DUALITY IN SPACES OF FINITE LINEAR COMBINATIONS OF ATOMS

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ABSTRACT. In this paper we describe the dual and the completion of the space of finite linear combinations of \((p, \infty)\)-atoms, \(0 < p \leq 1\). As an application, we show an extension result for operators uniformly bounded on \((p, \infty)\)-atoms, \(0 < p < 1\), whose analogue for \(p = 1\) is known to be false. Let \(0 < p < 1\) and let \(T\) be a linear operator defined on the space of finite linear combinations of \((p, \infty)\)-atoms, \(0 < p < 1\), which takes values in a Banach space \(B\). If \(T\) is uniformly bounded on \((p, \infty)\)-atoms, then \(T\) extends to a bounded operator from \(H^p(\mathbb{R}^n)\) into \(B\).

1. Introduction

For each \(0 < p \leq 1\) consider the space \(F^p\) of finite linear combinations of \((p, \infty)\)-atoms, endowed with its natural norm (or quasi-norm for \(p < 1\))

\[
\|f\|_{F^p} = \inf \left\{ \left( \sum_j |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_j \lambda_j a_j, a_j \text{ a } (p, \infty)\text{-atom}, \lambda_j \in \mathbb{C} \right\},
\]

where \(\sum'\) denotes a finite sum. Recall that \(a\) is a \((p, \infty)\)-atom if \(a\) is a measurable function supported on a ball \(B\), satisfying the cancellation condition

\[
\int a(x)x^\alpha \, dx = 0, \quad |\alpha| \leq n \left(\frac{1}{p} - 1\right),
\]

and the size condition

\[
|a| \leq \frac{1}{|B|^{\frac{1}{p}}}.\]

The space \(F^p\) is clearly contained in \(H^p = H^p(\mathbb{R}^n)\), the standard real Hardy space on \(\mathbb{R}^n\). The elements of \(H^p\) are the distributions that admit an atomic decomposition, \(f = \sum_{j=1}^{\infty} \lambda_j a_j\), converging in the sense of distributions, for some \((p, \infty)\)-atoms \(a_j\) and scalars \(\lambda_j\) with \(\sum_{j=1}^{\infty} |\lambda_j|^p < \infty\) (for \(p = 1\), \(H^1 \subset L^1\) and atomic sums converge in the \(L^1\)-norm). In [MTW] Meyer, Taibleson and Weiss observed that the \(F^p\)-norm is not comparable to the \(H^p\)-norm on \(F^p\). Recently, it was shown in [B] that the Meyer-Taibleson-Weiss result leads to the following conclusion in the case \(p = 1\): there exists a bounded linear functional on \(F^1\) which does not extend to a bounded linear functional on \(H^1\). In other words, there is a linear operator which is uniformly bounded on \((1, \infty)\)-atoms but does not extend to a bounded linear operator on \(H^1\).
In this paper we describe the structure of the completion \( \widetilde{F^p} \) of \( F^p \), \( 0 < p \leq 1 \), and of its dual space. We show in particular that, when \( p < 1 \), \( F^p \) and \( H^p \) have the same dual, and therefore no example like the one in [B] can be exhibited for \( p < 1 \). An immediate consequence of this is that if \( 0 < p < 1 \) and the linear operator
\[
T : F^p \to B,
\]
maps \( F^p \) into a Banach space \( B \) satisfying the inequality
\[
\|T(a)\|_B \leq C,
\]
for some positive constant \( C \) and all \((p, \infty)\)-atoms, then \( T \) extends to a bounded linear operator from \( H^p \) into \( B \). The argument proceeds by duality as follows. Take any \( u \) in the dual \( B^* \) of \( B \). Since \( u \circ T \in (F^p)^* = (H^p)^* \),
\[
|u(T(f))| \leq C \|u\| \|f\|_{H^p},
\]
and so, by the dual expression of the norm in a Banach space,
\[
\|T(f)\|_B \leq C \|f\|_{H^p}.
\]
We prove the following facts about \( \widetilde{F^p} \), \( 0 < p \leq 1 \).

(i) The closed subspace \( \widetilde{F^{p,c}} \) of \( \widetilde{F^p} \) spanned by the continuous \((p, \infty)\)-atoms is isomorphic to \( H^p \) as a Banach space, and \( \widetilde{F^p} \) splits as the direct sum of \( \widetilde{F^{p,c}} \) and a non-trivial complementary closed subspace \( N^p \).

(ii) Every element \( \xi \) of \( \widetilde{F^p} \) admits an atomic decomposition
\[
\xi = \sum_{j=1}^{\infty} \lambda_j a_j,
\]
for \((p, \infty)\)-atoms \( a_j \) and scalars \( \lambda_j \) with \( \sum_{j=1}^{\infty} |\lambda_j|^p < \infty \). Moreover, the \( \widetilde{F^p} \)-norm of \( \xi \) is equivalent to its atomic norm
\[
\inf \left\{ \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} : \sum_{j=1}^{\infty} \lambda_j a_j = \xi \text{ in } \widetilde{F^p} \right\}.
\]

(iii) If an atomic sum \( \sum_{j=1}^{\infty} \lambda_j a_j \), with \( \lambda_j \) and \( a_j \) as above, converges to 0 in \( \widetilde{F^p} \), it also converges to 0 in \( H^p \), but not vice versa. In fact, \( N^p \) consists of those elements of \( \widetilde{F^p} \) that are represented by atomic sums converging to 0 in \( H^p \).

In other words, \( H^p \) and \( \widetilde{F^p} \) are both quotients of the space of “formal series” of \((p, \infty)\)-atoms with \( l^p \) coefficients, but the equivalence relation defining \( \widetilde{F^p} \) is finer than that defining \( H^p \).

So, the reason why \( (F^1)^* \) is strictly larger than \( (H^1)^* \) is that it is the direct sum of \( (F^{1,c})^* = (H^1)^* \) and \( (N^1)^* \). Notice that \( (N^1)^* \) is non-trivial, as the dual of the non-trivial Banach space \( N^1 \). On the other hand, it turns out that \( (N^p)^* \) is trivial for \( p < 1 \).

To describe our results we need to introduce some notation and recall some basic classical facts in the theory of Banach algebras (see Section 3 for details).

Denote by \( L^\infty_0(\mathbb{R}^n) \) the space of bounded measurable functions on \( \mathbb{R}^n \) vanishing at infinity. Then \( L^\infty_0(\mathbb{R}^n) \) is a commutative \( C^* \)-algebra without unit, and its maximal ideal space is a locally compact, non-compact space, which we call \( \tilde{\mathbb{R}^n} \) (cf. [F]).
By the Gelfand-Naimark theorem, the Gelfand transform $f \mapsto \hat{f}$ establishes an isometric isomorphism between $L^\infty_0(\mathbb{R}^n)$ and the algebra $C_0(\mathbb{R}^n)$ of all continuous functions on $\mathbb{R}^n$ vanishing at $\infty$. On the other hand, $C_0(\mathbb{R}^n)$ is a closed subalgebra of $L^\infty_0(\mathbb{R}^n)$, and its maximal ideal space is $\mathbb{R}^n$. This embedding induces a continuous projection $\pi$ from $\mathbb{R}^n$ onto $\mathbb{R}^n$. Clearly, if $f \in C_0(\mathbb{R}^n)$, then $\hat{f} = f \circ \pi$.

In a similar way, given any ball $B$ in $\mathbb{R}^n$, the maximal ideal space of $L^\infty(B)$ is a compact space $\hat{B}$, endowed with a projection $\pi_B$ onto $\hat{B}$ induced by the inclusion of $C(\hat{B})$ in $L^\infty(B)$. Moreover, $L^\infty(B) \cong C(\hat{B})$, again by the Gelfand-Naimark theorem.

The restriction map $f \mapsto f|_B$ from $L^\infty_0(\mathbb{R}^n)$ to $L^\infty(B)$ induces a natural embedding $\iota_B : \hat{B} \to \mathbb{R}^n$, which is compatible with the projections $\pi$ and $\pi_B$, in the sense that

$$\pi_B = \pi \circ \iota_B.$$  

Similar embeddings $\iota_{B,B'} : \hat{B'} \to \hat{B}$ exist for pairs of balls $B, B'$ with $B' \subset B$, with the same compatibility with respect to the corresponding projections.

Denote by $m$ the Lebesgue measure on $\mathbb{R}^n$. The continuous linear functional $f \mapsto \int f\, dm$ on $L^\infty(B)$ is represented by a positive Borel measure $\hat{m}_B$ on $\hat{B}$, that is,

$$\int f\, dm = \int \hat{f}\, d\hat{m}_B, \quad f \in L^\infty(B). \quad (3)$$

If $B$ is contained in a second ball $B'$, then the restriction of $\hat{m}_{B'}$ to $\hat{B}$ is precisely $\hat{m}_B$ and thus we can define a positive Borel measure $\hat{m}$ globally on $\mathbb{R}^n$ by requiring that its restriction to $\hat{B}$ be $\hat{m}_B$ for each ball $B$.

We can now state our main result.

**Theorem.**  

(A) Let $\ell$ be a bounded linear functional on $F^1$. Then there exist a function $b \in BMO(\mathbb{R}^n)$ and a Radon measure $\mu$ on $\mathbb{R}^n$, singular with respect to $\hat{m}$, satisfying

$$|\mu|(\hat{B}) \leq Cm(B), \quad \text{for each ball } B \quad (4)$$

such that

$$\ell(f) = \int f\, b\, dm + \int \hat{f}\, d\mu, \quad f \in F^1. \quad (5)$$

Conversely, if $b$ and $\mu$ are as above, then the identity $\ell$ defines a bounded linear functional on $F^1$ and

$$\|\ell\|_{(F^1)^*} \equiv \|b\|_{BMO} + \sup_B \frac{|\mu|(\hat{B})}{m(B)}. \quad (6)$$

(B) Each bounded linear functional on $F^p$, $0 < p < 1$, extends uniquely to a bounded linear functional on $H^p(\mathbb{R}^n)$. Thus $(F^p)^* = H^p(\mathbb{R}^n)^*$, $0 < p < 1$.

It is clear that relation (6) determines the function $b$ and the measure $\mu$ uniquely. Therefore $(F^1)^*$ differs from $(H^1)^* = BMO$ by the presence of the complementary subspace $S$ of singular measures satisfying (4). We will show that $S$ is non-trivial; in fact, the Meyer, Taibleson and Weiss argument may be interpreted as the construction of a non-zero measure in $S$. The decomposition of $(F^1)^*$ as $BMO \oplus S$
is the dual counterpart of the decomposition of $\tilde{F}^1$ as $\tilde{F}^{1,c} \oplus N^1$, although $S$ and $BMO$ do not coincide with the annihilators of $F^{1,c}$ and $N^1$, respectively.

The nature of the elements of $N^p$, including $p = 1$, is somehow mysterious. It is not clear at all to us if they can be represented by concrete analytic objects.

Section 2 contains the discussion of the completion of $F^p$ and a constructive argument which proves the non-triviality of $N^p$. In Section 3 we prove the Theorem. We also give an example of a non-zero singular measure satisfying $\text{(4)}$.

We remark here that a variation of the main argument in the proof of the Theorem provides an alternative proof of some results in [MSV] and [YZ] on the equivalence of the finite and infinite atomic norms of $(1, q)$-atoms, $q < \infty$, and on extension of bounded operators defined on finite linear combinations of $(p, q)$-atoms with $1 < q < \infty$.

2. The completion of $F^p$

Let $F^{p,c}$ stand for the subspace of $H^p$ consisting of finite linear combinations of continuous $(p, \infty)$ atoms. A surprising recent result in [MSV] states that the $H^p$ and the $F^p$ norms are equivalent on $F^{p,c}$, $0 < p \leq 1$. Indeed, the result is proved in [MSV] only for $p = 1$, but, as suggested in Remark 3.2 there, the same argument extends to the case $0 < p < 1$. A complete proof may be found in [MSV2, section 3].

More precisely, we can quote Lemma 3.1 and Remark 3.2 in [MSV] as follows.

**Lemma 1.** The following norms are equivalent on $F^{p,c}$:

(a) the $H^p$-norm;
(b) the $F^p$-norm $\| \cdot \|_F$;
(c) the $F^{p,c}$-norm $\| \cdot \|_{F^{p,c}}$.

\[ \| f \|_{F^{p,c}} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j , \ a_j \text{ a continuous } (p, \infty)\text{-atom}, \ \lambda_j \in \mathbb{C} \right\}. \]

Since $F^{p,c}$ is dense in $H^p$, the natural inclusion of $F^{p,c}$ in $F^p$ extends uniquely to a continuous linear operator $T$ from $H^p$ to $\tilde{F}^p$. By Lemma 1 $T$ maps $H^p$ isomorphically onto the closure $\tilde{F}^{p,c}$ of $F^{p,c}$ in $\tilde{F}^p$. Notice that, again by Lemma 1 $\tilde{F}^{p,c}$ is the completion of $F^{p,c}$ endowed either with the norm $\| \cdot \|_{F^{p,c}}$ or with the norm inherited from $F^p$.

On the other hand, the inclusion of $F^p$ (endowed with its natural norm) into $H^p$ is continuous, and it extends to a continuous linear operator $U$ from $\tilde{F}^p$ to $H^p$. We then have the diagram

\[ H^p \xrightarrow{T} \tilde{F}^p \xrightarrow{U} H^p, \]

with $U \circ T$ being the identity map. In particular, $U$ is surjective. Set $P = T \circ U$, so that $P$ is a projection, that is, $P^2 = P$. The kernel of $P$ is the kernel of $U$, which we denote by $N^p$, and the kernel of $I - P$ is $T(H^p) = \tilde{F}^{p,c}$. Hence we get the topological direct sum decomposition

\[ \tilde{F}^p = \tilde{F}^{p,c} \oplus N^p. \]

Notice that $N^p$ is non-trivial, since otherwise the $H^p$ and the $F^p$ norms would be comparable on $F^p$.

To better understand the space $\tilde{F}^p$ we now prove the following.
**Proposition.** Given any sequence of $(p, \infty)$-atoms $a_j$ and any $\ell^p$-sequence of scalars $\lambda_j$, the series $\sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\tilde{F}^p$ to an element $\xi$ such that $\|\xi\|_{\tilde{F}^p}^p \leq \sum_{j=1}^{\infty} |\lambda_j|^p$. Conversely, each $\xi \in \tilde{F}^p$ can be written as

$$
(6) \quad \xi = \sum_{j=1}^{\infty} \lambda_j a_j,
$$

where each $a_j$ is a $(p, \infty)$-atom and the sum is convergent in $\tilde{F}^p$. Moreover,

$$
(7) \quad \|\xi\|_{\tilde{F}^p}^p = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j|^p \right\},
$$

where the infimum is taken over all decompositions (6) of $\xi$.

**Proof.** Let $\xi$ be an element of $\tilde{F}^p$. To prove (6), express $\xi$ as the limit in $\tilde{F}^p$ of a sequence $S_k$ of elements of $F^p$. Given $\epsilon > 0$, we may assume that $\|S_1\|_{\tilde{F}^p}^p < (1+\epsilon)\|\xi\|_{\tilde{F}^p}^p$ and that $\|S_k - S_{k+1}\|_{\tilde{F}^p}^p < \epsilon^k \|\xi\|_{\tilde{F}^p}^p$. Thus

$$
\xi = \lim_{k \to \infty} S_1 + (S_2 - S_1) + \cdots + (S_k - S_{k-1}).
$$

Set

$$
S_1 = \sum_{j=1}^{N_1} \lambda_j a_j,
$$

where the above expression has been chosen so that

$$
\sum_{j=1}^{N_1} |\lambda_j|^p < (1+\epsilon)\|\xi\|_{\tilde{F}^p}^p.
$$

Similarly, set

$$
S_\ell - S_{\ell-1} = \sum_{j=N_{\ell-1}+1}^{N_\ell} \lambda_j a_j, \quad \ell \geq 2,
$$

with

$$
\sum_{j=N_{\ell-1}+1}^{N_\ell} |\lambda_j|^p < \epsilon^\ell \|\xi\|_{\tilde{F}^p}^p.
$$

Then $\sum_{j=1}^{\infty} |\lambda_j|^p < (1-\epsilon)^{-1}\|\xi\|_{\tilde{F}^p}^p$ and the partial sums $\xi_m = \sum_{j=1}^{m} \lambda_j a_j$ form a Cauchy sequence in $F^p$. This shows that (6) holds.

Notice also that, for each $\xi \in \tilde{F}^p$, the inequality $\|\xi\|_{\tilde{F}^p}^p \leq \inf\{\sum_{j=1}^{\infty} |\lambda_j|^p\}$, where the infimum is taken over all possible expressions (6), is due to the fact that $\|\cdot\|_{\tilde{F}^p}^p$ satisfies the triangle inequality.

The atomic decomposition of elements of $\tilde{F}^p$ given above provides an explicit description of the operator $U$.

**Corollary.** Let $\xi \in \tilde{F}^p$ be represented by the sum (6). Then $U(\xi)$ is the sum of the same series in $H^p$. 

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We end this section by providing a constructive proof of the non-triviality of $N^p$. Let us first describe the Meyer-Tableau-Weiss construction as presented in [3]. Let $B$ denote the open ball centered at the origin with radius 1. Take a sequence of open disjoint balls $B_j, j \geq 1$, such that $\bigcup_j B_j$ is dense in $B$. Notice that we may also choose the $B_j$ so that the Lebesgue measure of their union $\sum_{j \geq 1} |B_j|$ is as small as we wish. As shown in [3], for each $j$ there exists a (non-continuous) $(p, \infty)$-atom $a_j$ supported on $B_j$ with the property that $|a_j| \geq c |B_j|^{-\frac{1}{p}}$, where $c$ is a small positive constant depending only on $n$. Thus, setting

$$f = \sum_{j \geq 1} |B_j|^\frac{1}{p} a_j,$$

we get $|f| \geq c$ on $\bigcup B_j$. From that it is not difficult to conclude (see [3]) that

$$\|f\|_{F^p} \geq c |B|^{\frac{1}{p}}.$$

On the other hand, we clearly have $\|f\|_{H^p} \leq \sum_{j \geq 1} |B_j|$, so that the ratio between $H^p$-norm and $F^p$-norm can be made as small as we wish.

We can now construct a sequence $\{f_m\}$ in $F^p$ satisfying

$$\|f_m\|_{F^p} \geq c^p |B|,$$

$$\|f_m - f_{m+1}\|_{F^p} \leq 2^p |B|^{1/2m},$$

$$\|f_m\|_{H^p} \leq \frac{|B|}{2^{m}}.$$

The first two conditions imply that $\{f_m\}$ has a non-zero limit $\xi \in \widetilde{F}^p$, whereas the third implies that $Uf_m = f_m$ tends to 0 in $H^p$. Hence $\xi \in N^p$.

The functions $f_m$ have the form (8); precisely,

$$f_m = \sum_{j \geq 1} |B_j|^m |a_j|^m,$$

where, for each $m$, $\{B_j^m\}_j$ is a disjoint family of balls contained in $B$ with dense union and small total measure, and each $a_j^m$ is a $(p, \infty)$-atom with $|a_j^m| \geq c |B_j^m|^{-\frac{1}{p}}$.

The first function $f_1$ can be any function as in (8) with, say, $\sum_{j \geq 1} |B_j^1| < |B|/2$. We then construct inductively $f_{m+1}$ from $f_m$ as follows.

Take $N$ so large that $\sum_{j > N} |B_j^m| < (1/4) \sum_{j \geq 1} |B_j^m|$. Inside each $B_j^m, 1 \leq j \leq N$, we take open disjoint balls $B_{j,l}^m, l \geq 1$, such that $\bigcup_{l \geq 1} B_{j,l}^m$ is dense in $B_j^m$ and $\sum_{l \geq 1} |B_{j,l}^m| < |B_j^m|/4$.

Then

$$\sum_{j=1}^N \sum_{l \geq 1} |B_{j,l}^m| + \sum_{j > N} |B_{j,l}^m| \leq \frac{1}{2} \sum_{j \geq 1} |B_j^m|.$$

Let $a_{j,l}$ be a $(p, \infty)$-atom supported on $B_{j,l}^m$ with $|a_{j,l}| \geq c |B_{j,l}^m|^{-\frac{1}{p}}$. Set

$$f_{m+1} = \sum_{j=1}^N \sum_{l \geq 1} |B_{j,l}^m|^\frac{1}{p} a_{j,l}^m + \sum_{j > N} |B_j^m|^{\frac{1}{p}} a_j^m.$$
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Since $|f_{m+1}| \geq c$ on an open dense subset of $B$, $\|f_{m+1}\|_F^p \geq c_p |B|$. Moreover,

$$f_m - f_{m+1} = \sum_{j=1}^{N} \left( |B_j^m|^{\frac{1}{p}} a_j^m - \sum_{l \geq 1} |B_j'_{jl}|^{\frac{1}{p}} a_j'^l \right).$$

For each $j$, the function

$$|B_j^m|^{\frac{1}{p}} a_j^m - \sum_{l \geq 1} |B_j'_{jl}|^{\frac{1}{p}} a_j'^l$$

is supported on $B_j^m$ and its absolute value is not greater than 2. Hence

$$\|f_m - f_{m+1}\|_F^p \leq \sum_{j=1}^{N} 2^p |B_j^m|.$$

Now, we relabel the balls in such a way that

$$\{B_j^{m+1}\}_{j \geq 1} = \{B_j^m\}_{j > N} \cup \{B_j'_{jl}\}_{j \leq N, l \geq 1},$$

and rename the atoms in $f_{m+1}$ as $a_j^{m+1}$ accordingly. Then, inductively from (10),

$$\sum_{j \geq 1} |B_j^m| \leq 2^{-m} |B|$$

for every $m$, and the required estimates can be easily verified.

3. PROOF OF THE THEOREM

We start by proving, for the reader’s sake, a few statements made (explicitly or not) in the last part of the introduction concerning the Gelfand spectrum $\hat{\mathbb{R}}^n$ and its projection $\pi$ on $\mathbb{R}^n$.

The first statement we want to prove is that $\pi$ is, in fact, well defined. Given $\phi$ in $\hat{\mathbb{R}}^n$, i.e., a non-trivial multiplicative functional on $L_0^\infty(\mathbb{R}^n)$, it is clear that its restriction to $C_0(\mathbb{R}^n)$ is also multiplicative. We must show that this restriction is “evaluation” at some point $x = \pi(\phi)$ of $\mathbb{R}^n$, or, equivalently, that it is not identically zero.

Since $L_0^\infty(\mathbb{R}^n)$ is a $C^*$-algebra, it is symmetric, so that $\phi(f) = \overline{\phi(f)}$ for every $f$. Therefore, $f \geq 0$ implies that $\phi(f) \geq 0$, so that $\phi$ is monotonic on real-valued functions. If $\phi$ vanishes identically on $C_0(\mathbb{R}^n)$, it also vanishes on characteristic functions of compact sets. By linearity and continuity, this would be a contradiction.

The second statement is that the mapping $\pi$ is surjective. We know that to each $\phi \in \hat{\mathbb{R}}^n$ we can associate a point $\pi(\phi)$ in $\mathbb{R}^n$. Given $y \in \mathbb{R}^n$, we can define a translate $\tau_y \phi \in \hat{\mathbb{R}}^n$ by

$$\tau_y \phi(f) = \phi(f(\cdot + y)).$$

It is quite clear that $\pi(\tau_y \phi) = \pi(\phi) + y$. Since $\hat{\mathbb{R}}^n$ is non-empty, $\pi$ is surjective.

The last statement which remained unproved in the introduction is that $\hat{\mathbb{R}}^n$ is the union of the $\hat{B}$ over all balls $B$. This is a direct consequence of (ii) in the following lemma.
Lemma 2. Let $B$ be an open ball in $\mathbb{R}^n$. Then

(i)

$$\hat{B} = \{ \phi \in \mathbb{R}^n : \phi(\chi_B) = 1 \} = \text{supp } \chi_B,$$

where $\hat{f}$ stands for the Gelfand transform of $f \in L_0^\infty(\mathbb{R}^n)$.

(ii)

$$\pi^{-1}(B) \subset \hat{B} \subset \pi^{-1}(\overline{B}).$$

Proof. To prove (12) notice that $\phi(\chi_B) = \phi(f \chi_B)$, $f \in L_0^\infty(\mathbb{R}^n)$, which means that $\phi$ factors through a character of $L_\infty(B)$. Thus $\phi \in \hat{B}$. The argument can be reversed, so (12) is proved.

Assume now that for some $\phi \in \mathbb{R}^n$ we have $\pi(\phi) \in B$. Let $f$ be a continuous function on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support contained in $B$. Then $f \chi_B = f$ and so

$$1 = \phi(f) = \phi(f \phi(\chi_B) = \phi(\chi_B).$$

Then $\phi \in \hat{B}$ because of (12).

If $\pi(\phi)$ is not in $\hat{B}$, then there is a continuous function $f$ on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support in $\mathbb{R}^n \setminus \overline{B}$. Thus $f \chi_B = 0$ and so $\phi(\chi_B) = 0$, that is, $\phi$ is not in $\hat{B}$. \qed

We now turn to the proof of the Theorem. We begin by discussing the converse statement in part (A) of the Theorem. Obviously, given $b \in BMO$, the linear functional $f \mapsto \int fbdm$ is bounded on $\hat{F}^1$ with a norm controlled from above by the BMO-norm of $b$. On the other hand, restriction of the functional to $\hat{F}^{1,c}$ gives a control from below by the same BMO-norm.

We first remark that (12) clearly implies that, given $f \in L_0^\infty(\mathbb{R}^n)$, the support of $f$ is contained in $B$ if and only if the support of $\hat{f}$ is contained in $\hat{B}$.

Let $\mu$ be a Radon measure on $\hat{\mathbb{R}}^n$ satisfying (i). For each $(1,\infty)$-atom $a$ supported on a ball $B$ one has

$$\left| \int a \, d\mu \right| \leq \|a\|_\infty |\mu|(\hat{B}) \leq \frac{|\mu|(\hat{B})}{m(B)} < C.$$ 

Hence $\mu$ determines a bounded linear functional on $\hat{F}^1$.

Assume now that $\ell$ is a bounded linear functional on $F^1$. Fix a ball $B$ and let $L_0^\infty(B)$ stand for the set of functions in $L_\infty(B)$ with zero integral. Given $f \in L_0^\infty(B)$,

$$\frac{1}{m(B)} \frac{f}{\|f\|_\infty}$$

is a $(1,\infty)$-atom. Thus

$$|\ell(f)| \leq \|\ell\| \|f\|_\infty m(B), \quad f \in L_0^\infty(B).$$

The restriction of $\ell$ to $L_0^\infty(B)$ extends to a bounded linear functional on $L_\infty(B) = C(\hat{B})$. Thus there exists a measure $\nu_B$ on $\hat{B}$ such that

$$\ell(f) = \int \hat{f} \, d\nu_B, \quad f \in L_0^\infty(B).$$
If \( f \in L^\infty(B) \), then clearly \( \hat{f}_B = f_B \), where \( g_E \) stands for the mean of the function \( g \) on the set \( E \) with respect to the underlying measure \( (\hat{m} \text{ or } m \text{ in the case at hand}) \). Then
\[
\ell(f - f_B) = \int_B (\hat{f} - \hat{f}_B) \, d\nu_B
\]
\[
= \int_B (\hat{f} - \hat{f}_B) \left( d\nu_B - \nu_B(\hat{B}) \frac{d\hat{m}}{\hat{m}(\hat{B})} \right)
\]
\[
= \int_B \hat{f} \left( d\nu_B - \frac{\nu_B(\hat{B})}{\hat{m}(\hat{B})} d\hat{m} \right).
\]
for each \( f \in L^\infty(B) \). Therefore, if \( \nu_B \) represents \( \ell \) on \( L^\infty_0(B) \), that is, if \( \text{(14)} \) holds, then \( d\nu_B - \nu_B(\hat{B}) \frac{d\hat{m}}{\hat{m}(\hat{B})} \) is uniquely determined.

Let \( B_N \) stand for the open ball with center at the origin and radius \( N, N = 1, 2, \ldots \). Take any measure \( \nu_1 \) on \( \hat{B}_1 \) that represents \( \ell \) on \( L^\infty_0(B_1) \). Every other such measure differs from \( \nu_1 \) by a constant multiple of \( \chi_{\hat{B}_1} \hat{m} \). By the preceding remark applied to \( B_N \) there exists a unique measure \( \nu_N \) on \( \hat{B}_N \) which represents \( \ell \) on \( L^\infty_0(B_N) \) and \( \nu_N(B_1) = \nu_1(B_1) \). Clearly \( \nu_N \) restricted to \( \hat{B}_{N-1} \) is precisely \( \nu_{N-1} \). Therefore we can define a measure \( \nu \) on \( \mathbb{R}^n \) by requiring that \( \nu \) restricted to \( \hat{B}_N \) be \( \nu_N \).

Given any ball \( B \) take \( N \) such that \( B \subset B_N \). Since the restriction of \( \nu \) to \( \hat{B}_N \) represents \( \ell \) on \( L^\infty_0(B_N) \), which contains \( L^\infty_0(B) \), the restriction of \( \nu \) to \( \hat{B} \) represents \( \ell \) on \( L^\infty_0(B) \) as well. By \( \text{(15)} \),
\[
\left| \int_B \hat{f} \left( d\nu - \nu(\hat{B}) \frac{d\hat{m}}{\hat{m}(\hat{B})} \right) \right| \leq 2 \| \ell \| \| f \|_\infty m(B), \quad f \in L^\infty(B)
\]
or
\[
\left\| \nu - \nu(\hat{B}) \frac{\hat{m}}{m(\hat{B})} \right\|_\hat{B} \leq 2 \| \ell \| m(B).
\]
Let us now consider the Radon-Nikodym decomposition of \( \nu \)
\[
\nu = g \hat{m} + \mu,
\]
where \( g \in L^1_{loc}(\hat{m}) \) and \( \mu \) is singular with respect to \( \hat{m} \). By \( \text{(16)} \),
\[
|\mu|(\hat{B}) \leq 2 \| \ell \| m(B)
\]
and
\[
\int_B |g - g_{\hat{B}} - \frac{\mu(\hat{B})}{\hat{m}(\hat{B})} | \, d\hat{m} \leq 2 \| \ell \| m(B).
\]
We are left with the task of finding the \( BMO \)-function \( b \).

Combining \( \text{(17)} \) and \( \text{(18)} \) we readily get
\[
\int_B |g - g_{\hat{B}} | \, d\hat{m} \leq 4 \| \ell \| m(B).
\]
We need a lemma.
Lemma 3. For each function $g \in L^1_{\text{loc}}(\hat{m})$ there exists a unique function $f \in L^1_{\text{loc}}(m)$ with the property that for each ball $B$, 

$$\int_B g \hat{\varphi} \, d\hat{m} = \int_B f \varphi \, dm, \quad \varphi \in L^\infty(B).$$

Such $f$ satisfies 

$$\int_B |g - g_B| \, d\hat{m} = \int_B |f - f_B| \, dm,$$

for each ball $B$.

Once the lemma is proved we complete the proof of part (A) of the Theorem by just calling $b$ the function $f$ associated with $g$ in Lemma 3. Inequality (19) tells us that $b \in BMO(\mathbb{R}^n)$ and that its $BMO(\mathbb{R}^n)$ norm is not greater than $4 \|\ell\|$.

Proof of Lemma 3. We will show that for each ball $B$ the Gelfand transform, which is an isometry between $L^\infty(B)$ and $C(\bar{B})$, extends to an isometry between $L^1(B, m)$ and $L^1(B, \hat{m})$. This immediately provides a further extension of the Gelfand transform to a topological isomorphism between $L^1_{\text{loc}}(m)$ and $L^1_{\text{loc}}(\hat{m})$.

We begin by showing that, for each ball $B$ in $\mathbb{R}^n$ and every $f \geq 0$ in $L^\infty(B)$,

$$(20) \quad \int_B \hat{f} \, d\hat{m} = \int_B f \, dm.\tag{20}$$

This follows from

$$\int_B f \, dm = \sup_\varphi \int_B f \varphi \, dm$$

$$= \sup_\varphi \int_B \hat{f} \hat{\varphi} \, d\hat{m}$$

$$= \int_B \hat{f} \, d\hat{m},$$

where the supremum is taken on the closed unit ball of $L^\infty(B)$.

By linearity, (20) provides an extension of the Gelfand transform to a topological isomorphisms $f \to \hat{f}$ of $L^1_{\text{loc}}(m)$ onto $L^1_{\text{loc}}(\hat{m})$. Given $g \in L^1_{\text{loc}}(\hat{m})$ take $f \in L^1_{\text{loc}}(m)$ with $g = \hat{f}$. The first identity in the statement of Lemma 3 follows by approximating $f \in L^1(B, m)$ by functions in $L^\infty(B)$ and the second follows from (20). \hfill $\square$

Before proving part (B) of the Theorem we give an explicit example, modeled on the Meyer-Taibleson-Weiss argument, of a non-zero measure which is singular with respect to $\hat{m}$ and satisfies (11).

Take an open set $U$ of $\mathbb{R}^n$, $U \subset B_0 = \{x : |x| \leq 1\}$, such that $U$ is dense in $B_0$ and $m(U) < m(B_0)$. Then the compact set $E = B_0 \setminus U$ has positive Lebesgue measure. Set $V = \pi^{-1}(U)$, so that $V \subset \hat{B}_0$ by Lemma 1. Then $U \subset \pi(V)$ and so $\pi(\partial V) = B_0$, because $U$ is dense in $B_0$. Hence $\pi(\partial V) = E$. Now, the boundary of each open set in $\hat{B}_0$ has zero $\hat{m}$ measure ([R, p. 286]). Therefore $\hat{m}(\partial V) = 0$ but $m(\pi(\partial V)) = m(E) > 0$. Identify $C(E)$ to the subspace $S$ of continuous functions on $\partial V$ of the form $f \circ \pi$, $f \in C(E)$. The bounded linear functional on $S$ defined
by \( f \to \int f \, dm \) extends by Hahn-Banach to a bounded linear functional on \( C(\partial V) \) with the same norm. Thus there exists a positive measure \( \mu \) on \( \partial V \) such that
\[
\int (f \circ \pi) \, d\mu = \int f \, dm, \quad f \in C(E).
\]
If \( B \) is an open ball, then by Lemma 21
\[
\mu(B) \leq \mu(\pi^{-1}(B)) = m(B \cap E) \leq m(B),
\]
and condition 41 is satisfied.

**Proof of part (B) of the Theorem.** The argument is analogous to the proof of part (A), except for minor technical details. If \( 0 < p < 1 \), then, as we will see, the singular measure \( \mu \) vanishes and so we will conclude that \( (F_p)^* = H^p(\mathbb{R}^n)^* \).

Let \( \ell \) be a bounded linear functional on \( F_p \), \( 0 < p < 1 \). Let \( d \) be the integer part of \( n(\frac{1}{p} - 1) \). Given a ball \( B \) let \( L_\infty^\circ(B) \) stand for the set of functions \( f \in L_\infty(B) \) such that
\[
\int f(x) \, x^\alpha \, dx = 0, \quad |\alpha| \leq d.
\]
For each \( f \in L_\infty^\circ(B) \),
\[
\frac{1}{m(B)^{\frac{1}{p}}} \frac{f}{\|f\|_\infty}
\]
is a \((p, \infty)\)-atom and so
\[
(21) \quad |\ell(f)| \leq \max(1, \|f\|_\infty) m(B)^{\frac{1}{p}}, \quad f \in L_\infty^\circ(B).
\]
For each \( f \in L_\infty(B) \) let \( P_B(f) \) be (the restriction to \( B \) of) the unique polynomial of degree not greater than \( d \) such that
\[
\int f(x) \, x^\alpha \, dx = \int_B P_B(f)(x) \, x^\alpha \, dx, \quad |\alpha| \leq d.
\]
Since \( P_B(f) \) is the orthogonal projection (in \( L^2(B) \)) of \( f \) into the subspace of polynomials of degree not greater than \( d \),
\[
\|P_B(f)\|_2 \leq \|f\|_2 \leq \|f\|_\infty,
\]
where the \( L^2 \) norms are taken with respect to the normalized Lebesgue measure on \( B \). Now, we want to compare the norms \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) on the space \( P_d(B) \) of restrictions to \( B \) of polynomials of degree not greater than \( d \). After appropriate translation and dilation we may assume that \( B \) has center 0 and radius 1. Since \( P_d(B) \) is finite dimensional, there is a constant \( C(d, n) \), depending only on \( d \) and \( n \), such that
\[
\|P\|_\infty \leq C(d, n) \|P\|_2, \quad P \in P_d(B),
\]
and so
\[
\|P_B(f)\|_\infty \leq C(d, n) \|f\|_\infty, \quad f \in L_\infty(B).
\]
Therefore by 21,
\[
(22) \quad |\ell(f - P_B(f))| \leq (1 + C(d, n)) \|\ell\| \|f\|_\infty m(B)^{\frac{1}{p}}, \quad f \in L_\infty(B).
\]
By 21 there is a measure \( \nu_B \) on \( \hat{B} \) such that
\[
(23) \quad \ell(f) = \int \hat{f} \, d\nu_B, \quad f \in L_\infty^\circ(B).
\]
Given a measure $\lambda$ on $\widehat{B}$ there is a unique polynomial $P_B(\lambda) \in P_d(B)$ such that
\[
\int_{\widehat{B}} (\pi(\phi))^\alpha d\lambda(\phi) = \int_{\widehat{B}} P_B(\lambda)(x) x^\alpha dx, \quad |\alpha| \leq d.
\]
Hence, for every polynomial $Q$ of degree $\leq d$,
\[
\int_{\widehat{B}} \hat{Q} d\lambda = \int_{\widehat{B}} P_B(\lambda) Q \, dm = \int_{\widehat{B}} \hat{P_B(\lambda)} \hat{Q} \, d\hat{m}.
\]
Therefore, by (23),
\[
\ell(f - P_B(f)) = \int_{\widehat{B}} (\hat{f} - \hat{P_B(f)}) \, d\nu_B
\]
(24)
\[
= \int_{\widehat{B}} (\hat{f} - \hat{P_B(f)}) (d\nu_B - \hat{P_B(\nu_B)} \, d\hat{m})
\]
\[
= \int_{\widehat{B}} \hat{f} (d\nu_B - \hat{P_B(\nu_B)} \, d\hat{m}),
\]
for each $f \in L^\infty(B)$. Hence the measure $\nu_B - \hat{P_B(\nu_B)} \, \hat{m}$ is determined by $\ell$.

As before, with $B_N$ denoting the ball of radius $N$ centered at the origin, we fix a measure $\nu_1$ on $\widehat{B_1}$ that represents $\ell$ on $L^\infty(\widehat{B_1})$ and then take the unique measure $\nu_N$ on $\widehat{B_N}$ which represents $\ell$ on $L^\infty(B_N)$ and such that $P_{B_1}(\nu_N) = P_{B_1}(\nu_1)$. Then $\nu_N$ restricted to $B_{N-1}$ is $\nu_{N-1}$, so we can define a measure $\nu$ on $\mathbb{R}^n$ by requiring that $\nu$ restricted to $B_N$ be $\nu_N$.

Given any ball $B$, take $N$ such that $B \subset B_N$. Then the restriction of $\nu$ to $L^\infty(B)$ is $\ell$, so, by (22) and (24),
\[
\left| \int_{\widehat{B}} \hat{f} (d\nu_B - \hat{P_B(\nu_B)} \, d\hat{m}) \right| \leq C \|f\|_\infty m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B).
\]
Hence
(25)
\[
|\nu - \hat{P_B(\nu)} \hat{m}|(\widehat{B}) \leq C m(B)^{\frac{1}{p}}.
\]
Now consider the Radon-Nikodym decomposition of $\nu$,
\[
\nu = g \hat{m} + \mu,
\]
with $\mu$ singular with respect to $\hat{m}$. We get, by (25) and Lemma 2
\[
|\mu|(\pi^{-1}(B)) \leq |\mu|(\widehat{B}) \leq C m(B)^{\frac{1}{p}},
\]
for each open ball $B$. Since $0 < p < 1$, we readily conclude that $\mu = 0$. Indeed, let $r$ be the radius of $B$. Covering $B$ by $A_n k^n$ balls of radius $r/k$, we see that the constant $C$ in the right-hand side of the above inequality can be replaced by $CA_n k^{n(1-\frac{1}{p})}$. Letting $k$ tend to $\infty$, we obtain the conclusion.

Now take $f \in L^1_{loc}(\mathbb{R}^n)$ with $g = \hat{f}$. Then
\[
\int_B |f - P_B(f)| \, dm \leq C m(B)^{\frac{1}{p}},
\]
which is precisely the condition that guarantees that $f$ determines a bounded linear functional on $H^p(\mathbb{R}^n)$ ([TW]). Thus $\ell$ is a bounded linear functional on $H^p(\mathbb{R}^n)$, and the proof is complete. \(\square\)
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