ON THE ZEROS OF FUNCTIONS IN DIRICHLET-TYPE SPACES

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Abstract. We study the sequences of zeros for functions in the Dirichlet spaces $D_s$. Using Carleson-Newman sequences we prove that there are great similarities for this problem in the case $0 < s < 1$ with that for the classical Dirichlet space.

1. Introduction and main results

The problem of describing the zero sets for the Dirichlet-type spaces $D_s$ is an old one, and to the best of our knowledge, is still an open problem whose best results are the ones given by Carleson in [8], [10], and by Shapiro and Shields in [39]. The purpose of this paper is to give some light on this difficult problem. Since the Dirichlet-type spaces are subclasses of the Hardy space $H^2$, any zero sequence $\{z_n\}$ satisfies the Blaschke condition $\sum (1 - |z_n|^2) < \infty$ ([8, p. 18]). However, this condition is far from being sufficient. Many examples of Blaschke sequences that are not $D_s$-zero sets can be found in the literature (see [12], [29] and [39]). When $0 < s < 1$, Carleson proved in [8] that the condition

$$\sum (1 - |z_n|^2)^s < \infty$$

implies that the Blaschke product $B$ with zeros $\{z_n\}$ belongs to the space $D_s$, and therefore, it is a sufficient condition for the sequence $\{z_n\}$ to be a $D_s$-zero set.

Concerning the Dirichlet space $D$ (the case $s = 0$), since it does not contain infinite Blaschke products, one must go in a different way. In [10], by constructing a function $g \in D$ with $gB \in D$, Carleson found the sufficient condition $\sum \left( \log \frac{1}{1 - |z_n|^2} \right)^{-1+\epsilon} < \infty$, for a sequence $\{z_n\}$ to be a zero set for the Dirichlet space. Using Hilbert space techniques, this was improved in [39] by Shapiro and Shields, who proved that the condition

$$\sum_n \left( \log \frac{1}{1 - |z_n|^2} \right)^{-1} < \infty$$

is sufficient for $\{z_n\}$ to be a Dirichlet zero set.

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Note that the spaces $D_s$ are Hilbert function spaces with the norm of the corresponding reproducing kernels $k_z$ comparable to $(\log(1/|z|))^{1/2}$ if $s = 0$, and to $(1 - |z|^2)^{-s/2}$ if $s > 0$. So, the corresponding sufficient conditions stated before can be restated as $\sum \|k_{zn}\|_{D_s}^{-2} < \infty$. On the other hand, if $\{r_n\} \subset (0,1)$ and $\sum \|k_{rn}\|_{D_s}^{-2} = \infty$, with $0 \leq s < 1$, in [29], Nagel, Rudin, and Shapiro constructed a sequence of angles $\{\theta_n\}$ such that $\{r_ne^{i\theta_n}\}$ is not the zero set of any function in $D_s$. Together with the previous sufficient condition, this implies that given $\{r_n\} \subset (0,1)$, then $\{r_ne^{i\theta_n}\}$ is a zero set for $D_s$ for any choice of angles $\{\theta_n\}$ if and only if

$$\sum_n \|k_{rn}\|_{D_s}^{-2} < \infty.$$  (1.1)

We also note that, in [7], Bogdan described the regions $\Omega \subset \mathbb{D}$ for which any Blaschke sequence of points in $\Omega$ must be a Dirichlet zero set. For example, it follows that any Blaschke sequence that lies in a region with finite order of contact with the unit circle must be a Dirichlet zero set.

What about conditions on the angles? Here we touch the notion of a Carleson set. Given a sequence of points $\{e^{i\theta_n}\}$, the sequence $\{r_ne^{i\theta_n}\}$ is a zero sequence of $D$ for any choice of radius $\{r_n\}$, $0 < r_n < 1$ with $\sum(1 - r_n) < \infty$ if and only if the closure of $\{e^{i\theta_n}\}$ is a Carleson set. Indeed, if the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set, Caughran proved in [15] that there is a function $f$ with all derivatives bounded in the unit disk vanishing at the points $\{r_ne^{i\theta_n}\}$. Conversely, if $\{e^{i\theta_n}\}$ is not a Carleson set, by modifying the construction in [12] Theorem 1], he obtained in [13] a sequence $\{r_n\}$ for which $\{r_ne^{i\theta_n}\}$ is not contained in the zero set of any function with finite Dirichlet integral. We will see that the same holds for the spaces $D_s$ when $0 < s < 1$.

In [26] Corollary 13, Marshall and Sundberg proved that the zero sets of the Dirichlet-type spaces $D_s$, $0 \leq s \leq 1$, coincide with the zero sets of its multiplier algebra (see also [2 Corollary 9.39]). From this follows the remarkable result that the union of two zero sets is also a zero set for $D_s$. Note that the corresponding result for the weighted Bergman spaces (the case $s > 1$) is not true; the first example was given by Horowitz in [22]. A complete description of the zeros of functions in Bergman spaces is still open, but the gap between the necessary and sufficient known conditions is small. We refer to [19 Chapter 4], [21 Chapter 4], [22, 25, 37] and [38] for more information on this interesting problem.

1.1. Main results. Let $\mathbb{D}$ denote the open unit disk of the complex plane, let $T$ denote the unit circle and let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. For $s \geq 0$, the weighted Dirichlet-type space $D_s$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{D_s}^2 \overset{\text{def}}{=} |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^s \, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} dx \, dy$ is the normalized area measure on $\mathbb{D}$. As usual, $D_0$ will be simply denoted by $D$.

Given a space $X$ of analytic functions in $\mathbb{D}$, a sequence $Z = \{z_n\} \subset \mathbb{D}$ is said to be an $X$-zero set if there exists a function in $X$ that vanishes on $Z$ and nowhere else.

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be separated if $\inf_{j\neq k} d(z_j,z_k) > 0$, where $d(z,w) = \frac{|z-w|}{1-\overline{z}w}$ denotes the pseudohyperbolic metric in $\mathbb{D}$. This condition is
equivalent to the fact that there is a positive constant \( \delta < 1 \) such that the pseudo-hyperbolic discs \( \Delta(z, \delta) = \{ z : g(z, z_j) < \delta \} \) are pairwise disjoint.

We denote by \( H^p (0 < p \leq \infty) \) the classical Hardy spaces of analytic functions on \( \mathbb{D} \) (see [18]). We remind the reader that \( \{ z_k \} \subset \mathbb{D} \) is an interpolating sequence if for each bounded sequence \( \{ w_k \} \) of complex numbers there exists \( f \in H^\infty \) such that \( f(z_k) = w_k \) for all \( k \). It is a classical result of Carleson (see e.g. [18]) that \( \{ z_k \} \subset \mathbb{D} \) is an interpolating sequence if and only if

\[
\inf_k \prod_{j \neq k} \varrho(z_j, z_k) > 0. \tag{1.2}
\]

Clearly a sequence satisfying (1.2) is separated. A finite union of interpolating sequences is usually called a Carleson-Newman sequence.

In this research on \( D_s \)-zero sets, \( 0 < s < 1 \), the additional hypothesis of being a Carleson-Newman sequence enables us to obtain better results. The key is the following one which moves the problem to a new situation on the boundary.

**Theorem 1.** Suppose that \( 0 < s < 1 \) and \( \{ z_k \} \) is a Carleson-Newman sequence. Then the following conditions are equivalent:

(i) \( \{ z_k \} \) is a \( D_s \)-zero set.

(ii) There exists an outer function \( g \in D_s \) such that

\[
\sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2)^s < \infty. \tag{1.3}
\]

(iii) There exists an outer function \( g \in D_s \) such that

\[
\sum_{k=1}^{\infty} (1 - |z_k|^2)^{1+s} \int_T |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2} < \infty.
\]

We recall that a function \( g \in H(\mathbb{D}) \) is called an outer function if \( \log |g| \) belongs to \( L^1(\mathbb{T}) \) and

\[
g(z) = \exp \left( \frac{1}{2\pi} \int_T \log |g(e^{it})| \frac{e^{it} + z}{e^{it} - z} \, dt \right).
\]

Although obviously there are \( D_s \)-zero sets that are not Carleson-Newman sequences, this additional assumption is not an obstacle in order to construct relevant examples, and to get analogous results for \( D_s \) to those known for \( D \). Combining ideas from [10], [12] and Theorem 1, the next result follows.

**Corollary 1.** Suppose that \( 0 < s < 1 \) and \( \{ z_k \} \) is a Carleson-Newman sequence. If \( \{ z_k \} \) is a \( D_s \)-zero set, then

\[
\int_T \log \left( \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt < \infty. \tag{1.4}
\]

We note that this result remains true for \( s = 0 \) without assuming that the sequence is Carleson-Newman (see [12]); that is, if \( \{ z_k \} \) is a \( D \)-zero set, then

\[
\int_T \log \left( \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|e^{it} - z_k|^2} \right) \, dt < \infty. \tag{1.5}
\]

Corollary 1 allows us to extend Theorem 1 of [12] to the case \( 0 < s < 1 \).
Theorem 2. Let $0 < s < 1$. Then there exists a Blaschke sequence $\{z_n\}$ which is not a $\mathcal{D}_s$-zero set and with 1 as a unique accumulation point.

Denote by $|E|$ the normalized Lebesgue measure of a subset $E$ of the unit circle $\mathbb{T}$. A Carleson set is a closed subset $E \subset \mathbb{T}$ of Lebesgue measure zero for which, if the intervals $\{I_k\}$ complementary to $E$ have lengths $|I_k|$, then $\sum_k |I_k| \log |I_k| > -\infty$. This notion was introduced in [3], and in [9] Carleson used it to describe the sets of uniqueness of some function spaces. Corollary [1] is also useful to obtain results on the angular distribution of the $\mathcal{D}_s$-zero sets.

Theorem 3. Let $0 < s < 1$, and $\{e^{i\theta_n}\} \subset \mathbb{T}$. The following are equivalent:

(i) the sequence $\{r_n e^{i\theta_n}\}$ is a $\mathcal{D}_s$-zero set for any choice of $\{r_n\} \subset (0,1)$ with $\sum (1 - r_n) < \infty$;

(ii) the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set.

As noted before, if $0 \leq s < 1$ and $\{r_n\} \subset (0,1)$ is a Blaschke sequence that does not satisfy (1.1), then there is a sequence of angles $\{\theta_n\}$ such that $Z = \{r_n e^{i\theta_n}\}$ is not a $\mathcal{D}_s$-zero set. The sequences doing that which have been constructed in [24] (and also the examples in [39]) satisfy that every $\xi \in \mathbb{T}$ is an accumulation point of $Z$. Ross, Richter and Sundberg proved in [35] that this can be done in $\mathcal{D}$ with a sequence $Z$ which accumulates to a single point in $\mathbb{T}$. We shall extend this result to the range $0 < s < 1$, which improves our Theorem 2 but whose proof is much more technical.

Theorem 4. Let $0 < s < 1$. Suppose that $\{r_n\} \subset (0,1)$ satisfies

$$\sum_{n=0}^{\infty} (1 - r_n)^s = \infty.$$ 

Then there exists a sequence $\{\theta_n\}$ such that $\{r_n e^{i\theta_n}\} \cap \mathbb{T} = \{1\}$ and $\{r_n e^{i\theta_n}\}$ is not a $\mathcal{D}_s$-zero set.

Let $X$ be a space of analytic functions in $\mathbb{D}$ contained in the Nevanlinna class (see [13]), so for every function $f \in X$ has nontangential limits a.e. on $\mathbb{T}$. Denote also by $f$ the function of boundary values of $f$ (taken as a nontangential limit). A closed set $E \subset \mathbb{T}$ is called a set of uniqueness for $X$ if it has the property that $f \equiv 0$ if $f \in X$ vanishes at all points $\xi \in E$. It is well known that $E \subset \mathbb{T}$ is a set of uniqueness for a Lipschitz class $\Lambda_\alpha$ if and only if $E$ is not a Carleson set. We remind the reader that $f \in H(\mathbb{D})$ belongs to $\Lambda_\alpha$, $0 < \alpha \leq 1$, if there is $C > 0$ such that

$$|f(z) - f(w)| \leq C |z - w|^\alpha, \quad \text{for all } z, w \in \overline{\mathbb{D}}.$$ 

In [9] Theorem 5], under a very weak additional assumption, the sets of uniqueness for the classical Dirichlet space are described.

If $\alpha > 0$, we denote by $C_\alpha(E)$ the $\alpha$-capacity of a subset of $\mathbb{T}$ (see Section 4 for a definition). The following result is an extension of Theorem 5 in [9].

Theorem 5. Let $0 \leq s < \alpha < 1$ and $E \subset \mathbb{T}$ with null Lebesgue measure. Suppose that there exists $m > 0$ such that for each interval $I \subset \mathbb{T}$ centered at a point of $E$,

$$C_\alpha(E \cap I) \geq m |I|.$$

Then $E$ is a set of uniqueness for $\mathcal{D}_s$ if and only if $E$ is not a Carleson set.
The paper is organized as follows. Section 2 is devoted to the study of Carleson-
Newman sequences as $D_s$-zero sets proving Theorem 1, Corollary 1, Theorem 2, and
Theorem 3. Theorem 3 is proved in Section 3, and Theorem 5 is proved in Section 4.
In Section 5, we shall give a new proof of a result of Bogdan [7] on the description
of Blaschke sets for $D$. Finally, in Section 6, between other results, we prove that
$D_s$-zero sets and the zero sets of their generated Moebius invariant spaces coincide.

In the sequel, the notation $A \asymp B$ will mean that there exist two positive con-
stants $C_1$ and $C_2$ which only depend on some parameters $p, \alpha, s, \ldots$ such that
$C_1 A \leq B \leq C_2 A$. Also, we remark that throughout the paper we shall be us-
ing the convention that the letter $C$ will denote a positive constant whose value
may depend on some parameters $p, \alpha, s, \ldots$, not necessarily the same at different
occurrences.

2. Carleson-Newman $D_s$-zero sets

We first recall some useful concepts and results. The Carleson square $S(I)$ of an
interval $I \subset \mathbb{T}$ is defined as

$$ S(I) = \{ re^{i\theta} : e^{i\theta} \in I, \quad 1 - |I| \leq r < 1 \} . $$

Given $s > 0$ and a positive Borel measure $\mu$ on $\mathbb{D}$, we say that $\mu$ is an $s$-
Carleson measure if there exists a positive constant $C$ such that

$$ \mu(S(I)) \leq C |I|^s, \quad \text{for every interval } I \subset \mathbb{T}. $$

If $s = 1$ we simply say that $\mu$ is a Carleson measure. We recall that a sequence
$\{z_n\} \subset \mathbb{D}$ is Carleson-Newman if and only if the measure $d\mu_{z_n} = \sum (1 - |z_n|) \delta_{z_n}$ is a Carleson measure (see [24] and [25]). Here, as usual, $\delta_{z_n}$ denotes the point mass
at $z_n$. A Blaschke product whose zero sequence is Carleson-Newman is called a
Carleson-Newman Blaschke product (a CN-Blaschke product, for short).

Let $P_z(e^{it})$ denote the Poisson kernel at a point $z \in \mathbb{D}$, so that

$$ P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad e^{it} \in \mathbb{T}, $$

and let

$$ \Psi(z, \phi) = \frac{1}{2\pi} \int_\mathbb{T} \phi(e^{it}) P_z(e^{it}) \, dt - \exp \left( \frac{1}{2\pi} \int_\mathbb{T} \log \phi(e^{it}) P_z(e^{it}) \, dt \right), \quad z \in \mathbb{D}, $$

where $\phi$ is a positive function which belongs to $L^1(\mathbb{T})$. Observe that the arithmetic-
geometric inequality implies that $\Psi(z, \phi) \geq 0$. If $\phi \in L^2(\mathbb{T})$, $\phi \geq 0$, we set

$$ \Phi(z, \phi) = \Psi(z, \phi^2). $$

We observe that for an outer function $g \in H^2$,

$$ \Phi(z, |g|) = P(|g|^2)(z) - |g(z)|^2, $$

where $P(|g|^2)$ is the Poisson integral of $|g|^2$.

The following result, Theorem 3.1 of [17] (see [6] for related results), characterizes
the membership in $D_s$ of an outer function in terms of its modulus on the boundary.

**Theorem A.** Suppose that $0 < s < 1$ and $f$ is an outer function. Then the
following are equivalent:

(i) $f \in D_s$.

(ii) $\int_{\mathbb{D}} \Phi(z, |f|) \frac{dA(z)}{(1 - |z|)^{2-s}} < \infty$. 

In order to prove Theorem 1 we need some lemmas. The following result is implicit in some places (see e.g. [33, Theorem 5] or [15, Theorem 8]). For completeness we sketch a proof here.

**Lemma 1.** Suppose that $0 < s < 1$, $f \in \mathcal{D}_s$ and let $B$ be a Carleson-Newman Blaschke product with zeros $\{z_k\} \subset \mathbb{D}$. Then $fB \in \mathcal{D}_s$ if and only if

$$\sum_{k=1}^\infty |f(z_k)|^2(1 - |z_k|^2)^s < \infty.$$ 

Moreover,

$$\|fB\|_{\mathcal{B}_s}^2 \leq \|f\|_{\mathcal{B}_s}^2 + \sum_{k=1}^\infty |f(z_k)|^2(1 - |z_k|^2)^s.$$ 

**Proof.** Suppose first that $fB \in \mathcal{D}_s$. By Theorem 4 of [16],

$$\|fB\|_{\mathcal{B}_s}^2 \leq \|f\|_{\mathcal{B}_s}^2 + \int_{\mathbb{D}} |f(z)|^2 (1 - |B(z)|^2) (1 - |z|^2)^{s-2} \, dA(z).$$

Since $B$ is a CN-Blaschke product, there is a positive constant $C$ such that (see e.g. [16, p. 15])

$$1 - |B(z)|^2 \geq C \sum_n \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - z_n z|^2}. $$

Therefore, if $\Delta_n = \{\theta(z, z_n) < 1/2\}$, the subharmonicity of $|f|^2$ gives

$$\sum_n |f(z_n)|^2(1 - |z_n|^2)^s \leq C \sum_n \int_{\Delta_n} |f(z)|^2 \frac{(1 - |z|^2)^s}{|1 - z_n z|^2} \, dA(z)$$

$$\leq C \sum_n (1 - |z_n|^2) \int_{\Delta_n} |f(z)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - z_n z|^2} \, dA(z)$$

$$\leq C \sum_n (1 - |z_n|^2) \int_{\mathbb{D}} |f(z)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - z_n z|^2} \, dA(z)$$

$$\leq C \int_{\mathbb{D}} |f(z)|^2 (1 - |B(z)|^2) (1 - |z|^2)^{s-2} \, dA(z).$$

For the converse we refer to [16, Proposition 3.2], where an elementary proof is given. \[\Box\]

Next, if $g \in H^2$ we shall see that the function $\Phi(z, |g|)$, although it is superharmonic, verifies a certain sub-mean-value property.

**Lemma 2.** Suppose that $g$ is an outer function which belongs to $H^2$. Then there is a constant $M > 1$ such that

$$\Phi(z, |g|) \leq \frac{M}{A(D(z, r))} \int_{D(z, r)} \Phi(w, |g|) \, dA(w), \quad \text{for all } r \in \left(0, \frac{1 - |z|}{2}\right),$$

where $D(z, r)$ is the Euclidean disk of center $z$ and radius $r$. 
Proof. Take $z \in \mathbb{D}$ and $r \in \left(0, \frac{1-|z|}{2}\right)$. Using the trivial but useful identity
\begin{equation}
\int_0^{2\pi} |g(e^{it}) - g(z)|^2 P_z(e^{it}) \frac{dt}{2\pi} = P(|g|^2)(z) - |g(z)|^2, \tag{2.3}
\end{equation}
the subharmonicity of the function $h_t(z) = |g(e^{it}) - g(z)|^2$, Fubini's theorem and (2.1), we obtain that
\begin{equation}
\Phi(z, |g|) = \int_0^{2\pi} h_t(z) P_z(e^{it}) \frac{dt}{2\pi} \leq \int_0^{2\pi} \left( \frac{1}{A(D(z, r))} \int_{D(z, r)} h_t(w) dA(w) \right) P_z(e^{it}) \frac{dt}{2\pi} \tag{2.4}
\end{equation}
which, together with (2.3) and (2.4), gives that
\begin{equation}
\Phi(z, |g|) \leq \frac{M}{A(D(z, r))} \int_{D(z, r)} \left( \int_0^{2\pi} |g(e^{it}) - g(w)|^2 P_w(e^{it}) \frac{dt}{2\pi} \right) dA(w) \tag{2.5}
\end{equation}
which finishes the proof.

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Let $B$ be a CN-Blaschke product with zeros $\{z_n\}$, where $\{z_n\}$ is a $\mathcal{D}_s$-zero set. Thus, there is $f \in \mathcal{D}_s$ whose zero sequence is $\{z_n\}$. Since $\mathcal{D}_s$ has the property of division by inner functions (see [6]), this implies that there is an outer function $g \in \mathcal{D}_s$ such that $g \cdot B \in \mathcal{D}_s$, which together with Lemma 1 gives that
\begin{equation}
\sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty.
\end{equation}

(ii) $\Rightarrow$ (i). Since $B$ is a CN-Blaschke product, this follows immediately from Lemma 1.

(iii) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). Without loss of generality we may assume that $\{z_k\}$ is separated. Therefore, there is a positive constant $\varepsilon < 1$ such that the pseudohyperbolic disks $\Delta(z_k, \varepsilon)$ are pairwise disjoint.

Suppose that there is an outer function $g$ which satisfies (1.3). It is observed that
\begin{equation}
\sum_k (1 - |z_k|^2)^{1+s} \int_0^1 |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2} \tag{2.5}
\end{equation}
\begin{equation}
\leq \sum_k \Phi(z_k, |g|)(1 - |z_k|^2)^s + \sum_{k=1}^\infty |g(z_k)|^2 (1 - |z_k|^2)^s.
\end{equation}
Next, bearing in mind Lemma 2 the separation of \(\{z_k\}\) and Theorem A we deduce that
\[
\sum_k \Phi(z_k, |g|)(1 - |z_k|^2)^s \leq C \sum_k (1 - |z_k|^2)^{s-2} \int_{\Delta(z_k, \varepsilon)} \Phi(z, |g|) \, dA(z)
\]
(2.6)
\[
\leq C \sum_k \int_{\Delta(z_k, \varepsilon)} (1 - |z|^2)^{s-2} \Phi(z, |g|) \, dA(z)
\]
\[
\leq C \int \sum_k (1 - |z|^2)^{s-2} \Phi(z, |g|) \, dA(z) < \infty.
\]

Finally, (iii) follows from (1.3), (2.6) and (2.5). \(\square\)

**Proof of Corollary 1.** By Theorem 1 there is an outer function \(g \in \mathcal{D}_s\) such that
\[
\int_T |g(e^{it})|^2 \left( \sum_k \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt < \infty,
\]
so bearing in mind that \(\log |g| \in L^1(T)\) and the geometric-arithmetic inequality, the result follows. \(\square\)

**Proof of Theorem 2.** The same sequence given in the proof of 12 Theorem 1 works. Choose a sequence \(\{\varepsilon_n\}\) such that \(0 < \varepsilon_n < 1\), \(\sum \varepsilon_n \leq 1\) and \(\sum \varepsilon_n \log \varepsilon_n = -\infty\). Next, take disjoint open arcs of \(T\) with \(|I_n| = \varepsilon_n\) converging to 1. Let \(r_n = 1 - \varepsilon_n\) and \(z_n = r_ne^{i\theta_n}\), where \(\theta_n\) is the center of \(I_n\). If \(I\) is an arc of \(T\), then
\[
\sum_{\varepsilon_n \in S(I)} (1 - |z_n|) \leq \sum_{I_n \subset 2I} |I_n| \leq 2|I|,
\]
proving that the measure \(\mu = \sum (1 - |z_n|) \delta_{z_n}\) is a Carleson measure. So, \(\{z_n\}\) is a Carleson-Newman sequence which accumulates only at \(\{1\}\). Moreover, since
\[
\int_T \log \left( \sum_{k=1}^\infty \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt \geq \sum_{j=1}^\infty \int_{I_j} \log \left( \sum_{k=1}^\infty \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt
\]
\[
\geq \sum_{j=1}^\infty \int_{I_j} \log \left( \frac{(1 - |z_j|^2)^{1+s}}{|e^{it} - z_j|^2} \right) \, dt
\]
\[
\geq \sum_{j=1}^\infty |I_j| \log (4|I_j|^{s-1}) = \infty,
\]
it follows from Corollary 1 that \(\{z_n\}\) is not a \(\mathcal{D}_n\)-zero set. The proof is complete. \(\square\)

**Proof of Theorem 3.** If \(\{e^{i\theta_n}\}\) is a Carleson set and \(\sum (1 - r_n) < \infty\), then it follows from 13 Theorem 2 that there is a function \(f\) with all derivatives bounded that vanishes only at \(\{r_ne^{i\theta_n}\}\).

Suppose now that \(E = \{e^{i\theta_n}\}\) is not a Carleson set. Let \(\{I_n\}\) be the complementary intervals of \(E\), with \(I_n = (e^{i\theta_n}, e^{i(\theta_n + |I_n|)})\). Set \(r_n = (1 - |I_n|)e^{i\theta_n},\)
which satisfies \( \sum (1 - r_n) < \infty \). Clearly, the sequence \( \{ z_n \} = \{ r_n e^{it_n} \} \) is Carleson-Newman, and arguing as in the proof of Theorem 2 we have
\[
\int \log \left( \sum_n \frac{(1 - |z_n|^2)^{1+\varepsilon}}{|e^{it} - z_n|^2} \right) dt \geq C \sum_n |I_n| \log (4|I_n|^{-1}) = \infty.
\]
Hence, by Corollary 1 the sequence \( \{ r_n e^{it_n} \} \) is not a \( \mathcal{D}_s \)-zero set. \( \square \)

3. Proof of Theorem 1

Some new concepts and preliminary results will be needed in the proof of Theorem 1. For \( 0 < s \leq 1 \), the \( s \)-dimensional Hausdorff capacity of \( E \subset \mathbb{T} \) is determined by
\[
\Lambda_s^\infty (E) = \inf \left\{ \sum_j |I_j|^s : E \subset \bigcup_j I_j \right\},
\]
where the infimum is taken over all coverings of \( E \) by countable families of open arcs \( I \subset \mathbb{T} \).

Although we think that the next result is known, a proof is included here since we were not able to find any clear reference.

Lemma 3. Let \( 0 < s \leq 1 \). Then there exists a universal constant \( C \) such that \( \Lambda_s^\infty (E) \geq C |E|^s \) for all \( E \subset \mathbb{T} \).

Proof. Let \( E \subset \mathbb{T} \). If \( |E| = 0 \), the result is clear. Suppose that \( |E| > 0 \) and take \( \varepsilon \in \left(0, \frac{|E|^s}{2}\right) \). Then there exists a covering \( \{ I_j \}_j \) of \( E \), such that
\[
\Lambda_s^\infty (E) \geq \sum_j |I_j|^s - \varepsilon \geq \left( \sum_j |I_j| \right)^s - \varepsilon \geq |E|^s - \frac{|E|^s}{2} = \frac{|E|^s}{2}.
\]
This finishes the proof. \( \square \)

The homogeneous \( \mathcal{D}_s \)-capacity of a set \( E \subset \mathbb{T} \) is defined by
\[
\mathop{\text{cap}} (E, \mathcal{D}_s) = \inf \left\{ \| f \|_{L^2_s}^2 : f \in L^2 (\mathbb{T}) \text{ and } f \geq 1 \text{ a.e. on } E \right\}.
\]

Lemma 4. Let \( J \subset \mathbb{T} \) be an open arc with center \( e^{i\theta_0} \). Suppose that \( F \in \mathcal{D}_s \) with
\[
E = \left\{ e^{it} \in J : |F(e^{it})| \geq 1 \right\}.
\]
If \( |E| \geq \frac{|J|}{2} \), then there exists a universal constant \( C \) such that
\[
\int_{S(J)} |F'(z)|^2 (1 - |z|^2)^s \, dA(z) \geq C |J|^s.
\]

Proof. Let \( z_0 = (1 - \frac{|J|}{2}) e^{i\theta_0} \). Arguing as in the proof of Lemma 3], we deduce that there is a universal constant \( C \) such that the harmonic measure of \( E \) with respect to \( Q := S(J) \) at \( z_0 \), \( \mu_{z_0}^Q (E) \), satisfies
\[
\mu_{z_0}^Q (E) \geq C.
\]
Consider a conformal map \( \varphi : \mathbb{D} \to Q \) with \( \varphi(0) = z_0 \) and take \( g = F \circ \varphi \). Then \( g \geq 1 \) on \( \varphi^{-1} (E) \) and \( |\varphi^{-1} (E)| = \mu_{z_0}^Q (E) \geq C \). Thus, putting together (5.1.3) of [1]...
and Lemma 3 we have
\begin{equation}
\|g\|_{L^2}^2 \geq \text{cap} (\varphi^{-1}(E), D_s) \geq C \left( \Lambda_\infty^s (\varphi^{-1}(E))^\gamma \right) \geq C \mu_\infty^Q (E)^s \gamma \geq C,
\end{equation}
where \( s' \in (s, 1) \) and \( \gamma \in (0, 1) \).

Next, since \( \varphi \) is a conformal map (see [34, Chapter 1]),
\begin{equation}
(1 - |z|^2)|\varphi'(z)| \simeq d(\varphi(z), \partial Q), \quad z \in \mathbb{D}.
\end{equation}

Moreover, since \( Q \) is convex, reasoning as in [20, Proposition 5] and bearing in mind (3.2) we obtain that
\begin{equation}
|\varphi'(z)| \geq \frac{1}{4} |\varphi'(0)| \geq C d(z_0, \partial Q) \geq C|J|,
\end{equation}
where \( d(z_0, \partial Q) \) is the Euclidean distance from \( z_0 \) to \( \partial Q \).

Taking into account (3.1), (3.2) and (3.3) we deduce that
\begin{align*}
\int_Q |F'(z)|^2 (1 - |z|^2)^s dA(z) &\geq \int_Q |F'(z)|^2 d(z, \partial Q)^s dA(z) \\
&\geq \int_D |g'(z)|^2 d(\varphi(z), \partial Q)^s dA(z) \\
&\geq C \int_D |g'(z)|^2 (1 - |z|^2)|\varphi'(z)|^s dA(z) \\
&\geq C |J|^s \int_D |g'(z)|^2 (1 - |z|^2)^s dA(z) \\
&\geq C |J|^s.
\end{align*}

This finishes the proof. \( \square \)

**Proof of Theorem 3.** Let \( \{r_n\} \subset (0, 1) \) be an increasing sequence such that
\[
\sum_n (1 - r_n)^s = \infty.
\]

We can find
\[1 \leq n_1 < m_1 < n_2 < m_2 < \cdots < n_k < m_k < \cdots\]
such that
\begin{equation}
(1 - r_n)^{1-s} < k^{-2} e^{-2k^2} \quad \text{if} \quad n \geq n_k, \quad k = 1, 2, \ldots
\end{equation}
and
\[ke^{2k^2} \leq \sum_{n=n_k}^{m_k} (1 - r_n)^s < ke^{2k^2} + 1, \quad k = 1, 2, \ldots\]

For each \( k \), lay out arcs \( J_{n_k}, J_{n_k+1}, \ldots, J_{m_k} \) on the unit circle end-to-end starting at \( e^{i\theta} = 1 \) and such that
\begin{equation}
|J_n| = (1 - r_n)^s k^{-2} e^{-2k^2}, \quad n_k \leq n \leq m_k.
\end{equation}

Observe that (3.4) together with (3.5) implies that
\begin{equation}
|J_n| > (1 - r_n).
\end{equation}
Let $e^{i\theta_n}$ be the center of $J_n$ and set $\lambda_n = (1 - r_n)e^{i\theta_n}$. Suppose that there is $F \in D_s$ with $F(\lambda_n) = 0$ for all $n_k \leq n \leq m_k$. By [6, Theorem 3.4] we may assume that $||F||_{H^\infty} \leq 1$. Set

$$A_k = \left\{ n : n_k \leq n \leq m_k \text{ and } |F| \geq e^{-k^2} \right\},$$
$$B_k = \{ n : n_k \leq n \leq m_k, \quad n \notin A_k \}.$$

Using Lemma 4 and 3.6 with $S(J_n)$, $n \in A_k$, we deduce that

$$\int_{S(J_n)} |F'(z)|^2 (1 - |z|^2)^\alpha dA(z) \geq Ce^{-2k^2} |J_n|^\alpha \geq Ce^{-2k^2}(1 - r_n)^\alpha.$$

Moreover if $n \in B_k$,

$$\int_{J_n} \log \frac{1}{|F(\xi)|} d\xi \geq \frac{1}{2} k^2 |J_n| = \frac{1}{2}(1 - r_n)^\alpha e^{-2k^2}.$$

So, bearing in mind 3.3,

$$\sum_{n \in A_k} \int_{S(J_n)} |F'(z)|^2 (1 - |z|^2)^\alpha dA(z) + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|F(\xi)|} d\xi$$

$$\geq Ce^{-2k^2} \sum_{n = n_k}^{m_k} (1 - r_n)^\alpha \geq Ck,$$

which together with the integrability of $\log |F|$ on the boundary (see Theorem 2.2 of [18]), implies that $F$ must be the zero function. Finally, arguing as in the proof of Theorem 2 of [9], the proof can be finished. \qed

4. Zeros on the boundary. Sets of uniqueness

In order to prove Theorem 5 the notion of $\alpha$-capacity must be introduced. We shall recall some definitions (see [11] and [8]). Given $E \subset [0, 2\pi)$, let $\mathcal{P}(E)$ be the set of all probability measures supported on $E$. If $\alpha > 0$ and $\sigma \in \mathcal{P}(E)$, the $\alpha$-potential associated to $\sigma$ is

$$U_{\alpha}\sigma(\tau) = \int_E \frac{d\sigma(\theta)}{|	heta - \tau|^\alpha}.$$

Let

$$V_{E,\alpha} = \inf \int_E U_{\alpha}\sigma(\tau) d\sigma(\tau),$$

where the infimum is taken over all $\sigma \in \mathcal{P}(E)$. If $V_{E,\alpha} < \infty$, there is $\mu \in \mathcal{P}(E)$ where the value $V_{E,\alpha}$ is attained, and that measure $\mu$ is called the equilibrium distribution for the $\alpha$-potentials of $E$. It is known that $U_{\alpha}\mu(\tau) = V_{E,\alpha}$ for a.e. $(\mu)$. The $\alpha$-capacity of $E$ is determined by

$$C_\alpha(E) = (V_{E,\alpha})^{-1}.$$

Proof of Theorem 5. Suppose that $E$ is a set of uniqueness for $D_s$. Then $E$ is also a set of uniqueness for any Lipschitz class $A_\beta$ with $\beta > \frac{1}{2}$, due to $A_\beta \subset D_s$. So, $E$ is not a Carleson set.

For the converse, we shall follow the argument in the proof of Theorem 5 in [9]. Let $\mu$ be the equilibrium distribution for the $\alpha$-potentials of $E$. Then, if $\{\gamma_n\}$ are
the Fourier-Stieltjes coefficients of $\mu$, there is a constant $C$ which only depends on $\alpha$ such that
\[
\sum_n n^{\alpha-1} |\gamma_n|^2 \leq CV_{E,\alpha}.
\]

Suppose that there is a bounded function $f \in D_s$, $f \neq 0$, that vanishes on $E$. We shall see that this leads to a contradiction. The function $h(\theta) = |f(e^{i\theta})|$ can be written as
\[
h(\theta) = \sum_n c_n e^{in\theta},
\]
where
\[
\sum_n n^{1-s} |c_n|^2 < \infty.
\]

For each $t \in (0, \pi)$, let us consider $h_t(\theta) = \frac{1}{2t} \int_{\theta-t}^{\theta+t} h(s) \, ds$. Integrating the Fourier series of $h$, it follows that the Fourier coefficients of $h_t$ are $\sin(nt) c_n$. Then by (4.1) and Schwarz’s inequality,
\[
\int_E h_t(\theta) \, d\mu(\theta) = \left| \int_E (h_t(\theta) - h(\theta)) \, d\mu(\theta) \right| = \left| \sum_n \left( 1 - \frac{\sin(nt)}{nt} \right) c_n \int_E e^{in\theta} \, d\mu(\theta) \right|
\leq C \sum_n \left( 1 - \frac{\sin(nt)}{nt} \right) |c_n| |\gamma_n|
\leq C \left( \sum_n \left( 1 - \frac{\sin(nt)}{nt} \right)^2 |c_n|^2 n^{1-\alpha} \right)^{\frac{1}{2}} \left( \sum_n n^{\alpha-1} |\gamma_n|^2 \right)^{\frac{1}{2}}.
\]

We claim that there is $C > 0$ such that
\[
n^{s-\alpha} \left( 1 - \frac{\sin(nt)}{nt} \right)^2 \leq C t^{\alpha-s}, \quad t > 0, \quad n = 1, 2, \ldots.
\]
If $nt \leq 1$, there is a positive constant $C$ which does not depend on $n$ or $t$, such that $1 - \frac{\sin(nt)}{nt} \leq C (nt)^2$, so
\[
n^{s-\alpha} \left( 1 - \frac{\sin(nt)}{nt} \right)^2 \leq C^2 n^{s-\alpha} (nt)^4 \leq C^2 n^{s-\alpha} (nt)_{\alpha-s} \leq C^2 t^{\alpha-s}.
\]

On the other hand, if $nt \geq 1$, bearing in mind that $1 - \frac{\sin(\theta)}{\theta}$ is a bounded function of $\theta$, we deduce that
\[
n^{s-\alpha} \left( 1 - \frac{\sin(nt)}{nt} \right)^2 \leq C n^{s-\alpha} \leq C t^{\alpha-s},
\]
which together with (4.4) gives (4.5).

Therefore, using (4.3), (4.4), (4.1) and (4.2), it follows that
\[
\int_E h_t(\theta) \, d\mu(\theta) \leq Ct^{\frac{\alpha-s}{2}} \left( \sum_n n^{1-s} |c_n|^2 \right)^{\frac{1}{2}} \left( \sum_n n^{\alpha-1} |\gamma_n|^2 \right)^{\frac{1}{2}}
\leq Ct^{\frac{\alpha-s}{2}} \|f\|_{D_s} V_{E,\alpha}^{1/2}.
\]
Now, let $k_n$ be the number of complementary intervals of $E$ whose lengths are in $[2^{-n}, 2^{-n+1})$. Since $E$ is not a Carleson set,

$$\sum \frac{n k_n}{2^n} = \infty. \tag{4.7}$$

Let $\{\omega_i\}_{i=1}^{k_n}$ be those intervals, and let $\{\theta_i\}_{i=1}^{2k_n}$ be the endpoints of $\{\omega_i\}_{i=1}^{k_n}$. We consider the open intervals $\{\delta_i\}_{i=1}^{2k_n}$ of length $2^{-n}$ with midpoints $\{\theta_i\}_{i=1}^{2k_n}$. Take $\gamma \in \left(0, \frac{\alpha - s}{2}\right)$ and let $S$ be the set of those $\delta_i$ such that

$$h_\gamma(\theta_i) > 2^{-\gamma n}, \quad \tau = 2^{-n}. \tag{4.8}$$

Observe that (4.8) implies that $h_{2\gamma}(\theta) > 2^{-\gamma n-1}$ holds for $\theta \in \delta_i$ whenever $\delta_i \in S$, which, together with the general relation (4.6), gives that for $\mu^s$ the equilibrium distribution for the $\alpha$-potentials of $E \cap S$, 

$$2^{-\gamma n-1} \leq \int_{E \cap S} h_\gamma(\theta) \, d\mu^s(\theta) \leq CV_{E \cap S}^{1/2} 2^{-n(\alpha - s)/2},$$

so

$$C_\alpha(E \cap S) \leq C 2^{(2\gamma - (\alpha - s))n}. \tag{4.9}$$

Let $N$ be the number of intervals $\delta_i$ which belong to $S$. We shall estimate $N$ using condition (1.6). Take $\mu_i$ to be the equilibrium distribution for the $\alpha$-potentials of $E \cap \delta_i$. Let us consider $\sigma = N^{-1} \sum_{\delta_i \subseteq S} \mu_i$ and $u$ the corresponding $\alpha$-potential. Suppose that $\tau \in \delta_i$, where $\delta_i \in S$, and let $\delta_{i-1}$ and $\delta_{i+1}$ be the intervals in $S$ which are on the left and on the right of $\delta_i$. We shall define $F = \{k - 1, k, k + 1\}$. Then bearing in mind that the intervals $\{\delta_j\}$ are disjoint, the distance between the intervals $\{\delta_j\}$, and condition (1.6) we deduce that

$$u(\tau) = \int_{E \cap S} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha} \leq \sum_{j \in F} \int_{\delta_j \cap S} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha} + \sum_{j=1, j \notin F}^N \int_{\delta_j \cap S} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha} \leq N^{-1} \left( \sum_{j \in F} \int_{\delta_j \cap S} \frac{d\mu_j(\theta)}{|\theta - \tau|^\alpha} + \sum_{j=1, j \notin F}^N \int_{\delta_j \cap S} \frac{d\mu_j(\theta)}{|\theta - \tau|^\alpha} \right) \leq CN^{-1} \left( 2^n + \sum_{j=1}^{N} \frac{1}{(j 2^{-n})^\alpha} \right) \leq CN^{-1} 2^n,$$

which together with (4.9) gives

$$N^{-1} 2^n \geq Cu \geq \frac{C}{C_\alpha(E \cap S)} \geq C 2^{(2\gamma - (\alpha - s))n},$$

so due to $\gamma < \frac{\alpha - s}{2}$, one obtains

$$N \leq C 2^{pn}, \quad \text{for some } p \in (0, 1). \tag{4.10}$$
If \( \omega = (\theta_{2^\nu-1}, \theta_{2^\nu}) \) and (4.8) does not hold for \( \theta_{2^\nu-1} \) and \( \theta_{2^\nu} \), then by the arithmetic-geometric inequality,

\[
\frac{1}{|\omega|} \int_{\omega} \log h(\theta) \, d\theta \leq \log \left( \frac{1}{|\omega|} \int_{\omega} h(\theta) \, d\theta \right)
\leq \log \left( \frac{1}{|\omega|} \left( \int_{\theta_{2^\nu-1}-2^{-n}}^{\theta_{2^\nu}+2^{-n}} h(\theta) \, d\theta + \int_{\theta_{2^\nu}-2^{-n}}^{\theta_{2^\nu-1}-2^{-n}} h(\theta) \, d\theta \right) \right)
\leq \log \left( 2^{-n+1} \frac{1}{|\omega|} |\omega(\theta_{2^\nu-1}) + \omega(\theta_{2^\nu})| \right)
\leq -\gamma n + C.
\]

By (4.10), the number of indices \( n \) for which the above inequality is true is greater than \( k_n - 2N \geq k_n - C2^n \). Hence

\[
\sum_{\nu=1}^{k_n} \int_{\omega_{2^\nu}} \log h(\theta) \, d\theta \leq -\gamma n 2^{-n}(k_n - C2^n) + C \sum_{\nu=1}^{k_n} |\omega_{2^\nu}|,
\]

which, joined to the fact that \( p < 1 \), gives

\[
\int_0^{2\pi} \log h(\theta) \, d\theta \leq -\gamma \sum_n n 2^{-n} k_n + C.
\]

Consequently, bearing in mind that \( \gamma > 0 \) and (4.7), this implies a contradiction. \( \square \)

5. B拉斯克 sets

A subset \( A \) of the unit disc \( \mathbb{D} \) is called a B拉斯克 set for \( \mathcal{D} \) if any Blaschke sequence with elements in \( A \) is a zero set of \( \mathcal{D} \). These sets were characterized by Bogdan in [7]. Here we shall give a new proof of that result.

**Theorem 6.** A \( \subset \mathbb{D} \) is a Blaschke set for \( \mathcal{D} \) if and only if

\[
\int T \log \operatorname{dist}(e^{it}, A) \, dt > -\infty.
\]

Some definitions and results will be introduced. A tent is an open subset \( T \) of \( \mathbb{D} \) bounded by an arc \( I \subset \mathbb{T} \) with \( |I| < \frac{1}{4} \) and two straight lines through the endpoints of \( I \) forming with \( I \) an angle of \( \frac{\pi}{4} \). The closed arc \( \overline{T} \) will be called the base of the tent \( T = T_I \). A tent is said to support \( A \) if \( T \cap A = \emptyset \) but \( \overline{T} \cap A \neq \emptyset \). A finite or countable collection of tents \( \{T_n\} \) is an A-belt if \( \{T_n\} \) are pairwise disjoint, \( A \)-supporting and \( T \setminus A \subset \bigcup_n \overline{T_n} \). The following result can be found in [24, Lemma 1].

**Lemma B.** Let \( A \subset \mathbb{D} \) such that \( T \setminus A \neq \emptyset \). Let \( \{T_n\} \) be an A-belt. Then (5.1) holds if and only if \( A \cap T \) has zero Lebesgue measure, and

\[
\sum_n |I_n| \log \left( \frac{c}{|I_n|} \right) < \infty.
\]

**Lemma 5.** Let \( \{z_n\} \) be a \( \mathcal{D} \)-zero set. If \( \{\lambda_n\} \subset \mathbb{D} \) satisfies that \( g(z_n, \lambda_n) < \delta < 1 \) for each \( n \), then \( \{\lambda_n\} \) is a \( \mathcal{D} \)-zero set.

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Proof. Since $Z = \{z_n\}$ is a $D$-zero set, there is a function $g$ in $D$ such that $gB_Z \in D$, where $B_Z$ is the Blaschke product with zeros $\{z_n\}$. By Carleson’s formula for the Dirichlet integral (see [11] and also [35]), we have

$$\|gB_\Lambda\|_D^2 = \|g\|_D^2 + \int_T \sum_n P_{\Lambda_n}(e^{it}) |g(e^{it})|^2 \, dt$$

$$\leq \|g\|_D^2 + C \int_T \sum_n P_{\Lambda_n}(e^{it}) |g(e^{it})|^2 \, dt$$

$$\leq C \|gB_Z\|_D^2 < \infty.$$ 

Hence, $\{\alpha_n\}$ is a $D$-zero set, and the proof is complete. \[\square\]

Remark 1. Note that this result implies that, if $A$ is a Blaschke set for $D$ and $\{w_k\}$ is a sequence such that $\rho(\{w_k\}, A) \leq C < 1$, then $A \cup \{w_k\}$ is also a a Blaschke set for $D$.

Proof of Theorem 6. Suppose that (5.1) holds, and let $Z$ be a Blaschke sequence of points in $A$. Then

$$\int_T \log \text{dist}(e^{it}, Z) \, dt > -\infty,$$

and by a result of Taylor and Williams in [40], $Z$ is a $\Lambda_\alpha$-zero set for any $\alpha$. Since $\Lambda_\alpha \subset D$ for $\alpha > \frac{1}{2}$, it follows that $A$ is a Blaschke set for $D$.

Suppose that $A$ is a Blaschke set for $D$. We shall use Lemma 3 to see that (5.1) holds. Suppose that $|A \cap T| > 0$. Then we can choose a sequence $\{\varepsilon_n\}$ of positive numbers satisfying

$$\sum_n \varepsilon_n \leq |A \cap T|,$$

$$\sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \infty,$$

and a collection of disjoint arcs $\{I_n\}$ in $T$ such that

$$|I_n| = \varepsilon_n, \quad I_n \cap \overline{A} \neq \emptyset, \quad n \geq 1.$$

In order to construct this sequence of subsets $\{I_n\}$, take $I_1$ with $|I_1| = \varepsilon_1$ and $I_1 \cap \overline{A} \neq \emptyset$, and once $I_n$ has been taken, choose $I_{n+1}$ such that $I_{n+1} \cap (\overline{A} \setminus \bigcup_{j=1}^n I_j) \neq \emptyset$ with $|I_{n+1}| = \varepsilon_{n+1}$.

Next, take a sequence $\{w_n\} \subset A$ such that $\text{dist}(w_n, I_n \cap \overline{A}) \leq \varepsilon_n$ and let $p_n$ be the integer part of $\frac{\varepsilon_n}{1 - |w_n|}$. Let $Z$ be the sequence of points in $A$ that consists of $p_n$ repetitions of each point $w_n$. Observe that $Z$ is a Blaschke sequence,

$$\sum_{z \in Z} (1 - |z|) = \sum_n p_n (1 - |w_n|) \leq \sum_n \varepsilon_n < \infty,$$
so that $Z$ must be a sequence of zeros of $D$. We also have
\[
\int_T \log \left( \sum_{z \in Z} \frac{1 - |z|^2}{|e^{it} - z|^2} \right) dt = \int_T \log \left( \sum_n p_n \frac{1 - |w_n|^2}{|e^{it} - w_n|^2} \right) dt
\]
\[
\geq \sum_k \int_{I_k} \log \left( p_k \frac{1 - |w_k|^2}{|e^{it} - w_k|^2} \right) dt
\]
\[
\geq \sum_k |I_k| \log \left( \frac{1}{8 \varepsilon_k} \right) = \infty,
\]
which gives a contradiction with condition (1.5). Therefore, $\overline{A} \cap T$ has zero Lebesgue measure.

Next, let $\{T_n\}$ be an $A$-belt. Then for each $n$ there is $w_n \in \overline{A} \cap \partial T_n$. We may assume that $w_n$ belongs to $A$. Indeed, if $w_n$ is an endpoint of the arc $I_n$, there is a point $\alpha_n \in A$ which is in the Stolz angle with vertex $w_n$ and aperture $\frac{\pi}{2}$. Consequently, if $\tilde{\alpha}_n$ is the closest point in $\partial T_n$ with the same modulus as $\alpha_n$, then $g(\alpha_n, \tilde{\alpha}_n) \leq C < 1$, where $C$ is independent of $n$, and now we can use the remark after Lemma 5.

Let $v_n$ be the vertex of the tent $T_n$. Since $\{I_n\}$ is a sequence of disjoint arcs, $\{v_n\}$ is a Blaschke sequence. We denote by $q_n$ the integer part of $(1 - |v_n|)/(1 - |w_n|)$ and we consider $Z$ to be the sequence of points in $A$ that consists of $q_n$ repetitions of each point $w_n$. Arguing as before, it follows that $Z$ is a Blaschke sequence, and moreover there is $C > 0$ such that
\[
|w_n - e^{it}|^2 \leq C|v_n - e^{it}|^2, \quad \text{for each } n \text{ and } e^{it} \in T.
\]

So, bearing in mind that $A$ is a Blaschke set for $D$, (1.5) and (5.2), we have that
\[
\infty > \int_T \log \left( \sum_{z \in Z} \frac{1 - |z|^2}{|e^{it} - z|^2} \right) dt = \int_T \log \left( \sum_n q_n \frac{1 - |w_n|^2}{|e^{it} - w_n|^2} \right) dt
\]
\[
\geq \int_T \log \left( \sum_n q_n \frac{1 - |w_n|^2}{1 - |v_n|^2} \frac{1 - |v_n|^2}{|e^{it} - v_n|^2} \right) dt
\]
\[
\geq \int_T \log \left( \sum_n C \frac{1 - |v_n|^2}{|e^{it} - v_n|^2} \right) dt
\]
\[
\geq \sum_k \int_{I_k} \log \left( \frac{C}{|I_k|} \right).
\]

This finishes the proof. 

\[\square\]

6. Other results

6.1. Other necessary angular conditions on $D_\alpha$-zero sets. First we shall prove the following result of its own interest.
Proof. Bearing in mind (2.2), the result follows from the identity (see [3, p. 191])

\[ \frac{1}{1 - |z|^2} = \sum_k |B_k(z)|^2 \frac{1 - |z_k|^2}{|1 - \overline{z_k}z|^2}, \quad z \in \mathbb{D}. \]

We also obtain different conditions from (1.4) (which can work for any Blaschke sequence) on the angular distribution of a Blaschke sequence \( \{ z_k \} \) to be a \( \mathcal{D}_s \)-zero set, \( 0 < s < 1 \).

**Proposition 1.** Suppose that \( 0 < s < 1 \) and \( \{ z_k \} \subset \mathbb{D} \). If there exists \( r_0 \in (0,1) \) such that

\[ M(\{ z_k \}) \overset{\text{def}}{=} \inf_{r_0 \leq |z| < 1} \left( \sum_k \frac{(1 - |z_k|^2)(1 - |z|^2)^s}{|1 - \overline{z_k}z|^2} \right) > 0, \]

then \( \{ z_k \} \) is not a \( \mathcal{D}_s \)-zero set.

**Proof.** Suppose that \( \{ z_k \} \) is a \( \mathcal{D}_s \)-zero set and satisfies (6.1). Then, there exists \( F \in \mathcal{D}_s \) which vanishes uniquely on \( \{ z_k \} \), so \( F = f \cdot B \), where \( f \in \mathcal{D}_s \) and \( B \) is the Blaschke product with zeros \( \{ z_k \} \). Thus, Lemma 6 and (6.1) imply that

\[ \infty > \sum_k (1 - |z_k|^2) \int_{\mathbb{D}} \frac{|f(z)|^2|B_k(z)|^2}{|1 - \overline{z_k}z|^2} \frac{dA(z)}{(1 - |z|^2)^{1-s}} \]
\[ \geq \int_{\mathbb{D}} |f(z)|^2|B(z)|^2 \left( \sum_k \frac{(1 - |z_k|^2)(1 - |z|^2)^s}{|1 - \overline{z_k}z|^2} \right) \frac{dA(z)}{(1 - |z|^2)} \]
\[ \geq M(\{ z_k \}) \int_{\mathbb{D}} |F(z)|^2 \frac{dA(z)}{(1 - |z|^2)}; \]

consequently \( F \equiv 0 \). This finishes the proof. \( \square \)

This result allows us to make constructions of Blaschke sequences which are not \( \mathcal{D}_s \)-zero sets.

**Corollary 2.** For \( 0 < s < 1 \), set

\[ z_{k,j}^{(s)} \overset{\text{def}}{=} \left( 1 - 2^{-\frac{s}{1-s}} \right) \exp \left( \frac{2\pi j}{2^k-1} \right), \quad k = 0, 1, 2, \ldots, \]
\[ j = 0, 1, \ldots, 2^k - 1. \]

The sequence \( \{ z_{k,j}^{(s)} \} \) is not a \( \mathcal{D}_s \)-zero set.

**Proof.** There is \( \beta = \beta(s) > 0 \) such that for each \( z \in \mathbb{D} \) we can find a pair \( (k(z), j(z)) \) with \( 1 - |z| \asymp 1 - |z_{k(z), j(z)}^{(s)}| \), and

\[ |1 - \overline{z_{k(z), j(z)}^{(s)}}z| \leq \beta(1 - |z|^2)^{1+s}. \]
Therefore
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} \frac{(1 - |z_{k,j}|^2)(1 - |z|^2)^s}{|1 - \overline{z}_{k,j} z|^2} \geq \frac{(1 - |z_{k(z),j(z)}|^2)(1 - |z|^2)^s}{|1 - \overline{z}_{k(z)} z|^2} \geq C \beta^{-1},
\]
so, by Proposition 2, \( \{z^{(s)}\} \) is not a \( D_s \)-zero set. \[\square\]

6.2. Möbius invariant spaces generated by \( D_s \). The space \( Q_s \), \( 0 \leq s < \infty \), is the Möbius invariant space generated by \( D_s \), that is, \( f \in Q_s \) if
\[
\sup_{a \in \mathbb{D}} \| f \circ \varphi_a - f(a) \|_{D_s}^2 < \infty.
\]

It is known that \( Q_1 \) coincides with \( BMOA \). However, if \( 0 < s < 1 \), \( Q_s \) is a proper subspace of \( BMOA \) and has many interesting properties (see the detailed monograph [42]).

As usual, for a space of analytic functions \( X \), we shall write \( M(X) \) for the algebra of (pointwise) multipliers of \( X \), that is,
\[
M(X) \overset{\text{def}}{=} \{ g \in H(\mathbb{D}) : gf \in X \text{ for all } f \in X \}.
\]

**Theorem 7.** Suppose that \( 0 < s \leq 1 \). Then \( D_s \), \( Q_s \), \( Q_s \cap H^\infty \) and \( M(D_s) \) have the same zero sets.

**Proof.** If \( s = 1 \), the result is well known because \( D_1 = H^2 \), \( M(H^2) = H^\infty \) and \( Q_1 = BMOA \). If \( 0 < s < 1 \), by [25] Corollary 13 the zeros sets of \( D_s \) and \( M(D_s) \) coincide, so the result follows from the chain of embeddings (see [4] Lemma 5.1)
\[
M(D_s) \subset Q_s \cap H^\infty \subset Q_s \subset D_s.
\]

This finishes the proof. \[\square\]

Since from different values of \( s \in (0,1) \), the \( D_s \)-zero sets are not the same, we obtain directly the following result.

**Corollary 3.** Suppose that \( 0 \leq s < p < 1 \). Then there exists \( Z \subset \mathbb{D} \), which is a \( Q_p \)-zero set but not a \( Q_s \)-zero set.

A stronger result, in the following sense, can be proved. A sequence \( \{z_n\} \) is interpolating for \( Q_p \cap H^\infty \), \( 0 < p < 1 \), if for each bounded sequence \( \{w_k\} \) of complex numbers, there exists \( f \in Q_p \cap H^\infty \) such that \( f(z_k) = w_k \) for all \( k \). A characterization of these sequences in terms of \( p \)-Carleson measures is given in [30]. It is clear that each interpolating sequence for \( Q_p \cap H^\infty \) is a \( D_p \)-zero set.

**Theorem 8.** Suppose that \( 0 < s < p < 1 \). Then, there exists \( Z = \{z_n\}_{n=0}^{\infty} \subset \mathbb{D} \) which is an interpolating sequence for \( Q_p \cap H^\infty \) and such that it is not a \( D_s \)-zero set.

**Proof.** Set
\[
z_n = \left(1 - \frac{1}{n^{1/s}}\right) e^{i\theta_n}, \quad n = 2, 3, \ldots,
\]
where
\[
\theta_n = \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{2n}, \quad n = 2, 3, \ldots.
\]

The proof of [29] Theorem 5.10 gives that \( \{z_n\} \) is not a \( D_s \)-zero set. Moreover, borrowing the argument of the proof of [32] Theorem 2, we have that \( \{z_n\} \) is
separated and \( \mu_{z_n,p} = \sum_n (1 - |z_n|)^p \delta_{z_n} \) is a \( p \)-Carleson measure. So [30] Theorem 1.3] gives that \( \{z_n\} \) is an interpolating sequence for \( Q_p \cap H^\infty \). This finishes the proof.

Finally, we note that in a recent paper [31], the algebra of (pointwise) multipliers of \( Q_s, \ 0 < s < 1 \), has been characterized in terms of \( \alpha \)-logarithmic \( s \)-Carleson measures. Using Corollary 3 as a main tool we shall prove the following result.

**Corollary 4.** Suppose that \( 0 < s < p < 1 \). Then

\[
M(Q_p, Q_s) \overset{\text{def}}{=} \{ g \in H(\mathbb{D}) : gf \in Q_s \text{ for all } f \in Q_p \} = \{0\}.
\]

**Proof.** Suppose that \( M(Q_p, Q_s) \neq \{0\} \). Let \( g \in M(Q_p, Q_s) \), \( g \neq 0 \) and denote by \( W \) its zero set. By Corollary 3 there exists \( f \in Q_p, f \neq 0 \), whose sequence of zeros \( Z \) is not a \( Q_s \)-zero set. It is clear that \( Z \cup W \) is the zero set of \( fg \in Q_s \), and since \( g \in Q_s \), \( W \) satisfies the Blaschke condition. Now, taking \( B \) to be the Blaschke product with zeros \( W \) and bearing in mind that \( Q_s \) has the \( f \)-property (see Corollary 1 of [14] or Corollary 5.4.1 of [42]), we obtain that \( \frac{f}{g} \in Q_s \), whose zero set is \( Z \). This finishes the proof. \( \square \)

### 7. Further remarks

We would like to emphasize that conditions (ii) and (iii) of Theorem 11 are equivalent when \( \{z_n\} \) is a finite union of separated Blaschke sequences. So, it seems natural to ask whether or not for finite unions of separated Blaschke sequences, condition (ii) implies that \( \{z_n\} \) is a \( D_s \)-zero set. Although we are not able to answer this question, if the function \( g \) has some additional regularity properties, one can prove that condition (ii) implies that \( \{z_n\} \) is a \( D_s \)-zero set, as the following result shows.

**Proposition 2.** Let \( \{z_n\} \subset \mathbb{D} \) be a Blaschke sequence, \( 0 < s < 1 \) and \( \alpha > \frac{1-s}{s} \). If there exists a function \( g \in \Lambda_\alpha \) such that

\[
\sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty,
\]

then \( \{z_n\} \) is a \( D_s \)-zero set.

**Proof.** Let \( B \) be the Blaschke product with zeros \( \{z_n\} \). We shall prove that \( gB \in D_s \). Using the fact that \( g \in \Lambda_\alpha \), and [43] Lemma 4.2.2, one has

\[
\sum_n (1 - |z_n|^2) \int_D |g(z) - g(z_n)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}z_n|^2} \ dA(z)
\]

\[
\leq C \sum_n (1 - |z_n|^2) \int_D \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}z_n|^2 - 2\alpha} \ dA(z)
\]

\[
\leq C \sum_n (1 - |z_n|^2) < \infty.
\]

Also, by our assumption and [43] Lemma 4.2.2,

\[
\sum_n (1 - |z_n|^2) |g(z_n)|^2 \int_D \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}z_n|^2} \ dA(z)
\]

\[
\leq C \sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty.
\]
Now, since $\Lambda_\alpha \subset D_s$ for $\alpha > \frac{1-s}{2}$, it follows easily from (7.1) and (7.2) that
\[
\|gB\|_{D_s}^2 \leq C\|g\|_{D_s}^2 + C\int_D |(gB')(z)|^2 (1-|z|^2)^s \, dA(z) < \infty.
\]
\[\square\]

In view of all this, we state the following related problem.

**Problem.** For $0 < s < 1$, describe those separated Blaschke sequences $\{z_n\} \subset \mathbb{D}$ such that there is $g \in D_s$, $g \neq 0$, with
\[
\sum_n |g(z_n)|^2 (1-|z_n|^2)^s < \infty.
\]

Another interesting problem is to find sufficient conditions in order for a sequence $\{z_n\}$ to be a zero set for the analytic Besov space $B_p$, $1 < p < \infty$ (see [43, Chapter 5]). Since the point evaluations are bounded linear functionals in $B_p$, there are reproducing kernels $k_z \in B_{p'}$, where $p'$ is the conjugate exponent of $p$. Also, it is well known that
\[
\|k_z\|_{B_{p'}}^{-p} \asymp \left(\log \frac{1}{1-|z|}\right)^{-(p-1)}.
\]

So, bearing in mind (1.1), it seems natural to ask the following.

**Question.** Let $1 < p < \infty$, and let $\{z_n\} \subset \mathbb{D}$ such that
\[
\sum_n \left(\log \frac{1}{1-|z_n|^2}\right)^{-(p-1)} < \infty.
\]
Is the sequence $\{z_n\}$ a $B_p$-zero set?

In order to answer that question, it seems that a more constructive proof of the case $p = 2$ (the Shapiro-Shields result [39]) must be given, not relying so heavily on Hilbert space techniques.

**References**


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