

SCHUR FUNCTORS AND DOMINANT DIMENSION

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ABSTRACT. The dominant dimension of an algebra A provides information about the connection between A -mod and B -mod for $B = eAe$, a certain centralizer subalgebra of A . Well-known examples of such a situation are the connection (given by Schur-Weyl duality) between Schur algebras and group algebras of symmetric groups, and the connection (given by Soergel's 'Struktursatz') between blocks of the category \mathcal{O} of a complex semisimple Lie algebra and the coinvariant algebra. We study cohomological aspects of such connections, in the framework of highest weight categories. In this setup we characterize the dominant dimension of A by the vanishing of certain extension groups over A , we determine the range of degrees, for which certain cohomology groups over A and over eAe get identified, we show that Ringel duality does not change dominant dimensions and we determine the dominant dimension of Schur algebras.

1. INTRODUCTION

Let A be an algebra and $e \in A$ a non-zero idempotent. Then there is an exact covariant functor $f = eA \otimes_A - : A\text{-mod} \rightarrow eAe\text{-mod}$, called the *Schur functor*, which relates the two module categories. In general, the two algebras A and eAe may have nothing in common, and the two module categories $A\text{-mod}$ and $eAe\text{-mod}$ may be completely different. On the other hand, there are well-known and important situations, where such a setup is useful or even fundamental. Most prominently, Schur-Weyl duality [15] relates Schur algebras $S(n, r)$ (for $n \geq r$), and thus representation theory of algebraic groups GL_n , with group algebras $k\Sigma_r$ of finite symmetric groups. Similarly, Soergel's 'Struktursatz' [26] relates blocks of the Bernstein-Gelfand-Gelfand category \mathcal{O} of a complex semisimple Lie algebra with (subalgebras of) the coinvariant algebra, which is the cohomology algebra of a flag variety. These two results serve as paradigms in many areas of mathematics or mathematical physics, and there are plenty of generalizations.

What makes these classical situations different from the general one? A first and crucial point is the existence of a double centralizer property. In both cases, the A -module Ae is faithful projective and injective, and there are isomorphisms $\text{End}_A(Ae) \cong eAe$ (obvious) and $\text{End}_{eAe}(Ae) \cong A$ (highly non-trivial). In abstract terms, this property can be characterized, according to classical work of Morita and Tachikawa [22, 28, 29, 30], by A having *dominant dimension* at least two. This

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means that A has an injective coresolution $0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$ such that I_0 and I_1 are also projective. In [19] it has been shown that this point of view is also feasible in a practical way; indeed, Schur-Weyl duality and Soergel's result have been reproved there in a brief and general way by checking that the dominant dimension of the respective algebra A is at least two.

So, the dominant dimension of A at least two implies a very close relationship between the two algebras A and eAe . But what kind of relationship can be expected between the two module categories $A\text{-mod}$ and $eAe\text{-mod}$ or interesting subcategories thereof? An old result of B. Müller [23] settles the case of the subcategory $\text{add}(A)$ of projective modules, which by the Schur functor gets identified with $\text{add}(eA) \subset eAe\text{-mod}$. Müller has shown that this identification also works for many (but definitely not for all) extension groups, the range of degrees being fixed by the dominant dimension of A . Recent results of Hemmer and Nakano [16] suggest considering much larger subcategories that are of great interest for $A\text{-mod}$ being a highest weight category. They have shown that under mild assumptions, the Schur functor $S(n, r)\text{-mod} \rightarrow k\Sigma_r\text{-mod}$ provides an equivalence between Weyl-filtered modules (of the algebraic group) and dual Specht-filtered modules (of the finite symmetric group). Moreover, they have shown that certain extension groups get identified by this equivalence; their approach does not use dominant dimension. Similar equivalences between subcategories have been found for quantized Schur algebras, Brauer algebras and many other classes of algebras; but in these cases, comparison results on extension groups are still missing.

These results raise questions on finite dimensional algebras A having dominant dimension at least two:

(1) Can one characterize the dominant dimension of an algebra A in terms of vanishing of certain extension groups over A (while Müller's result uses extension groups over eAe)?

(2) Suppose $A\text{-mod}$ is a highest weight category, that is, A is a quasi-hereditary algebra (for instance, a Schur algebra or a block of \mathcal{O}). What are the dominant dimensions of standard (that is, Weyl or Verma) modules and of the characteristic tilting module? How are they related to the dominant dimension of A itself?

(3) What is the range of degrees of extension groups that get identified by the Schur functor, and how does it depend on dominant dimension?

(4) How does dominant dimension behave under taking Ringel duals or passing to quasi-hereditary quotient algebras?

(5) What is the dominant dimension of a Schur algebra $S(n, r)$ (assuming $n \geq r$, since otherwise there is, in general, no faithful projective and injective module)?

In this article we will provide answers to these questions under the general assumption of the algebra A being quasi-hereditary and carrying a special kind of duality (an anti-automorphism preserving pointwise a full set of primitive orthogonal idempotents). Our assumptions cover classical and generalized Schur algebras, blocks of \mathcal{O} and many other known classes of examples.

Our answer to question (1) is part (2) of Corollary 3.7; it describes $\text{domdim}(A)$ in terms of the vanishing of extensions between the dual of A and A itself, in degrees ranging from 1 to $\text{domdim}(A) - 2$; $\text{domdim } A \geq n$ if and only if $\text{Ext}_A^i({}^{\natural}A, A) = 0$ for $1 \leq i \leq n - 2$. This result complements the well-known fact that the global dimension of A , if finite, is the largest degree in which there is a non-vanishing extension group between the dual of A and A itself.

Our answer to question (2) states that the dominant dimension of the characteristic tilting module T is the minimum of the dominant dimensions of standard modules (Corollary 3.7 (3)), and it equals exactly one half of the dominant dimension of A (Theorem 4.3), which therefore must be an even number. This result is a counterpart to a recent theorem by Mazorchuk and Ovsienko [20], which states in our situation that the global dimension of A equals twice the projective dimension of T . Our results imply that standard modules have strictly positive dominant dimension and therefore are torsionless; this result (Corollary 4.4) generalizes results by James [17] and by Donkin [10] in the case of classical and quantized Schur algebras.

The answer to question (3) depends, of course, on the choice of modules whose extensions are to be considered. We always assume that the first variable of the extension group may be any module. We then describe (in Theorem 3.9) degree ranges in which extension groups are identified when either the second variable is required to be projective, or more generally when the second variable has a filtration by standard modules. These degree ranges are shown to be the largest possible such degree intervals starting from one.

Question (4) has a complete answer (Theorem 4.3, (3) and (4)) with respect to taking Ringel duals; we show that A and its Ringel dual have the same dominant dimension. We also prove (Theorem 4.9) that when passing from A to a quasi-hereditary quotient modulo a heredity ideal, the dominant dimension either is kept or it drops by two; both cases may happen.

Finally, we give a full answer to question (5) (in Theorem 5.1) by proving that the dominant dimension of a classical Schur algebra $S(n, r)$ (for $n \geq r \geq p$) equals $2(p - 1)$. So, in contrast to the global dimension [31], it only depends on the characteristic p of the underlying field, but not on n or r . This is in contrast to a result in [13] that blocks of \mathcal{O} , unless semisimple, always have dominant dimension exactly two.

2. PRELIMINARIES

In this section we fix notation and recall the main concepts to be used.

Throughout, k is a field, all algebras are finite dimensional k -algebras, split over k , and modules are finite dimensional left modules, unless stated otherwise. Let A be an algebra and let $\{L(x) : x \in X\}$ be a complete set of pairwise non-isomorphic simple A -modules. We denote by $P(x)$ and $I(x)$, respectively, the projective cover and injective hull of the simple A -module $L(x)$. By $\text{add}(M)$ of a module M , we mean the full subcategory of direct summands of finite direct sums of M . We write $D(M) = \text{Hom}_k(M, k)$.

2.1. Quasi-hereditary algebras. Quasi-hereditary algebras have been introduced by Cline, Parshall and Scott when studying highest weight categories [6]. By definition, an algebra A is said to be *quasi-hereditary* over a poset X if there is a bijection from X to the set of isomorphism classes of simple A -modules and for each $x \in X$, there is a quotient module $\Delta(x)$ of $P(x)$, called a *standard module*, satisfying

- (1) the kernel of the canonical epimorphism $\Delta(x) \twoheadrightarrow L(x)$ is filtered by $L(y)$ with $y < x$;

- (2) the kernel of the canonical epimorphism $P(x) \twoheadrightarrow \Delta(x)$ is filtered by $\Delta(z)$ with $z > x$.

The costandard module $\nabla(x)$ is defined to be the k -dual of the standard module $\Delta_{A^{op}}(x)$ of the quasi-hereditary algebra (A^{op}, X) . By $\mathcal{F}(\Delta)$ (respectively $\mathcal{F}(\nabla)$) we denote the full subcategory of A -mod which has objects filtered by the standard A -modules (resp. costandard modules). Given a module M , its Δ -filtration dimension, which we denote by $\dim_{\mathcal{F}(\Delta)}(M)$ according to [20], is the minimal length n of the exact sequence $0 \rightarrow K_n \rightarrow \cdots \rightarrow K_1 \rightarrow K_0 \rightarrow M \rightarrow 0$, where $K_i \in \mathcal{F}(\Delta)$ for all i . Note that this dimension is finite because of the ordering conditions on composition factors of standard modules, which imply that $\text{Hom}_A(\Delta(x), \Delta(y)) = 0$ unless $x \leq y$ and all non-zero endomorphisms of the standard modules are isomorphisms.

Quasi-hereditary algebras have a ring-theoretical definition in terms of heredity ideals; see [10, appendix] for more details. Here we list some familiar results for later use:

- (1) Ringel has shown that for each $x \in X$, there exists a unique (up to isomorphism) indecomposable module $T(x)$, called *tilting module* in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. The direct sum $T := \bigoplus_{x \in X} T(x)$ is called the *characteristic tilting module* of (A, X) which satisfies $\text{add}(T) = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.
- (2) Let Y be a *cosaturated* subset of X (i.e., $x > y \in Y$ implies $x \in Y$). Let e be the cosaturated idempotent of A corresponding to Y . Then $I = AeA$ is an ideal in the heredity chain of A and $(A/I, X \setminus Y)$, (eAe, Y) are quasi-hereditary algebras. The standard modules of A/I are $\{\Delta(x) : x \in X \setminus Y\}$ and the tilting modules of A/I are $\{T(x) : x \in X \setminus Y\}$. The standard modules of eAe are $\{e\Delta(x) : x \in Y\}$, and the tilting modules of eAe are $\{eT(x) : x \in Y\}$.

Classical and quantized Schur algebras are known to be quasi-hereditary, and blocks of category \mathcal{O} are so, too. There are large classes of further examples, such as hereditary algebras and algebras of global dimension two, but these in general do not have a duality as we will require below.

A motivation for our question (2) and also a useful tool in some of our proofs is a result by Mazorchuk and Ovsienko, stated here in a special case:

Theorem 2.1 (Mazorchuk and Ovsienko [20]). *Let A be a quasi-hereditary algebra and suppose A -mod has a duality \diamond preserving simples (up to isomorphism). Then $\text{gldim } A = 2 \text{projdim } T$. For a module M , if $\dim_{\mathcal{F}(\Delta)}(M) = t < \infty$, then $\text{Ext}^{2t}(M, \diamond M) \neq 0$.*

(The notion of duality used here is slightly more general than the definition used throughout this article.)

2.2. Dominant dimension. Much of the theory of dominant dimensions is due to Morita and Tachikawa; see [2, 22, 28, 29, 30, 32] for more information and [3] for a treatment of some of the topics to be discussed below in terms of homological algebra.

The *dominant dimension* of a module M , which we denote by $\text{domdim } M$, is the maximal number t (or ∞) having the property that if $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_t \rightarrow \cdots$ is a minimal injective resolution of M , then I_j is projective for all $j < t$ (or ∞). Here we set $\text{domdim } M = 0$ if the injective hull I_0 of M is not projective, or equivalently if M is not a submodule of a module that is both projective and

injective. We write $\text{domdim } A = \text{domdim } {}_A A$. So, $\text{domdim } A = 0$ if and only if A does not have a faithful module that is both projective and injective. Algebras of dominant dimension 1 and 2 are sometimes called QF-3 algebras and maximal quotient algebras, respectively; see [32].

Suppose A has dominant dimension at least two. Then by classical results of Morita, Tachikawa and others (see [30]) there is a double centralizer property between A and its centralizer subalgebra eAe : $\text{End}_A(Ae) = eAe$ and $\text{End}_{eAe}(Ae) = A$. In [19] it has been shown that classical Schur-Weyl duality between the Schur algebra $S(n, r)$ (for $n \geq r$) and the group algebra $k\Sigma_r$ of the symmetric group as well as Soergel's 'Struktursatz' for the Bernstein-Gelfand-Gelfand category \mathcal{O} , providing a double centralizer property between a block and a subalgebra of the coinvariant algebra, both are special cases of this situation. In particular, those Schur algebras and also blocks of \mathcal{O} have dominant dimension at least two.

In some vague sense, one expects the connection between A -mod and eAe -mod to be all the better the larger the dominant dimension of A is. In this article we are trying to turn this expectation into precise statements.

We will make crucial use of the following characterization of dominant dimension over A in terms of cohomology over eAe .

Theorem 2.2 (Müller [23]). *Assume that $\text{domdim } A \geq 2$ and Ae is a projective, injective and faithful module. Let M be an A -module. Then $\text{domdim } M \geq n \geq 2$ if and only if $M \cong \text{Hom}_{eAe}(eA, eM)$ canonically and $\text{Ext}_{eAe}^i(eA, eM) = 0$ for $1 \leq i \leq n - 2$.*

2.3. Algebras with a duality. Let A be an algebra with an anti-automorphism ω . We shall call ω a *duality* if $\omega^2 = 1$ and ω fixes a complete set of primitive orthogonal idempotents.

Lemma 2.3. *Let A be an algebra with a duality ω . Let B be a basic algebra which is Morita equivalent to A . Then B also has a duality.*

Proof. By definition, there is a decomposition of the unit of A into primitive orthogonal idempotents, $1 = e_1 + \cdots + e_n$, such that each e_i is fixed by ω . Now B is Morita equivalent to A and basic. There exists a projective A -module $P = P_1 \oplus \cdots \oplus P_m$ with $\{P_i : 1 \leq i \leq m\}$ being mutually non-isomorphic and indecomposable such that $B = \text{End}_A(P)$. As a left A -module, $P \cong Ae$ for some idempotent e that can be chosen to be a sum $e = e_{i_1} + \cdots + e_{i_m}$ of some of the given e_i . So $B = \text{End}_A(P) \cong eAe$. Since each e_i is fixed by ω , e is fixed by ω . As a result, the restriction of the duality ω on A to eAe gives a duality on eAe and hence an induced duality on B . \square

Let M be an A -module. By twisting with ω we turn M into a right A -module, denoted by ${}^\omega M$. Similarly, for a right A -module N , we shall write N^ω for the associated left module. We set ${}^{\natural}M = \text{Hom}_k({}^\omega M, k)$, a left module, and $N^{\natural} = \text{Hom}_k(N^\omega, k)$, a right module. A left module M is said to be *self-dual* if ${}^{\natural}M \cong M$ and a right module N is said to be *self-dual* if $N^{\natural} \cong N$.

For self-dual modules, we have the following construction; see [5]. Let M be a self-dual A -module and $\iota : M \rightarrow {}^{\natural}M$ an isomorphism of left modules. There is an isomorphism of algebras $\alpha : \text{End}_A(M) \rightarrow \text{End}_A({}^{\natural}M)$, which is conjugation by ι , and an anti-isomorphism of algebras $\beta : \text{End}_A(M) \rightarrow \text{End}_A({}^{\natural}M)$, which is

basically precomposition; these maps are respectively defined by

$$\begin{aligned} \alpha(f) &= \iota \circ f \circ \iota^{-1}, & \forall f \in \text{End}_A(M), \\ \beta(f)(m^*)(n) &= m^*(f(n)), & \forall f \in \text{End}_A(M), \quad m^* \in {}^{\natural}M, \quad n \in M. \end{aligned}$$

Let $\tau = \beta^{-1} \circ \alpha$ and consider the non-degenerate bilinear form $(-, -) : M \times M \rightarrow k$ given by $(m, n) = \iota(m)(n)$. We get $(m, \tau(h)n) = \iota(m)(\beta^{-1}\alpha(h)n) = \alpha(h)\iota(m)(n) = \iota h \iota^{-1}(\iota(m))(n) = (hm, n)$ for any $m, n \in M$ and $h \in \text{End}_A(M)$. (To prove the second equality, one may write $\alpha(h) = \beta(h')$ for some h' . Then $\beta^{-1}\alpha(h) = h'$ and on the left hand side of the equation we get $\iota(m)(h'n)$, while on the right hand side we have $\beta(h')\iota(m)(n)$, which by definition of β equals the left hand side.)

Altogether, we have

Proposition 2.4. *Let M be a self-dual A -module. Then there is a non-degenerate bilinear form $(-, -) : M \times M \rightarrow k$ and an anti-isomorphism τ of $\text{End}_A(M)$ such that*

$$(af(m), n) = (am, \tau(f)n) = (f(m), \omega(a)n) \quad \forall a \in A, \quad f \in \text{End}_A(M), \quad m, n \in M.$$

In particular, M is self-dual as an $(A, \text{End}_A(M)^{op})$ -bimodule.

3. CHARACTERIZATIONS AND COMPARISONS OF DOMINANT DIMENSION

In this section we will answer question (1) by characterizing $\text{domdim } A$ in terms of the vanishing of extension groups over A , as well as question (3) by identifying extension groups mapped isomorphically under Schur functors, and also the first part of question (2) about dominant dimensions of A , of standard modules and of the characteristic tilting module. First we fix the class of algebras to be considered.

Definition 3.1. The class \mathcal{A} consists of finite dimensional k -algebras A , split over k and satisfying the following properties:

- (1) A is a quasi-hereditary algebra over a poset X ,
- (2) A has a duality,
- (3) $\text{domdim } A \geq 2$.

As examples, (quantized) Schur algebras $S_q(n, r)$ with $n \geq r$ and block algebras of Bernstein-Gelfand-Gelfand category \mathcal{O} are in \mathcal{A} ; see [13, 15, 19]. Indeed, each such algebra has a set of idempotents fixed by the involution defining the duality and generating the ideals in a heredity chain. Moreover, the idempotent with the largest index is primitive, so by induction one can subtract primitive idempotents in order to find a full set of primitive idempotents fixed by the involution.

On the other hand, it is clear that \mathcal{A} contains all semi-simple k -algebras and is closed under tensor product. In Theorem 4.3, we will see that \mathcal{A} is also closed under taking Ringel-duals.

3.1. Basic properties of algebras in \mathcal{A} . We will be interested in properties of A that are Morita invariant. Thus we shall represent an algebra in the class \mathcal{A} by a tuple (A, X, ω, e) such that $A/\text{rad}(A) \cong k \times \cdots \times k$, ω is the duality, e is an idempotent such that Ae is a basic projective, injective and faithful A -module. Throughout this section, A is of this form.

Lemma 3.2. *Let (A, X, ω, e) be in \mathcal{A} . Then*

- (1) ${}^{\natural}P(x) \cong I(x)$, ${}^{\natural}\Delta(x) \cong \nabla(x)$, ${}^{\natural}L(x) \cong L(x)$, ${}^{\natural}T(x) \cong T(x)$ for each $x \in X$.
- (2) *A projective A -module is injective if and only if it is tilting.*

- (3) Ae is self-dual as an A -module.
- (4) eAe is a symmetric subalgebra of A with a duality τ the restriction of ω to eAe .
- (5) Ae is self-dual as an (A, eAe) -bimodule; i.e., $Ae \cong D({}^\omega(Ae)^\tau)$ as (A, eAe) -bimodules.
- (6) $\text{Hom}_k(eA, k) \cong Ae$ as (A, eAe) -bimodules. eA is self-dual as an (eAe, A) -bimodule.

We note that the duality τ will be seen to be the same as τ in Proposition 2.4, so there is no contradiction in our notation.

Proof. (1), (2) and (3) are consequences of the fact that the anti-automorphism ω identifies the left A -module structure on A with the right A -module structure on ${}^\omega A$.

(4) Let Ae_i be an indecomposable direct summand of Ae . Since Ae_i is projective and injective, Ae_i is tilting. Hence, $Ae_i = T(x)$ for some $x \in X$. Let ae_i be an element of Ae_i which generates the submodule $\Delta(x)$ of $T(x)$. The indecomposable tilting module $Ae_i = T(x)$ has highest weight x and therefore $e(x)Ae_i$ is one-dimensional, where $e(x)$ generates the projective cover $Ae(x)$ of $L(x)$. We may assume that $ae_i \in e(x)Ae_i$, that is, $ae_i = e(x)ae_i$. Moreover, $Ae_i = T(x)$ is self-dual by (1) and it follows that $(ae_i, ae_i) \neq 0$ where $(-, -) : Ae_i \times Ae_i \rightarrow k$ is the non-degenerate bilinear form in Proposition 2.4; indeed, $(e(x)ae_i, e(y)be_i) = 0$ for any b and any $y \neq x$. In particular, as Ae_i is projective and injective, the element $e_i\omega(a)ae_i$ spans the 1-dimensional socle of Ae_i , which is fixed by ω . For any non-zero element be_i in Ae_i , there exists some element $e_i c$ in A such that $e_i bce_i = e_i\omega(a)ae_i$. Therefore,

$$\begin{aligned} (be_i, \omega(c)e_i) &= (e_i, e_i\omega(cb)e_i) = (e_i, e_i\omega(a)ae_i) \\ &= (ae_i, ae_i) = (\omega(c)e_i, be_i) = (e_i, e_i cbe_i). \end{aligned}$$

Note that for any $b, c \in A$, $(be_i, ce_i) = (e_i, e_i\omega(b)ce_i)$ is zero unless $e_i\omega(b)ce_i$ has a summand in the socle of Ae_i ; this summand must be a scalar multiple of $e_i\omega(a)ae_i$, which is fixed by ω . Consequently, $(be_i, ce_i) = (ce_i, be_i)$. Consider the bilinear form $\langle -, - \rangle : eAe \times eAe \rightarrow k$ defined by $\langle eae, ebe \rangle = (eae, e\omega(b)e)$. It is straightforward to check that this bilinear form is associative, symmetric and non-degenerate. So eAe is symmetric with the obvious duality τ , which is the restriction of ω to eAe .

(5) Since Ae is self-dual by (3), we have by Proposition 2.4 that eAe has an anti-automorphism τ' such that for all $a, b, x \in A$, $(ae, b\tau'(exe)) = (ae(exe), be)$. Let $\langle -, - \rangle$ be the symmetric non-degenerate bilinear form on eAe defined in (4). We have

$$\langle eae, \omega\tau'(ebe) \rangle = (eae, \tau'(ebe)) = (ebe, e\omega(a)e) = \langle ebe, eae \rangle = \langle eae, ebe \rangle.$$

Hence $\tau' = \tau$, the restriction of ω to eAe , and by Proposition 2.4 again, Ae is self-dual as an (A, eAe) -bimodule.

(6) Consider the k -linear map $\varphi : eA \rightarrow {}^\omega(Ae)^\tau$ defined by $\varphi(ea) = \omega(a)e$. It is easy to check that φ is indeed an isomorphism of (eAe, A) -bimodules. Therefore by Proposition 2.4 and (5) $\text{Hom}_k(eA, k) \cong \text{Hom}_k({}^\omega(Ae)^\tau, k) \cong Ae$ as (A, eAe) -bimodules. In particular, eA is self-dual as an (eAe, A) -bimodule. \square

In the following, we fix the notation τ to be the restriction of ω to eAe . For an eAe -module N , we can do the same construction as in Section 2.3. For brevity, we shall write ${}^{\natural}N = \text{Hom}_k({}^{\tau}N, k)$ without causing any confusion.

Lemma 3.3. *Let A be in \mathcal{A} . Then for any A -module M ,*

- (1) ${}^{\natural}eM \cong e{}^{\natural}M$ as eAe -modules,
- (2) $\text{Ext}_{eAe}^i(eM, eA) \cong {}^{\omega}\text{Ext}_{eAe}^i(eA, e{}^{\natural}M)$ as right A -modules for each $i \geq 0$,
- (3) $Ae \otimes_{eAe} N \cong {}^{\natural}\text{Hom}_{eAe}(eA, {}^{\natural}N)$ as A -modules for any eAe -module N .

Proof. (1) Consider the pairing $B : {}^{\natural}M \times M \rightarrow k$ defined by $B(f, m) = f(m)$ for any $f \in {}^{\natural}M$ and $m \in M$. Since obviously $B(ef, (1 - e)m) = 0$ and B is non-degenerate, it follows that the restriction of B to $e{}^{\natural}M \times eM$ is also non-degenerate. By definition, $B(exe \cdot f, m) = (exe \cdot f)(m) = f(e\omega(x)em) = f(\tau(exe)m) = B(f, \tau(exe)m)$. Therefore $e{}^{\natural}M \cong {}^{\natural}eM$ as eAe -modules.

By Lemma 3.2(6), eA is self-dual as an (eAe, A) -bimodule, i.e., $\text{Hom}_k({}^{\tau}eA^{\omega}, k) \cong eA$ as (eAe, A) -bimodules. In particular, $\text{Hom}_k(eA, k)^{\tau} \cong eA^{\omega}$ as (eAe, A^{op}) -bimodules. Hence

$$\begin{aligned} \text{Ext}_{eAe}^i(eM, eA) &\cong \text{Ext}_{eAe}^i(\text{Hom}_k(eA, k)^{\tau}, \text{Hom}_k(eM, k)^{\tau}) = {}^{\omega}\text{Ext}_{eAe}^i(eA, e{}^{\natural}M), \\ {}^{\natural}(Ae \otimes_{eAe} {}^{\natural}N) &\cong \text{Hom}_k(Ae \otimes_{eAe} {}^{\natural}N, k)^{\omega} \cong \text{Hom}_{eAe}({}^{\natural}N, eA)^{\omega} \\ &\cong \text{Hom}_{eAe}(\text{Hom}_k(eA, k)^{\tau}, N)^{\omega} \cong \text{Hom}_{eAe}(eA, N) \end{aligned}$$

for any A -module M and any eAe -module N . In particular, (2) and (3) hold. \square

Lemma 3.4. *Let A be as above. Let M be an A -module. If $\text{domdim } {}^{\natural}M \geq 2$, then $\text{DTr } M \simeq \Omega^2(M)$; that is, the Auslander-Reiten translate coincides with the second syzygy.*

Proof. Consider the minimal projective presentation of M :

$$0 \rightarrow \Omega^2(M) \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$

Since $\text{domdim } {}^{\natural}M \geq 2$, it follows that $P_0, P_1 \in \text{add}(Ae)$. Applying $\text{D Hom}_A(-, A)$ to this sequence, we get by the definition of AR-translation (see [2]),

$$0 \rightarrow \text{DTr } M \rightarrow \text{D Hom}_A(P_1, A) \rightarrow \text{D Hom}_A(P_0, A) \rightarrow \text{D Hom}_A(M, A) \rightarrow 0.$$

By Lemma 3.2(6), we have $\text{D Hom}_A(Ae, A) \cong Ae$ and in particular $\text{D Hom}_A(P_i, A) \cong P_i$ for $i = 0, 1$. Moreover, by Lemma 3.3(2) and Theorem 2.2 we have

$$\text{D Hom}_A(M, A) \cong \text{D Hom}_{eAe}(eM, eA) \cong \text{D } {}^{\omega}\text{Hom}_{eAe}(eA, e{}^{\natural}M) \cong \text{D } {}^{\omega}({}^{\natural}M) \cong M.$$

So $\Omega^2(M) \cong \text{DTr } M$, as claimed. \square

3.2. Higher extension groups and dominant dimension. Fix an algebra (A, X, ω, e) in the class \mathcal{A} . We set $X_{sg} = \{x \in X \mid eL(x) = 0\}$ and call its elements *singular*. So, an index is non-singular (= regular) if and only if it corresponds to a simple module isomorphic to the socle of some projective and injective module. We denote by \mathcal{G} the right adjoint of the Schur functor $f = eA \otimes_A -$, which sends M to $\text{Hom}_{eAe}(eA, eM)$ and by $R^i \mathcal{G}$ the i -th derived functor of \mathcal{G} . For each A -module M , let ξ_M be the canonical homomorphism from M to $\text{Hom}_{eAe}(eA, eM)$ via the identification of M and $\text{Hom}_A(A, M)$. We define β_M to be the composition of the A -module isomorphisms (ξ_A is an isomorphism)

$$\text{Hom}_A(M, A) \cong \text{Hom}_A(M, \text{Hom}_{eAe}(eA, eA)) \cong \text{Hom}_{eAe}(eM, eA).$$

Clearly, $\beta_M(f)(em) = ef(m)$ for any $f \in \text{Hom}_A(M, A)$ and $m \in M$. Let δ_M be the composition of the following A -module homomorphisms:

$$M \xrightarrow{\xi_M} \text{Hom}_{eAe}(eA, eM) \xrightarrow{\cong} \text{Hom}_{eAe}(e^{\natural}M, eA)^{\omega} \xrightarrow{\theta} \text{Hom}_A({}^{\natural}M, A)^{\omega},$$

where $\theta = \beta_M^{-1}$ and the inverse of the second isomorphism exists by Lemma 3.3(2). We denote by δ'_M the composition of the first two isomorphisms defining δ_M . It is straightforward to check that δ'_M satisfies: $(\delta'_M(m)(f), ea) = f(eam)$ for any $m \in M, f \in e^{\natural}M$ and $a \in A$. Here $(-, -) : eA \times eA \rightarrow k$ is the bilinear form in Proposition 3.2 and Proposition 2.4. Now we are going to recall some known results.

Lemma 3.5. *Let A be in \mathcal{A} and let N be an A -module.*

- (1) *Then $\text{domdim } N \geq 1$ if and only if ξ_N is a monomorphism; $\text{domdim } N \geq 2$ if and only if ξ_N is an isomorphism.*
- (2) *If $R^i \mathcal{G}(eN) = 0$ for $1 \leq i \leq q$, then for any A -module M , there are canonical isomorphisms $\text{Ext}_A^i(M, \mathcal{G}(eN)) \cong \text{Ext}_{eAe}^i(eM, eN)$ for $0 \leq i \leq q$ and an exact sequence*

$$\begin{aligned} 0 \longrightarrow \text{Ext}_A^{q+1}(M, \mathcal{G}(eN)) &\longrightarrow \text{Ext}_{eAe}^{q+1}(eM, eN) \longrightarrow \text{Hom}_A(M, R^{q+1} \mathcal{G}(eN)) \\ &\longrightarrow \text{Ext}_A^{q+2}(M, \mathcal{G}(eN)) \longrightarrow \text{Ext}_{eAe}^{q+2}(eM, eN). \end{aligned}$$

Proof. For (1), see [22, Theorem 1.2]. For (2), see [13, Proposition 3.1] or [18]. \square

Proposition 3.6. *Let A be in \mathcal{A} and let M be an A -module. Then*

- (1) *$\text{domdim } M \geq n$ if and only if $\text{Ext}_A^i(L(x), M) = 0$ for any $x \in X_{sg}$ and $0 \leq i \leq n - 1$.*
- (2) *Let $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ be a short exact sequence of A -modules. Let $n = \text{domdim } M$ and $n_i = \text{domdim } M_i$ for $i = 1, 2$. Then $n \geq \min(n_1, n_2)$. Moreover,*
 - (a) $n_1 < n \Rightarrow n_2 = n_1 - 1$.
 - (b) $n_1 = n \Rightarrow n_2 \geq n - 1$; $n_1 = n + 1 \Rightarrow n_2 \geq n$; $n_1 \geq n + 2 \Rightarrow n_2 = n$.
 - (c) $n < n_2 \Rightarrow n_1 = n$.
 - (d) $n = n_2 \Rightarrow n_1 \geq n_2$; $n = n_2 + 1 \Rightarrow n_1 \geq n_2 + 1$; $n \geq n_2 + 2 \Rightarrow n_1 = n_2 + 1$.
- (3) *Assume that $\text{domdim } M \leq \text{domdim } A$. Then $\text{domdim } M \geq n \geq 2$ if and only if $\delta_M : M \rightarrow \text{Hom}_A({}^{\natural}M, A)^{\omega}$ is an isomorphism and $\text{Ext}_A^i({}^{\natural}M, A) = 0$ for each $1 \leq i \leq n - 2$.*

Proof. (1) is clear by the definition of dominant dimension in Section 2.2; see also [19].

To prove (2), let $U = \bigoplus_{x \in X_{sg}} L(x)$ and apply $\text{Hom}_A(U, -)$ to the exact sequence above to get a long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}_A^i(U, M_1) \rightarrow \text{Ext}_A^i(U, M) \rightarrow \text{Ext}_A^i(U, M_2) \\ \rightarrow \text{Ext}_A^{i+1}(U, M_1) \rightarrow \text{Ext}_A^{i+1}(U, M) \rightarrow \cdots \end{aligned}$$

By (1), it is clear that $n \geq \min(n_1, n_2)$. If $n_1 < n$, then by the long exact sequence, $\text{Ext}_A^i(U, M_2) = 0$ for $0 \leq i \leq n_1 - 2$ and $\text{Ext}_A^{n_1-1}(U, M_2) \cong \text{Ext}_A^{n_1}(U, M_1) \neq 0$. So $n_2 = n_1 - 1$. Similarly, (b), (c) and (d) hold.

(3) If $\text{domdim } M \geq n \geq 2$, then ξ_M and hence δ_M are isomorphisms. Furthermore, Lemma 3.3(2) combined with Theorem 2.2 yields $\text{Ext}_{eAe}^i(e^{\natural}M, eA) \cong {}^{\omega}\text{Ext}_{eAe}^i(eA, eM) = 0$ for $1 \leq i \leq n - 2$. Note that $\text{domdim } A \geq \text{domdim } M \geq n$. Theorem 2.2 and Lemma 3.5(2) imply $\text{Ext}_A^i({}^{\natural}M, A) \cong \text{Ext}_{eAe}^i(e^{\natural}M, eA) = 0$ for $1 \leq i \leq n - 2$.

Conversely, if δ_M is an isomorphism, then by Lemma 3.3(2), ξ_M being the composition of

$$M \xrightarrow{\delta_M} \text{Hom}_A({}^{\natural}M, A)^{\omega} \xrightarrow{\beta_{\natural}M} \text{Hom}_{eAe}(e^{\natural}M, eA)^{\omega} \cong \text{Hom}_{eAe}(eA, eM)$$

is an isomorphism. So $\text{domdim } M \geq 2$ according to Lemma 3.5(1). If $m := \text{domdim } M < n$, we get by Theorem 2.2 that $\text{Ext}_{eAe}^i(eA, eM) = 0$ for $1 \leq i \leq m - 2$ and $\text{Ext}_{eAe}^{m-1}(e^{\natural}M, eA) \cong \text{Ext}_{eAe}^{m-1}(eA, eM) \neq 0$. Since $\text{domdim } A \geq m$ by assumption, we obtain by Lemma 3.5(2) an exact sequence

$$0 \rightarrow \text{Ext}_A^{m-1}({}^{\natural}M, A) \rightarrow \text{Ext}_{eAe}^{m-1}(e^{\natural}M, eA) \rightarrow \text{Hom}_A({}^{\natural}M, R^m \mathcal{G}(eA)).$$

Note that $\text{domdim } M \geq 2$ implies that $\text{Hom}_A(L(x), M) = 0$ for any $x \in X_{sg}$ and by [11, Proposition 2.6(A)], all composition factors of $R^m \mathcal{G}(eA)$ are of the form $L(x)$ with x singular. So $\text{Hom}_A({}^{\natural}M, R^m \mathcal{G}(eA)) = 0$ and $\text{Ext}_A^{m-1}({}^{\natural}M, A) \cong \text{Ext}_{eAe}^{m-1}(e^{\natural}M, eA) \neq 0$, which contradicts the vanishing of $\text{Ext}_A^i({}^{\natural}M, A)$ for $1 \leq i \leq n - 2$. \square

For any $x \in X$, we define the *depth* of x in X to be the length t of the longest chain $x = x_0 < x_1 < \dots < x_t$ in X and denote it by $\text{dept}(X; x)$. Let $d(X)$ be the maximum of $\text{dept}(X; x)$ for all $x \in X$. Clearly, if $x \in X$ is maximal, then $\text{dept}(X; x) = 0$.

Corollary 3.7. *Let A be in \mathcal{A} and let T be the characteristic tilting A -module.*

- (1) $\text{domdim } A = \min\{\text{domdim } P(x) \mid x \in X\}$.
- (2) $\text{domdim } A \geq n$ if and only if $\text{Ext}_A^i({}^{\natural}A, A) = 0$ for $1 \leq i \leq n - 2$.
- (3) $\text{domdim } T = \min\{\text{domdim } \Delta(x) \mid x \in X\} \leq \text{domdim } A$.
- (4) $\text{domdim } T = n \geq 2$ if and only if δ_M induces an isomorphism $D(T) \cong \text{Hom}_A(T, A)$ and $\text{Ext}_A^i(T, A) = 0$ for $1 \leq i \leq n - 2$.
- (5) For any $x \in X$, $\text{domdim } \Delta(x) \geq \text{domdim } A - \text{dept}(X; x)$. In particular,
 - (a) $\text{Ext}_A^i(\nabla(x), A) = 0$ for $1 \leq i \leq \text{domdim } A - \text{dept}(X; x) - 2$,
 - (b) $\text{Ext}_A^i(\Delta(x), A) = 0$ for $i \geq 1 + \text{dept}(X; x)$.

Proof. (1) is clear by Theorem 2.2. (2) and (4) are special cases of Proposition 3.6(3) since T is self-dual, i.e., $D(T) \cong {}^{\omega}T$.

To prove (3), we first note that by Proposition 3.6(2), $\min\{\text{domdim } T, \text{domdim } A\} \geq \min\{\text{domdim } \Delta(x) \mid x \in X\} := c$. Choose $x \in X$ to be minimal such that $\text{domdim } \Delta(x) = c$ and consider the short exact sequence

$$(1) \quad 0 \longrightarrow \Delta(x) \longrightarrow T(x) \longrightarrow K \longrightarrow 0,$$

where K has a filtration by the standard A -modules $\Delta(y)$ with $y < x$. By the choice of x and by Proposition 3.6(2), it follows that $\text{domdim } K \geq c + 1$ and hence $\text{domdim } T(x) = c$ since $\text{domdim } \Delta(x) = c$ and $\text{domdim } T(x) \geq c$.

(5) We proceed by induction on $x \in X$. If x is maximal, then $\text{dept}(X; x) = 0$ and $\Delta(x)$ is projective. Hence all statements hold true by (1) and Proposition 3.6(3). Suppose the statements hold true for all $y > x$. Consider the exact sequence

$$(2) \quad 0 \longrightarrow Y \longrightarrow P(x) \longrightarrow \Delta(x) \longrightarrow 0,$$

where Y has a filtration by some standard A -modules $\Delta(z)$ with $z > x$. By Proposition 3.6(2), $\text{domdim } Y \geq \text{domdim } A - \text{dept}(X, x) + 1$ and hence $\text{domdim } \Delta(x) \geq \text{domdim } A - \text{dept}(X; x)$. Thus (a) holds by Proposition 3.6(3). For (b), applying $\text{Hom}_A(-, A)$ to the sequence above and using induction, we get that $\text{Ext}_A^i(\Delta(x), A) = 0$ for $i \geq \text{dept}(X; x) + 1$. \square

This characterization of dominant dimension may be compared with the following general characterization of global dimension, which has been used (and proved) in Totaro’s computation [31] of global dimensions of Schur algebras $S(n, r)$ with $n \geq r$. Let A be any algebra of finite global dimension. Then the global dimension of A is the largest integer n such that $\text{Ext}_A^n(I, P) \neq 0$ for some injective module I and some projective module P . The (well-known) proof of this characterization goes as follows. By definition, $\text{gldim}(A)$ is the largest integer n such that $\text{Ext}_A^n(M, N) \neq 0$. Choose an injective envelope of M and a corresponding exact sequence $0 \rightarrow M \rightarrow I \rightarrow C \rightarrow 0$. The long exact cohomology sequence then implies that one of the extension groups $\text{Ext}_A^n(I, N)$ and $\text{Ext}_A^{n+1}(C, N)$ must be non-zero. By n being maximal, we get $\text{Ext}_A^n(I, N) \neq 0$. Now choose a projective cover P of N and apply a similar argument to show that $\text{Ext}_A^n(I, P) \neq 0$.

Corollary 3.8. *Let A be in \mathcal{A} , let T be the characteristic tilting A -module and let i be a positive integer. Then*

- (1) $\text{Ext}_A^i(\natural A, A) \neq 0$ implies $\text{domdim } A - 1 \leq i \leq \text{gldim } A$;
- (2) $\text{Ext}_A^i(T, A) \neq 0$ implies $\text{domdim } T - 1 \leq i \leq \text{projdim } T$.

We note that when A is semi-simple, then $\text{domdim } A = \infty$ and $\text{gldim } A = 0$. On the other hand, if $\text{domdim } A = \infty$, using that $\text{gldim } A < \infty$ (since A is quasi-hereditary), we have by Corollary 3.8(1), $\text{Ext}_A^i(\natural A, A) = 0$ for all $i \geq 1$. Thus $\text{gldim } A = 0$, i.e., A is semisimple. Recall Nakayama’s conjecture (see [32]): if an algebra has infinite dominant dimension, then it is self-injective. So we have shown that *Nakayama’s conjecture holds true for algebras in the class \mathcal{A}* .

For any full subcategory \mathcal{C} , we define $n(\mathcal{C})$ to be the maximal number such that the Schur functor f induces isomorphisms $\text{Ext}_A^i(M, N) \cong \text{Ext}_{eAe}^i(eM, eN)$ for all $M, N \in \mathcal{C}$ and $0 \leq i \leq n(\mathcal{C})$. Sometimes, $n(\mathcal{C})$ is called the *Hemmer-Nakano dimension* of \mathcal{C} .

By \mathcal{P} we denote the full subcategory of $A\text{-mod}$ consisting of all projective A -modules.

Theorem 3.9. *Let (A, X, ω, e) be in the class \mathcal{A} and M be any A -module. Then the Schur functor induces canonical isomorphisms for any projective A -module P and any $K \in \mathcal{F}(\Delta)$:*

$$\begin{aligned} \text{Ext}_A^i(M, P) &\cong \text{Ext}_{eAe}^i(eM, eP), & 0 \leq i \leq \text{domdim } A - 2, \\ \text{Ext}_A^i(M, K) &\cong \text{Ext}_{eAe}^i(eM, eK), & 0 \leq i \leq \text{domdim } T - 2. \end{aligned}$$

Furthermore, there are equalities $n(\mathcal{P}) = \text{domdim } A - 2$ and $n(\mathcal{F}(\Delta)) = \text{domdim } T - 2$.

Proof. This follows by Theorem 2.2, Lemma 3.5, Corollary 3.7 and the non-vanishing of cohomology groups $\text{Ext}_{eAe}^{\text{domdim } A - 1}(eA, eA) \neq 0$, $\text{Ext}_{eAe}^{\text{domdim } T - 1}(eA, eT) \neq 0$. \square

As a consequence of Theorem 3.9, we see that when $\text{domdim } T \geq 2$ (or equivalently $\text{domdim } A \geq 4$ by Theorem 4.3(4) below), the Schur functor f induces an equivalence from $\mathcal{F}(\Delta)$ to the full subcategory $\mathcal{F}(e\Delta)$ of eAe -mod consisting of those modules which are filtered by $\{e\Delta(x) : x \in X\}$, namely

$$f : \mathcal{F}(\Delta) \xrightarrow{\cong} \mathcal{F}(e\Delta).$$

Moreover, this equivalence preserves Ext-groups up to degree $\text{domdim } T - 2$. This extends Hemmer and Nakano’s results [16] from (quantized) Schur algebras to all algebras in \mathcal{A} that have large enough dominant dimension. We have now shown that dominant dimension controls the existence of such relative equivalences and their properties of preserving extensions.

We remark that the dominant dimension of A does not depend on the quasi-hereditary structure. By Theorem 4.3(4) below, the dominant dimension of T is precisely half of that of A , hence also does not depend on the choice of the quasi-hereditary structure.

4. RINGEL DUALITY AND TRUNCATING QUASI-HEREDITARY STRUCTURES

In this section we discuss the behaviour of dominant dimension under two natural operations that produce new quasi-hereditary algebras from given ones; these are Ringel duality, which we show to preserve dominant dimension, and forming quotient algebras modulo heredity ideals. This discussion provides our answer to question (4). We are also going to complete our answer of question (2) by showing that $\text{domdim } A = 2 \text{ domdim } T$, where T is the characteristic tilting module.

4.1. Ringel duality. Under our standard assumptions on A , we will show that the Ringel dual of A also satisfies these conditions.

Fix an algebra (A, X, ω, e) in the class \mathcal{A} . Let T be the characteristic tilting module and $R = \text{End}_A(T)^{op}$ be the Ringel-dual of A , which is quasi-hereditary over X^{op} ; see [25].

The functor $\text{Hom}_A(T, -)$ restricts to an exact equivalence between the subcategories $\mathcal{F}(\nabla_A)$ and $\mathcal{F}(\Delta_R)$, sending injective modules to tilting and tilting to projective modules (see [25, 21]). Moreover, $\text{Hom}_A(T, -)$ induces a derived equivalence, which identifies higher extension groups, too.

Lemma 4.1. *The algebra R has a duality θ such that T is self-dual as an (A, R) -bimodule. The R -module $\text{Hom}_A(T, Ae)$ is projective and injective. Moreover any projective and injective R -module belongs to $\text{add}(\text{Hom}_A(T, Ae))$.*

Proof. By Lemma 3.2(1), each direct summand of T is self-dual as an A -module. Proposition 2.4 then yields that $R = \text{End}_A(T)^{op}$ has a duality θ such that T is self-dual as an (A, R) -bimodule.

The R -module $\text{Hom}_A(T, Ae)$ is projective and injective by Lemma 3.2(2), because it is tilting and projective as the $\text{Hom}(T, -)$ -image of Ae , which is injective and tilting. Conversely, for any projective and injective R -module F , $T \otimes_R F$ is tilting and $\text{Hom}_A(T, T \otimes_R F) \cong F$. Again using that $\text{Hom}_A(T, -)$ is a full embedding of $\mathcal{F}(\nabla_A)$ onto $\mathcal{F}(\Delta_R)$, we get that $T \otimes_R F$ is injective, since it is the preimage of F , and hence it belongs to $\text{add}(Ae)$. So F belongs to $\text{add}(\text{Hom}_A(T, Ae))$. \square

Lemma 4.2. *Let A be a quasi-hereditary algebra with a duality ω . Let M be an A -module. If $\text{projdim } M \leq 1$, then $M \in \mathcal{F}(\Delta)$.*

Proof. Since $\text{projdim } M \leq 1$, we have a projective resolution of M as $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$. If $M \notin \mathcal{F}(\Delta)$, then by definition, $\dim_{\mathcal{F}(\Delta)}(M) = 1$. Hence Theorem 2.1 yields $\text{Ext}_A^2(M, {}^bM) \neq 0$, which contradicts the assumption $\text{projdim } M \leq 1$. \square

Theorem 4.3. *Let (A, X, ω, e) be in the class \mathcal{A} and let T be the characteristic tilting module. Then:*

- (1) R has characteristic tilting module $D(T)$; $\text{domdim}_A T \geq 1$ and $\text{domdim}_R D(T) \geq 1$;
- (2) $R = \text{End}_A(T)^{op}$ belongs to \mathcal{A} ;
- (3) $\text{domdim}_R D(T) = \text{domdim}_A T$;
- (4) $\text{domdim } A = 2 \text{ domdim}_A T$. In particular, $\text{domdim } A = \text{domdim } R$.

(Here and in general we set $2 \times \infty = \infty$.)

Proof. All statements hold true when A is semisimple. Next we assume that A is not semisimple.

(1) $\text{domdim } A \geq 1$ yields a short exact sequence $0 \rightarrow A \rightarrow E \rightarrow K \rightarrow 0$ with $E \in \text{add}(Ae)$. Clearly $\text{projdim } K = 1$; hence by Lemma 4.2, $K \in \mathcal{F}(\Delta_A)$. Since $D(T)$ is the characteristic tilting R -module and $D(T) \otimes_A -$ induces a full embedding of $\mathcal{F}(\Delta_A)$ onto $\mathcal{F}(\nabla_R)$ (see [21]), we obtain the exact sequence $0 \rightarrow D(T) \rightarrow D(T) \otimes_A E \rightarrow D(T) \otimes_A K \rightarrow 0$, and $D(T) \otimes_A E$ is projective, injective and faithful. In particular, $\text{domdim } R \geq 1$ and $\text{domdim}_R D(T) \geq 1$. Since R also has a duality, the above arguments, but with A and R exchanging their roles, yield $\text{domdim}_A T \geq 1$.

(2) By Lemma 4.1, it remains to show that $\text{domdim } R \geq 2$. From (1), we see that $\text{domdim } R \geq 1$ and $\text{domdim}_R D(T) \geq 1$. Let F be the injective hull of R . It follows that F is projective, injective and faithful (since $D(T)$ is faithful). Moreover, by Lemma 4.2 the quotient F/R belongs to $\mathcal{F}(\Delta_R)$ and hence has dominant dimension at least $\text{domdim}_R D(T) \geq 1$ (note that the arguments used in the previous section to establish this inequality only use assumptions valid here). As a result, $\text{domdim } R \geq 2$.

(3) First we consider the case that $\text{domdim}_A T \geq 2$. To compute $\text{domdim}_R D(T)$, we note that $\text{domdim } A \geq 2$ yields a short exact sequence $0 \rightarrow A \rightarrow E_0 \rightarrow E_1$ with $E_0, E_1 \in \text{add}(Ae)$. Applying $\text{Hom}_A(T, -)$ to this sequence and using that $\text{Hom}_A(T, A) \cong D(T)$ by Corollary 3.7(4) and that $\text{Hom}_A(T, Ae)$ is projective and injective over R , we get $\text{domdim}_R D(T) \geq 2$. Moreover $D(A) = {}^bA$ and there are isomorphisms

$$\begin{aligned} \text{Ext}_R^i(D(T), R) &\cong \text{Ext}_R^i(\text{Hom}_A(T, D(A)), \\ \text{Hom}_A(T, T)) &\cong \text{Ext}_A^i({}^bA, T) \cong \text{Ext}_A^i(T, A). \end{aligned}$$

So by Corollary 3.7(4) again, $\text{domdim}_A T = \text{domdim}_R D(T)$.

In the second case, $\text{domdim}_A T = 1$, we will show $\text{domdim}_R D(T) < 2$ by (1) and (2). Indeed, by the same arguments as above, $\text{domdim}_R D(T) \geq 2$ would imply $\text{domdim}_A T = \text{domdim}_R D(T) \geq 2$.

(4) Since the Ringel-dual of R is isomorphic to A (when choosing the right multiplicities of direct summands) and $\text{domdim}_A T = \text{domdim}_R D(T)$ by (3), it suffices to show $\text{domdim } R = 2 \text{ domdim } T$. We will first prove that $\text{domdim } R \geq 2 \text{ domdim } T$. Let $\text{domdim}_A T = n$. Consider a minimal injective resolution of T ,

$$0 \rightarrow T \rightarrow E_0 \rightarrow \cdots \rightarrow E_{n-2} \xrightarrow{\pi} E_{n-1} \rightarrow I_n \rightarrow \cdots,$$

where $E_i \in \text{add}(Ae)$. Since $\mathcal{F}(\nabla_A)$ is closed under cokernels of monomorphisms, it follows by induction that $\text{cok}(\pi) \in \mathcal{F}(\nabla_A)$. As $\text{Hom}_A(T, -)$ is exact on $\mathcal{F}(\nabla_A)$, we get

$$0 \longrightarrow R \longrightarrow \text{Hom}_A(T, E_0) \longrightarrow \cdots \longrightarrow \text{Hom}_A(T, E_{n-1}) \longrightarrow \text{Hom}_A(T, \text{cok}(\pi)) \longrightarrow 0$$

with $\text{Hom}_A(T, \text{cok}(\pi)) \in \mathcal{F}(\Delta_R)$. Combining Proposition 3.6(2) and Corollary 3.7(3) implies $\text{domdim}_R \text{Hom}_A(T, \text{cok}(\pi)) \geq \text{domdim}_R D(T) = \text{domdim}_A T = n$ and in particular $\text{domdim } R \geq 2n$.

Next, we will prove that $\text{domdim } R \leq 2n$. Let $M = \text{Hom}_A(T, \text{cok}(\pi)) \in \mathcal{F}(\Delta_R)$ and consider a minimal injective presentation of M :

$$(*) \quad 0 \longrightarrow M \longrightarrow F \longrightarrow N \longrightarrow 0.$$

Since $n \geq 1$ by (1), we get $\text{domdim}_R M \geq 1$. So F is a projective and injective R -module. We claim that $N \notin \mathcal{F}(\Delta_R)$. Otherwise, applying $T \otimes_R -$ to the sequence (*) and using that $T \otimes_R -$ is exact on $\mathcal{F}(\Delta_R)$ and inverse to $\text{Hom}_A(T, -)$, we get an exact sequence

$$0 \longrightarrow \text{cok}(\pi) \longrightarrow T \otimes_R F \longrightarrow T \otimes_R N \longrightarrow 0$$

with $T \otimes_R F \in \text{add}(Ae)$ by Lemma 4.1. So $\text{domdim}_A \text{cok}(\pi) \geq 1$, which in turn implies that $\text{domdim}_A T \geq n + 1$, a contradiction.

On the other hand, by (*), $\dim_{\mathcal{F}(\Delta)}(N) \leq 1$. So $\dim_{\mathcal{F}(\Delta)}(N) = 1$ and then by Theorem 2.1, we get that $\text{Ext}_R^2(N, {}^{\natural}N) \neq 0$. In particular, $\text{Ext}_R^1(N, {}^{\natural}M) \cong \text{Ext}_R^1(M, {}^{\natural}N) \neq 0$.

We now consider the case $n \geq 2$. Then $\text{domdim } {}^{\natural}M \geq 2$ and by Lemma 3.4,

$$\text{Ext}_R^{2n-1}(D(R), R) \cong \text{Ext}_R^{n-1}({}^{\natural}M, R) \cong \text{Ext}_R^1({}^{\natural}M, \Omega^2 M) \cong \text{Ext}_R^1({}^{\natural}M, D\text{Tr } M).$$

Using the Auslander-Reiten formula [2] and noting that M is not projective, we have

$$\begin{aligned} \text{Ext}_R^1({}^{\natural}M, D\text{Tr } M) &\cong D \overline{\text{Hom}}_R(D\text{Tr } M, D\text{Tr } {}^{\natural}M) \\ &\cong D \underline{\text{Hom}}_R(M, {}^{\natural}M) \cong D \text{Ext}_R^1(N, {}^{\natural}M) \neq 0. \end{aligned}$$

Hence $\text{Ext}_R^{2n-1}(D(R), R) \neq 0$ and by Corollary 3.7(2), $\text{domdim } R \leq 2n$.

We are left with the second case, $n = 1$, and we have to show that in this case $\text{domdim } R = 2$ when $n = 1$. Indeed, if $\text{domdim } R \geq 3$, we get a presentation of N , namely $0 \rightarrow N \rightarrow F' \rightarrow L \rightarrow 0$ with F' being projective and injective. Then $\text{Ext}_R^2(N, {}^{\natural}N) \neq 0$ yields $\text{Ext}_R^3(N, {}^{\natural}L) \neq 0$, which is a contradiction to $\text{projdim } N \leq 2$. □

The first statement in (4) is our analogue of Theorem 2.1 by Mazorchuk and Ovsienko.

Corollary 4.4. *For any $x \in X$, the standard module $\Delta(x)$ has dominant dimension at least 1. In particular, all standard A -modules are torsionless (i.e., submodules of projectives).*

This generalizes results of James and of Donkin [10, 17] on the (quantum) Weyl modules of (quantum) Schur algebras being torsionless; see also [16]. For blocks of category \mathcal{O} , it is well known that all Verma modules are submodules of the projective Verma module.

Corollary 4.5. *If $\text{domdim } A \geq d(X) + 2 + s$ for some $s \geq 0$, then $n(\mathcal{F}(\Delta)) \geq s$.*

Proof. By a well-known result of Dlab and Ringel [7], $\text{gldim } A \leq 2d(X)$. So by Corollary 3.8, Theorem 4.3 and Theorem 3.9, we have $d(X) \geq s+1$ and $n(\mathcal{F}(\Delta)) = \text{domdim } T - 2 = 1/2 \text{ domdim } A - 2 \geq (d(X) + s + 2)/2 - 2 \geq (2s + 3)/2 - 2 = s - 1/2$. \square

For algebras in \mathcal{A} , this corollary recovers the (more general) main theorem in [13].

Copying the proof of (4) for $T(x)$ instead of T we get a finer inequality.

Corollary 4.6. *Let T be the characteristic tilting A -module and let R be the Ringel dual of A . Then $\text{domdim } P_R(x) \geq \text{domdim } T_A(x) + \text{domdim}_A T$. In particular, if $\text{domdim } P_R(x) = \text{domdim } A$, then $\text{domdim } T_A(x) = \text{domdim } T = \text{domdim } \Delta_A(x)$.*

4.2. Truncation process. Fix an algebra (A, X, ω, e) in the class \mathcal{A} . Let x_{\max} be a maximal element in the poset X and let I be the corresponding heredity ideal of A . So, A/I again is a quasi-hereditary algebra and we wish to compare its dominant dimension with that of A . In general, the quotient algebra A/I may not have a faithful projective and injective module, so it may not be in the class \mathcal{A} . For instance, let A be the block algebra of the principal block of the Bernstein-Gelfand-Gelfand category \mathcal{O} of the complex simple Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Then $\text{domdim } A = 2$ and $\text{domdim } A/I = 0$; see e.g. [13]. On the other hand, as we shall see shortly, this problem only happens when $\text{domdim } A = 2$. Indeed, we will show that when truncating, the dominant dimension either is left unchanged or it drops by two.

To proceed, we observe first that the duality ω on A induces a duality on the quasi-hereditary algebra A/I . Since $\Delta(x_{\max})$ can be embedded into a projective and injective module, the tilting module $T(x_{\max})$ must be projective and injective for which we write $P(z) = I(z)$ for some index z . As before, we set $X_{sg} = \{x \in X \mid eL(x) = 0\}$. Given an A -module M , we denote by $\text{hd}(M)$ and $\text{soc}(M)$ respectively its head and socle and by $[M : L(x)]$, we denote the composition multiplicity of $L(x)$ in M .

Lemma 4.7. *With the notation as above, $e\Delta(x_{\max}) = eL(z)$.*

Proof. Since $I(z) = T(x_{\max})$, we get $[\Delta(x_{\max}) : L(z)] = \dim_k \text{Hom}_A(\Delta(x_{\max}), I(z)) = \dim_k \text{Hom}_A(\Delta(x_{\max}), T(x_{\max})) = 1$. Moreover, $\text{soc}(\Delta(x_{\max})) \subseteq \text{soc}(T(x_{\max})) = L(z)$. It follows that $e\Delta(x_{\max}) = eL(z)$ by the self-duality of $e\Delta(x_{\max}) = eP(x_{\max}) \subseteq eA$ as an eAe -module. \square

Corollary 4.8. *Let P be a projective A -module. Then $L(z) \subseteq \text{soc}(P)$ if and only if $\Delta(x_{\max}) \subseteq P$.*

Proof. If $\Delta(x_{\max}) \subseteq P$, then by $\text{soc}(\Delta(x_{\max})) = L(z)$, we get $L(z) \subseteq P$. Conversely, consider the short exact sequence $0 \rightarrow L(z) \rightarrow P \rightarrow N \rightarrow 0$. Applying the Schur functor f and then its right adjoint \mathcal{G} to the sequence, we get a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L(z) & \longrightarrow & P & \longrightarrow & N \longrightarrow 0 \\
 & & \xi_{L(z)} \downarrow & & \xi_P \downarrow & & \xi_N \downarrow \\
 0 & \longrightarrow & \mathcal{G}(eL(z)) & \longrightarrow & \mathcal{G}(eP) & \longrightarrow & \mathcal{G}(eN) \longrightarrow \dots
 \end{array}$$

By Lemma 4.7 and $\text{domdim } A \geq 2$, $\mathcal{G}(eP) \cong P$ and $\mathcal{G}(eL(z)) \cong \mathcal{G}(e\Delta(x_{\max})) \cong \Delta(x_{\max})$. In particular, $\Delta(x_{\max}) \subseteq P$. \square

Theorem 4.9. *Let (A, X, ω, e) be an indecomposable (= connected) algebra in the class \mathcal{A} .*

- (1) *If $T(x_{\max}) = \Delta(x_{\max})$, then A is simple and isomorphic to k .*
- (2) *If $T(x_{\max})/\Delta(x_{\max})$ is tilting, then $\text{domdim } A = 2$ and $\text{domdim } A/I = 2$ or ∞ .*
- (3) *If $\text{domdim } A = n \geq 4$, then $\text{domdim}_{A/I} A/I = n$ or $n - 2$.*

Proof. (1) An equality $T(x_{\max}) = \Delta(x_{\max})$ implies that $\Delta(x_{\max})$ is projective and injective. Hence, $\Delta(x_{\max}) = L(x_{\max})$ gives an algebra direct summand of A .

(2) Let $T(x_{\max})/\Delta(x_{\max}) = T(x)$ for some $x \in X$. Consider the short exact sequence $0 \rightarrow \Delta(x_{\max}) \rightarrow T(x_{\max}) \rightarrow T(x) \rightarrow 0$ and apply $\text{Hom}_A(\Delta(x), -)$ to it. We get

$$0 \longrightarrow \text{Hom}_A(\Delta(x), \Delta(x_{\max})) \longrightarrow \text{Hom}_A(\Delta(x), T(x_{\max})) \\ \longrightarrow \text{Hom}_A(\Delta(x), T(x)) \longrightarrow \text{Ext}_A^1(\Delta(x), \Delta(x_{\max})) \longrightarrow 0$$

On the other hand, the dimension (over $\text{End}_A(\Delta(x))$ of $\text{Hom}_A(\Delta(x), T(x_{\max}))$ is given by $[T(x_{\max}) : \nabla(x)] = [T(x) : \nabla(x)] = 1$, which also gives the dimension of $\text{Hom}_A(\Delta(x), T(x))$. So $\dim_k \text{Ext}_A^1(\Delta(x), \Delta(x_{\max})) = \dim_k \text{Hom}_A(\Delta(x), \Delta(x_{\max}))$ and the extension space is non-zero by the universal construction of $T(x_{\max})$, see [25], since x is one of the nearest neighbours of x_{\max} . Since $T(x)$ is self-dual with the head and hence the socle $\text{hd}(T(x_{\max})) = L(z)$, we have $\text{soc}(\Delta(x)) = L(z)$. Combining this fact with the equality $[\Delta(x) : L(z)] = [P(z) : \Delta(x)] = [T(x_{\max}) : \Delta(x)] = 1$ implies that $\Delta(x)$ is embedded into $\Delta(x_{\max})$. In particular, $e\Delta(x) = e\Delta(x_{\max}) = eL(z)$ by Lemma 4.7. If $\text{domdim } \Delta(x_{\max}) = 2$, then $\text{domdim } A = 2$ by Corollary 3.7(1). If $\text{domdim } \Delta(x_{\max}) \geq 3$, then by Lemma 3.5(2), we get $\text{Ext}_{eAe}^1(e\Delta(x_{\max}), e\Delta(x_{\max})) \cong \text{Ext}_{eAe}^1(e\Delta(x), e\Delta(x_{\max})) \cong \text{Ext}_A^1(\Delta(x), \Delta(x_{\max})) \neq 0$ and hence a contradiction.

Next, we compute the dominant dimension of the quotient algebra A/I . Note that by Lemma 3.2(2), the projective and tilting A/I -module $T(x) = T(x_{\max})/IT(x_{\max})$ is injective. Given an indecomposable projective and injective module P different from $T(x_{\max})$, it is clear that $IP = 0$ since P is tilting. As a result, Ae/Ie is a projective, injective and faithful A/I -module and A/I is a quasi-hereditary algebra on $Y := X - \{x_{\max}\}$ with an induced duality from that of A and it has dominant dimension at least 1.

To continue, we need the following injective resolution of $\Delta(x_{\max})$ deduced from the exact sequence $0 \rightarrow \Delta(x_{\max}) \rightarrow T(x_{\max}) \rightarrow T(x) \rightarrow 0$ and the dual sequence,

$$0 \longrightarrow \Delta(x_{\max}) \xrightarrow{\alpha} T(x_{\max}) \xrightarrow{\quad\quad\quad} T(x_{\max}) \longrightarrow \nabla(x_{\max}) \longrightarrow 0 \\ \qquad\qquad\qquad \searrow \qquad\qquad\qquad \nearrow \beta \\ \qquad\qquad\qquad T(x)$$

The two long exact cohomology sequences associated with the two parts of this sequence combined with the facts that $T(x_{\max})$ is projective and injective, that

$\Delta(x_{\max})$ is projective and that $\nabla(x_{\max})$ is injective determine the following extension groups:

$$(3) \quad \text{Ext}_A^1(L(w), \Delta(x_{\max})) \neq 0 \iff w = z,$$

$$(4) \quad \text{Ext}_A^2(L(w), \Delta(x_{\max})) \neq 0 \iff w = x_{\max},$$

$$(5) \quad \text{Ext}_A^3(L(w), \Delta(x_{\max})) = 0 \quad \forall w \in X.$$

To see that A/I has dominant dimension at least 2, for each $y \in Y_{sg} = X_{sg} - \{x_{\max}\}$, we apply $\text{Hom}_A(L(y), -)$ to the short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ and get

$$0 = \text{Ext}_A^1(L(y), A) \rightarrow \text{Ext}_A^1(L(y), A/I) \rightarrow \text{Ext}_A^2(L(y), I) \rightarrow \dots$$

Note that I is a direct sum of copies of $\Delta(x_{\max})$ and $\text{Ext}_{A/I}^i(M, N) \cong \text{Ext}_A^i(M, N)$ for any A/I -modules M and N and any $i \geq 0$. We have $\text{Ext}_{A/I}^1(L(y), A/I) \cong \text{Ext}_A^1(L(y), A/I) = 0$. So by Proposition 3.6(1), $\text{domdim}_{A/I} A/I \geq 2$.

We divide the remaining part of the proof into two cases.

Case 1. $x = z$. By the arguments above, $\text{soc}(\Delta(x)) = L(z)$ and $\text{hd}(\Delta(x)) = L(x)$ and $[\Delta(x) : L(x)] = 1$; thus $x = z$ implies $\Delta(z) = L(z)$. $T(z) = T(x_{\max})/\Delta(x_{\max})$ has the simple head $L(z)$, since $T(x_{\max}) = P(z)$. But $T(z)$ is self-dual and has only one composition factor $L(z)$, so it must equal $L(z)$. Since $T(x_{\max})$ is self-dual, $P(x_{\max})$ and $P(z)$ have the following Loewy series:

$$P(x_{\max}) = \Delta(x_{\max}) = \begin{matrix} L(x_{\max}) \\ L(z) \end{matrix}, \quad P(z) = T(x_{\max}) = \begin{matrix} L(z) \\ L(x_{\max}) \\ L(z) \end{matrix}.$$

By assumption, A is indecomposable, so $A = P(x_{\max}) \oplus P(z)$ and $A/I \cong k$.

Case 2. $x \neq z$. Since $\Delta(x)$ is embedded into $\Delta(x_{\max})$, by Lemma 4.7, we have $x \in Y_{sg}$. Let $W = \Delta(x_{\max})/\Delta(x)$. Then all composition factors of W are singular again by Lemma 4.7. If $\text{soc}(W) = L(x_{\max})$, then $W = L(x_{\max})$ and $\text{Ext}_A^1(L(x_{\max}), L(x)) \neq 0$. Apply $\text{Hom}_A(L(x), -)$ to the short exact sequence $0 \rightarrow \Delta(x) \rightarrow \Delta(x_{\max}) \rightarrow L(x_{\max}) \rightarrow 0$ and note that $\text{Ext}_A^i(L(x), \Delta(x_{\max})) = 0$ for $i = 1, 2$ by the equations (3), (4) above. We have $\text{Ext}_A^2(L(x), \Delta(x)) \cong \text{Ext}_A^1(L(x), L(x_{\max})) \neq 0$. Applying $\text{Hom}_A(L(x), -)$ to the short exact sequence $0 \rightarrow \Delta(x) \rightarrow T(x) \rightarrow T(x)/\Delta(x) \rightarrow 0$ and noting $T(x)$ is projective and injective as an A/I -module, we get $\text{Ext}_{A/I}^1(L(x), T(x)/\Delta(x)) \cong \text{Ext}_A^1(L(x), T(x)/\Delta(x)) \cong \text{Ext}_A^2(L(x), \Delta(x)) \neq 0$. Hence by Proposition 3.6(2), Corollary 3.7(3) and Theorem 4.3(4), $\text{domdim}_{A/I} A/I = 2$.

If $\text{soc}(W) \neq L(x_{\max})$, there exists $u \in Y_{sg}$ such that $L(u)$ is in the socle of W and $\text{Ext}_A^1(L(u), \Delta(x)) \neq 0$. In particular, $\text{Ext}_{A/I}^1(L(u), \Delta(x)) \neq 0$. So $\text{domdim}_{A/I} A/I = 2$ by Proposition 3.6(2), Corollary 3.7(3) and Theorem 4.3(4).

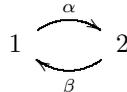
To prove (3), note that I being projective and $\text{domdim} A = n \geq 4$ implies that $\text{domdim}_A A/I \geq n-1 \geq 3$ by Proposition 3.6(2). So, by Lemma 4.7 and Lemma 3.5,

$$\begin{aligned} \text{Hom}_A(L(z), A/I) &\cong \text{Hom}_A(eL(z), e(A/I)) = \text{Hom}_A(e\Delta(x_{\max}), e(A/I)) \\ &\cong \text{Hom}_A(\Delta(x_{\max}), A/I) = 0, \\ \text{Ext}_A^i(L(z), A/I) &\cong \text{Ext}_A^i(eL(z), e(A/I)) = \text{Ext}_A^i(e\Delta(x_{\max}), e(A/I)) \\ &\cong \text{Ext}_A^i(\Delta(x_{\max}), A/I) = 0 \quad (1 \leq i \leq n-3). \end{aligned}$$

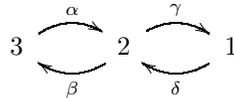
As a result, the composition factors of $\text{soc}(A/I)$ as an A -module are of the form $L(u)$ with $u \in X - X_{sg} - \{z\}$ and by Proposition 3.6(1), $\text{Ext}_{A/I}^i(L(x), A/I) \cong \text{Ext}_A^i(L(x), A/I) = 0$ for all $x \in X_{sg} \cup \{z\} - \{x_{\max}\}$ and $1 \leq i \leq n - 3$. Again using Proposition 3.6(1), it follows that the direct sum of tilting modules, whose head is indexed by $w \in X - X_{sg} - \{z\}$ forms a projective, injective and faithful A/I -module (note that $T(x_{\max})$ does not occur, since its head is indexed by z) and $\text{domdim}_{A/I} A/I \geq n - 2$.

Note that $T' = \bigoplus_{w \neq x_{\max}} T(w)$ is the characteristic tilting A/I -module and $T(x_{\max})$ is both projective and injective; hence it has infinite dominant dimension. We have by Theorem 4.3(4), $\text{domdim} A = 2 \text{domdim}_A T'$ and $\text{domdim}_{A/I} T' \geq \text{domdim}_A T' - 1$. On the other hand, by the above description of a projective, injective and faithful A/I -module as a direct sum of tilting modules, it is clear that $\text{domdim}_{A/I} T' \leq \text{domdim}_A T'$. So $\text{domdim}_{A/I} A/I = n$ or $n - 2$. \square

We remark that in the theorem above, both cases of (2) can happen. For example, let A be the k -algebra given by quiver



and relations $\beta\alpha = 0$. Then A is a quasi-hereditary algebra on $\{1 > 2\}$ with the obvious duality sending α to β . By direct computations, $\text{domdim} A = 2$ and $A/I \cong k$; see also [13]. Let B be the k -algebra given by quiver



and relations $\beta\alpha = 0, \delta\gamma\alpha = \beta\delta\gamma = 0, \alpha\beta = \delta\gamma$. The Loewy series of the indecomposable B -modules are

$$P(3) = \begin{array}{c} 3 \\ 2 \\ 1 \end{array}, \quad P(2) = \begin{array}{ccc} 2 & & \\ 3 & 1 & \\ 2 & & \\ 1 & & \end{array}, \quad P(1) = \begin{array}{ccc} 1 & & \\ 2 & & \\ 3 & 1 & \\ 2 & & \\ 1 & & \end{array}.$$

It is straightforward to check that B is a quasi-hereditary algebra over the poset $\{3 > 2 > 1\}$ with the obvious duality sending α to β and γ to δ . Moreover, $\text{domdim} B = 2$ and $B/I \cong A$.

5. DOMINANT DIMENSION OF SCHUR ALGEBRAS $S(n, r)$, $n \geq r$

The results of the previous sections indicate that it is of interest to determine explicitly the dominant dimension of classes of quasi-hereditary algebras with a duality. The two fundamental examples are Schur algebras and blocks of category \mathcal{O} . The dominant dimension of blocks of category \mathcal{O} has been shown to equal two (unless the block is a simple algebra) in [13]. Schur algebras $S(n, r)$ with $n \leq r$ do not in general have a faithful projective and injective module (and in this case, the results of [19] suggest developing a relative theory of dominant dimension based on tilting modules instead of using the classical theory). Therefore, we are left with Schur algebras $S(n, r)$ with $n \geq r$. In this section, we shall compute the dominant

dimension of $S_k(n, r)$ for $n \geq r$. (We assume $r \geq p \neq 0$, since otherwise the Schur algebra is semisimple.) It will turn out that this dominant dimension only depends on the characteristic p of the underlying field and not on n or r (as long as $n \geq r$). This answers question (5); this answer is quite different in nature from the known results on global dimension of Schur algebras, which heavily depend on n or r and not just on p .

We first recall some basic information about Schur algebras. Let p be the characteristic of the field k and let E be an n -dimensional k -vector space. Let Σ_r be the symmetric group on r letters. Consider the natural action of $GL(E)$ on $E^{\otimes r}$ (r -tensor space) from the left and the action of Σ_r on $E^{\otimes r}$ from the right by place permutations. By definition, the Schur algebra $S_k(n, r) = \text{End}_{\Sigma_r}(E^{\otimes r})$. See [10, 15] for more information on classical or quantized Schur algebras and their representation theory.

Theorem 5.1. *Suppose $n \geq r \geq p$. Then $\text{domdim } S_k(n, r) = 2(p - 1)$.*

Proof. We fix $n \geq r \geq p$. We divide the proof into two parts, dealing with lower and upper bounds separately.

First part: $2(p - 1)$ is a lower bound.

We will show that $2(p - 1)$ is a lower bound for the dominant dimension of $S(n, r)$.

First we note that all Schur algebras $S(n, r)$ with $n \geq r$ satisfy a double centralizer property and thus have dominant dimension at least two. Indeed, in this case, the tensor space $E^{\otimes r}$ is a faithful projective and injective module; hence Schur-Weyl duality is the required double centralizer property.

Suppose $p = 2$. Then the observation we just made gives $2 = 2(p - 1)$ as a lower bound.

Suppose $p = 3$. Then we need four to be a lower bound. This is implied by known results on the cohomology of Schur algebras or symmetric groups. For instance, we may argue as follows. By Lemma 3.5(1), we have to check that ξ_T is an isomorphism. Since the algebra has dominant dimension at least two, we know already that ξ_T is injective. We need to show that $\xi_T : T \rightarrow \text{Hom}_{eAe}(eA, eT)$ also is surjective. Doty and Nakano's theorem [12, Sec. 6.2] states that, in our notation, $\mathcal{G}(eT) = T$. Thus maps can be lifted, and ξ_T is surjective.

Suppose $p \geq 5$. Again we claim that $2(p - 1)$ is a lower bound for the dominant dimension of $A = S_k(n, r)$. This follows from Theorem 2.2 and Hemmer-Nakano's result [16, Corollary 3.9.2], based on work of Kleshchev and Nakano [18], which states that the Schur functor preserves extension groups of Weyl filtered modules in degrees up to $p - 2$. Alternatively, we may use the results on the vanishing of group cohomology of symmetric groups given in [18] (which, by our results, again imply the result of Hemmer and Nakano).

Second part: $2(p - 1)$ is an upper bound.

In order to prove equality, we need to establish $2(p - 1)$ as an upper bound by finding a non-zero self-extension in degree $2(p - 1) - 1$ of the module eA over eAe .

By [15], the Schur algebras $S(n, r)$ (with $n \geq r$) and $S(r, r)$ are Morita equivalent; thus we may assume $n = r$.

The endomorphism ring eAe equals the group algebra $k\Sigma_r$ of the symmetric group and Ae as an eAe -module is a direct sum of permutation modules taken with respect to Young subgroups; the regular module and the trivial module are among the direct summands. By Theorem 2.2, it is enough to find a non-vanishing

self-extension of eA over eAe in the correct degree $2(p-1)-1$. We will find such a self-extension of some direct summand of eA .

Suppose for a moment that $n = r = p$. Then the group algebra $k\Sigma_p$ is a direct sum of the principal block and some simple blocks. The simple modules of $k\Sigma_p$ are indexed by p -regular partitions of p . The p -core of a partition λ is λ itself, unless λ is a hook, and then the core is the empty partition, and λ has p -weight one. Thus, all the hook partitions (except for the one p -singular partition) belong to one block, the principal block, and all other partitions belong to simple blocks. Since p divides the group order exactly once, the principal block is a Brauer tree algebra, and its Brauer tree is known to be a line. There are exactly p regular hooks, so the tree has p edges, one of which belongs to the trivial module; this edge is situated at one end of the tree. Projective resolutions of cell modules of Brauer tree algebras are given by Green's walk around the Brauer tree (see, for instance, Section 4.18 in [4]), and the trivial module is such a cell module. Thus a non-trivial self-extension exists precisely in those degrees when the walk meets the starting vertex again, that is, in degree 0, $2(p-1)-1$, $2(p-1)$, $4(p-1)-1$, $4(p-1)$, and so on. Thus our result follows in this case.

Now we return to the general case and reduce it to the special one just finished, by using the Mackey decomposition theorem and Frobenius reciprocity. We are going to compare extensions of modules over the group $G = \Sigma_r$ with extensions over the Young subgroup $H = \Sigma_p \times \Sigma_1 \times \cdots \times \Sigma_1 \cong \Sigma_p$. Since kG is free over kH , there is a homological version of the Mackey decomposition theorem (see [4], Corollary 3.3.5): $\text{Ext}_{kG}^n(kG \otimes_{kK} M, kG \otimes_{kH} N) = \bigoplus_{HgK} (\text{Ext}_{H \cap {}^g K}^n({}^g M, N))$, where K and H are subgroups of G , M and N are modules over kK and kH , respectively, g runs through double cosets, and on the right hand side, M and N are restricted to the subgroup $H \cap {}^g K$. We set $K = H$, the Young subgroup just fixed, and we set $M = N = kG \otimes_H k$, a permutation module that is isomorphic to a direct summand of eA , since H is a Young subgroup of G . Then we get that the self-extension of k that we just found in the special case induces a non-zero self-extension of M in the same degree. \square

As we remarked already in the proof, we do not need to use the result of Hemmer and Nakano; it is sufficient to use Kleshchev and Nakano's result on the vanishing of cohomology over symmetric groups. Theorem 3.9 then recovers the result by Hemmer and Nakano. We note that our result also extends the theorem of Hemmer and Nakano to the small prime numbers two and three, excluded in [16]. In characteristic two, our result has just the negative consequence that there are no equalities of homomorphisms beyond Schur-Weyl duality, and no equalities of extension spaces. In characteristic three (when T has dominant dimension two) we do, however, get equalities of homomorphism spaces, and we have shown that this is best possible in the sense that there are no equalities of extension spaces.

It is also possible to completely avoid in the above proof the use of results on the cohomology of symmetric groups, and instead to use the cohomology of general linear groups and thus of Schur algebras. By results of Donkin [9], the tilting module T is a direct sum of tensor powers of exterior powers of the natural module. Tensor powers of symmetric powers of the natural module provide a full set of injective modules over the Schur algebra. It is possible to get the required vanishing and non-vanishing of higher extensions between injective modules and tilting modules

by first using results of Akin [1] and then of Franjou, Friedlander, Scorichenko and Suslin [14].

We leave it as an open problem to determine the dominant dimension of quantized Schur algebras.

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