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CHARACTERIZING COMPLETE CAT(κ)-SPACES, $\kappa < 0$, WITH GEODESIC BOUNDARY

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Abstract. We investigate the Bourdon and Hamenstädt boundaries of complete CAT(κ)-spaces, $\kappa < 0$, and characterize those with geodesic Hamenstädt boundary up to isometry.

1. Introduction

In metric geometry two notions of negative upper curvature bounds have been intensively studied within the last decades, namely Gromov's notion of hyperbolicity and the CAT(κ)-condition, $\kappa < 0$.

Whereas the CAT(κ)-condition is a condition on all scales of the space, Gromov's notion of hyperbolicity is an asymptotically global concept. Indeed, it is a rough-isometry invariant.

While in metric geometry one thinks of two spaces as being indistinguishable if and only if they are isometric to each other in the coarse theory of Gromov hyperbolic spaces, two spaces should be identified with each other if and only if they are rough-isometric to each other. Thus, the objects of interest in this theory are rough-isometry classes of spaces, and the invariants to be studied are rough-isometry invariants.

An example of such an invariant is, for instance, the (negative) asymptotic upper curvature of a metric space, which has been introduced and studied in [BF1]. Among all metric spaces, the Gromov hyperbolic spaces turn out to be precisely those that are of negative asymptotic upper curvature. It is in this sense that we refer to the notion of Gromov hyperbolicity as a concept of negative curvature.

Being interested in rough-isometry classes of Gromov hyperbolic spaces, it is natural to ask whether within such a class it is possible to pass to some particularly nice representative, say a CAT(κ)-space for some $\kappa < 0$. Note that it follows from Theorem 1.1 in [BS] that under certain growth restrictions this is indeed the case.

Considering the infimum

$$\overline{R}(X) \ := \ \inf \left\{ \kappa < 0 \;\middle|\; X \overset{\text{rough}}{\cong} \mathrm{CAT}(\kappa) \right\}$$

over all $\kappa < 0$, such that the Gromov hyperbolic space X in question is rough-isometric to some CAT(κ)-space, one obtains another interesting rough-isometry invariant, which, for a Gromov hyperbolic space of so-called 'bounded growth at some scale', coincides with its asymptotic upper curvature bound ([BF1]).

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With this notation, our question raised above may be formulated as whether it is possible to characterize such Gromov hyperbolic spaces X for which $\overline{R}(X)$ is finite.

It turns out that, in order to attack this question, an understanding of the boundaries at infinity of $\mathrm{CAT}(\kappa)$-spaces is very desirable. Indeed, a crucial ingredient in the proof of Theorem 1.1 in [BS] is the fact that the Euclidean spaces $\mathbb{E}^n$ can be realized as certain metric boundaries at infinity of particular $\mathrm{CAT}(\kappa)$-spaces, namely the real hyperbolic spaces $\mathbb{H}^{n+1}_\kappa$ of dimension $n+1$ and constant sectional curvature $\kappa < 0$.

The goal of this paper is to contribute to the understanding of the boundaries at infinity of $\mathrm{CAT}(\kappa)$-spaces with the following theorems.

Theorem 1.1. *Let $\kappa < 0$. Then every complete* $\mathrm{CAT}(0)$*-space can be realized as a so-called Hamenstädt boundary of a complete* $\mathrm{CAT}(\kappa)$*-space.*

In other words, the $\mathrm{CAT}(0)$-condition is a sufficient condition for a metric space to be realized as a so-called Hamenstädt boundary of a $\mathrm{CAT}(\kappa)$-space. Moreover, if the Hamenstädt boundary of a $\mathrm{CAT}(\kappa)$-space, $\kappa < 0$, is geodesic, it necessarily has to be $\mathrm{CAT}(0)$.

Theorem 1.2. *The* $\mathrm{CAT}(0)$*-condition is a necessary and sufficient condition for a geodesic metric space to be realized as a Hamenstädt boundary of a* $\mathrm{CAT}(\kappa)$*-space.*

Finally, we precisely characterize the complete $\mathrm{CAT}(\kappa)$-spaces, $\kappa < 0$, with geodesic Hamenstädt boundary, up to isometry, and show that these are very specific metric warped products.

Theorem 1.3. *Let X be a* $\mathrm{CAT}(\kappa)$*-space, $\kappa < 0$, $o \in X$ and $\omega \in \partial_\infty X$ such that X is ω-visual and such that its Hamenstädt boundary $Y = Y(X, o, \omega)$ is geodesic. Then X is isometric to the warped product $\mathbb{R} \times_{\exp\{-\sqrt{-\kappa}t\}} Y$.*

For a more precise statement of Theorems 1.2 and 1.3, compare Theorem 4.1 in Section 4.

The $\mathrm{CAT}(\kappa)$-spaces, $\kappa < 0$, as constructed in the proof of Theorem 1.1, are certain special metric warped products, and all that really has to be done is to prove that a Hamenstädt boundary of such a warped product is isometric to its fiber. The basic ingredient here and also in the proofs of the other theorems is that the Euclidean space, the comparison space in the $\mathrm{CAT}(0)$-setting, is the corresponding metric warped product fiber of the real hyperbolic spaces, which are the comparison spaces in the $\mathrm{CAT}(\kappa)$-settings, $\kappa < 0$. That this fact can be used to build up an appropriate comparison theory is basically due to Theorem 3.1(b) in [AB1]. For the proofs of Theorems 1.2 and 1.3, the other basic ingredient is the rigidity part of Theorem 1.1 in [FS2], which claims that certain three-point configurations in a Hamenstädt boundary of $\mathrm{CAT}(\kappa)$-spaces guarantee the existence of isometrically embedded real hyperbolic ideal quadrilaterals in the spaces themselves.

Theorem 1.1 above has a certain analogue when considering a Bourdon boundary instead of a Hamenstädt boundary. Together with recent results, as obtained in [LS] and [BF2], our theorems also yield some corollaries important to note in the context of our original question.

The paper is structured as follows. In the preliminary Section 2 we recall the notions of Hamenstädt and Bourdon boundaries of Gromov hyperbolic and $\mathrm{CAT}(\kappa)$-spaces as well as Theorem 1.1 of [FS2], the rigidity part of which will essentially be used in the characterization of complete $\mathrm{CAT}(\kappa)$-spaces, $\kappa < 0$, with geodesic

Hamenstädt boundary. We also recall the other essential ingredient, namely the notion of metric warped products as introduced and deeply investigated in [C], [AB1] and [AB2]. In Section 3 we collect easy observations concerning the boundaries of very special metric warped products, which will frequently be used in Sections 3 and 4 when proving our Theorems 1.1, 1.2 and 1.3. Section 5 contains the analogue of Theorem 1.1 when considering a Bourdon boundary instead of a Hamenstädt boundary. Finally, in Section 6 we describe the aforementioned corollaries our theorems yield when combined with Theorem 1.3 in [LS] and the main theorem in [BF2], respectively.

2. Preliminaries

In this section we set up our notation and gather some material, which will frequently be used in the remainder of the paper.

2.1. **Boundaries of Gromov hyperbolic spaces.**

2.1.1. CAT(κ)-*spaces,* $\kappa < 0$. For Riemannian manifolds the impact of upper (or lower) sectional curvature bounds on the geometry of the space is rather well understood. Here one knows that a Riemannian manifold is of sectional curvature bounded above by $\kappa \in \mathbb{R}$ if and only if sufficiently small geodesic triangles are in an appropriate sense not thicker than their comparison triangles in the two-dimensional model space $\mathcal{M}^2_\kappa$, the complete, simply connected Riemannian surface of constant sectional curvature κ. This fact can be used as a basis for a definition of curvature bounds of geodesic metric spaces and has led to the theory of metric spaces of curvature bounded above in the sense of Alexandrov. If such a comparison condition holds for all and not merely sufficiently small geodesic triangles, the space is called a CAT(κ)-space, $\kappa < 0$.

More precisely, let (X, d) be a geodesic metric space and let $\kappa < 0$. Consider the hyperbolic plane $\mathbb{H}^2_\kappa$ of constant curvature κ and denote the metric on $\mathbb{H}^2_\kappa$ by $\tilde{d}$. If Δ is a geodesic triangle in X, then, by definition, a comparison triangle in $\mathbb{H}^2_\kappa$ is a geodesic triangle $\tilde{\Delta}$ in $\mathbb{H}^2_\kappa$, the side-lengths of which are the same as those of Δ. Now the space (X, d) is a CAT(κ)-space if the following comparison condition is satisfied. Suppose x, y and z are the vertices of Δ, and $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ the corresponding vertices of $\tilde{\Delta}$. If u is an arbitrary point on the side $[y, z]$ of Δ and $\tilde{u}$ is the corresponding point of the side $[\tilde{y}, \tilde{z}]$ of $\tilde{\Delta}$ such that $d(y, u) = \tilde{d}(\tilde{y}, \tilde{u})$ and $d(u, z) = \tilde{d}(\tilde{u}, \tilde{z})$, then $d(x, u) \leq \tilde{d}(\tilde{x}, \tilde{u})$.

2.1.2. *Gromov hyperbolic spaces.* In contrast to the CAT(κ)-condition, which is a condition on geodesic triangles of all scales, Gromov's hyperbolicity condition is of an asymptotically global nature. Recall that a geodesic metric space is said to be Gromov hyperbolic if arbitrary geodesic triangles are uniformly thin, i.e., there exists $\delta \geq 0$ such that every side of every geodesic triangle is contained in the union of the δ-neighborhoods of the other two sides. This hyperbolicity condition, which captures some typical global property of negatively curved spaces, may be generalized to arbitrary (not necessarily geodesic) metric spaces. Namely, a metric space X is Gromov hyperbolic if there exists $\delta \geq 0$ such that

$$(1) \qquad (x \cdot y)_o \ \geq\ \min\Big\{(x \cdot z)_o, (z \cdot y)_o\Big\} \ -\ \delta \qquad \forall\, x, y, z, o \in X,$$

where, for instance, $(x \cdot y)_o$ denotes the so-called Gromov product of x and y with respect to the base-point o, i.e., $(x \cdot y)_o := \frac{1}{2}[|xo| + |yo| - |xy|]$.

This property of being Gromov hyperbolic is a rough-isometry invariant. Recall that a map $f : X \longrightarrow Y$ between metric spaces is called a k-rough-isometric embedding, $k \geq 0$, if

$$|xx'| - k \leq |f(x)f(x')| \leq |xx'| + k \qquad \forall\, x, x' \in X.$$

If, moreover, the image of X under f is k-dense in Y, i.e., for all $y \in Y$ there exists $x \in X$ such that $|f(x)y| \leq k$, then f is called a k-rough-isometry. Two metric spaces are said to be rough-isometric to each other if for some $k \geq 0$ there exists a k-rough-isometry between them. The corresponding equivalence classes of metric spaces are referred to as rough-isometry classes.

2.1.3. *Quasi-metrics on the boundary at infinity.* Many of the large scale geometric features of Gromov hyperbolic spaces are encoded in their boundaries at infinity as introduced by Gromov in [Gr]. The boundary of a Gromov hyperbolic space X consists of equivalence classes of sequences in X which converge at infinity. More precisely, fix a base-point $o \in X$; then a sequence $\{x_i\}_i$ in X is said to converge at infinity if $\liminf_{i,j \to \infty} (x_i \cdot x_j)_o = \infty$. Denote the set of sequences in X converging at infinity by S and define an equivalence relation on S through $\{x_i\}_i \sim \{y_j\}_j :\Longleftrightarrow \liminf_{i,j \to \infty} (x_i \cdot y_j) = \infty$. Then the boundary at infinity $\partial_\infty X$ of X is the quotient space $\partial_\infty X := S/\sim$. Note that this concept is independent of the particular choice of base-point $o \in X$.

In order to endow $\partial_\infty X$ with more structure, recall that one can generalize the Gromov product $(x \cdot y)_o$, originally defined for $x, y \in X$, to points at infinity, by setting

$$(\xi \cdot \xi')_o := \inf \liminf_{i \longrightarrow \infty} (x_i \cdot y_i)_o, \tag{2}$$

where the infimum is taken over all sequences $\{x_i\}_i$ and $\{y_i\}_i$ converging to ξ and ξ' in $\partial_\infty X$, respectively.

Recall that a map $q : A \times A \longrightarrow \mathbb{R}_0^+$ is called a quasi-metric on the set A if it satisfies the conditions of a metric on A with the triangle inequality replaced by the weaker condition that there exists some $\lambda \geq 1$ such that $q(a, b) \leq \lambda \max\{q(a, c), q(c, b)\}$ for all $a, b, c \in A$. Given a pointed Gromov hyperbolic space (X, o), the map $\rho_o : \partial_\infty X \times \partial_\infty X \longrightarrow \mathbb{R}_0^+$, defined through $\rho_o(\xi, \xi') := e^{-(\xi \cdot \xi')_o}$, is a quasi-metric, which follows immediately from inequality (1).

For different base-points, the associated quasi-metrics are bi-Lipschitz equivalent. Furthermore, note that a rough-isometry class of Gromov hyperbolic spaces determines a bi-Lipschitz class of quasi-metrics on $\partial_\infty X$. A crucial property of this quasi-metric boundary $(\partial_\infty X, \rho_o)$ of (X, o), which we refer to as a Bourdon boundary in the following, is, that under a certain visuality assumption on X, the canonical bi-Lipschitz class of quasi-metrics on $\partial_\infty X$ already determines the rough-isometry class of X. Recall that a Gromov hyperbolic space is called visual if there exist $o \in X$ as well as some $k \geq 0$ such that every $x \in X$ lies on some k-rough-geodesic ray initiating in $o \in X$. Here, a k-rough-geodesic ray is a k-rough-isometric embedding of $[0, \infty)$ into X.

It is well-known that to every such bi-Lipschitz class $[q]$ of complete, bounded quasi-metric spaces there exists a rough-isometry class $[X]$ of visual, Gromov hyperbolic spaces, such that the Bourdon quasi-metrics on the boundaries at infinity of the representatives of $[X]$ are elements of $[q]$.

2.1.4. *Boundary continuity of* $\mathrm{CAT}(\kappa)$*-spaces.* A well-known but nevertheless important feature of $\mathrm{CAT}(\kappa)$-spaces, $\kappa < 0$, is their so-called *boundary continuity* (see p. 32 in [BuS] and Definition 3.5 below). This means that in definitions such as the one in (2), the extension of the Gromov product to points at infinity, or in the definition of the Busemann functions (see below), the infima of limits involving sequences converging to certain points at infinity do not depend on the actual choice of sequence. For instance, for $\mathrm{CAT}(\kappa)$-spaces, $\kappa < 0$, the extension of the Gromov product to points at infinity may as well be replaced by $(\xi \cdot \xi')_o = \lim_{i\to\infty}(x_i \cdot x_i')_o$, where $\{x_i\}_i$ and $\{x_i'\}_i$ are any sequences converging at infinity to ξ and ξ', respectively.

2.1.5. *Bourdon metrics.* For $\mathrm{CAT}(\kappa)$-spaces X, $\kappa < 0$, Bourdon proved that the associated quasi-metrics $\rho_o^{\sqrt{-\kappa}}$ on $\partial_\infty X$, $o \in X$, indeed satisfy the triangle inequality. It follows that a necessary condition for a visual Gromov hyperbolic space to be rough-isometric to, say, a $\mathrm{CAT}(-1)$-space, is that its associated bi-Lipschitz class of boundary quasi-metrics contains a metric d, i.e. for $o \in X$ there exists $\Lambda \geq 1$ such that

$$\frac{1}{\Lambda}\,\rho_o(\xi,\xi') \;\leq\; d(\xi,\xi') \;\leq\; \Lambda\rho_o(\xi,\xi') \qquad \forall\, \xi,\xi' \in \partial_\infty X.$$

If X is a complete $\mathrm{CAT}(\kappa)$-space, $\kappa < 0$, then for every $o \in X$ and every $\xi \in \partial_\infty X$ there exists a unique geodesic ray $\gamma : [0,\infty) \longrightarrow X$ such that $\gamma(0) = o$ and $\gamma(\infty) := [\{\gamma(n)\}] = \xi$. Let $o \in X$, $\xi, \xi' \in \partial_\infty X$, and denote the unique geodesic rays connecting o to ξ and ξ' by γ and γ', respectively. The asymptotic κ-comparison angle $\Theta_o^\kappa(\xi,\xi')$ of ξ and ξ' in o is defined as $\Theta_o^\kappa(\xi,\xi') = \lim_{i\to\infty} \angle_o^\kappa(\gamma(i),\gamma'(i))$, where $\angle_o^\kappa(x,x')$, $x,x' \in X$, denotes the κ-comparison angle of x and x' in o, i.e. the angle of $\overline{x}$ and $\overline{x'}$ in $\overline{o}$, where $\overline{o},\overline{x}$ and $\overline{x'}$ are comparison points of o, x and x' in $\mathbb{H}^2_\kappa$. The Bourdon metric $\rho_o^{\sqrt{-\kappa}}$ may be written as $\rho_o^{\sqrt{-\kappa}}(\xi,\xi') = \sin\frac{1}{2}\Theta_o^\kappa(\xi,\xi')$ for all $\xi,\xi' \in \partial_\infty X$.

Consider, for example, the real hyperbolic space $\mathbb{H}^{n+1}_\kappa$ in the Poincaré ball model and choose the base-point o as the center of the ball. Then the Bourdon quasi-metric $\rho_o^{\sqrt{-\kappa}}$ is precisely half the chordal metric on $\partial_\infty\mathbb{H}^{n+1}_\kappa = \mathbb{S}^n \subset \mathbb{E}^{n+1}$.

2.1.6. *Hamenstädt metrics.* The model of the real hyperbolic spaces we are mainly going to stress throughout this paper is the upper halfspace model. In this model, the boundary at infinity is given by $\partial_\infty\mathbb{H}^{n+1} = \mathbb{R}^n \cup \{\omega\}$, where ω denotes the upper end-point of the vertical geodesics in $\mathbb{H}^{n+1}$. Following [FS2], for an arbitrary $\mathrm{CAT}(\kappa)$-space X, $\kappa < 0$, we fix some $o \in X$ as well as some $\omega \in \partial_\infty X$ and consider the map $\rho_{\omega,o}^{\sqrt{-\kappa}} : \partial_\infty X \setminus \{\omega\} \times \partial_\infty X \setminus \{\omega\} \longrightarrow \mathbb{R}_0^+$,

$$\rho_{\omega,o}^{\sqrt{-\kappa}}(\xi,\xi') \;:=\; \frac{\rho_o^{\sqrt{-\kappa}}(\xi,\xi')}{\rho_o^{\sqrt{-\kappa}}(\omega,\xi)\rho_o^{\sqrt{-\kappa}}(\omega,\xi')}, \tag{3}$$

i.e., the involution of the Bourdon metric $\rho_o^{\sqrt{-\kappa}}$ on $\partial_\infty X$ at the point $\omega \in \partial_\infty X$. Once again, this is indeed a metric on $\partial_\infty X \setminus \{\omega\}$ (see the next subsection), which we refer to as the Hamenstädt metric in the following.

Note that for the real hyperbolic spaces $\mathbb{H}^{n+1}_\kappa$, the Hamenstädt boundary $(\partial_\infty \mathbb{H}^{n+1}_\kappa \setminus \{\omega\}, \rho^{\sqrt{-\kappa}}_{\omega,o})$ is isometric to the n-dimensional Euclidean space $\mathbb{E}^n$. Thus, contrary to the Bourdon boundaries, the Hamenstädt boundaries of the model spaces are geodesic.

It is worthwhile mentioning that for general Gromov hyperbolic spaces, one may obtain the Hamenstädt quasi-metric in analogy to the Bourdon quasi-metric, replacing the distances to the base-point $o \in X$ in the definition of the Bourdon quasi-metric, through appropriately 'scaled distances', i.e. Busemann functions, to the base-point $\omega \in \partial_\infty X$. Namely, for $x \in X$, define the Busemann function $b_{\omega,o}(x)$ at x through

$$b_{\omega,o}(x) \ := \ \inf \liminf_{i \longrightarrow \infty} \{|xz_i| - |oz_i|\},$$

where the infimum is taken over all sequences $\{z_i\}_i$ converging at infinity to ω. Now the analogue of the Gromov product is defined as

$$(x \cdot x')_{\omega,o} \ := \ \frac{1}{2}[b_{\omega,o}(x) \ + \ b_{\omega,o}(x') \ - \ |xx'|] \qquad \forall\, x, x' \in X$$

and as

$$(\xi \cdot \xi')_{\omega,o} \ := \ \inf \liminf_{i \longrightarrow \infty} \, (x_i \cdot x'_i)_{\omega,o} \qquad \forall\, \xi, \xi' \in \partial_\infty X \setminus \{\omega\},$$

where the infimum is taken over all sequences $\{x_i\}_i$ and $\{x'_i\}_i$ converging at infinity to ξ and ξ', respectively. Then the Hamenstädt quasi-metric $\rho^{\sqrt{-\kappa}}_{\omega,o}$ may be written as $\rho^{\sqrt{-\kappa}}_{\omega,o}(\xi, \xi') = e^{-\sqrt{-\kappa}(\xi \cdot \xi')_{\omega,o}}$ for all $\xi, \xi' \in \partial_\infty X \setminus \{\omega\}$.

2.2. **The Ptolemy inequality.** Let X be a complete CAT(κ)-space, $\kappa < 0$, $o \in X$ and $\omega \in \partial_\infty X$. On $\partial_\infty X$ we consider the Bourdon metric $\rho^{\sqrt{-\kappa}}_o$, while on $\partial_\infty X \setminus \{\omega\}$ we consider the Hamenstädt metric $\rho^{\sqrt{-\kappa}}_{\omega,o}$. In order to treat both cases simultaneously, we write Y for $\partial_\infty X$ or $\partial_\infty X \setminus \{\omega\}$, respectively, and $|\cdot|$ for $\rho^{\sqrt{-\kappa}}_o$ or $\rho^{\sqrt{-\kappa}}_{\omega,o}$, respectively.

As mentioned above, Bourdon proved in [B] that $\rho^{\sqrt{-\kappa}}_o$ satisfies the triangle inequality. Now the Hamenstädt metric $\rho^{\sqrt{-\kappa}}_{\omega,o}$ is obtained from the Bourdon metric $\rho^{\sqrt{-\kappa}}_o$ by involution at the point ω. In general, the involution of a metric need not satisfy the triangle inequality. However, from equality (3) and the following four-point property, one easily obtains the validity of the triangle inequality for $\rho^{\sqrt{-\kappa}}_{\omega,o}$.

Theorem 2.1 (Theorem 1.1 in [FS2]). *Let Y be the boundary of a complete* CAT(κ)*-space, $\kappa < 0$, endowed with a Bourdon or a Hamenstädt metric $|\cdot|$. Let $y_1, y_2, y_3, y_4 \in Y$. Then*

$$|y_1y_3|\,|y_2y_4| \ \leq \ |y_1y_2|\,|y_3y_4| \ + \ |y_2y_3|\,|y_1y_4|.$$

Equality holds if and only if the convex hull of the four points is isometric to an ideal quadrilateral in the hyperbolic plane $\mathbb{H}^2_\kappa$ such that the geodesics $\overline{y_1y_3}$ and $\overline{y_2y_4}$ are the diagonals.

The rigidity part of Theorem 2.1 will play an essential role in our proofs of Theorems 1.2 and 1.3. Therefore, note that the triangle inequality

$$\rho^{\sqrt{-\kappa}}_{\omega,o}(y, y'') \ \leq \ \rho^{\sqrt{-\kappa}}_{\omega,o}(y, y') \ + \ \rho^{\sqrt{-\kappa}}_{\omega,o}(y', y'') \tag{4}$$

is equivalent to the Ptolemy inequality

(5)
$$\rho_o^{\sqrt{-\kappa}}(y,y'')\rho_o^{\sqrt{-\kappa}}(y',\omega) \;\leq\; \rho_o^{\sqrt{-\kappa}}(y,y')\rho_o^{\sqrt{-\kappa}}(y'',\omega) \;+\; \rho_o^{\sqrt{-\kappa}}(y',y'')\rho_o^{\sqrt{-\kappa}}(y,\omega),$$

and the equality in (4) holds if and only if the equality holds in (5).

Thus, the rigidity part of Theorem 2.1 implies that if y' is a point in between y and y'' in $\partial_\infty X \setminus \{\omega\}$, i.e., that equality holds in (4), then the convex hull of the points y, y', y'', ω is isometric to an ideal quadrilateral in the real hyperbolic plane $\mathbb{H}^2_\kappa$. From this observation one immediately derives the

Corollary 2.2. *Let X be a complete* $\mathrm{CAT}(\kappa)$*-space, $\kappa < 0$, $o \in X$, $\omega \in \partial_\infty X$, and suppose that there exists a geodesic segment $c : [\alpha,\beta] \longrightarrow \partial_\infty X \setminus \{\omega\}$ in $(\partial_\infty X \setminus \{\omega\}, \rho_{\omega,o}^{\sqrt{-\kappa}})$. Then the convex hull $C_{\mathrm{hull}}(c([\alpha,\beta]) \cup \{\omega\})$ of ω and $c([\alpha,\beta])$ is isometric to an ideal sector in the real hyperbolic plane $\mathbb{H}^2_\kappa$.*

Motivated by the upper halfspace model of $\mathbb{H}^2_\kappa$, we also refer to such an ideal sector as an ideal strip in $\mathbb{H}^2_\kappa$.

2.3. **Metric warped products.** Riemannian warped products were introduced by Bishop and O'Neill in [BO'N] in order to produce many examples of Riemannian manifolds of negative sectional curvature. This concept has been generalized by Alexander, Bishop and Chen in [AB1], [AB2] and [C] to the setting of geodesic metric spaces.

Here, we will recall the definition of metric warped products as well as some facts, which our proofs rely on.

Let (B, d_B) and (F, d_F) be complete geodesic metric spaces and assume that (B, d_B) is locally compact. Further, let $f : B \longrightarrow \mathbb{R}^+$ be a continuous positive function. For a curve $\gamma = (\alpha,\beta) : [0,1] \longrightarrow B \times F$, one defines the length $l(\gamma)$ of γ by

$$l(\gamma) \;=\; \lim_\tau \sum_{i=1}^n \sqrt{d_B^2(\alpha(t_{i-1}),\alpha(t_i)) \;+\; f^2(\alpha(t_{i-1}))d_F^2(\beta(t_{i-1}),\beta(t_i))},$$

where the limit is taken with respect to the refinement ordering of partitions $\tau : 0 = t_0 < t_1 < \ldots < t_n = 1$, $n \in \mathbb{N}$.

As is shown in Theorem 4.1 and Lemma 4.1 in [C], $B \times F$ endowed with the associated length metric d is a complete, geodesic metric space. This space $B \times_f F := (B \times F, d)$ is called the metric warped product with base B and fiber F with respect to the warping function f.

Theorem 1.1 in [AB2] provides sufficient (and actually necessary) conditions on the basis, the fiber and the warping function, in order for the metric warped product $B \times_f F$ to be a $\mathrm{CAT}(\kappa)$-space. Note that the real hyperbolic spaces $\mathbb{H}^n_\kappa$ are given through the warped products $\mathbb{H}^n_\kappa = \mathbb{E} \times_{e^{-\sqrt{-\kappa}t}} \mathbb{E}^{n-1}$.

Actually, for our purposes, it will suffice to restrict on the special case $B = \mathbb{E}$, the one-dimensional Euclidean space, as well as $f = e^{-\sqrt{-\kappa}t}$. As a special case of Theorem 1.1 in [AB2], which is also easy to verify by hand, the warped product $\mathbb{E} \times_{e^{-\sqrt{\kappa}t}} F$ is a $\mathrm{CAT}(\kappa)$-space if and only if the fiber F is a $\mathrm{CAT}(0)$-space.

The following facts on metric warped products are essential in what follows.

Theorem 2.3 (Theorem 3.1 (a) and (b) in [AB1])**.** *For a geodesic $\gamma = (\alpha,\beta)$ in $B \times_f F$,*

(1) *β is a pre-geodesic in F, i.e. β is a geodesic up to parameterization.*

(2) *α is independent of F, except for the total height, i.e. the length of β. Precisely: for another geodesic metric space $\bar{F}$ and pre-geodesic $\bar{\beta}$ in $\bar{F}$ with the same length and speed as β, $(\alpha, \bar{\beta})$ is a geodesic in $B \times_f \bar{F}$.*

3. Gromov hyperbolicity of cones and their Gromov boundaries

Let F be a complete, geodesic metric space. Then its κ-cone $C_\kappa(F)$ is defined as the metric warped product $C_\kappa(F) := \mathbb{E} \times_{e^{-t\sqrt{-\kappa}}} F$. Thus, the κ-cone of the Euclidean space $\mathbb{E}^n$ is the real hyperbolic space $\mathbb{H}^{n+1}_\kappa$ of constant curvature κ. As mentioned in Section 2, the κ-cone of a complete, geodesic metric space is a CAT(κ)-space if and only if F is a CAT(0)-space.

In this section we will first observe that the κ-cone $C_\kappa(F)$ of any complete, geodesic metric space F is Gromov hyperbolic (Proposition 3.1). We denote the equivalence class of end-points of geodesic rays $\gamma_A : [0, \infty) \longrightarrow C_\kappa(F)$, $t \mapsto (t, a)$, $A = (t_A, a)$, $t_A \in \mathbb{E}$, $a \in F$, by $\omega := [\gamma_a(\infty)] \in \partial_\infty(F)$, observe that $\partial_\infty C_\kappa(F)$ can naturally be identified with $F \cup \{\omega\}$, and prove that for some (and therefore any) $O = (\frac{\log \sqrt{-\kappa}}{\sqrt{-\kappa}}, o_F)$ the Hamenstädt boundary $(\partial_\infty C_\kappa(F) \setminus \{\omega\}, \rho^{\sqrt{-\kappa}}_{\omega,O})$ is isometric to the fiber F (Theorem 3.4). As a corollary, we obtain the validity of Theorem 1.1.

3.1. **Gromov hyperbolicity of κ-cones.**

The following proposition is neither surprising nor hard to verify.

Proposition 3.1. *Let F be a complete, geodesic metric space and $\kappa < 0$. Then its κ-cone $C_\kappa(F)$ is a complete, geodesic Gromov hyperbolic space.*

Before we embark on the proof, we set up some notation and explain, given three points $A, B, C \in C_\kappa(F)$ in the κ-cone of a complete, geodesic metric space F, how to construct corresponding comparison points in $\mathbb{H}^3_\kappa$.

Therefore, let $A = (t_A, a), B = (t_B, b), C = (t_C, c) \in C_\kappa(F)$ be given. Let, for instance, $\gamma_{AB} : [0, |AB|] \longrightarrow C_\kappa(F)$ denote a geodesic segment in $C_\kappa(F)$ connecting A to B, i.e. $\gamma_{AB}(0) = A$ and $\gamma_{AB}(|AB|) = B$, and, as before, let $\gamma_A : [0, \infty) \longrightarrow C_\kappa(F)$ denote the geodesic segment with $\gamma_A(t) = (t_A + t, a)$ initiating in $A = (t_A, a) \in C_\kappa(F)$. We denote the projections of $C_\kappa(F)$ onto its basis and fiber by $P_\mathbb{E} : C_\kappa(F) \longrightarrow \mathbb{E}$ and $P_F : C_\kappa(F) \longrightarrow F$, respectively, i.e. $P_\mathbb{E}(A) = t_A$ and $P_F(A) = a$.

Now let $\Delta = \{\gamma_{AB}, \gamma_{AC}, \gamma_{BC}\}$ be a geodesic triangle in $C_\kappa(F)$ with vertices $A = (t_A, a)$, $B = (t_B, b)$ and $C = (t_C, c)$. Further, let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{E}^2$ be comparison points to $a, b, c \in F$. We claim that $\bar{A} = (t_A, \bar{a})$, $\bar{B} = (t_B, \bar{b})$ and $\bar{C} = (t_C, \bar{c})$ are comparison points for A, B and C in the Gromov hyperbolic space $\mathbb{H}^3_\kappa = C_\kappa(\mathbb{E}^2)$.

Indeed, let $\alpha : [0, |AB|] \longrightarrow \mathbb{E}$, $\alpha := P_\mathbb{E} \circ \gamma_{AB}$ and $\beta : [0, |AB|] \longrightarrow F$, $\beta := P_F \circ \gamma_{AB}$, i.e. $\gamma_{AB} = (\alpha, \beta)$. Then, due to Theorem 2.3 (1), the curve β is a pregeodesic in F which connects a to b, and, due to Theorem 2.3 (2), the curve $\gamma_{\bar{A}\bar{B}} = (\alpha, \bar{\beta})$ is a geodesic in $\mathbb{H}^3_\kappa$, where $\bar{\beta}$ is a pregeodesic in $\mathbb{E}$ with the same speed as β in F, connecting the points $\bar{a}$ and $\bar{b}$. By construction, the lengths of γ_{AB} and $\gamma_{\bar{A}\bar{B}}$ agree, and we derive that $|AB| = |\bar{A}\bar{B}|$ and similarly that $|AC| = |\bar{A}\bar{C}|$ and $|BC| = |\bar{B}\bar{C}|$.

For a geodesic $\gamma : I \longrightarrow C_\kappa(F)$ we define $T_\gamma \in \mathbb{E}$ as $T_\gamma := \max\{(P_\mathbb{E} \circ \gamma)(t) \,|\, t \in I\}$, the parameter value Σ_γ implicitly as $(P_\mathbb{E} \circ \gamma)(\Sigma_\gamma) = T_\gamma$, and $M_\gamma \in C_\kappa(F)$ as $M_\gamma := \gamma(\Sigma_\gamma)$. Note that Σ_γ is uniquely defined, due to Theorem 2.3.

We can now formulate the following easy application of Theorem 2.3.

Lemma 3.2. *Let $\Delta = \Delta_\kappa \geq 0$ be such that every side of every geodesic triangle in $\mathbb{H}^2_\kappa$ is contained in the Δ-neighborhood of the union of the other two sides. Then, for every geodesic triangle $\{\gamma_{AB}, \gamma_{AC}, \gamma_{BC}\}$ in $C_\kappa(F)$ we have the inequality*

$$T_{\gamma_{AB}} \;\leq\; \max\{T_{\gamma_{AC}}, T_{\gamma_{BC}}\} \;+\; \Delta.$$

Proof. Let $\gamma_{\bar{A}\bar{B}}$ be as above. Then $P_{\mathbb{E}} \circ \gamma_{AB} = P_{\mathbb{E}} \circ \gamma_{\bar{A}\bar{B}}$. Thus, in particular, $T_{\gamma_{AB}} = T_{\gamma_{\bar{A}\bar{B}}}$ and similarly $T_{\gamma_{AC}} = T_{\gamma_{\bar{A}\bar{C}}}$ and $T_{\gamma_{BC}} = T_{\gamma_{\bar{B}\bar{C}}}$. Therefore, we only have to verify the claim for $\mathbb{H}^3_\kappa$, where for $T_{\gamma_{\bar{A}\bar{B}}} \geq \max\{T_{\gamma_{\bar{A}\bar{C}}}, T_{\gamma_{\bar{B}\bar{C}}}\}$ we find

$$\begin{aligned} |T_{\gamma_{\bar{A}\bar{B}}} - \max\{T_{\gamma_{\bar{A}\bar{C}}}, T_{\gamma_{\bar{B}\bar{C}}}\}| &= |(P_{\mathbb{E}} \circ \gamma_{\bar{A}\bar{B}})(\Sigma_{\gamma_{\bar{A}\bar{B}}}) \;-\; \max\{P_{\mathbb{E}}(\gamma_{\bar{A}\bar{C}} \cup \gamma_{\bar{B}\bar{C}})\}| \\ &\leq \operatorname{dist}\Big(\gamma_{\bar{A}\bar{B}}(\Sigma_{\gamma_{\bar{A}\bar{B}}}), \gamma_{\bar{A}\bar{C}} \cup \gamma_{\bar{B}\bar{C}}\Big) \\ &\leq \Delta. \end{aligned}$$

□

In the next easy lemma we basically observe that ideal triangles in κ-cones are uniformly thin.

Lemma 3.3. *Let F be a complete, geodesic metric space. Then for every geodesic γ_{AB} connecting $A = (t_A, a)$ to $B = (t_B, b)$ in $C_\kappa(F)$ we have*

$$d\Big(\gamma_A(P_{\mathbb{E}}(\gamma_{AB}(t)) - t_A), \gamma_{AB}(t)\Big) \;\leq\; \frac{2}{\sqrt{-\kappa}} \log(1 + \sqrt{2}) \;=:\; \Delta' \qquad \forall\, t \in [0, \Sigma_{\gamma_{AB}}].$$

The proof of this lemma is another easy application of Theorem 2.3 and the fact that the statement holds true in $\mathbb{H}^2_\kappa$. Note therefore that with the angle of parallelism in $\mathbb{H}^2_\kappa$ (see for instance p. 184 in [BuS]) it follows that all ideal (and therefore all) geodesic triangles in $\mathbb{H}^2_\kappa$ are $\frac{\Delta'}{2}$-slim.

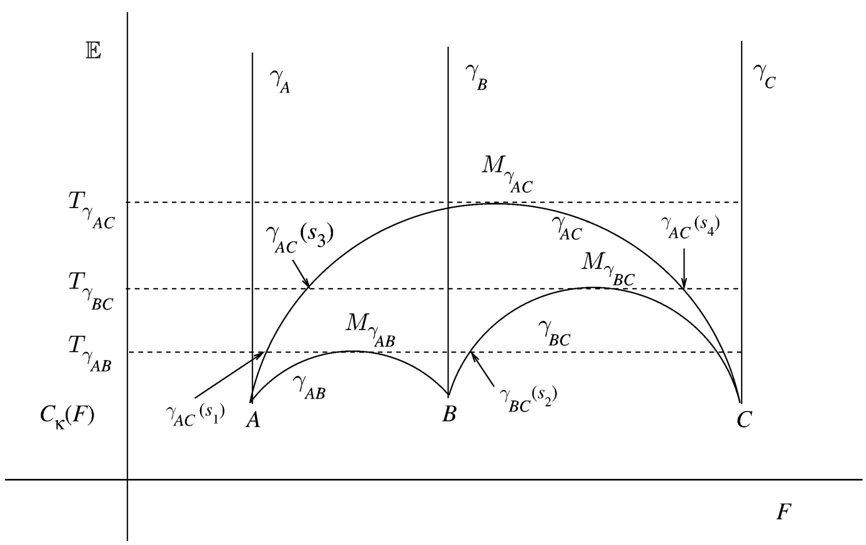

FIGURE 1. This figure shows a geodesic triangle in $C_\kappa(F)$ and clarifies the notation used in the proof of Proposition 3.1.

Proof of Proposition 3.1. The proof is a simple application of Lemma 3.3. We show that geodesic triangles in κ-cones are $(\Delta + 4\Delta')$-slim, where Δ and Δ' are as in Lemmata 3.2 and 3.3. Therefore, we divide the three sides into appropriate pieces, which is illustrated in Figure 1. In the following, we say that two geodesic segments γ and γ' are δ-slim if their images are in finite Hausdorff distance δ to each other and write $\gamma \doteq_\delta \gamma'$.

We subsequently treat the different geodesic segments:

- the geodesic segment $\gamma_{AB} : [0, |AB|] \longrightarrow C_\kappa(F)$
  - $\gamma_{AB}|_{[0,\Sigma_{\gamma_{AB}}]} \doteq_{\Delta'} \gamma_A|_{[0,T_{\gamma_{AB}}-t_A]} \doteq_{\Delta'} \gamma_{AC}|_{[0,s_1]}$,
  - $\gamma_{AB}|_{[\Sigma_{\gamma_{AB}},|AB|]} \doteq_{\Delta'} \gamma_B|_{[0,T_{\gamma_{AB}}-t_B]} \doteq_{\Delta'} \gamma_{BC}|_{[0,s_2]}$,
- the geodesic segment $\gamma_{BC} : [0, |BC|] \longrightarrow C_\kappa(F)$
  - $\gamma_{BC}|_{[0,s_2]} \doteq_{2\Delta'} \gamma_{AB}|_{[\Sigma_{\gamma_{AB}},|AB|]}$ (see above),
  - $\gamma_{BC}|_{[s_2,\Sigma_{\gamma_{BC}}]} \doteq_{\Delta'} \gamma_B|_{[T_{\gamma_{AB}}-t_B,T_{\gamma_{BC}}-t_B]} \doteq_{2\Delta'} \gamma_A|_{[T_{\gamma_{AB}}-t_A,T_{\gamma_{BC}}-t_A]}$ $\doteq_{\Delta'} \gamma_{AC}|_{[s_1,s_3]}$,
  - $\gamma_{BC}|_{[\Sigma_{\gamma_{BC}},|BC|]} \doteq_{\Delta'} \gamma_C|_{[0,T_{\gamma_{BC}}-t_C]} \doteq_{\Delta'} \gamma_{AC}|_{[s_4,|AC|]}$,
- the remaining part of the geodesic segment γ_{AC}
  - $\gamma_{AC}|_{[s_3,\Sigma_{\gamma_{AC}}]} \doteq_{\Delta'} \gamma_A|_{[T_{\gamma_{BC}}-t_A,T_{\gamma_{AC}}-t_A]} \doteq_{\Delta} \gamma_A(T_{\gamma_{BC}} - t_A)$ $\doteq_{2\Delta'} \gamma_B(T_{\gamma_{BC}} - t_B) \doteq_{\Delta'} \gamma_{BC}(\Sigma_{\gamma_{BC}})$
  - $\gamma_{AC}|_{[\Sigma_{\gamma_{AC}},s_4]} \doteq_{\Delta'} \gamma_C|_{[T_{\gamma_{BC}}-t_C,T_{\gamma_{AC}}-t_C]} \doteq_{\Delta} \gamma_C(T_{\gamma_{BC}} - t_C)$ $\doteq_{\Delta'} \gamma_{BC}(\Sigma_{\gamma_{BC}})$.

□

3.2. **The Hamenstädt boundary of a κ-cone.** In this section we will obtain Theorem 1.1 as a corollary of

Theorem 3.4. *Let (F, d_F) be a complete, geodesic metric space and $\kappa < 0$. Further, let $O := (\frac{\log\sqrt{-\kappa}}{\sqrt{-\kappa}}, o_F)$, $o_F \in F$ and let $\omega := [\gamma_O(\infty)] \in \partial_\infty C_\kappa(F)$. Then $(\partial_\infty C_\kappa(F) \setminus \{\omega\}, \rho^{\sqrt{-\kappa}}_{\omega,O})$ is isometric to (F, d_F).*

Note that, a priori, $\rho^{\sqrt{-\kappa}}_{\omega,O}$ denotes the Hamenstädt quasi-metric on the boundary of a Gromov hyperbolic space, which, as in the case of CAT(κ)-spaces, turns out to satisfy the triangle inequality on boundaries of κ-cones.

Theorem 3.4 basically follows from its validity for $F = \mathbb{E}$ and Theorem 2.3. To make this more precise, we need some technical preparation.

Consider the κ-cone $C_\kappa(F)$ of a complete, geodesic metric space F. Let, as before, $\omega := [\{(j, f)\}_j] \in \partial_\infty C_\kappa(F)$, $f \in F$, and set $F^\infty := \{[\{(-j, f)\}_j] \mid f \in F\}$. Then, just as for the real hyperbolic spaces, there is a natural bijection between $\partial_\infty C_\kappa(F)$ and $F^\infty \cup \{\omega\}$.

We have remarked in Section 2 that CAT(κ)-spaces, $\kappa < 0$, are boundary continuous, and just as for the real hyperbolic spaces, we also obtain the boundary continuity of arbitrary κ-cones. To make this more precise, let us recall the

Definition 3.5 (see p. 32 in [BuS])**.** A Gromov hyperbolic space X is called boundary continuous if the Gromov product on X extends continuously onto the boundary at infinity, i.e. if for all sequences $\{x_i\}_i, \{x'_i\}_i$ in X converging to $\xi, \xi' \in \partial_\infty X$, the limit $\lim_{i\to\infty}(x_i \cdot x'_i)_o$ converges to $(\xi \cdot \xi')_o$.

We state the observation above as

Lemma 3.6. *The κ-cone $C_\kappa(F)$ of a complete, geodesic metric space F is boundary continuous.*

Remark 3.7. As a consequence of Lemma 3.6, the Busemann functions $b_{\omega,o}$, $o \in C_\kappa(F)$, satisfy

$$b_{\omega,o}(x) \;=\; \lim_{j\to\infty}\left(|xz_j| - |oz_j|\right) \text{ for every sequence } \{z_j\}_j \in \omega,\; x \in C_\kappa(F).$$

Moreover, one easily deduces that

$$b_{\omega,o}(x) \;=\; b_{\omega,o'}(x) \text{ for every } o' \in C_\kappa(F) \text{ with } P_{\mathbb{E}}(o) = P_{\mathbb{E}}(o'),\; x \in C_\kappa(F),$$

and that for another complete, geodesic metric space F' with distinguished point at infinity $\omega' \in \partial_\infty C_\kappa(F')$ and $o' \in C_\kappa(F')$ with $P_{\mathbb{E}}(o) = P_{\mathbb{E}}(o')$,

$$b_{\omega,o}(x) \;=\; b_{\omega',o'}(x') \quad \text{for all } x \in C_\kappa(F) \text{ and } x' \in C_\kappa(F') \text{ with } P_{\mathbb{E}}(x) = P_{\mathbb{E}}(x').$$

With this preparation we are able to provide the

Proof of Theorem 3.4. The validity of the claim for $\mathbb{H}^2_\kappa = C_\kappa(\mathbb{E})$ is well known and easy to compute by hand. For arbitrary F we basically apply Theorem 2.3, Lemma 3.6 and Remark 3.7.

Let (F, d_F) be a complete, geodesic metric space, let ω denote the distinguished point in $\partial_\infty C_\kappa(F)$ and choose a base-point $O = (\frac{\log\sqrt{-\kappa}}{\sqrt{-\kappa}}, o_F)$ in $C_\kappa(F)$ for some $o_F \in F$. Let $\Phi : F \longrightarrow F^\infty = \partial_\infty C_\kappa(F) \setminus \{\omega\}$ be the bijection defined through $\Phi(f) := [\{(-j, f)\}_j]$ for all $f \in F$. We verify that $\Phi : (F, d_F) \longrightarrow (F^\infty, \rho^{\sqrt{-\kappa}}_{\omega,O})$ is distance-preserving and hence the desired isometry.

Therefore, let $\mu, \nu \in F^\infty$, set $m := \Phi^{-1}(\mu)$, $n := \Phi^{-1}(\nu)$ and let $\gamma^F_{mn} : [0, d_F(m,n)] \longrightarrow F$ denote a geodesic in (F, d_F) connecting m to n. From Theorem 2.3 it follows that $C_\kappa(\gamma^F_{mn})$ isometrically embeds into $\mathbb{H}^2_\kappa = C_\kappa(\mathbb{E})$ via $g : (t, \gamma^F_{mn}(s)) \mapsto (t, s)$.

Consider the sequences $\{\bar m_i\}_i, \{\bar n_i\}_i \in {\mathbb{H}^2_\kappa}^{\mathbb{N}}$ of images $\bar m_i := g(m_i)$ and $\bar n_i := g(n_i)$ of $m_i := (-i, m)$ and $n_i := (-i, n)$ in $C_\kappa(\gamma^F_{mn})$, and write $\bar\mu = [\{\bar m_i\}]$ and $\bar\nu = [\{\bar n_i\}]$, i.e. $\bar\mu, \bar\nu \in \partial_\infty \mathbb{H}^2_\kappa \setminus \xi$, where ξ denotes the distinguished point in $\partial_\infty C_\kappa(\mathbb{E})$.

From Remark 3.7 we deduce that $b_{\omega,o}(m_i) = b_{\xi,\bar o}(\bar m_i)$ and $b_{\omega,o}(n_i) = b_{\xi,\bar o}(\bar n_i)$ for all $i \in \mathbb{N}$, where $\bar o := (P_{\mathbb{E}}(o), 0)$. Thus, the boundary continuity of $C_\kappa(F)$ and $C_\kappa(\mathbb{E})$ yields

$$\begin{aligned}
\exp\{-\sqrt{-\kappa}(\mu\cdot\nu)_{\omega,o}\} &= \lim_{i\longrightarrow\infty} \exp\Big\{-\frac{\sqrt{-\kappa}}{2}\left[b_{\omega,o}(m_i) + b_{\omega,o}(n_i) - |m_i n_i|\right]\Big\} \\
&= \lim_{i\longrightarrow\infty} \exp\Big\{-\frac{\sqrt{-\kappa}}{2}\left[b_{\xi,\bar o}(\bar m_i) + b_{\xi,\bar o}(\bar n_i) - |\bar m_i \bar n_i|\right]\Big\} \\
&= \exp\{-\sqrt{-\kappa}\,(\bar\mu\cdot\bar\nu)_{\xi,\bar o}\}.
\end{aligned}$$

Hence, the validity of the claim follows from its validity for $\mathbb{H}^2_\kappa$. □

As a corollary of Theorem 3.4, we obtain Theorem 1.1.

Proof of Theorem 1.1. Given $\kappa < 0$ and any complete CAT(0)-space F, its κ-cone $C_\kappa(F)$ is a complete CAT(κ)-space, the boundary at infinity of which, endowed with the Hamenstädt metric as in Theorem 3.4, is isometric to F. □

4. Hamenstädt boundaries of CAT(κ)-spaces, $\kappa < 0$

In this section, we prove Theorems 1.2 and 1.3. We first rephrase and state more precisely the claims of these theorems in

Theorem 4.1 (Theorems 1.2 and 1.3)**.** *Let X be a* CAT(κ)*-space, $\kappa < 0$, $o \in X$ and $\omega \in \partial_\infty X$ such that its Hamenstädt boundary $Y = Y(X, \omega, o) := (\partial_\infty X \setminus \{\omega\}, \rho_{\omega,o}^{\sqrt{-\kappa}})$ is a geodesic metric space.*

(1) *Then Y is a* CAT(0)*-space.*
(2) *If, moreover, X is ω-visual, i.e. each geodesic ray $\gamma : [0, \infty) \longrightarrow X$ with $[\{\gamma(i)\}_i] = \omega$ admits an extension to all of $\mathbb{R}$, then X is isometric to $C_\kappa(Y)$.*

Remark 4.2. Note that the visuality assumption in Theorem 4.1 is a typical necessary condition on a space in asymptotic geometry in order to guarantee that all relevant geometry of the space is already encoded in its boundary at infinity (compare, for instance, Theorem 1.1 in [BS] and Chapter 7 in [BuS]).

Note also that the visuality assumption in (2) implies the completeness of X. This follows, for instance, from the validity of the theorem above. In order to see this directly, take a Cauchy sequence $\{x_i\}$ in X and consider the complete geodesics through ω and x_i. Their end-points in Y form a Cauchy sequence there, which converges to some $y \in Y$. Moreover, the Busemann functions $b_{\omega,o}(x_i)$ do converge to some b. It follows that $\{x_i\}$ converges to the point with Busemann function b on the geodesic connecting y to ω. Hence X is complete.

4.1. **Proof of Theorem 1.2.** We first provide the proof of Theorem 1.2 (Theorem 4.1 (1)). In order to reach a contradiction, we suppose that the boundary Y is not a CAT(0)-space and, using Theorem 2.1, construct a four-point configuration in X which does not satisfy the CAT(κ)-condition.

Therefore let $y_1, y_2, y_3 \in Y$, and let $\gamma^Y_{y_1 y_2}$, $\gamma^Y_{y_2 y_3}$ and $\gamma^Y_{y_3 y_1}$ denote geodesic segments in Y connecting y_1 to y_2, y_2 to y_3 and y_3 to y_1, respectively. Let $m_Y \in Y$ be a midpoint of y_2 and y_3 in Y, i.e. $m_Y \in \gamma^Y_{y_2 y_3}$ with $|y_2 m_Y| = \frac{1}{2}|y_2 y_3| = |m_Y y_3|$, and suppose that for the comparison points $\bar{y}_1, \bar{y}_2, \bar{y}_3$ and $\bar{m}_Y$ in $\mathbb{E}^2$ we have $|y_1 m_Y| > |\bar{y}_1 \bar{m}_Y|$.

From Corollary 2.2 we deduce that there exist isometries $f_i : C_{\mathrm{hull}}(\gamma^Y_{y_i y_{i+1}} \cup \{\omega\}) \longrightarrow S_i$ onto strips S_i in $\mathbb{H}^3_\kappa$, which we choose such that $(f_i \circ \gamma_{y_i \omega})(\mathbb{R}) = \gamma_{\bar{y}_i \xi}(\mathbb{R})$ and $(f_i \circ \gamma_{y_{i+1}\omega})(\mathbb{R}) = \gamma_{\bar{y}_{i+1}\xi}$, where the indices are taken modulo 3. Here, as before, ξ denotes the distinguished point in $\partial_\infty C_\kappa(\mathbb{E}^2)$.

Let $x_i \in \gamma_{y_i \omega}(\mathbb{R})$ with $b_{\omega,o}(x_i) = 0$. By construction, the points $\bar{x}_i := f_i(x_i) \in \mathbb{H}^3_\kappa$, $i = 1, 2, 3$, constitute a comparison configuration for $x_1, x_2, x_3 \in X$. Note that $P_{\mathbb{E}}(\bar{x}_i) = \frac{\log \sqrt{-\kappa}}{\sqrt{-\kappa}}$, $i = 1, 2, 3$.

The geodesic triangle $\bar{\Delta} = \{\gamma_{\bar{x}_1 \bar{x}_2}, \gamma_{\bar{x}_2, \bar{x}_3}, \gamma_{\bar{x}_3, \bar{x}_1}\}$ in $\mathbb{H}^3_\kappa$ is a comparison triangle for $\Delta = \{\gamma_{x_1 x_2}, \gamma_{x_2, x_3}, \gamma_{x_3, x_1}\}$ in X. Let m_X denote the midpoint of x_2 and x_3 in X. Then $\bar{m}_X := \gamma_{\bar{x}_2 \bar{x}_3} \cap \gamma_{\bar{m}_Y \omega} \in \mathbb{H}^3_\kappa$ is its corresponding comparison point on $\bar{\Delta}$.

Now let $f : C_{\mathrm{hull}}(\gamma^Y_{y_1, m_Y} \cup \{\omega\}) \hookrightarrow S \subset C_{\mathrm{hull}}(\mathrm{span}_{\mathbb{E}^2}\{\bar{y}_1, \bar{m}_Y\}, \xi) \subset \mathbb{H}^3_\kappa$ denote the isometric embedding onto a strip S in $\mathbb{H}^3_\kappa$, defined through $f(\gamma_{y_1 \omega}(\mathbb{R})) = \gamma_{\bar{y}_1 \xi}(\mathbb{R})$, $f(x_1) = \bar{x}_1$ and $\bar{m}_X \in f(C_{\mathrm{hull}}(\gamma^Y_{y_1 m_Y} \cup \{\omega\}))$. Since $|y_1 m_Y| > |\bar{y}_1 \bar{m}_Y|$, we find $|x_1 m_X| = |f(x_1) f(m_X)| > |\bar{x}_1 \bar{m}_X|$. Hence, X is not a CAT(κ)-space, which yields the desired contradiction. $\square$

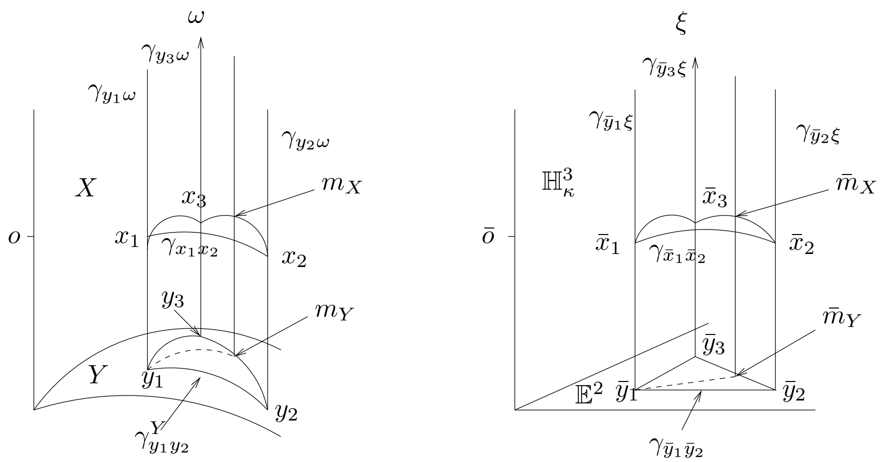

FIGURE 2. This figure visualizes the notation used in the proof of Theorem 1.2. Here $\bar{o} = (\frac{\log\sqrt{-\kappa}}{\sqrt{-\kappa}}, \mathfrak{o})$, where $\mathfrak{o}$ denotes the origin in $\mathbb{E}^2$.

4.2. **Proof of Theorem 1.3.** We now provide the proof of Theorem 1.3 (Theorem 4.1 (2)). Therefore, fix $\tilde{o} \in C_\kappa(Y)$ with $P_{\mathbb{E}}(\tilde{o}) = \frac{\log\sqrt{-\kappa}}{\sqrt{-\kappa}}$ and let $\tilde{\omega}$ denote the distinguished point in $\partial_\infty C_\kappa(Y)$. By Theorem 3.4 we may fix an isometry $f : Y = (\partial_\infty X \setminus \{\omega\}, \rho_{\omega,o}^{\sqrt{-\kappa}}) \longrightarrow (\partial_\infty C_\kappa(Y) \setminus \{\tilde{\omega}\}, \rho_{\tilde{\omega},\tilde{o}}^{\sqrt{-\kappa}})$. Let π ($\tilde{\pi}$, resp.) denote the canonical projection of X ($C_\kappa(Y)$, resp.) onto Y (($\partial_\infty C_\kappa(Y) \setminus \{\tilde{\omega}\}$, resp.), i.e. for instance, $\pi(x) := [\{\hat{\gamma}_{x\omega}(-i)\}_i]$, where $\hat{\gamma}_{x\omega}$ denotes the extension of the ray $\gamma_{x\omega}$ to all of $\mathbb{R}$. Note that π is well defined, since X is ω-visual and Y is geodesic. Thus, assume that our extension of the ray was not unique. Then its limit points in the boundary would be joined there by a boundary geodesic. But the convex hull of this boundary geodesic and the distinguished point ω are isometric to a strip in real hyperbolic space, which yields the desired contradiction, and the extension, which exists due to the ω-visuality, is unique indeed.

Now we define the map $F : X \longrightarrow C_\kappa(Y)$ through

$$F(x) \in C_\kappa(Y) \quad \text{such that} \quad b_{\omega,o}(x) = b_{\tilde{\omega},\tilde{o}}(F(x)) \quad \text{and} \quad (\tilde{\pi} \circ F)(x) = (f \circ \pi)(x).$$

By construction, F is the desired isometry. □

5. THE BOURDON BOUNDARIES OF COMPLETE CAT(κ)-SPACES, $\kappa < 0$

In this section we consider the analogue of Theorem 1.1 when considering a Bourdon boundary instead of a Hamenstädt boundary of a CAT(κ)-space.

Now let F be a bounded, complete, geodesic metric space, $\kappa < 0$, and set $\mathcal{C}_\kappa(F) := \mathbb{R}_0^+ \times_{\frac{1}{\sqrt{-\kappa}}\sinh(t\sqrt{-\kappa})} F$. Note that $\mathcal{C}_\kappa(\mathbb{S}_1^{n-1}) = \mathbb{H}_\kappa^n$, where $\mathbb{S}_1^{n-1}$ denotes the round sphere of radius 1. Moreover, as a special case of Theorem 1.1 in [AB2], it follows that for any CAT(1)-space F, the cone $\mathcal{C}_\kappa(F)$ is a CAT(κ)-space.

Before we state and prove the analogue of Proposition 3.1, let us recall another simple but useful fact.

Lemma 5.1. *Let X be a Gromov hyperbolic metric space, which satisfies the δ-inequality* (1) *with respect to the base-point $o \in X$. Then for arbitrary $\xi, \eta \in \partial_\infty X$*

and arbitrary sequences $\{x_i\}_{i\in\mathbb{N}}$ *and* $\{y_i\}_{i\in\mathbb{N}}$ *in* X *converging at infinity to* ξ *and* η*, respectively,*

$$(\xi\cdot\eta)_o \;\le\; \liminf_{i\longrightarrow\infty}(x_i\cdot y_i)_o \;\le\; \limsup_{i\longrightarrow\infty}(x_i\cdot y_i)_o \;\le\; (\xi\cdot\eta)_o\; + 2\delta.$$

The analogue of Proposition 3.1 when considering the warped product $\mathcal{C}_\kappa(F)$ reads as follows.

Proposition 5.2. *Let* F *be a bounded, complete, geodesic metric space. Then* $\mathcal{C}_\kappa(F)$*,* $\kappa<0$*, is (complete, geodesic and) Gromov hyperbolic.*

Proof. Since F is bounded we find $R>0$ satisfying

$$R \;\ge\; \max\{\frac{3}{2\pi}\operatorname{diam}F, 1\}.$$

Here $R\ge\frac{3}{2\pi}\operatorname{diam}F$ guarantees that for all triples (a,b,c) in F there exists a comparison triple in $\mathbb{S}^2_R$, while $R\ge 1$ ensures that the warped product $\mathcal{C}_\kappa(\mathbb{S}^2_R)$ is a $\mathrm{CAT}(\kappa)$-space and therefore Gromov hyperbolic.

In order to prove the Gromov hyperbolicity of $\mathcal{C}_\kappa(F)$, let $A=(t_A,a)$, $B=(t_B,b)$ and $C=(t_C,c)$ be three arbitrary points in $\mathcal{C}_\kappa(F)$. We want to show that there exists some $\delta\ge 0$ such that for each such choice of points,

$$(A\cdot C)_o \;\ge\; \min\{(A\cdot B)_o,(B\cdot C)_o\} \;-\; \delta,$$

where o denotes the origin in $\mathcal{C}_\kappa(F)$. Note that without loss of generality we may assume $t_A,t_B,t_C>0$.

Let o' denote the origin in $\mathcal{C}_\kappa(\mathbb{S}^2_R)$, which serves as a comparison space for $\mathcal{C}_\kappa(F)$ in the following way. Let the triple (a',b',c') in $\mathbb{S}^2_R$ be a comparison triple for (a,b,c) and consider the points $A':=(t_A,a')$, $B':=(t_B,b')$ and $C':=(t_C,c')$ in $\mathcal{C}_\kappa(\mathbb{S}^2_R)$. Then we find

$$|oA|=t_A=|o'A'|,\quad |oB|=t_B=|o'B'|\quad\text{and}\quad |oC|=t_C=|o'C'|. \tag{6}$$

Let $\gamma=(\alpha,\beta):I\longrightarrow\mathcal{C}_\kappa(F)$ be a geodesic connecting for instance A and B in $\mathcal{C}_\kappa(F)$. Suppose first that $\min\{\alpha(I)\}>0$. Then, due to Theorem 2.3, for an appropriate pre-geodesic β' in $\mathbb{S}^2_R$ connecting a' to b', the curve $\gamma=(\alpha,\beta'):I\longrightarrow\mathcal{C}_\kappa(\mathbb{S}^2_R)$ is a geodesic in $\mathcal{C}_\kappa(\mathbb{S}^2_R)$. Moreover, since $\mathcal{C}_\kappa(\mathbb{S}^2_R)$ is a $\mathrm{CAT}(\kappa)$-space, it is the unique geodesic connecting A' to B' in $\mathcal{C}_\kappa(\mathbb{S}^2_R)$.

On the other hand, if $\min\{\alpha(I)\}=0$, it similarly follows from Theorem 2.3 that $o'\in\operatorname{im}\{\gamma'\}$ for the unique geodesic γ' in $\mathcal{C}_\kappa(\mathbb{S}^2_R)$ connecting A' to B'.

Now the definition of the warped product distance immediately yields $|AB|=|A'B'|$ and similarly $|AC|=|A'C'|$ and $|BC|=|B'C'|$. This together with the equalities (6) implies

$$(A\cdot B)_o \;=\; (A'\cdot B')_{o'},\quad (A\cdot C)_o \;=\; (A'\cdot C')_{o'}\quad\text{and}\quad (B\cdot C)_o \;=\; (B'\cdot C')_{o'}.$$

Since $\mathcal{C}_\kappa(\mathbb{S}^2_R)$ is δ-hyperbolic for some $\delta>0$, we deduce that $\mathcal{C}_\kappa(F)$ also is δ-hyperbolic. □

With this proposition at hand we are able to prove the analogue of Theorem 3.4 for Bourdon boundaries.

Theorem 5.3. *Let (F, d_F) be a bounded, complete, geodesic metric space. Then $(\partial_\infty \mathcal{C}_\kappa(F), \rho_o^{\sqrt{-\kappa}})$, $\kappa < 0$, is bi-Lipschitz equivalent to F.*

Proof. We fix some $\kappa < 0$. Note that the projection along geodesic rays originating in the origin o of $\mathcal{C}_\kappa(F)$ yields a natural identification of F with $\partial_\infty \mathcal{C}_\kappa(F)$. We have to verify that there exists $C \geq 1$ such that

$$\frac{1}{C} e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \;\leq\; d_F(\xi,\eta) \;\leq\; C e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \qquad \forall \xi,\eta \in \partial_\infty(\mathcal{C}_\kappa(F)). \tag{7}$$

Indeed, Theorem 2.3 implies the following observations:

- **(A)** Let $\gamma : I \longrightarrow F$ be a geodesic in (F, d_F) of length $l(\gamma) \leq \pi$. Then $\mathcal{C}_\kappa(\gamma(I))$ lies isometrically embedded in $\mathbb{H}^2_\kappa = \mathcal{C}_\kappa(\mathbb{S}^1_1)$.
- **(B)** Moreover, if $l(\gamma) \geq \pi$, then the origin $o \in \mathcal{C}_\kappa(\gamma(I))$ lies on every geodesic connecting points in $\mathcal{C}_\kappa(\gamma(I))$ with fiber projections coinciding with the end-points of γ.

Since $\mathcal{C}_\kappa(F)$ is Gromov hyperbolic, Lemma 5.1 yields the existence of a constant $C_1 \geq 1$ such that

$$\frac{1}{C_1} \liminf_{i\longrightarrow\infty} e^{-\sqrt{-\kappa}(x_i\cdot y_i)_o} \;\leq\; e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \;\leq\; C_1 e^{-\sqrt{-\kappa}(x_i\cdot y_i)_o}$$

for all sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ converging at infinity to ξ and η, respectively. Thus, since the metric on $\mathbb{S}^1_1$ is a visual metric on $\partial_\infty \mathbb{H}^2_\kappa$ of visual parameter $e^{-\sqrt{-\kappa}}$, it follows that there exists $C_2 \geq 1$ such that

$$\frac{1}{C_2} e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \;\leq\; d_F(\xi,\eta) \;\leq\; C_2 e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \tag{8}$$

for all $\xi, \eta \in \partial_\infty \mathcal{C}_\kappa(F)$ satisfying $d_F(\xi,\eta) \leq \pi$. Moreover, the observations **(A)** and **(B)** together with Lemma 5.1 imply that there exists C_3 such that

$$e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \;\leq\; \frac{1}{C_3} \qquad \Longrightarrow \qquad d_F(\xi,\eta) \;\leq\; \pi. \tag{9}$$

Finally, since F is bounded, for $e^{-\sqrt{\kappa}(\xi\cdot\eta)_o} \geq \frac{1}{C_3}$,

$$d_F(\xi,\eta) \;\leq\; C_4 e^{-\sqrt{-\kappa}(\xi\cdot\eta)_o} \qquad \text{with} \qquad C_4 \;:=\; C_3 \operatorname{diam} F. \tag{10}$$

Now (8), (9) and (10) imply the inequalities (7) with $C := \max\{C_2, C_4\}$. □

The usual correspondence between boundary morphisms and morphisms of the spaces themselves, immediately yields the

Corollary 5.4. *Let X be a Gromov hyperbolic metric space, $o \in X$, $\kappa < 0$, such that $(\partial_\infty X, \rho_o^{\sqrt{-\kappa}})$ is bi-Lipschitz equivalent to a* CAT(1)*-space. Then X is rough-isometric to a* CAT(κ)*-space.*

6. Some corollaries

6.1. **Spaces of bounded growth at some scale and constant asymptotic curvature.**

6.1.1. *Asymptotic upper curvature bounds.* In [BF1], the authors introduce the following notion of asymptotic upper curvature bounds for metric spaces.

Definition 6.1. Let X be a metric space and $\kappa \in [-\infty, 0)$. Then X has an asymptotic upper curvature bound κ if there exists $p \in X$ and a constant $c \geq 0$ such that for all $z, z' \in X$ and all chains $z = x_0, x_1, ..., x_n = z'$ in X,

$$(z \cdot z')_p \;\geq\; \min_{i=1,\ldots,n} (x_{i-1} \cdot x_i)_p \;-\; \frac{1}{\sqrt{-\kappa}} \log n \;-\; c.$$

Here we interpret $1/\sqrt{\infty} = 0$. For convenience we say that X is an $\mathrm{AC}_u(\kappa)$-space if X has an asymptotic upper curvature bound $\kappa < 0$. Note that every $\mathrm{AC}_u(\kappa)$-space is Gromov hyperbolic. Conversely, every Gromov hyperbolic space is an $\mathrm{AC}_u(\kappa)$-space for some $\kappa < 0$.

Definition 6.2. Let X be a Gromov hyperbolic metric space. Then

$$K_u(X) \;:=\; \inf\{\kappa \,|\, X \text{ is an } \mathrm{AC}_u(\kappa)\text{-space}\} \;\in\; [-\infty, 0)$$

is called the asymptotic upper curvature of X.

One should best think of this asymptotic upper curvature bound as measuring the degree of hyperbolicity of a Gromov hyperbolic space. Unlike the $\mathrm{CAT}(\kappa)$-condition, the asymptotic upper curvature bound is an asymptotically global notion. Indeed, it is a rough-isometry invariant.

For a geometric interpretation of the concept of geodesic $\mathrm{AC}_u(\kappa)$-spaces in terms of slim geodesic polygons, see [BF1], where the authors also explore some basic features of this concept.

Note that $\mathrm{CAT}(\kappa)$-spaces are $\mathrm{AC}_u(\kappa)$-spaces (cf. Proposition 1.4 in [BF1]).

Recall that a k-rough-geodesic in a metric space, $k \geq 0$, is a k-rough-geodesic embedding of an interval $I \subset \mathbb{R}$. A metric space is called rough-geodesic if there exists some $k \geq 0$, such that all pairs of points can be joined by a k-rough-geodesic. If $I = [0, \infty)$ we refer to the corresponding rough-geodesic as a rough-geodesic ray. A rough-geodesic metric space X is called visual if there exists $k \geq 0$ such that for some base-point $o \in X$ every $x \in X$ is in distance $\leq k$ to a k-rough-geodesic ray. Under this assumption the finiteness of the Assouad dimension of a Bourdon boundary of a Gromov hyperbolic space is equivalent to a growth condition on the space itself, which Bonk and Schramm refer to as being of bounded growth at some scale (see [BS]).

Now, and this immediately proves that our κ-cones are of asymptotic upper curvature $\leq \kappa$, for visual Gromov hyperbolic spaces the asymptotic upper curvature admits an interpretation in terms of a critical exponent.

Theorem 6.3 (Theorem 1.5 in [BF1]). *Let X be a visual Gromov hyperbolic metric space. If there exists a visual metric on $\partial_\infty X$ with parameter $\epsilon > 0$, then X is an $\mathrm{AC}_u(\kappa)$-space with $\kappa = -\epsilon^2$. Conversely, if X is an $AC_u(\kappa)$-space, then for every $0 < \epsilon < \sqrt{-\kappa}$ there exists a visual metric on $\partial_\infty X$ with visual parameter ϵ. In particular,*

$$K_u(X) \;=\; -a^2,$$

where

$$a := \sup\{\epsilon \mid \textit{there exists a visual metric on } \partial_\infty X \textit{ with parameter } \epsilon\}.$$

For a rough-geodesic Gromov hyperbolic metric space X we define the rough-isometry invariants $\underline{R}(X)$ and $\overline{R}(X)$ as the infima over such $\kappa < 0$, for which X admits a rough-isometric embedding into a CAT(κ)-space or is rough-isometric to some CAT(κ)-space, respectively. With this notation one has

$$[-\infty, 0) \ni K_u(X) \ \leq \ \underline{R}(X) \ \leq \ \overline{R}(X) \in \{[-\infty, 0) \cup \{\infty\}\}.$$

It seems rather likely that even $\underline{R}(X)$ can take the value ∞, but, in order to prove this, a better understanding of the boundaries at infinity of arbitrary CAT(κ)-spaces, $\kappa < 0$, than obtained in [FS2] needs to be achieved.

6.1.2. *Spaces of constant asymptotic curvature.* Now there is a corresponding notion of so-called $\mathrm{AC}_l(\kappa)$-spaces, $\kappa < 0$, admitting a lower asymptotic curvature bound. For a precise definition of this concept we refer the reader to [BF2], where the authors also characterize the spaces of bounded growth at some scale and constant asymptotic curvature through a condition on their Bourdon boundaries.

Theorem 6.4 (see [BF2])**.** *Let X be a visual, Gromov hyperbolic space of bounded growth at some scale. Then X is of constant asymptotic curvature $\kappa < 0$ if and only if $(\partial_\infty X, \rho_o^{\sqrt{-\kappa}})$ is a metric space which is 1-Lipschitz connected, i.e. that there exists $L \geq 1$ such that for all $\xi, \eta \in \partial_\infty X$ there exists a curve $c_{\xi,\eta} : I \longrightarrow \partial_\infty X$ connecting ξ to η of length $l(c_{\xi\eta}) \leq L\rho_o^{\sqrt{-\kappa}}(\xi, \eta)$.*

The κ-cones $\mathcal{C}_\kappa$ are indeed spaces of constant asymptotic curvature κ. As a corollary of Theorem 6.4 and Theorem 5.3 we obtain that for a certain class of Gromov hyperbolic spaces they are, up to rough-isometry equivalence, the only such spaces.

Corollary 6.5. *Let X be a visual, Gromov hyperbolic metric space of bounded growth at some scale and constant asymptotic curvature $\kappa < 0$. Then X is rough-isometric to the κ-cone $\mathcal{C}_\kappa(\tilde{Y})$, where $\tilde{Y}$ denotes the (geodesic!) length structure associated to the Bourdon boundary $Y = (\partial_\infty X, \rho_o^{\sqrt{-\kappa}})$ for some $o \in X$.*

Note that, for instance, the complex hyperbolic spaces are spaces of constant asymptotic curvature, where K_u takes the value of the upper $\frac{1}{4}$-pinched curvature bound.

6.2. **Gromov hyperbolic spaces with boundaries of finite Assouad Nagata dimension.** Note that every bounded, complete metric space (Z, d) can appear as the boundary of a visual, geodesic Gromov-hyperbolic space, where the metric d is a visual metric to an arbitrary but fixed visual parameter. As a consequence, every bi-Lipschitz or snowflake embedding theorem of metric spaces yields a corresponding rough-isometry or rough-similar embedding theorem for Gromov hyperbolic spaces.

In order to give an interesting example of such an embedding theorem, we have to prepare with some definitions. Let $\mathcal{U} = (U_i)_{i \in I}$ be a family of subsets of a metric space (X, d). The family $\mathcal{U}$ is called D-bounded for some $D \geq 0$ if all the sets in $\mathcal{U}$ have diameter bounded above by D. The multiplicity of the family $\mathcal{U}$ is the infimum of all integers $n \geq 0$ such that each intersection of $n + 1$ different sets in $\mathcal{U}$ is empty. For $s > 0$, the s-multiplicity of $\mathcal{U}$ is the infimum of all integers $n \geq 0$

such that every subset of X of diameter $\leq s$ intersects at most n members of the family.

Definition 6.6 (Assouad-Nagata dimension)**.** For a metric space X the Assouad-Nagata dimension $\dim_N X$ of X is the infimum of all integers $n \geq 0$ with the following property: There exists a constant $c > 0$ such that for all $s > 0$, X has a cs-bounded covering with s-multiplicity at most $n + 1$.

Note that such metric spaces of finite Assouad-Nagata dimension include all doubling spaces (those metric spaces of finite Assouad dimension), metric trees, Euclidean buildings and homogeneous or pinched negatively curved Hadamard manifolds.

In [LS] the authors prove the

Theorem 6.7 (Lang/Schlichenmaier embedding theorem, cf. Theorem 1.3 in [LS])**.** *Let (X, d) be a metric space with $\dim_N X \leq n < \infty$. Then for all sufficiently small exponents $p \in (0, 1)$, there exists a bi-Lipschitz embedding of (X, d^p) into the product of $n + 1$ metric trees.*

With the remarks above on the morphism equivalences of Gromov hyperbolic spaces on the one hand and their boundaries at infinity on the other hand, we obtain the

Corollary 6.8. *Let X be a Gromov hyperbolic metric space with some Hamenstädt boundary Y of finite Assouad-Nagata dimension. Then there exists $\kappa < 0$ such that X embeds rough-isometrically into a $\mathrm{CAT}(\kappa)$-space.*

As a consequence, we obtain

$$K_u(X) \;\leq\; \underline{R}(X) \;<\; 0. \tag{11}$$

Recall that for a visual, Gromov hyperbolic metric space of bounded growth at some scale, i.e. with boundary of finite Assouad dimension, the first inequality in (11) is an equality, and it is an interesting question whether or not this actually remains true for Gromov hyperbolic spaces with boundaries of finite Assouad-Nagata dimension.

References

[AB1] S.B. Alexander & R.L. Bishop, *Warped products of Hadamard spaces*, Manuscripta Math. 96 (1998), 487–505 MR1639844 (99g:53070)

[AB2] S.B. Alexander & R.L. Bishop, *Curvature bounds for warped products of metric spaces*, Geom. Funct. Anal., vol. 14 (2004), 1143–1181 MR2135163 (2007d:53061)

[BF1] M. Bonk & T. Foertsch, *Asymptotic upper curvature bounds in coarse geometry*, Math. Z. 253 (2006), 753–785 MR2221098 (2007e:53042)

[BF2] M. Bonk & T. Foertsch, *Asymptotic lower curvature bounds in coarse geometry*, preprint 2007

[BO'N] R.L. Bishop & B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969) 1–49 MR0251664 (40:4891)

[BS] M. Bonk & O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. 10, no. 2 (2000), 266–306 MR1771428 (2001g:53077)

[B] M. Bourdon, *Structure conforme au bord et flot géodésique d'un* $\mathrm{CAT}(-1)$*-espace*, Enseign. Math. (2) no. 41 (1995), 63–102. MR1341941 (96f:58120)

[BuS] S. Buyalo & V. Schroeder, *Elements of asymptotic geometry*, EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007. xii+200 pp. MR2327160 (2009a:53068)

[C] C.H. Chen, *Warped products of metric spaces of curvature bounded from above*, Trans. Amer. Math. Soc., vol. 351, no. 12 (1999), 4727–4740 MR1466944 (2000c:53031)

[FLS] T. Foertsch & A. Lytchak & V. Schroeder, *Nonpositive curvature and the Ptolemy inequality*, Int. Math. Res. Not. 2007, No. 22 (2007). MR2376212 (2009d:53049a)

[FS1] T. Foertsch & V. Schroeder, *A product construction for hyperbolic metric spaces*, Illinois Journal of Mathematics, vol. 49, no. 3 (2005), 793–810 MR2210259 (2006m:53059)

[FS2] T. Foertsch & V. Schroeder, *Hyperbolicity,* CAT(-1)*-spaces and the Ptolemy Inequality*, to appear in Math. Ann.

[Gr] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987, 75–263 MR919829 (89e:20070)

[H] U. Hamenstädt, *A new description of the Bowen-Margulis measure*, Ergodic Theory Dynam. Systems 9 (1989), no. 3, 455–464. MR1016663 (91b:58191)

[LS] U. Lang & T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, Int. Math. Res. Not., no. 58 (2005), 3625–3655 MR2200122 (2006m:53061)

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