n \geq 0 \mid \sigma^n(x) \in Z\} = \{n \geq 0 \mid x \in \sigma^{-n}(Z)\}$$

is infinite. For any such n , we have that $B_n I \subseteq J$. Thus

$$\{r \in B \mid rI \subseteq J\}/J$$

is infinite-dimensional, giving a contradiction.

Thus $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is finite. □

The left-hand side is very different. If \mathcal{R} is left noetherian, then so is R ; but R can only be strongly left noetherian if Z is a divisor. In this case, R is both a left and a right idealizer, so the strong left noetherian property will follow from the left-handed version of Proposition 7.2.

Proposition 7.3. *Assume Assumption-Notation 7.1. If $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse, then $R = R(X, \mathcal{L}, \sigma, Z)$ is left noetherian.*

Proof. By Proposition 6.2 and Corollary 5.5, we have that $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, Z)$ is left noetherian and that $\{(\mathcal{R}_n)_{\sigma^n}\}$ is a left ample sequence. Thus by Theorem 2.9, the section ring $R(X, \mathcal{L}, \sigma, Z)$ is also left noetherian. □

Unfortunately, we cannot prove the converse to Proposition 7.3 in full generality. We do give several special cases below where the converse does hold.

Proposition 7.4. *Assume Assumption-Notation 7.1. Suppose that $\{\sigma^n(Z)\}_{n \geq 0}$ is not critically transverse, and either*

- (1) *there is some σ -invariant subscheme Y that is not homologically transverse to Z or*
- (2) *all irreducible components of Z have codimension 1.*

Then $R = R(X, \mathcal{L}, \sigma, Z)$ is not left noetherian.

Before giving the proof, we give a preliminary lemma.

Lemma 7.5. *Let $X = X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \dots$ be the singular stratification of X . Suppose that Z is a subscheme of pure codimension 1 such that for all j , $\text{Tor}_1^X(\mathcal{O}_Z, \mathcal{O}_{X^{(j)}}) = 0$. Then Z is locally principal.*

Proof. Fix $x \in Z$; we will show that Z is locally principal at x . Let $\mathcal{O} = \mathcal{O}_{X,x}$.

Let j be maximal so that $x \in X^{(j)}$, and let J be the ideal of $X^{(j)}$ in \mathcal{O} . Let I be the defining ideal of Z in \mathcal{O} . By Lemma 5.7, $I \not\subseteq J$. Thus $(I + J)/J$ locally defines a hypersurface in $X^{(j)}$. Since \mathcal{O}/J is a regular local ring, $(I + J)/J$ is principal in \mathcal{O}/J , and so there is $f \in I$ such that $(f) + J = I + J$.

By homological transversality, $I \cap J = IJ$. Thus

$$\frac{I}{(f)} \otimes_{\mathcal{O}} \frac{\mathcal{O}}{J} \cong \frac{I}{(f) + IJ} = \frac{I}{(f) + I \cap J}.$$

But

$$(f) + I \cap J = I \cap ((f) + J) = I \cap (I + J) = I.$$

Thus

$$\frac{I}{(f)} \otimes_{\mathcal{O}} \frac{\mathcal{O}}{J} \cong \frac{I}{I} = 0.$$

Let K be the residue field of \mathcal{O} . Since $I/(f) \otimes_{\mathcal{O}} (\mathcal{O}/J)$ surjects onto $(I/(f)) \otimes_{\mathcal{O}} K$, we see that $(I/(f)) \otimes_{\mathcal{O}} K = 0$. Nakayama's Lemma implies that $I = (f)$. \square

Proof of Proposition 7.4. Suppose (1) holds. Let Y be a σ -invariant subscheme that is not homologically transverse to Z , and let $j \geq 1$ be such that

$$\text{Tor}_j^X(\mathcal{O}_Z, \mathcal{O}_Y) \neq 0.$$

Let $\mathcal{J} = \mathcal{I}_Y$, and let $J = \Gamma_*(\mathcal{J})$ be the right ideal of B generated by sections that vanish on Y . Since $\sigma Y = Y$, J is a two-sided ideal of B . We claim that $\text{Tor}_j^B(B/I, B/J)_n \neq 0$ for $n \gg 0$.

Form a graded projective resolution

$$\dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow B/I \rightarrow 0$$

of the right B -module B/I , where each P^i is a finitely generated graded free right B -module. Thus for each $i \leq 0$, there is a finite multiset A_i of integers such that

$$P^i = \bigoplus_{a \in A_i} B[a].$$

Now, for each i let $\mathcal{P}^i = \widetilde{P}^i$. Since the functor $\widetilde{}$ is exact, the complex

$$\dots \rightarrow \mathcal{P}^{-1} \rightarrow \mathcal{P}^0$$

is a resolution of $\mathcal{O}_Z = \widetilde{B/I}$. Furthermore, by the σ -invariance of Y and the σ -ampleness of \mathcal{L} , for $-j - d \leq i \leq -j + 1$ and for $n \gg 0$, we have that

$$(7.3) \quad H^0(X, \mathcal{P}^i \otimes \mathcal{L}_n \otimes \mathcal{O}_Y) = \bigoplus_{a \in A_i} (B/J)_{n+a} = (P^i \otimes_B B/J)_n.$$

Fix n and let $\mathcal{Q}^\bullet = \mathcal{P}^\bullet \otimes \mathcal{L}_n \otimes \mathcal{O}_Y$. Reasoning as in Proposition 6.2, from a Cartan-Eilenberg resolution $\mathcal{C}^{\bullet,\bullet}$ of \mathcal{Q}^\bullet we obtain two spectral sequences,

$$(7.4) \quad {}^I E_{pq}^2 = H^p(H^q(X, \mathcal{Q}^\bullet))$$

and

$$(7.5) \quad {}^{II} E_{pq}^2 = H^p(X, \mathcal{T}or_{-q}^X(\mathcal{O}_Z, \mathcal{O}_Y) \otimes \mathcal{L}_n),$$

both of which converge (since X has finite cohomological dimension) to the hypercohomology $\mathbb{H}^{p+q}(\mathcal{C}^{\bullet,\bullet})$.

By σ -ampleness of \mathcal{L} , by taking $n \gg 0$ we may assume that

$$H^p(X, \mathcal{T}or_{-q}(\mathcal{O}_Z, \mathcal{O}_Y) \otimes \mathcal{L}_n) = 0 \quad \text{for } p \geq 1 \text{ and } -j - d \leq q \leq -j - 1$$

and that

$$H^q(X, \mathcal{Q}^p) = 0 \quad \text{for } q \geq 1 \text{ and } -j - d \leq p \leq -j - 1.$$

Thus for $p + q = -j$, both (7.4) and (7.5) collapse, and we obtain that

$$(7.6) \quad H^0(X, \mathcal{T}or_j^X(\mathcal{O}_Z, \mathcal{O}_Y) \otimes \mathcal{L}_n) = H^{-j}(H^0(X, \mathcal{Q}^\bullet)).$$

Since $\mathcal{T}or_j^X(\mathcal{O}_Z, \mathcal{O}_Y) \neq 0$ and \mathcal{L} is σ -ample, for $n \gg 0$ the left-hand side of (7.6) is nonzero. However, (7.3) implies that for $n \gg 0$, the right-hand side is equal to

$$H^{-j}(P^\bullet \otimes_B B/J)_n = \text{Tor}_j^B(B/I, B/J)_n.$$

Thus $\text{Tor}_j^B(B/I, B/J)_n \neq 0$.

But if R is left noetherian, then, using Proposition 4.3 and a similar argument to that used in the proof of Lemma 5.4, for any finitely generated left B -module M and for any $j \geq 1$, we must have that $\text{Tor}_j^B(B/I, M)$ is torsion. Since we have shown this is false for $M = B/J$, R is not left noetherian.

Now suppose that (2) holds. Consider the singular stratification

$$X = X^{(0)} \supset X^{(1)} \supset \dots$$

of X . If Z is not homologically transverse to some $X^{(i)}$, then by (1) R is not left noetherian. If Z is homologically transverse to all $X^{(i)}$, then by Lemma 7.5 Z is locally principal. By Lemma 5.4, there is some reduced and irreducible subscheme Y such that $\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0$ for infinitely many $n \geq 0$. Our assumptions on Y and Z imply that $\mathcal{T}or_1^X(\mathcal{O}_{\sigma^n Z}, \mathcal{O}_Y) \neq 0$ if and only if $\sigma^n Z \supseteq Y$.

Thus $\sigma^n(Z) \supseteq Y$ for infinitely many $n \geq 0$. Let \mathcal{I} be the ideal sheaf defining Y and let

$$A = \{n \geq 0 \mid Y \subseteq \sigma^n Z\} = \{n \geq 0 \mid \mathcal{I}^{\sigma^n} \supseteq \mathcal{I}\}.$$

Let $R' = \mathbb{k} \oplus H^0(X, \mathcal{I}B_+)$. It is sufficient to show that R' is not left noetherian. Let

$$J = \bigoplus_{n \geq 0} H^0(X, (\mathcal{I} \cap \mathcal{I}^{\sigma^n}) \mathcal{L}_n).$$

We will show that the left ideal J of R' is not finitely generated.

Fix an integer $k \geq 1$. By σ -ampleness of \mathcal{L} , we may choose $n > k$ so that $n \in A$ and $(\mathcal{I} \cap \mathcal{I}^{\sigma^n}) \mathcal{L}_n = \mathcal{I} \mathcal{L}_n$ is globally generated. Then

$$(R' \cdot J_{\leq k})_n \subseteq H^0(X, \mathcal{I} \mathcal{I}^{\sigma^n} \mathcal{L}_n) \subsetneq J_n$$

and ${}_R J$ is not finitely generated. □

As a corollary to Proposition 7.4, we obtain that if we are in characteristic 0 and if σ is an element of an algebraic group acting on X , then the converse to Proposition 7.3 holds.

Corollary 7.6. *Suppose that $\text{char } \mathbb{k} = 0$. Let X be a projective variety and let σ be an element of an algebraic group G that acts on X . Let \mathcal{L} be a σ -ample invertible sheaf on X . Let Z be a closed subscheme of X such that the components of Z^{red} have infinite order under σ . Then the following are equivalent:*

- (1) *the geometric idealizer $R = R(X, \mathcal{L}, \sigma, Z)$ is noetherian;*
- (2) *the set $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse;*
- (3) *Z is homologically transverse to all reduced σ -invariant subschemes of X .*

Proof. First suppose that there is $x \in X$ so that $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is infinite. Then by Proposition 7.2, R is not right noetherian. Furthermore, $\{\sigma^n Z\}_{n \in \mathbb{Z}}$ is certainly not critically transverse, and so by Theorem 5.11 there is a σ -invariant subscheme that is not homologically transverse to Z , and $\{\sigma^n Z\}_{n \geq 0}$ is not critically transverse. Thus (1), (2), and (3) all fail, and the result holds.

Thus we may assume that no such x exists; by Proposition 7.2, R is right noetherian. Note also that by Lemma 4.2, Assumption-Notation 4.1 is satisfied. Then (1) \Rightarrow (3) is Proposition 7.4. (3) \Rightarrow (2) is Theorem 5.11. (2) \Rightarrow (1) is Proposition 7.3. □

Since the geometric condition required for a right idealizer to be left noetherian is fairly subtle, it is not surprising that right idealizers are almost never strongly left noetherian. To show this, we use the concept of *generic flatness* as defined in [2]. Let C be a commutative noetherian domain. We say that a C -module M is *generically flat* if there is some $f \neq 0 \in C$ such that M_f is flat over C_f . If R is a finitely generated commutative C -algebra, then by Grothendieck’s generic freeness theorem [13, Theorem 6.9.1], every finitely generated R -module is a generically flat C -module.

Artin, Small, and Zhang have generalized this result to strongly noetherian non-commutative rings. They prove:

Theorem 7.7 ([2, Theorem 0.1]). *Let R be a strongly noetherian algebra over an excellent Dedekind domain C . Then every finitely generated right R -module is generically flat over C .* □

Lemma 7.8. *Assume Assumption-Notation 7.1. If Z' is a component of Z such that $\text{codim } Z' \geq 2$ and such that $\bigcup_{m \geq 0} \sigma^m Z'$ is Zariski dense in X , then for every open affine $U \subseteq X$, the finitely generated left $R \otimes_{\mathbb{k}} \mathcal{O}(U)$ -module*

$$M = \bigoplus_{n \geq 0} \mathcal{R}(\sigma^{-n}U)$$

is not a generically flat $\mathcal{O}(U)$ -module.

Proof. We first verify that M is a left R -module. By [3, Equation 2.5], the multiplication rule in \mathcal{R} acts on sections via

$$\mathcal{R}_n(V) \times \mathcal{R}_m(\sigma^n V) \rightarrow \mathcal{R}_{n+m}(V)$$

or, writing $V = \sigma^{-n-m}U$,

$$\mathcal{R}_n(\sigma^{-n-m}U) \times \mathcal{R}_m(\sigma^{-m}U) \rightarrow \mathcal{R}_{n+m}(\sigma^{-n-m}U).$$

Thus we have a map

$$R_n \times M_m = \mathcal{R}(X) \times \mathcal{R}_m(\sigma^{-m}U) \xrightarrow{\text{res}} \mathcal{R}_n(\sigma^{-n-m}U) \times \mathcal{R}_m(\sigma^{-m}U) \rightarrow M_{m+n}.$$

Verifying associativity is trivial, and so M is a left R -module.

Let $C = \mathcal{O}(U)$. By identifying C with C^{op} , consider the right action of C on M given by $g \star f = g \cdot f^{\sigma^n} = g \cdot (f \circ \sigma^n)$, where $g \in M_n$, $f \in C$. Note that $f \circ \sigma^n$ acts on $\sigma^{-n}U$ and so acts naturally on elements of M_n .

Now since for $n \gg 0$ the sheaves $\mathcal{I} \otimes \mathcal{L}_n$ are globally generated, the restriction map $R \rightarrow M$ is surjective in degree $\geq m$ for some m . But since $M_{<m}$ is a finitely generated C -module, M is a finitely generated R_C module.

Now let f be an arbitrary element of C and let $M' = M_f$. Since $\bigcup_{m \geq 0} \sigma^m(Z')$ is Zariski dense, there is some m such that $\sigma^m Z'$ meets U_f , say at a point p . But then $(M'_m)_p = (\mathcal{I}\mathcal{L}_m)_{\sigma^{-m}p}$, which is not flat over C_p , since $\text{codim } Z' \geq 2$. Thus M_f is not flat over C_f . \square

Corollary 7.9. *Assume Assumption-Notation 7.1. Then R is strongly left noetherian if and only if Z has pure codimension 1 and $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse in X .*

Proof. If $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse, then in particular Z is homologically transverse to the singular stratification of X . If Z has pure codimension 1, then by Lemma 7.5, Z is locally principal and $\mathcal{I} = \mathcal{I}_Z$ is invertible. Now, letting $\mathcal{L}' = \mathcal{I}\mathcal{L}(\mathcal{I}^{-1})^\sigma$, we have that $\mathcal{I}\mathcal{L}_n = (\mathcal{L}')_n \mathcal{I}^{\sigma^n}$. Since \mathcal{L}' is clearly also σ -ample, we see that R is also the left idealizer at Z inside the twisted homogeneous coordinate ring $B(X, \mathcal{L}', \sigma)$. By assumption on critical transversality, we have in particular that for any $p \in X$, the set $\{n \leq 0 \mid \sigma^n(p) \in Z\}$ is finite. Thus by Proposition 7.2, R is strongly left noetherian.

If Z has pure codimension 1 and $\{\sigma^n Z\}_{n \geq 0}$ is not critically transverse, then by Proposition 7.4(2), R is not left noetherian and so is certainly not strongly left noetherian.

Now suppose that Z has a component Z' such that $\text{codim } Z' > 1$. Let Y be the Zariski closure of $\{\sigma^n(Z')\}_{n \geq 0}$. Note that there is a chain of containments

$$Y \supseteq \sigma Y \supseteq \sigma^2 Y \supseteq \dots$$

As this chain terminates, for some j we have $\sigma^j Y = \sigma^{j+1} Y$; applying σ^{-j} we see that Y itself is σ -invariant. If $Y \neq X$, then Z is not homologically transverse to Y by Lemma 5.7, and so by Proposition 7.4(1), R is not left noetherian.

We have thus reduced to considering the case that Z has a component Z' such that $\text{codim } Z' > 1$ and that $\bigcup_{n \geq 0} \sigma^n(Z')$ is Zariski-dense in X . Fix an open affine $U \subseteq X$ such that $X \setminus U$ has codimension 1. Let M be the module from Lemma 7.8. As M is not a generically flat left $\mathcal{O}(U)$ -module, by Theorem 7.7 $R \otimes_{\mathbb{k}} \mathcal{O}(U)$ is not strongly left noetherian. Thus R is not strongly left noetherian. \square

To end this section, we use Proposition 7.4 to show that the idealizers $R = R(X, \mathcal{L}, \sigma, Z)$ have the unusual property that $R \otimes_{\mathbb{k}} R$ is not noetherian. The idealizers constructed by Rogalski in [21] were the first known examples of noetherian rings with this property.

Proposition 7.10. *Assume Assumption-Notation 7.1. Assume in addition that Z is not of pure codimension 1. Then $R \otimes_{\mathbb{k}} R$ is not left noetherian.*

Proof. Let $T = R \otimes_{\mathbb{k}} R$. Consider the \mathbb{Z} -grading on T given by

$$T_n = \sum_{i \in \mathbb{N}} R_i \otimes R_{n+i}.$$

If T is left noetherian, then T_0 must also be left noetherian. Thus we will show that T_0 is not left noetherian.

Let $S = T_0$. The ring S is also known as the *Segre product* of R with itself and is sometimes written $S = R \overset{s}{\otimes} R$. It has a natural \mathbb{N} -grading, given by $S_n = R_n \otimes R_n$. Let $Y = (Z \times X) \cup (X \times Z)$. It can readily be seen that

$$S = R(X \times X, \mathcal{L} \boxtimes \mathcal{L}, \sigma \times \sigma, Y).$$

Let $\Delta \subseteq X \times X$ be the diagonal. We claim that Y is not homologically transverse to Δ . As R is left noetherian, by Proposition 7.4 Z is homologically transverse to the singular stratification of X . In particular, there is some component Z' of Z , of codimension $d \geq 2$, so that Z' is not contained in the singular stratification of X . Thus, if η is the generic point of Z' , the local ring $\mathcal{O} = \mathcal{O}_{X,\eta}$ is a regular local ring of dimension d . Let J be the ideal of Z in \mathcal{O} . By assumption, J is η -primary and is not principal.

Let $\mathcal{O}' = \mathcal{O}_{X \times X, \eta \times \eta}$. There is a natural embedding $\mathcal{O} \otimes_{\mathbb{k}} \mathcal{O} \rightarrow \mathcal{O}'$. Let x_1, \dots, x_d be a system of parameters for the maximal ideal η_η of \mathcal{O} . The maximal ideal \mathfrak{m} of \mathcal{O}' is generated by the system of parameters $x_1 \otimes 1, \dots, x_d \otimes 1, 1 \otimes x_1, \dots, 1 \otimes x_d$. Let K be the ideal of Y in \mathcal{O}' ; note that K is generated by the image of $J \otimes J$.

Let $H = (\{x_i \otimes 1 - 1 \otimes x_i \mid 1 \leq i \leq d\})$ be the ideal of Δ in \mathcal{O}' . Now, by [14, Theorem A.1.1], the intersection product on $X \times X$ satisfies

$$i(Y, \Delta; \eta) = i(X \times Z, \Delta; \eta) + i(Z \times X, \Delta; \eta).$$

This is easily seen to be

$$2 \operatorname{len}_\eta(\mathcal{O}/J).$$

On the other hand, consider $\mathcal{O}'/K \otimes_{\mathcal{O}'} \mathcal{O}'/H \cong \mathcal{O}'/(K + H)$. Since $\mathcal{O}'/H \cong \mathcal{O}$, this is isomorphic to \mathcal{O}/J^2 . As J is not principal, we have

$$(7.7) \quad \operatorname{len}_\eta(\mathcal{O}/J^2) > 2 \operatorname{len}_\eta(\mathcal{O}/J).$$

By (5.1),

$$2 \operatorname{len}_\eta(\mathcal{O}/J) = i(Y, \Delta; \eta) = \sum_{i \geq 0} (-1)^i \operatorname{len}_\eta \mathcal{T}or_i^{X \times X}(\mathcal{O}_Y, \mathcal{O}_\Delta).$$

Using (7.7), we obtain

$$\begin{aligned} \sum_{i \geq 1} (-1)^i \operatorname{len}_\eta \mathcal{T}or_i^{X \times X}(\mathcal{O}_Y, \mathcal{O}_\Delta) &= i(Y, \Delta; \eta) - \operatorname{len}_\eta(\mathcal{O}_Y \otimes \mathcal{O}_\Delta) \\ &= 2 \operatorname{len}_\eta \mathcal{O}/J - \operatorname{len}_\eta \mathcal{O}/J^2 < 0. \end{aligned}$$

Thus Y and Δ are not homologically transverse.

Now, Δ is $\sigma \times \sigma$ -invariant, and the proof of [16, Corollary 3.5] shows that $\mathcal{L} \boxtimes \mathcal{L}$ is $\sigma \times \sigma$ -ample. Thus, by Proposition 7.4,

$$T_0 = S \cong R(X \times X, \mathcal{L} \boxtimes \mathcal{L}, \sigma \times \sigma, Y)$$

is not left noetherian. Thus T is not left noetherian. □

We remark that if Z is reduced at η (so that $J = \eta$), then it is easy to directly see that $\mathcal{T}or_1^{X \times X}(\mathcal{O}_Y, \mathcal{O}_\Delta) \neq 0$. Indeed, the element

$$x_1 \otimes x_2 - x_2 \otimes x_1 = (x_1 \otimes 1 - 1 \otimes x_1)(x_2 \otimes 1) + (1 \otimes x_2 - x_2 \otimes 1)(x_1 \otimes 1)$$

is in $H \cap K \setminus HK$; note that $HK \subseteq \mathfrak{m}^3$.

In the case that Z is a point in \mathbb{P}^d that moves in a critically dense orbit, Rogalski [21, Theorem 1.1] shows that $R \otimes_k R$ is right noetherian and $R \otimes_k R^{\text{op}}$ is left and right noetherian. Rogalski’s proof depends crucially on the fact that Z is 0-dimensional. For higher-dimensional Z , we expect that analogous results hold, although we have not been able to generalize Rogalski’s proof.

8. THE χ CONDITIONS FOR IDEALIZERS

In this section, we determine the homological properties of graded idealizers; specifically, we investigate the Artin-Zhang χ conditions, as defined in the Introduction.

We first recall Rogalski’s result that a right idealizer will fail χ_1 and all higher χ_j on the left.

Proposition 8.1 (Rogalski). *Assume Assumption-Notation 7.1. Then R fails left χ_1 .*

Proof. This is proved in [21, Proposition 4.2]. To see it directly, note that changing R by a finite-dimensional vector space does not affect the χ conditions, so without loss of generality we have $R = \mathbb{k} + I$. Now B/R is infinite-dimensional and is killed on the left by I ; as we have an injection $B/R \hookrightarrow \text{Ext}_R^1(\mathbb{k}, R)$, we see that $\text{Ext}_R^1(\mathbb{k}, R)$ is infinite-dimensional. \square

To analyze the right χ conditions, our key result is the following, due to Rogalski:

Proposition 8.2 ([21, Proposition 4.1]). *Let B be a noetherian ring that satisfies right χ . Let I be a right ideal of B , and let $R = \mathbb{I}_B(I)$. Assume that B/I is infinite-dimensional, that B_R is finitely generated, and that R/I is finite-dimensional. Then R satisfies right χ_i for some $i \geq 0$ if and only if $\underline{\text{Ext}}_B^j(B/I, M)$ is finite-dimensional for all $0 \leq j \leq i$ and all $M \in \text{gr-}B$.* \square

Rogalski proved that the right idealizer of a point in \mathbb{P}^d satisfies right χ_{d-1} and fails right χ_d if the orbit of the point is critically dense. Here we extend Rogalski’s result to idealizers at higher-dimensional subschemes.

Lemma 8.3. *Let X be a projective variety, let $\sigma \in \text{Aut}_{\mathbb{k}} X$, and let \mathcal{L} be a σ -ample invertible sheaf on X . Let Z and Y be closed subschemes of X , and let $B = B(X, \mathcal{L}, \sigma)$. Let J be the right ideal of B consisting of sections vanishing along Y , and let I be the right ideal of B consisting of sections vanishing along Z . For $n \gg 0$, there is an isomorphism of \mathbb{k} -vector spaces*

$$\underline{\text{Ext}}_{\text{gr-}B}^j(B/I, B/J)_n \cong \text{Ext}_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

Proof. The twisted homogeneous coordinate ring B satisfies χ as a consequence of [31, Theorem 7.3] and [29, Theorem 6.3] (or alternately, [32, Theorem 4.2]). Thus the natural map from $\underline{\text{Ext}}_{\text{gr-}B}^j(B/I, B/J)$ to $\underline{\text{Ext}}_{\text{Qgr-}B}^j(\pi(B/I), \pi(B/J))$ has right

bounded kernel and cokernel by [4, Proposition 3.5]. It therefore suffices to show that for $n \gg 0$, we have

$$\underline{\text{Ext}}_{\text{Qgr-}B}^j(\pi(B/I), \pi(B/J))_n \cong \text{Ext}_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

In fact, we show that we have this isomorphism for all $n \geq 0$.

Using the equivalence between $\text{qgr-}B$ and $\mathcal{O}_X\text{-mod}$, we have that

$$\begin{aligned} \underline{\text{Ext}}_{\text{Qgr-}B}^j(\pi(B/I), \pi(B/J))_n &\cong \text{Ext}_{\text{Qgr-}B}^j(\pi(B/I), \pi((B/J)[n])) \\ &\cong \text{Ext}_X^j(\widetilde{B/I}, (\widetilde{B/J})[n]). \end{aligned}$$

Now, $\widetilde{B/I} = \mathcal{O}_Z$, and by (2.1),

$$(\widetilde{B/J})[n] \cong (\mathcal{O}_Y \mathcal{L}_n)^{\sigma^{-n}} \cong \mathcal{O}_{\sigma^n Y} \mathcal{L}_n^{\sigma^{-n}}.$$

The result follows. □

We have seen that for R to be right noetherian is relatively straightforward, but the left noetherian property for R depends on the critical transversality of $\{\sigma^n Z\}$. It turns out that the right χ_j properties, for $j \geq 1$, also depend on the critical transversality of $\{\sigma^n Z\}$. In particular, we have:

Proposition 8.4. *Assume Assumption-Notation 7.1. Let $d = \text{codim } Z$.*

- (1) *If $\{\sigma^n Z\}_{n \leq 0}$ is critically transverse, and either*
 - (a) *X is nonsingular and Z is Gorenstein or*
 - (b) *Z is 0-dimensional,*

then R satisfies right χ_{d-1} but fails right χ_d .

- (2) *More generally, if Z contains an irreducible component of codimension d that is not contained in the singular locus of X , then R fails right χ_d . In particular, if R is left noetherian, then R fails right χ_d .*

Proof. By Proposition 8.2, R satisfies right χ_i if and only if for all finitely generated M_B we have $\dim_{\mathbb{k}} \underline{\text{Ext}}_{\text{gr-}B}^j(B/I, M) < \infty$ for all $j \leq i$. Furthermore, using the equivalence of categories between $\text{qgr-}B$ and $\mathcal{O}_X\text{-mod}$, without loss of generality we may assume that $M = B/J$, where J is a right ideal of B consisting of sections vanishing along a reduced, irreducible subscheme Y of X .

By Lemma 8.3, for $n \gg 0$ we have isomorphisms

$$\underline{\text{Ext}}_{\text{gr-}B}^j(B/I, B/J)_n \cong \text{Ext}_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

Thus we have

$$(8.1) \quad R \text{ satisfies right } \chi_i \iff \text{for all } Y \subseteq X, \text{Ext}_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}) = 0 \text{ for all } j \leq i \text{ and } n \gg 0.$$

By [12, Prop 4.2.1], for any coherent sheaves \mathcal{E} and \mathcal{F} there is a spectral sequence

$$(8.2) \quad H^p(X, \mathcal{E}xt_X^q(\mathcal{E}, \mathcal{F})) \Rightarrow \text{Ext}_X^{p+q}(\mathcal{E}, \mathcal{F}).$$

We consider the special case

$$(8.3) \quad E^{pq} = H^p(X, \mathcal{E}xt_X^q(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}})) \Rightarrow \text{Ext}_X^{p+q}(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}).$$

We first suppose that (1)(a) holds, and show that R satisfies right χ_{d-1} .

Fix a closed subscheme Y of X and consider the sheaf $\mathcal{E}xt_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y})$. This is supported on Z ; we compute it by working locally at some closed point $x \in Z$.

Gorenstein rings are Cohen-Macaulay and therefore locally equi-dimensional [9, Corollary 18.11], so we may assume that Z is pure-dimensional of codimension $d' \geq d$. Let $J \subseteq \mathcal{O}$ be the ideal defining Z locally at x .

By [9, Corollary 21.16], \mathcal{O}/J has a self-dual free resolution as an \mathcal{O} -module

$$0 \rightarrow Q_{d'} \rightarrow \cdots \rightarrow Q_0 \rightarrow \mathcal{O}/J.$$

We write this resolution as $Q_\bullet \rightarrow \mathcal{O}/J$.

For a given n , let $K \subseteq \mathcal{O}$ be the ideal defining $\sigma^n Y$ at P . Let $M = \mathcal{O}/K$. Then we have isomorphisms of complexes

$$\mathrm{Hom}_{\mathcal{O}}(Q_\bullet, M) \cong \mathrm{Hom}_{\mathcal{O}}(Q_\bullet, \mathcal{O}) \otimes M \cong Q_\bullet \otimes M,$$

where the final isomorphism follows from the fact that Q_\bullet is self-dual. The right-hand complex of this equation computes $\mathrm{Tor}_{d'-j}^{\mathcal{O}}(\mathcal{O}/J, M)$. Thus we obtain isomorphisms

$$(8.4) \quad \mathcal{E}xt_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) \cong \mathrm{Tor}_{d'-j}^X(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) \cong \mathrm{Tor}_{d'-j}^X(\mathcal{O}_{\sigma^{-n}Z}, \mathcal{O}_Y)^{\sigma^{-n}}$$

for all j .

We return to the Grothendieck spectral sequence (8.3). By [14, III.6.7],

$$\mathcal{E}xt_X^q(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}) \cong \mathcal{E}xt_X^q(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) \otimes \mathcal{L}_n^{\sigma^{-n}}.$$

Using critical transversality and (8.4), choose n_0 such that $\mathcal{E}xt_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y}) = 0$ for all $n \geq n_0$ and $j < d \leq d'$. Then $E^{pq} = 0$ for $q < d$; so we see that if $p + q = j < d$, then (8.3) collapses to 0 and we have $\mathrm{Ext}_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n^{\sigma^{-n}}) = 0$ for $n \gg 0$. By (8.1), R satisfies χ_{d-1} .

Let X^{sing} be the singular locus of X . We now suppose that (2) holds; that is, Z contains an irreducible component of codimension d that is not contained in X^{sing} . We show that in this situation R fails right χ_d .

We consider the special case of (8.3) where $Y = X$:

$$(8.5) \quad H^p(X, \mathcal{E}xt_X^q(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}}).$$

Let $x \in Z$ be a nonsingular point of X such that the codimension of Z at x is d . Since X is nonsingular at x , by [7, Theorem 1.2.5]

$$(8.6) \quad \mathcal{E}xt_X^j(\mathcal{O}_Z, \mathcal{O}_X)_x = 0 \quad \text{for } j < d$$

and

$$(8.7) \quad \mathcal{E}xt_X^d(\mathcal{O}_Z, \mathcal{O}_X)_x \neq 0.$$

Now (8.6) implies that for $p + q = d$, (8.5) collapses, and we obtain that

$$\mathrm{Ext}_X^d(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}}) \cong H^0(X, \mathcal{E}xt_X^d(\mathcal{O}_Z, \mathcal{O}_X) \otimes \mathcal{L}_n^{\sigma^{-n}}) \cong H^0(X, \mathcal{E}xt_X^d(\mathcal{O}_Z, \mathcal{O}_X)^{\sigma^n} \otimes \mathcal{L}_n).$$

This is nonzero for $n \gg 0$ by (8.7) and by σ -ampleness of \mathcal{L} . Thus by (8.1), R fails right χ_d .

We have seen that if (2) holds, then R fails right χ_d . We note that if $\{\sigma^n Z\}_{n \leq 0}$ is critically transverse, then Z is homologically transverse to all σ -invariant subschemes, and certainly no component of Z is contained in X^{sing} . If R is left noetherian, then using Proposition 7.4 and Lemma 5.7, we again have that no component of Z is contained in the singular locus of X . Thus if (1)(a) or (1)(b) hold, or if R is left noetherian, then (2) holds and R fails right χ_d .

It remains to show that if (1)(b) holds, then R satisfies right χ_{d-1} . We have seen that X is nonsingular at all points of Z , and so (8.6) holds. Let $j \leq d - 1$. By (8.5) we have that $\text{Ext}_X^j(\mathcal{O}_Z, \mathcal{L}_n^{\sigma^{-n}}) = 0$. On the other hand, if $Y \subset X$ is a proper subvariety, then critical transversality of $\{\sigma^n Z\}_{n \leq 0}$ and Corollary 5.8 show that $\sigma^n Y$ and Z are disjoint for $n \gg 0$, and so certainly $\text{Ext}_X^j(\mathcal{O}_Z, \mathcal{O}_{\sigma^n Y} \otimes \mathcal{L}_n) = 0$ for $n \gg 0$. By (8.1), R satisfies right χ_{d-1} . \square

9. PROJ OF GRADED IDEALIZER RINGS AND COHOMOLOGICAL DIMENSION

Assume Assumption-Notation 7.1. We end this paper by investigating the cohomological dimension of the (right) noncommutative projective scheme associated to R ; we briefly review the definitions here.

Let R be a (noncommutative) \mathbb{N} -graded ring, and recall that the category $\text{Qgr-}R$ is the noncommutative analogue of Proj of a commutative graded ring. In [4], Artin and Zhang define $\text{Proj-}R$ to be the pair $(\text{Qgr-}R, \pi R)$, where $\pi : \text{Gr-}R \rightarrow \text{Qgr-}R$ is the quotient functor. The cohomology groups on $\text{Proj-}R$ are defined by setting

$$H^i(\text{Proj-}R, \mathcal{M}) = \text{Ext}_{\text{Qgr-}R}^i(\pi R, \mathcal{M})$$

for any $\mathcal{M} \in \text{Qgr-}R$. The *cohomological dimension* of $\text{Proj-}R$ or the *right cohomological dimension* of R is

$$\text{cd}(\text{Proj-}R) = \max\{i \mid H^i(\text{Proj-}R, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \text{Qgr-}R\}.$$

That is, $\text{cd}(\text{Proj-}R)$ is the cohomological dimension of the functor $H^0(\text{Proj-}R, _)$. We similarly define the *left cohomological dimension* of R , or $\text{cd}(R\text{-Proj})$.

If R is a finitely generated commutative graded \mathbb{k} -algebra, then its cohomological dimension is finite and in fact bounded by the dimension of $\text{Proj } R$. The proofs of this are geometric, for example relying on Čech cohomology calculations, and do not generalize to the noncommutative situation. Stafford and Van den Bergh have asked [26, page 194] if every connected graded noetherian ring has finite left and right cohomological dimension.

In this section, we answer Stafford and Van den Bergh’s question for geometric idealizers. We prove:

Theorem 9.1. *Assume Assumption-Notation 7.1. If $R = R(X, \mathcal{L}, \sigma, Z)$ is noetherian, then R has finite left and right cohomological dimension.*

Thus, while we have seen that the χ conditions and the strong noetherian property are quite asymmetrical for geometric idealizers, cohomological dimension appears to behave better, at least in the (two-sided) noetherian case. This, unsurprisingly, breaks down for non-noetherian rings, and we give an example of a right, but not left, noetherian ring with infinite right cohomological dimension. Amusingly, this ring has finite left cohomological dimension.

To begin, we review Rogalski’s results on the cohomological dimension of idealizers.

Proposition 9.2 ([21, Lemma 3.2]). *Let B be a noetherian connected graded finitely \mathbb{N} -graded \mathbb{k} -algebra, and let I be a graded right ideal of B such that R/I is infinite-dimensional. Assume that B_R is finitely generated and R/I is finite-dimensional. Then there are isomorphisms of pairs*

$$(9.1) \quad R\text{-Proj} = (R\text{-Qgr}, \pi R) \cong (B\text{-Qgr}, \pi B) = B\text{-Proj}$$

and

$$(9.2) \quad \text{Proj-}R = (\text{Qgr-}R, \pi R) \cong (\text{Qgr-}B, \pi I).$$

□

Because of (9.1), it is clear that $\text{cd}(R\text{-Proj}) = \text{cd}(B\text{-Proj}) = \dim X$, and this was observed by Rogalski. We thus focus on calculating $\text{cd}(\text{Proj-}R)$.

Lemma 9.3. *Assume Assumption-Notation 7.1. Then $\text{cd}(\text{Proj-}R)$ is infinite if and only if $\text{hd}_X(\mathcal{O}_Z)$ is infinite.*

Proof. Let $I = \Gamma_*(\mathcal{I}) \subseteq B$. Since $(\text{Qgr-}B, \pi I) \cong (\mathcal{O}_X\text{-Mod}, \mathcal{I})$, by (9.2) $\text{cd}(\text{Proj-}R)$ is infinite if and only if for any $k \geq 0$, there is some quasi-coherent \mathcal{F} such that $\text{Ext}_X^k(\mathcal{I}, \mathcal{F}) \neq 0$.

Suppose that $\text{hd}_X(\mathcal{O}_Z)$ and therefore $\text{hd}_X(\mathcal{I})$ are infinite. Thus for any $k > 0$, there is some \mathcal{G} such that $\mathcal{E}xt_X^k(\mathcal{I}, \mathcal{G}) \neq 0$. But let $\mathcal{O}(1)$ be any very ample invertible sheaf on X . By [14, III.6.9] we may choose n so that $\text{Ext}_X^k(\mathcal{I}, \mathcal{G}(n)) = H^0(X, \mathcal{E}xt_X^k(\mathcal{I}, \mathcal{G}) \otimes \mathcal{O}(n)) \neq 0$. Thus $\text{cd}(\text{Proj-}R) \geq k$, and since k was arbitrary, $\text{cd}(\text{Proj-}R)$ is infinite.

Now suppose that $\text{hd}_X(\mathcal{I})$ is finite, say equal to N , and let \mathcal{G} be an arbitrary coherent sheaf. We apply (8.2) to obtain a spectral sequence

$$(9.3) \quad H^p(X, \mathcal{E}xt_X^q(\mathcal{I}, \mathcal{G})) \Rightarrow \text{Ext}_X^{p+q}(\mathcal{I}, \mathcal{G}).$$

The left-hand side has nonzero terms only for $0 \leq p \leq \dim X$ and $0 \leq q \leq N$. Thus if $p + q$ is large (in particular $p + q > N + \dim X$), then all the groups on the left-hand side are 0, and so (9.3) collapses to 0. Thus $\text{cd}(\text{Proj-}R) \leq N + \dim X$. □

Proof of Theorem 9.1. If $R(X, \mathcal{L}, \sigma, Z)$ is left noetherian, then by Proposition 7.4 we have that $\{\sigma^n Z\}_{n \geq 0}$ is homologically transverse to all σ -invariant subvarieties of X , and in particular, to the singular stratification of X . Thus by Lemma 5.3, $\text{hd}_X(\mathcal{O}_Z)$ is finite. By Lemma 9.3, $\text{cd}(\text{Proj-}R)$ is finite. □

We now give the promised example of a right noetherian ring with infinite right cohomological dimension.

Example 9.4. Assume that $\text{char } k = 0$. Let Y be the cuspidal cubic, and let $X = Y \times \mathbb{P}^1$. Let $\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the automorphism $\tau([x : y]) = [x + y : y]$, and let $\sigma = 1 \times \tau \in \text{Aut}_k X$. Let P be the singular point of Y , and let $Z = P \times [0 : 1] \in X$. Let \mathcal{L} be any ample invertible sheaf on X , and let $R = R(X, \mathcal{L}, \sigma, Z)$. Since the numerical action of σ is trivial, by [15, Theorem 1.2] \mathcal{L} is σ -ample.

Now Z is certainly of infinite order under σ , and R is right noetherian by Proposition 7.2. On the other hand, Z is contained in the singular locus of X , and so Proposition 7.4(1) and Lemma 5.7 imply that R is not left noetherian. Since X is not regular at Z , we have that $\text{hd}_X(\mathcal{O}_Z)$ is infinite. Lemma 9.3 implies that $\text{cd}(\text{Proj-}R) = \infty$.

We note that Proposition 9.2 implies that the left cohomological dimension of R is 2.

Remark. Suppose that $R = R(X, \mathcal{L}, \sigma, Z)$ is a left noetherian idealizer. Together, Lemma 9.3 and Lemma 5.3 imply that the right cohomological dimension of R is bounded by $2 \dim X - 1$. We conjecture that in fact the left cohomological dimension of R is precisely $\dim X$. It is easy to see that $\text{cd}(\text{Proj-}R) \geq \dim X$.

10. CONCLUSION

Here we collect our results on geometric idealizers, and prove Theorem 1.8 and its promised generalization. Throughout, we make the following assumptions.

Assumptions 10.1. *Let X be a projective variety, let σ be an automorphism of X , and let \mathcal{L} be a σ -ample invertible sheaf on X . Let Z be a closed subscheme of X such that for any irreducible component Y of Z ,*

$$\sigma^n(Y^{\text{red}}) \not\subseteq Z$$

for $n \gg 0$.

Given this data, we let

$$R = R(X, \mathcal{L}, \sigma, Z).$$

Let $\mathcal{I} = \mathcal{I}_Z$ be the ideal sheaf of Z on X .

We note that since by Theorem 3.3 any noetherian right idealizer is up to a finite extension an idealizer at a scheme whose defining data satisfies Assumptions 10.1, these assumptions are not unduly restrictive.

We now summarize our results.

Theorem 10.2. *Assume Assumptions 10.1.*

(1) *R is right noetherian if and only if for any $x \in X$, the set $\{n \geq 0 \mid \sigma^n(x) \in Z\}$ is finite.*

(2) *If R is right noetherian, then R is strongly right noetherian.*

(3) *R fails left χ_1 .*

(4) *If $\{\sigma^n(Z)\}_{n \geq 0}$ is critically transverse, then $\{(\mathcal{I}\mathcal{L}_n)\sigma^n\}_{n \geq 0}$ is a left and right ample sequence of bimodules, and R is left noetherian.*

(5) *R is strongly left noetherian if and only if Z has pure codimension 1 and $\{\sigma^n Z\}_{n \geq 0}$ is critically transverse.*

(6) *Let $d = \text{codim } Z$. If $\{\sigma^n Z\}_{n \leq 0}$ is critically transverse and either $d = \dim X$ or X and Z are both smooth, then R satisfies right χ_{d-1} . If R is noetherian, then R fails right χ_d .*

(7) *If R is noetherian, then R has finite left and right cohomological dimension.*

(8) *If Z does not have pure codimension 1, then $R \otimes_{\mathbb{k}} R$ is not left noetherian.*

We note that Theorem 1.8 is a special case of Theorem 10.2.

Proof. By Lemma 4.2, the defining data of R satisfy Assumption-Notation 4.1. Then (1) and (2) are Proposition 7.2. (3) is Proposition 8.1. (4) is Lemma 6.1, Proposition 6.2 and Proposition 7.3. (5) is Corollary 7.9. (6) is a special case of Proposition 8.4, and (7) is Theorem 9.1. (8) is Proposition 7.10. \square

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