ON THE ZEROS OF FUNCTIONS IN DIRICHLET-TYPE SPACES

JORDI PAU AND JOSÉ ÁNGEL PELÁEZ

Abstract. We study the sequences of zeros for functions in the Dirichlet spaces $D_s$. Using Carleson-Newman sequences we prove that there are great similarities for this problem in the case $0 < s < 1$ with that for the classical Dirichlet space.

1. Introduction and main results

The problem of describing the zero sets for the Dirichlet-type spaces $D_s$ is an old one, and to the best of our knowledge, is still an open problem whose best results are the ones given by Carleson in [8], [10], and by Shapiro and Shields in [39]. The purpose of this paper is to give some light on this difficult problem. Since the Dirichlet-type spaces are subclasses of the Hardy space $H^2$, any zero sequence $\{z_n\}$ satisfies the Blaschke condition $\sum (1 - |z_n|^2)^s < \infty$ ([18, p. 18]). However, this condition is far from being sufficient. Many examples of Blaschke sequences that are not $D_s$-zero sets can be found in the literature (see [12], [29] and [39]). When $0 < s < 1$, Carleson proved in [8] that the condition

$$\sum (1 - |z_n|^2)^s < \infty$$

implies that the Blaschke product $B$ with zeros $\{z_n\}$ belongs to the space $D_s$, and therefore, it is a sufficient condition for the sequence $\{z_n\}$ to be a $D_s$-zero set. Concerning the Dirichlet space $D$ (the case $s = 0$), since it does not contain infinite Blaschke products, one must go in a different way. In [10], by constructing a function $g \in D$ with $gB \in D$, Carleson found the sufficient condition $\sum \left( \log \frac{1}{1 - |z_n|^2} \right)^{-1+\epsilon} < \infty$, for a sequence $\{z_n\}$ to be a zero set for the Dirichlet space. Using Hilbert space techniques, this was improved in [39] by Shapiro and Shields, who proved that the condition

$$\sum_n \left( \log \frac{1}{1 - |z_n|^2} \right)^{-1} < \infty$$

is sufficient for $\{z_n\}$ to be a Dirichlet zero set.
Note that the spaces $D_s$ are Hilbert function spaces with the norm of the corresponding reproducing kernels $k_z$ comparable to $(\log \frac{1}{1-|z|^2})^{1/2}$ if $s = 0$, and to $(1 - |z|^2)^{-s/2}$ if $s > 0$. So, the corresponding sufficient conditions stated before can be restated as \( \sum \|k_z^n\|^2_{D^s_n} < \infty \). On the other hand, if \( \{r_n\} \subset (0,1) \) and \( \sum \|k_r^n\|^2_{D^s_n} = \infty \), with $0 < s < 1$, in \cite{29}, Nagel, Rudin, and Shapiro constructed a sequence of angles \( \{\theta_n\} \) such that \( \{r_ne^{i\theta_n}\} \) is not the zero set of any function in $D_s$. Together with the previous sufficient condition, this implies that given \( \{r_n\} \subset (0,1) \), then \( \{r_n e^{i\theta_n}\} \) is a zero set for $D_s$ for any choice of angles \( \{\theta_n\} \) if and only if
\[
\sum_n \|k_{r_n}\|^2_{D^s_n} < \infty.
\] (1.1)

We also note that, in \cite{7}, Bogdan described the regions $\Omega \subset \mathbb{D}$ for which any Blaschke sequence of points in $\Omega$ must be a Dirichlet zero set. For example, it follows that any Blaschke sequence that lies in a region with finite order of contact with the unit circle must be a Dirichlet zero set.

What about conditions on the angles? Here we touch the notion of a Carleson set. Given a sequence of points \( \{e^{i\theta_n}\} \), the sequence \( \{r_ne^{i\theta_n}\} \) is a zero sequence of $D$ for any choice of radius \( \{r_n\} \), \( 0 < r_n < 1 \) with \( \sum(1 - r_n) < \infty \) if and only if the closure of \( \{e^{i\theta_n}\} \) is a Carleson set. Indeed, if the closure of \( \{e^{i\theta_n}\} \) in the unit circle is a Carleson set, Caughran proved in \cite{15} that there is a function $f$ with all derivatives bounded in the unit disk vanishing at the points \( \{r_ne^{i\theta_n}\} \). Conversely, if \( \{e^{i\theta_n}\} \) is not a Carleson set, by modifying the construction in \cite[Theorem 1]{12}, he obtained in \cite{13} a sequence \( \{r_n\} \) for which \( \{r_ne^{i\theta_n}\} \) is not contained in the zero set of any function with finite Dirichlet integral. We will see that the same holds for the spaces $D_s$ when \( 0 < s < 1 \).

In \cite[Corollary 13]{26}, Marshall and Sundberg proved that the zero sets of the Dirichlet-type spaces $D_s$, \( 0 \leq s \leq 1 \), coincide with the zero sets of its multiplier algebra (see also \cite[Corollary 9.39]{2}). From this follows the remarkable result that the union of two zero sets is also a zero set for $D_s$. Note that the corresponding result for the weighted Bergman spaces (the case $s > 1$) is not true; the first example was given by Horowitz in \cite{22}. A complete description of the zeros of functions in Bergman spaces is still open, but the gap between the necessary and sufficient known conditions is small. We refer to \cite[Chapter 4]{19}, \cite[Chapter 4]{21}, \cite{22}, \cite{25}, \cite{37} and \cite{38} for more information on this interesting problem.

1.1. Main results. Let $\mathbb{D}$ denote the open unit disk of the complex plane, let $T$ denote the unit circle and let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. For $s \geq 0$, the weighted Dirichlet-type space $D_s$ consists of those functions $f \in H(\mathbb{D})$ for which
\[
\|f\|_s^2 \overset{\text{def}}{=} |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^s\ dA(z) < \infty,
\]
where $dA(z) = \frac{1}{\pi} dx\ dy$ is the normalized area measure on $\mathbb{D}$. As usual, $D_0$ will be simply denoted by $D$.

Given a space $X$ of analytic functions in $\mathbb{D}$, a sequence $Z = \{z_n\} \subset \mathbb{D}$ is said to be an $X$-zero set if there exists a function in $X$ that vanishes on $Z$ and nowhere else.

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be separated if $\inf_{j \neq k} \varrho(z_j, z_k) > 0$, where $\varrho(z, w) = |\frac{z - w}{1 - \bar{z}w}|$ denotes the pseudohyperbolic metric in $\mathbb{D}$. This condition is
equivalent to the fact that there is a positive constant \( \delta < 1 \) such that the pseudo-hyperbolic discs \( \Delta(z_j, \delta) = \{ z : g(z, z_j) < \delta \} \) are pairwise disjoint.

We denote by \( H^p \) (\( 0 < p \leq \infty \)) the classical Hardy spaces of analytic functions on \( \mathbb{D} \) (see [18]). We remind the reader that \( \{ z_k \} \subset \mathbb{D} \) is an interpolating sequence if for each bounded sequence \( \{ w_k \} \) of complex numbers there exists \( f \in H^\infty \) such that \( f(z_k) = w_k \) for all \( k \). It is a classical result of Carleson (see e.g. [18]) that \( \{ z_k \} \subset \mathbb{D} \) is an interpolating sequence if and only if

\[
\inf_k \prod_{j \neq k} \rho(z_j, z_k) > 0.
\]

Clearly a sequence satisfying (1.2) is separated. A finite union of interpolating sequences is usually called a Carleson-Newman sequence.

In this research on \( D_s \)-zero sets, \( 0 < s < 1 \), the additional hypothesis of being a Carleson-Newman sequence enables us to obtain better results. The key is the following one which moves the problem to a new situation on the boundary.

**Theorem 1.** Suppose that \( 0 < s < 1 \) and \( \{ z_k \} \) is a Carleson-Newman sequence. Then the following conditions are equivalent:

(i) \( \{ z_k \} \) is a \( D_s \)-zero set.

(ii) There exists an outer function \( g \in D_s \) such that

\[
\sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2)^s < \infty.
\]

(iii) There exists an outer function \( g \in D_s \) such that

\[
\sum_{k=1}^{\infty} (1 - |z_k|^2)^{1+s} \int_{\mathbb{T}} |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2} < \infty.
\]

We recall that a function \( g \in H(D) \) is called an outer function if \( \log |g| \) belongs to \( L^1(\mathbb{T}) \) and

\[
g(z) = \exp \left( \frac{1}{2\pi} \int_{\mathbb{T}} \log |g(e^{it})| \frac{e^{it} + z}{e^{it} - z} dt \right).
\]

Although obviously there are \( D_s \)-zero sets that are not Carleson-Newman sequences, this additional assumption is not an obstacle in order to construct relevant examples, and to get analogous results for \( D_s \) to those known for \( D \). Combining ideas from [10], [12] and Theorem 1, the next result follows.

**Corollary 1.** Suppose that \( 0 < s < 1 \) and \( \{ z_k \} \) is a Carleson-Newman sequence. If \( \{ z_k \} \) is a \( D_s \)-zero set, then

\[
\int_{\mathbb{T}} \log \left( \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) dt < \infty.
\]

We note that this result remains true for \( s = 0 \) without assuming that the sequence is Carleson-Newman (see [12]); that is, if \( \{ z_k \} \) is a \( D \)-zero set, then

\[
\int_{\mathbb{T}} \log \left( \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|e^{it} - z_k|^2} \right) dt < \infty.
\]

Corollary 1 allows us to extend Theorem 1 of [12] to the case \( 0 < s < 1 \).
Theorem 2. Let $0 < s < 1$. Then there exists a Blaschke sequence $\{z_n\}$ which is not a $D_s$-zero set and with 1 as a unique accumulation point.

Denote by $|E|$ the normalized Lebesgue measure of a subset $E$ of the unit circle $\mathbb{T}$. A Carleson set is a closed subset $E \subset \mathbb{T}$ of Lebesgue measure zero for which, if the intervals $\{I_k\}$ complementary to $E$ have lengths $|I_k|$, then $\sum |I_k| \log |I_k| > -\infty$. This notion was introduced in [3], and in [9] Carleson used it to describe the sets of uniqueness of some function spaces. Corollary [1] is also useful to obtain results on the angular distribution of the $D_s$-zero sets.

Theorem 3. Let $0 < s < 1$, and $\{e^{i\theta_n}\} \subset \mathbb{T}$. The following are equivalent:

(i) the sequence $\{r_n e^{i\theta_n}\}$ is a $D_s$-zero set for any choice of $\{r_n\} \subset (0, 1)$ with $\sum (1 - r_n) < \infty$;

(ii) the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set.

As noted before, if $0 \leq s < 1$ and $\{r_n\} \subset (0, 1)$ is a Blaschke sequence that does not satisfy [1.1], then there is a sequence of angles $\{\theta_n\}$ such that $Z = \{r_n e^{i\theta_n}\}$ is not a $D_s$-zero set. The sequences doing that which have been constructed in [24] (and also the examples in [39]) satisfy that every $\xi \in \mathbb{T}$ is an accumulation point of $Z$. Ross, Richter and Sundberg proved in [35] that this can be done in $\mathcal{D}$ with a sequence $Z$ which accumulates to a single point in $\mathbb{T}$. We shall extend this result to the range $0 < s < 1$, which improves our Theorem 2 but whose proof is much more technical.

Theorem 4. Let $0 < s < 1$. Suppose that $\{r_n\} \subset (0, 1)$ satisfies

$$\sum_{n=0}^{\infty} (1 - r_n)^s = \infty.$$ 

Then there exists a sequence $\{\theta_n\}$ such that $\{r_n e^{i\theta_n}\} \cap \mathbb{T} = \{1\}$ and $\{r_n e^{i\theta_n}\}$ is not a $D_s$-zero set.

Let $X$ be a space of analytic functions in $\mathbb{D}$ contained in the Nevanlinna class (see [18]), so every function $f \in X$ has nontangential limits a.e. on $\mathbb{T}$. Denote also by $f$ the function of boundary values of $f$ (taken as a nontangential limit). A closed set $E \subset \mathbb{T}$ is called a set of uniqueness for $X$ if it has the property that $f \equiv 0$ if $f \in X$ vanishes at all points $\xi \in E$. It is well known that $E \subset \mathbb{T}$ is a set of uniqueness for a Lipschitz class $\Lambda_\alpha$ if and only if $E$ is not a Carleson set. We remind the reader that $f \in H(\mathbb{D})$ belongs to $\Lambda_\alpha$, $0 < \alpha \leq 1$, if there is $C > 0$ such that

$$|f(z) - f(w)| \leq C|z - w|^\alpha, \quad \text{for all } z, w \in \overline{\mathbb{D}}.$$ 

In [9, Theorem 5], under a very weak additional assumption, the sets of uniqueness for the classical Dirichlet space are described.

If $\alpha > 0$, we denote by $C_\alpha(E)$ the $\alpha$-capacity of a subset of $\mathbb{T}$ (see Section 4 for a definition). The following result is an extension of Theorem 5 in [9].

Theorem 5. Let $0 \leq s < \alpha < 1$ and $E \subset \mathbb{T}$ with null Lebesgue measure. Suppose that there exists $m > 0$ such that for each interval $I \subset \mathbb{T}$ centered at a point of $E$,

$$C_\alpha(E \cap I) \geq m |I|.$$ 

Then $E$ is a set of uniqueness for $D_s$ if and only if $E$ is not a Carleson set.
The paper is organized as follows. Section 2 is devoted to the study of Carleson-Newman sequences as $D_s$-zero sets proving Theorem 1, Corollary 1, Theorem 2, and Theorem 3. Theorem 3 is proved in Section 3, and Theorem 5 is proved in Section 4. In Section 5, we shall give a new proof of a result of Bogdan [7] on the description of Blaschke sets for $D$. Finally, in Section 6 between other results, we prove that $D_s$-zero sets and the zero sets of their generated Möbius invariant spaces coincide.

In the sequel, the notation $A \asymp B$ will mean that there exist two positive constants $C_1$ and $C_2$ which only depend on some parameters $p, \alpha, s, \ldots$ such that $C_1A \leq B \leq C_2A$. Also, we remark that throughout the paper we shall be using the convention that the letter $C$ will denote a positive constant whose value may depend on some parameters $p, \alpha, s, \ldots$, not necessarily the same at different occurrences.

2. Carleson-Newman $D_s$-zero sets

We first recall some useful concepts and results. The *Carleson square* $S(I)$ of an interval $I \subset \mathbb{T}$ is defined as

$$S(I) = \{ re^{i\theta} : e^{i\theta} \in I, \quad 1 - |I| \leq r < 1 \}.$$ 

Given $s > 0$ and a positive Borel measure $\mu$ on $\mathbb{D}$, we say that $\mu$ is an $s$-Carleson measure if there exists a positive constant $C$ such that

$$\mu(S(I)) \leq C|I|^s, \quad \text{for every interval } I \subset \mathbb{T}.$$ 

If $s = 1$ we simply say that $\mu$ is a Carleson measure. We recall that a sequence $\{z_n\} \subset \mathbb{D}$ is Carleson-Newman if and only if the measure $d\mu_{z_n} = \sum(1 - |z_n|)\delta_{z_n}$ is a Carleson measure (see [24] and [28]). Here, as usual, $\delta_{z_n}$ denotes the point mass at $z_n$. A Blaschke product whose zero sequence is Carleson-Newman is called a Carleson-Newman Blaschke product (a CN-Blaschke product, for short).

Let $P_z(e^{it})$ denote the Poisson kernel at a point $z \in \mathbb{D}$, so that

$$P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad e^{it} \in \mathbb{T},$$

and let

$$\Psi(z, \phi) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(e^{it})P_z(e^{it}) \, dt - \exp \left( \frac{1}{2\pi} \int_{\mathbb{T}} \log \phi(e^{it})P_z(e^{it}) \, dt \right), \quad z \in \mathbb{D},$$

where $\phi$ is a positive function which belongs to $L^1(\mathbb{T})$. Observe that the arithmetic-geometric inequality implies that $\Psi(z, \phi) \geq 0$. If $\phi \in L^2(\mathbb{T}), \quad \phi \geq 0$, we set

$$\Phi(z, \phi) = \Psi(z, \phi^2).$$

We observe that for an outer function $g \in H^2$,

$$\Phi(z, |g|) = P(|g|^2)(z) - |g(z)|^2,$$

(2.1)

where $P(|g|^2)$ is the Poisson integral of $|g|^2$.

The following result, Theorem 3.1 of [17] (see [6] for related results), characterizes the membership in $D_s$ of an outer function in terms of its modulus on the boundary.

**Theorem A.** Suppose that $0 < s < 1$ and $f$ is an outer function. Then the following are equivalent:

(i) $f \in D_s$,

(ii) $\int_{\mathbb{D}} \Phi(z, |f|) \frac{dA(z)}{(1-|z|^2)^{s+\varepsilon}} < \infty.$
In order to prove Theorem 1 we need some lemmas. The following result is implicit in some places (see e.g. [33, Theorem 5] or [15, Theorem 8]). For completeness we sketch a proof here.

**Lemma 1.** Suppose that 0 < s < 1, f ∈ D, and let B be a Carleson-Newman Blaschke product with zeros \( \{z_k\} \subset \mathbb{D} \). Then \( kB \in D \) if and only if
\[
\sum_{k=1}^{\infty} |f(z_k)|^2 (1 - |z_k|^2)^s < \infty.
\]

Moreover,
\[
||kB||^2_{D_s} \leq ||f||^2_{D_s} + \sum_{k=1}^{\infty} |f(z_k)|^2 (1 - |z_k|^2)^s.
\]

**Proof.** Suppose first that \( kB \in D \). By Theorem 4 of [16],
\[
(2.2) \quad ||kB||^2_{D_s} \leq ||f||^2_{D_s} + \int_{\mathbb{D}} |f(z)|^2 (1 - |B(z)|^2) (1 - |z|^2)^{s-2} dA(z).
\]
Since \( B \) is a CN-Blaschke product, there is a positive constant \( C \) such that (see e.g. [16, p. 15])
\[
1 - |B(z)|^2 \geq C \sum_n (1 - |z_n|^2)(1 - |z|^2)
\]
Therefore, if \( \Delta_n = \{ \theta(z, z_n) < 1/2 \} \), the subharmonicity of \( |f|^2 \) gives
\[
\sum_n |f(z_n)|^2 (1 - |z_n|^2)^s \leq C \sum_n \int_{\Delta_n} |f(z)|^2 \frac{(1 - |z|^2)^s}{|1 - \bar{z}_n z|^2} dA(z)
\]
\[
\leq C \sum_n (1 - |z_n|^2) \int_{\Delta_n} |f(z)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2} dA(z)
\]
\[
\leq C \sum_n (1 - |z_n|^2) \int_{\mathbb{D}} |f(z)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2} dA(z)
\]
\[
\leq C \int_{\mathbb{D}} |f(z)|^2 (1 - |B(z)|^2) (1 - |z|^2)^{s-2} dA(z).
\]
For the converse we refer to [3] Proposition 3.2, where an elementary proof is given. \( \square \)

Next, if \( g \in H^2 \) we shall see that the function \( \Phi(z, |g|) \), although it is superharmonic, verifies a certain sub-mean-value property.

**Lemma 2.** Suppose that \( g \) is an outer function which belongs to \( H^2 \). Then there is a constant \( M > 1 \) such that
\[
\Phi(z, |g|) \leq \frac{M}{A(D(z, r))} \int_{D(z, r)} \Phi(w, |g|) dA(w), \quad \text{for all } r \in \left(0, \frac{1 - |z|}{2}\right),
\]
where \( D(z, r) \) is the Euclidean disk of center \( z \) and radius \( r \).
Proof. Take $z \in \mathbb{D}$ and $r \in \left(0,\frac{1-|z|}{2}\right)$. Using the trivial but useful identity

$$\int_0^{2\pi} \left|g(e^{it}) - g(z)\right|^2 P_z(e^{it}) \frac{dt}{2\pi} = P(|g|^2)(z) - |g(z)|^2,$$

the subharmonicity of the function $h_t(z) = |g(e^{it}) - g(z)|^2$, Fubini’s theorem and (2.1), we obtain that

$$\Phi(z, |g|) = \int_0^{2\pi} h_t(z) P_z(e^{it}) \frac{dt}{2\pi},$$

$$\leq \int_0^{2\pi} \left( \frac{1}{A(D(z, r))} \int_{D(z, r)} h_t(w) dA(w) \right) P_z(e^{it}) \frac{dt}{2\pi}
= \frac{1}{A(D(z, r))} \int_{D(z, r)} \int_0^{2\pi} \left|g(e^{it}) - g(w)\right|^2 P_z(e^{it}) \frac{dt}{2\pi} dA(w).$$

Now, by the Harnack inequality, there is a constant $M > 1$ (we can take $M = 3$) such that

$$P_z(e^{it}) \leq MP_w(e^{it}) \quad \text{for} \ w \in D(z, r),$$

which, together with (2.3) and (2.4), gives that

$$\Phi(z, |g|) \leq \frac{M}{A(D(z, r))} \int_{D(z, r)} \int_0^{2\pi} \left|g(e^{it}) - g(w)\right|^2 P_w(e^{it}) \frac{dt}{2\pi} dA(w)
= \frac{M}{A(D(z, r))} \int_{D(z, r)} \left(P(|g|^2)(w) - |g(w)|^2 \right) dA(w)
= \frac{M}{A(D(z, r))} \int_{D(z, r)} \Phi(w, |g|) dA(w),$$

which finishes the proof. \qed

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Let $B$ be a CN-Blaschke product with zeros $\{z_n\}$, where $\{z_n\}$ is a $\mathcal{D}_s$-zero set. Thus, there is $f \in \mathcal{D}_s$ whose zero sequence is $\{z_n\}$. Since $\mathcal{D}_s$ has the property of division by inner functions (see [16]), this implies that there is an outer function $g \in \mathcal{D}_s$ such that $g \cdot B \in \mathcal{D}_s$, which together with Lemma 1, gives that

$$\sum_n |g(z_n)|^2(1 - |z_n|^2)^s < \infty.$$

(ii) $\Rightarrow$ (i). Since $B$ is a CN-Blaschke product, this follows immediately from Lemma 1.

(iii) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). Without loss of generality we may assume that $\{z_k\}$ is separated. Therefore, there is a positive constant $\varepsilon < 1$ such that the pseudohyperbolic disks $\Delta(z_k, \varepsilon)$ are pairwise disjoint.

Suppose that there is an outer function $g$ which satisfies (1.3). It is observed that

$$\sum_k (1 - |z_k|^2)^{1+s} \int_0^{2\pi} |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2}
\leq \sum_k \Phi(z_k, |g|)(1 - |z_k|^2)^s + \sum_{k=1}^{\infty} |g(z_k)|^2(1 - |z_k|^2)^s.$$
Next, bearing in mind Lemma 2, the separation of \( \{z_k\} \) and Theorem A, we deduce that

\[
\sum_k \Phi(z_k, |g|)(1 - |z_k|^2)^s \leq C \sum_k (1 - |z_k|^2)^{s-2} \int_{\Delta(z_k, \varepsilon)} \Phi(z, |g|) \, dA(z)
\]

(2.6)

\[
\leq C \sum_k \int_{\Delta(z_k, \varepsilon)} (1 - |z|^2)^{s-2} \Phi(z, |g|) \, dA(z)
\]

\[
\leq C \int_{\mathbb{D}} (1 - |z|^2)^{s-2} \Phi(z, |g|) \, dA(z) < \infty.
\]

Finally, (iii) follows from (1.3), (2.6) and (2.5).

\[ \square \]

Proof of Corollary 1. By Theorem 1 there is an outer function \( g \in \mathcal{D}_s \) such that

\[
\int_{\mathbb{T}} |g(e^{it})|^2 \left( \sum_k \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt < \infty,
\]

so bearing in mind that \( \log |g| \in L^1(\mathbb{T}) \) and the geometric-arithmetic inequality, the result follows.

\[ \square \]

Proof of Theorem 2. The same sequence given in the proof of Theorem 1 works. Choose a sequence \( \{\varepsilon_n\} \) such that \( 0 < \varepsilon_n < 1 \), \( \sum \varepsilon_n \leq 1 \) and \( \sum \varepsilon_n \log \varepsilon_n = -\infty \). Next, take disjoint open arcs of \( \mathbb{T} \) with \( |I_n| = \varepsilon_n \) converging to 1. Let \( r_n = 1 - \varepsilon_n \) and \( z_n = r_ne^{i\theta_n} \), where \( \theta_n \) is the center of \( I_n \). If \( I \) is an arc of \( \mathbb{T} \), then

\[
\sum_{z_n \in S(I)} (1 - |z_n|) \leq \sum_{I_n \subset 2I} |I_n| \leq 2|I|,
\]

proving that the measure \( \mu = \sum (1 - |z_n|)\delta_{z_n} \) is a Carleson measure. So, \( \{z_n\} \) is a Carleson-Newman sequence which accumulates only at \( \{1\} \). Moreover, since

\[
\int_{\mathbb{T}} \log \left( \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt \geq \sum_{j=1}^{\infty} \int_{I_j} \log \left( \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt
\]

\[
\geq \sum_{j=1}^{\infty} \int_{I_j} \log \left( \frac{(1 - |z_j|^2)^{1+s}}{|e^{it} - z_j|^2} \right) \, dt
\]

\[
\geq \sum_{j=1}^{\infty} |I_j| \log (4|I_j|^{s-1}) = \infty,
\]

it follows from Corollary 1 that \( \{z_n\} \) is not a \( \mathcal{D}_s \)-zero set. The proof is complete.

\[ \square \]

Proof of Theorem 3. If \( \{e^{i\theta_n}\} \) is a Carleson set and \( \sum (1 - r_n) < \infty \), then it follows from Theorem 2 that there is a function \( f \) with all derivatives bounded that vanishes only at \( \{r_ne^{i\theta_n}\} \).

Suppose now that \( E = \{e^{i\theta_n}\} \) is not a Carleson set. Let \( \{I_n\} \) be the complementary intervals of \( E \), with \( I_n = (e^{i\theta_n}, e^{i(\theta_n + |I_n|)}) \). Set \( r_n = (1 - |I_n|)e^{i\theta_n} \).
which satisfies $\sum (1 - r_n) < \infty$. Clearly, the sequence $\{z_n\} = \{r_n e^{i\theta_n}\}$ is Carleson-Newton, and arguing as in the proof of Theorem 2 we have

$$\int_{\mathbb{T}} \log \left( \sum_n \frac{(1 - |z_n|^2)^{1+s}}{|e^{it} - z_n|^2} \right) \, dt \geq C \sum_n |I_n| \log (4|I_n|^{s-1}) = \infty.$$ 

Hence, by Corollary 11 the sequence $\{r_n e^{i\theta_n}\}$ is not a $\mathcal{D}_s$-zero set. \hfill \Box

### 3. Proof of Theorem 4

Some new concepts and preliminary results will be needed in the proof of Theorem 4. For $0 < s \leq 1$, the $s$-dimensional Hausdorff capacity of $E \subset \mathbb{T}$ is determined by

$$\Lambda^\infty_s(E) = \inf \left\{ \sum_j |I_j|^s : E \subset \bigcup_j I_j \right\},$$

where the infimum is taken over all coverings of $E$ by countable families of open arcs $I \subset \mathbb{T}$.

Although we think that the next result is known, a proof is included here since we were not able to find any clear reference.

**Lemma 3.** Let $0 < s \leq 1$. Then there exists a universal constant $C$ such that $\Lambda^\infty_s(E) \geq C|E|^s$ for all $E \subset \mathbb{T}$.

**Proof.** Let $E \subset \mathbb{T}$. If $|E| = 0$, the result is clear. Suppose that $|E| > 0$ and take $\varepsilon \in \left(0, \frac{|E|^s}{2}\right)$. Then there exists a covering $\{I_j\}_j$ of $E$, such that

$$\Lambda^\infty_s(E) \geq \sum_j |I_j|^s - \varepsilon \geq \left( \sum_j |I_j| \right)^s - \varepsilon \geq |E|^s - \frac{|E|^s}{2} = \frac{|E|^s}{2}.$$

This finishes the proof. \hfill \Box

The homogeneous $\mathcal{D}_s$-capacity of a set $E \subset \mathbb{T}$ is defined by

$$\text{cap}(E, \mathcal{D}_s) = \inf \left\{ ||f||_{\mathcal{D}_s}^2 : f \in L^2(\mathbb{T}) \text{ and } f \geq 1 \text{ a.e. on } E \right\}.$$

**Lemma 4.** Let $J \subset \mathbb{T}$ be an open arc with center $e^{i\theta_0}$. Suppose that $F \in \mathcal{D}_s$ with $E = \{e^{it} \in J : |F(e^{it})| \geq 1\}$.

If $|E| \geq \frac{|J|}{2}$, then there exists a universal constant $C$ such that

$$\int_{S(J)} |F'(z)|^2 (1 - |z|^2)^s \, dA(z) \geq C|J|^s.$$

**Proof.** Let $z_0 = (1 - \frac{|J|}{2}) e^{i\theta_0}$. Arguing as in the proof of [30, Lemma 3], we deduce that there is a universal constant $C$ such that the harmonic measure of $E$ with respect to $Q := S(J)$ at $z_0$, $\mu^Q_{z_0}(E)$, satisfies

$$\mu^Q_{z_0}(E) \geq C.$$

Consider a conformal map $\varphi : \mathbb{D} \to Q$ with $\varphi(0) = z_0$ and take $g = F \circ \varphi$. Then $g \geq 1$ on $\varphi^{-1}(E)$ and $|\varphi^{-1}(E)| = \mu^Q_{z_0}(E) \geq C$. Thus, putting together (5.1.3) of [1]
and Lemma 3 we have

\[(3.1) \quad \|g\|_{D_s}^2 \geq \text{cap}\left(\phi^{-1}(E), D_s\right) \geq C \left(\Lambda^{\infty}_s \left(\phi^{-1}(E)\right)\right)^\gamma \geq C \mu_s^Q(E)^s \gamma \geq C,\]

where \(s' \in (s, 1)\) and \(\gamma \in (0, 1)\).

Next, since \(\phi\) is a conformal map (see [34, Chapter 1]),

\[(3.2) \quad (1 - |z|^2)|\phi'(z)| \approx d(\phi(z), \partial Q), \quad z \in \mathbb{D}.
\]

Moreover, since \(Q\) is convex, reasoning as in [20, Proposition 5] and bearing in mind (3.2) we obtain that

\[(3.3) \quad |\phi'(z)| \geq \frac{1}{4} |\phi'(0)| \geq C d(z_0, \partial Q) \geq C |J|,
\]

where \(d(z_0, \partial Q)\) is the Euclidean distance from \(z_0\) to \(\partial Q\).

Taking into account (3.1), (3.2) and (3.3) we deduce that

\[
\int_Q |F'(z)|^2 (1 - |z|^2)^s \, dA(z) \geq \int_Q |F'(z)|^2 d(z, \partial Q)^s \, dA(z) 
\]

\[
\geq \int_{\mathbb{D}} |g'(z)|^2 d(\phi(z), \partial Q)^s \, dA(z) 
\]

\[
\geq C \int_{\mathbb{D}} |g'(z)|^2 \left( (1 - |z|^2)|\phi'(z)| \right)^s \, dA(z) 
\]

\[
\geq C |J|^s \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2)^s \, dA(z) 
\]

\[
\geq C |J|^s.
\]

This finishes the proof. \(\square\)

**Proof of Theorem 4.** Let \(\{r_n\} \subset (0, 1)\) be an increasing sequence such that

\[
\sum_n (1 - r_n)^s = \infty.
\]

We can find

\[
1 \leq n_1 < m_1 < n_2 < m_2 < \cdots < n_k < m_k < \cdots
\]

such that

\[(3.4) \quad (1 - r_n)^{1-s} < k^{-2}e^{-2k^2} \quad \text{if} \quad n \geq n_k, \quad k = 1, 2, \ldots \]

and

\[
ke^{2k^2} \leq \sum_{n=n_k}^{m_k} (1 - r_n)^s < ke^{2k^2} + 1, \quad k = 1, 2, \ldots.
\]

For each \(k\), lay out arcs \(J_{n_k}, J_{n_k+1}, \ldots, J_{m_k}\) on the unit circle end-to-end starting at \(e^{i\theta} = 1\) and such that

\[(3.5) \quad |J_n| = (1 - r_n)^{k-2}e^{-2k^2}, \quad n_k \leq n \leq m_k.
\]

Observe that (3.4) together with (3.5) implies that

\[(3.6) \quad |J_n| > (1 - r_n).
\]
Let $e^{it_n}$ be the center of $J_n$ and set $\lambda_n = (1 - r_n)e^{it_n}$. Suppose that there is $F \in \mathcal{D}_s$ with $F(\lambda_n) = 0$ for all $n_k \leq n \leq m_k$. By [6, Theorem 3.4] we may assume that $||F||_H \leq 1$. Set

$$A_k = \left\{ n : n_k \leq n \leq m_k \text{ and } |F| \geq e^{-k^2} \text{ on a set } E_n \subset J_n \text{ with } |E_n| \geq \frac{|J_n|}{2} \right\},$$

$$B_k = \left\{ n : n_k \leq n \leq m_k, \ n \notin A_k \right\}.$$

Using Lemma [4] and [3,6] with $S(J_n)$, $n \in A_k$, we deduce that

$$\int_{S(J_n)} |F'(z)|^2 (1 - |z|^2)^s \, dA(z) \geq Ce^{-2k^2}|J_n|^s \geq Ce^{-2k^2}(1 - r_n)^s.$$

Moreover if $n \in B_k$,

$$\int_{J_n} \log \frac{1}{|F(\xi)|} \, d\xi \geq \frac{1}{2}k^2|J_n| = \frac{1}{2}(1 - r_n)^se^{-2k^2}.$$ 

So, bearing in mind [3],

$$\sum_{n \in A_k} \int_{S(J_n)} |F'(z)|^2 (1 - |z|^2)^s \, dA(z) + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|F(\xi)|} \, d\xi \geq Ce^{-2k^2} \sum_{n=n_k}^{m_k} (1 - r_n)^s \geq Ck,$$

which together with the integrability of $\log |F|$ on the boundary (see Theorem 2.2 of [18]), implies that $F$ must be the zero function. Finally, arguing as in the proof of Theorem 2 of [30], the proof can be finished. 

\[\square\]

4. Zeros on the boundary. Sets of uniqueness

In order to prove Theorem 3 the notion of $\alpha$-capacity must be introduced. We shall recall some definitions (see [41] and [8]). Given $E \subset [0, 2\pi)$, let $\mathcal{P}(E)$ be the set of all probability measures supported on $E$. If $\alpha > 0$ and $\sigma \in \mathcal{P}(E)$, the $\alpha$-potential associated to $\sigma$ is

$$U_\alpha \sigma(\tau) = \int_{E} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha}.$$ 

Let

$$V_{E,\alpha} = \inf \int_{E} U_\alpha \sigma(\tau) \, d\sigma(\tau),$$

where the infimum is taken over all $\sigma \in \mathcal{P}(E)$. If $V_{E,\alpha} < \infty$, there is $\mu \in \mathcal{P}(E)$ where the value $V_{E,\alpha}$ is attained, and that measure $\mu$ is called the equilibrium distribution for the $\alpha$-potentials of $E$. It is known that $U_\alpha \mu(\tau) = V_{E,\alpha}$ for a.e. $(\mu)$. The $\alpha$-capacity of $E$ is determined by

$$C_\alpha(E) = (V_{E,\alpha})^{-1}.$$

Proof of Theorem 5. Suppose that $E$ is a set of uniqueness for $\mathcal{D}_s$. Then $E$ is also a set of uniqueness for any Lipschitz class $\Lambda_\beta$ with $\beta > \frac{1-s}{2}$, due to $\Lambda_\beta \subset D_s$. So, by Theorem 1 of [39], $E$ is not a Carleson set.

For the converse, we shall follow the argument in the proof of Theorem 5 in [39]. Let $\mu$ be the equilibrium distribution for the $\alpha$-potentials of $E$. Then, if $\{\gamma_n\}$ are
the Fourier-Stieltjes coefficients of \( \mu \), there is a constant \( C \) which only depends on \( \alpha \) such that

\[
\sum_n n^{\alpha-1} |\gamma_n|^2 \leq CV_{E,\alpha}.
\]  

Suppose that there is a bounded function \( f \in \mathcal{D}_s, f \neq 0 \), that vanishes on \( E \). We shall see that this leads to a contradiction. The function \( h(\theta) = |f(e^{i\theta})| \) can be written as

\[
h(\theta) = \sum_n c_n e^{in\theta},
\]

where

\[
\sum_n n^{1-s} |c_n|^2 < \infty.
\]

For each \( t \in (0, \pi) \), let us consider \( h_t(\theta) = \frac{1}{2t} \int_{\theta-t}^{\theta+t} h(s) \, ds \). Integrating the Fourier series of \( h \), it follows that the Fourier coefficients of \( h_t \) are \( \sin(nt) c_n \). Then by (4.1) and Schwarz’s inequality,

\[
\int_E h_t(\theta) \, d\mu(\theta) = \left| \int_E (h_t(\theta) - h(\theta)) \, d\mu(\theta) \right|
\geq \sum_n \left( 1 - \frac{\sin(nt)}{nt} \right) c_n \int_E e^{in\theta} \, d\mu(\theta)
\leq C \sum_n \left( 1 - \frac{\sin(nt)}{nt} \right) |c_n||\gamma_n|
\leq C \left( \sum_n \left( 1 - \frac{\sin(nt)}{nt} \right)^2 |c_n|^2 n^{1-\alpha} \right)^{\frac{1}{2}} \left( \sum_n n^{\alpha-1} |\gamma_n|^2 \right)^{\frac{1}{2}}.
\]  

We claim that there is \( C > 0 \) such that

\[
n^{s-\alpha} \left( 1 - \frac{\sin(nt)}{nt} \right)^2 \leq Ct^{\alpha-s}, \quad t > 0, \quad n = 1, 2, \ldots.
\]

If \( nt \leq 1 \), there is a positive constant \( C \) which does not depend on \( n \) or \( t \), such that

\[
1 - \frac{\sin(nt)}{nt} \leq C(nt)^2,
\]

so

\[
n^{s-\alpha} \left( 1 - \frac{\sin(nt)}{nt} \right)^2 \leq C^2 n^{s-\alpha} (nt)^4 \leq C^2 n^{s-\alpha} (nt)^{\alpha-s} \leq C^2 t^{\alpha-s}.
\]

On the other hand, if \( nt \geq 1 \), bearing in mind that \( 1 - \frac{\sin(\theta)}{\theta} \) is a bounded function of \( \theta \), we deduce that

\[
n^{s-\alpha} \left( 1 - \frac{\sin(nt)}{nt} \right)^2 \leq C n^{s-\alpha} \leq C t^{\alpha-s},
\]

which together with (4.4) gives (4.6).

Therefore, using (4.3), (4.4), (4.1) and (4.2), it follows that

\[
\int_E h_t(\theta) \, d\mu(\theta) \leq Ct^{\frac{\alpha-s}{\alpha}} \left( \sum_n n^{1-s} |c_n|^2 \right)^{\frac{1}{2}} \left( \sum_n n^{\alpha-1} |\gamma_n|^2 \right)^{\frac{1}{2}}
\leq Ct^{\frac{\alpha-s}{\alpha}} ||f||_{D_s V_{E,\alpha}}^{1/2}.
\]
Now, let \( k_n \) be the number of complementary intervals of \( E \) whose lengths are in \([2^{-n}, 2^{-n+1}]\). Since \( E \) is not a Carleson set,

\[
\sum \frac{n k_n}{2^n} = \infty.
\]

Let \( \{\omega_i\}_{i=1}^{k_n} \) be those intervals, and let \( \{\theta_i\}_{i=1}^{2k_n} \) be the endpoints of \( \{\omega_i\}_{i=1}^{k_n} \). We consider the open intervals \( \delta_i \) of length \( 2^{-n} \) with midpoints \( \theta_i \). Take \( \gamma \in (0, \frac{\alpha-\delta}{2}) \) and let \( S \) be the set of those \( \delta_i \) such that

\[
h_{\gamma}(\theta_i) > 2^{-\gamma}, \quad \tau = 2^{-n}.
\]

Observe that \((4.8)\) implies that \( h_{2\gamma}(\theta) > 2^{-\gamma n-1} \) holds for \( \theta \in \delta_i \) whenever \( \delta_i \in S \), which, together with the general relation \((4.9)\), gives that for \( \mu^* \) the equilibrium distribution for the \( \alpha \)-potentials of \( E \cap S \),

\[
2^{-\gamma n-1} \leq \int_{E \cap S} h_{\gamma}(\theta) \, d\mu^*(\theta) \leq CV_{E \cap S}^{1/2} 2^{-n(\alpha-s)/2},
\]

so

\[
C_\alpha(E \cap S) \leq C 2^{(2\gamma-(\alpha-s))n}.
\]

Let \( N \) be the number of intervals \( \delta_i \) which belong to \( S \). We shall estimate \( N \) using condition \((1.6)\). Take \( \mu_i \) to be the equilibrium distribution for the \( \alpha \)-potentials of \( E \cap \delta_i \). Let us consider \( \sigma = N^{-1} \sum_{\delta_i \subset S} \mu_i \), and let \( u \) be the corresponding \( \alpha \)-potential. Let \( \tau \in \delta_i \) where \( \delta_i \in S \), and let \( \delta_{i-1} \) and \( \delta_{i+1} \) be the intervals in \( S \) which are on the left and on the right of \( \delta_i \). We shall define \( F = \{k-1, k, k+1\} \). Then bearing in mind that the intervals \( \{\delta_j\} \) are disjoint, the distance between the intervals \( \{\delta_j\} \), and condition \((1.6)\) we deduce that

\[
u(\tau) = \int_{E \cap S} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha}
\leq \sum_{j \in F} \int_{\delta_j \cap S} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha} + \sum_{j=1}^{N} \int_{\delta_j \cap S} \frac{d\sigma(\theta)}{|\theta - \tau|^\alpha}
\leq N^{-1} \left( \sum_{j \in F} \int_{\delta_j \cap S} \frac{d\mu_j(\theta)}{|\theta - \tau|^\alpha} + \sum_{j=1}^{N} \int_{\delta_j \cap S} \frac{d\mu_j(\theta)}{|\theta - \tau|^\alpha} \right)
\leq CN^{-1} \left( 2^n + \sum_{j=1}^{N} \frac{1}{(j2^{-n})^\alpha} \right)
\leq CN^{-1/2} 2^n,
\]

which together with \((4.9)\) gives

\[
N^{-1/2} 2^n \geq C u \geq \frac{C}{C_\alpha(E \cap S)} \geq C 2^{(2\gamma-(\alpha-s))n},
\]

so due to \( \gamma < \frac{\alpha-\delta}{2} \), one obtains

\[
N \leq C 2^{pn}, \quad \text{for some } p \in (0, 1).
\]
If \( \omega = (\theta_{2\nu}, \theta_{2\nu}) \) and (1.8) does not hold for \( \theta_{2\nu-1} \) and \( \theta_{2\nu} \), then by the arithmetic-geometric inequality,

\[
\frac{1}{|\omega|} \int_{\omega} \log h(\theta) \, d\theta \leq \log \left( \frac{1}{|\omega|} \int_{\omega} h(\theta) \, d\theta \right)
\leq \log \left( \frac{1}{|\omega|} \left( \int_{\theta_{2\nu-1}}^{\theta_{2\nu+1}} h(\theta) \, d\theta + \int_{\theta_{2\nu-2}}^{\theta_{2\nu-2}} h(\theta) \, d\theta \right) \right)
= \log \left( \frac{2^{-(n+1)}}{|\omega|} (h_r(\theta_{2\nu-1}) + h_r(\theta_{2\nu})) \right)
\leq -\gamma n + C.
\]

By (1.10), the number of indices \( n \) for which the above inequality is true is greater than \( kn - 2N \geq \gamma n - C \). Hence

\[
\sum_{\nu=1}^{k_n} \int_{\omega_{\nu}} \log h(\theta) \, d\theta \leq -\gamma n 2^{-n}(k_n - C2^{2n}) + C \sum_{\nu=1}^{k_n} |\omega_{\nu}|,
\]

which, joined to the fact that \( p < 1 \), gives

\[
\int_{0}^{2\pi} \log h(\theta) \, d\theta \leq -\gamma \sum_{n} n 2^{-n} k_n + C.
\]

Consequently, bearing in mind that \( \gamma > 0 \) and (4.7), this implies a contradiction. \( \square \)

5. Blaschke sets

A subset \( A \) of the unit disc \( \mathbb{D} \) is called a Blaschke set for \( \mathcal{D} \) if any Blaschke sequence with elements in \( A \) is a zero set of \( \mathcal{D} \). These sets were characterized by Bogdan in [7]. Here we shall give a new proof of that result.

**Theorem 6.** \( A \subset \mathbb{D} \) is a Blaschke set for \( \mathcal{D} \) if and only if

\[
\int_{\mathbb{T}} \log \text{dist}(e^{it}, A) \, dt > -\infty.
\]

Some definitions and results will be introduced. A tent is an open subset \( T \) of \( \mathbb{D} \) bounded by an arc \( I \subset \mathbb{T} \) with \( |I| < \frac{1}{4} \) and two straight lines through the endpoints of \( I \) forming with \( I \) an angle of \( \frac{\pi}{4} \). The closed arc \( \overline{T} \) will be called the base of the tent \( T = T_I \). A tent \( T \) is said to support \( A \) if \( T \cap A = \emptyset \) but \( T \cap \overline{A} \neq \emptyset \). A finite or countable collection of tents \( \{T_n\} \) is an A-belt if \( \{T_n\} \) are pairwise disjoint, \( A \)-supporting and \( \mathbb{T} \setminus \overline{A} \subset \bigcup_n \overline{T_n} \). The following result can be found in [24, Lemma 1].

**Lemma B.** Let \( A \subset \mathbb{D} \) such that \( \overline{A} \neq \emptyset \). Let \( \{T_{I_n}\} \) be an A-belt. Then (5.1) holds if and only if \( \overline{A} \cap \mathbb{T} \) has zero Lebesgue measure, and

\[
\sum_{n} |I_n| \log \left( \frac{e}{|I_n|} \right) < \infty.
\]

**Lemma 5.** Let \( \{z_n\} \) be a \( \mathcal{D} \)-zero set. If \( \{\lambda_n\} \subset \mathbb{D} \) satisfies that \( g(z_n, \lambda_n) < \delta < 1 \) for each \( n \), then \( \{\lambda_n\} \) is a \( \mathcal{D} \)-zero set.
Proof. Since \( Z = \{ z_n \} \) is a \( D \)-zero set, there is a function \( g \) in \( D \) such that \( gB_Z \in D \), where \( B_Z \) is the Blaschke product with zeros \( \{ z_n \} \). By Carleson’s formula for the Dirichlet integral (see [11] and also [35]), we have

\[
\| gB_A \|_D^2 = \| g \|_D^2 + \int \sum_n P_{\lambda_n}(e^{it}) |g(e^{it})|^2 \, dt \\
\leq \| g \|_D^2 + C \int \sum_n P_{\lambda_n}(e^{it}) |g(e^{it})|^2 \, dt \\
\leq C \| gB_Z \|_D^2 < \infty.
\]

Hence, \( \{ \alpha_n \} \) is a \( D \)-zero set, and the proof is complete. \( \square \)

Remark 1. Note that this result implies that, if \( A \) is a Blaschke set for \( D \) and \( \{ w_k \} \) is a sequence such that \( \rho(\{ w_k \}, A) \leq C < 1 \), then \( A \cup \{ w_k \} \) is also a a Blaschke set for \( D \).

Proof of Theorem 6. Suppose that (5.1) holds, and let \( Z \) be a Blaschke sequence of points in \( A \). Then

\[
\int_T \log \text{dist}(e^{it}, Z) \, dt > -\infty,
\]

and by a result of Taylor and Williams in [40], \( Z \) is a \( \Lambda_\alpha \)-zero set for any \( \alpha \). Since \( \Lambda_\alpha \subset D \) for \( \alpha > \frac{1}{2} \), it follows that \( A \) is a Blaschke set for \( D \).

Suppose that \( A \) is a Blaschke set for \( D \). We shall use Lemma B to see that (5.1) holds. Suppose that \( |A \cap T| > 0 \). Then we can choose a sequence \( \{ \varepsilon_n \} \) of positive numbers satisfying

\[
\sum_n \varepsilon_n \leq |A \cap T|, \quad \sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \infty,
\]

and a collection of disjoint arcs \( \{ I_n \} \) in \( T \) such that

\[
|I_n| = \varepsilon_n, \quad I_n \cap \overline{A} \neq \emptyset, \quad n \geq 1.
\]

In order to construct this sequence of subsets \( \{ I_n \} \), take \( I_1 \) with \( |I_1| = \varepsilon_1 \) and \( I_1 \cap \overline{A} \neq \emptyset \), and once \( I_n \) has been taken, choose \( I_{n+1} \) such that \( I_{n+1} \cap (\overline{A} \setminus \bigcup_{j=1}^n I_j) \neq \emptyset \) with \( |I_{n+1}| = \varepsilon_{n+1} \).

Next, take a sequence \( \{ w_n \} \subset A \) such that \( \text{dist}(w_n, I_n \cap \overline{A}) \leq \varepsilon_n \) and let \( p_n \) be the integer part of \( \varepsilon_n/(1 - |w_n|) \). Let \( Z \) be the sequence of points in \( A \) that consists of \( p_n \) repetitions of each point \( w_n \). Observe that \( Z \) is a Blaschke sequence,

\[
\sum_{z \in Z} (1 - |z|) = \sum_n p_n(1 - |w_n|) \leq \sum_n \varepsilon_n < \infty,
\]
so that $Z$ must be a sequence of zeros of $D$. We also have
\[ \int_{T} \log \left( \sum_{z \in Z} \frac{1 - |z|^2}{|e^{it} - z|^2} \right) dt = \int_{T} \log \left( \sum_{n} p_n \frac{1 - |w_n|^2}{|e^{it} - w_n|^2} \right) dt \]
\[ \geq \sum_{k} \int_{I_k} \log \left( p_k \frac{1 - |w_k|^2}{|e^{it} - w_k|^2} \right) dt \]
\[ \geq \sum_{k} |I_k| \log \left( \frac{1}{8 \epsilon_k} \right) = \infty, \]
which gives a contradiction with condition (1.5). Therefore, $\bar{A} \cap T$ has zero Lebesgue measure.

Next, let $\{T_n\}$ be an $A$-belt. Then for each $n$ there is $w_n \in \bar{A} \cap \partial T_n$. We may assume that $w_n$ belongs to $A$. Indeed, if $w_n$ is an endpoint of the arc $I_n$, there is a point $\alpha_n \in A$ which is in the Stolz angle with vertex $w_n$ and aperture $\frac{\pi}{2}$. Consequently, if $\tilde{\alpha}_n$ is the closest point in $\partial T_n$ with the same modulus as $\alpha_n$, then $g(\alpha_n, \tilde{\alpha}_n) \leq C < 1$, where $C$ is independent of $n$, and now we can use the remark after Lemma 5.

Let $v_n$ be the vertex of the tent $T_n$. Since $\{I_n\}$ is a sequence of disjoint arcs, $\{v_n\}$ is a Blaschke sequence. We denote by $q_n$ the integer part of $(1 - |v_n|)/(1 - |w_n|)$ and we consider $Z$ to be the sequence of points in $A$ that consists of $q_n$ repetitions of each point $w_n$. Arguing as before, it follows that $Z$ is a Blaschke sequence, and moreover there is $C > 0$ such that
\[ |w_n - e^{it}|^2 \leq C |v_n - e^{it}|^2, \quad \text{for each } n \text{ and } e^{it} \in T. \]

So, bearing in mind that $A$ is a Blaschke set for $D$, (1.5) and (5.2), we have that
\[ \infty > \int_{T} \log \left( \sum_{z \in Z} \frac{1 - |z|^2}{|e^{it} - z|^2} \right) dt = \int_{T} \log \left( \sum_{n} q_n \frac{1 - |w_n|^2}{|e^{it} - w_n|^2} \right) dt \]
\[ \geq \int_{T} \log \left( C \sum_{n} q_n \frac{1 - |w_n|^2}{1 - |v_n|^2} \frac{1 - |v_n|^2}{|e^{it} - v_n|^2} \right) dt \]
\[ \geq \int_{T} \log \left( \sum_{n} C \frac{1 - |v_n|^2}{|e^{it} - v_n|^2} \right) dt \]
\[ \geq \sum_{k} \int_{I_k} \log \left( \frac{C}{|I_k|} \right). \]

This finishes the proof. \qed

6. Other results

6.1. Other necessary angular conditions on $D$-zero sets. First we shall prove the following result of its own interest.
Lemma 6. Suppose that $0 < s < 1$, $B$ is a Blaschke product with ordered sequence of zeros $\{z_k\}_{k=1}^\infty$ and $f \in D_s$. Then
\[
\|fB\|_{D_s}^2 \simeq \|f\|_{D_s}^2 + \sum_{k=1}^\infty (1 - |z_k|^2) \int_D \frac{|f(z)|^2|B_k(z)|^2}{|1 - \overline{z}z|^2} \frac{dA(z)}{(1 - |z|^2)^{1-s}},
\]
where $B_k(z)$ is the Blaschke product of the first $k - 1$ zeros.

Proof. Bearing in mind (2.2), the result follows from the identity (see [3, p. 191])
\[
\frac{1}{1 - |z|^2} - \frac{1}{1 - |B(z)|^2} = \sum_k |B_k(z)|^2 \frac{1 - |z_k|^2}{|1 - \overline{z}z|^2}, \quad z \in \mathbb{D}.
\]

We also obtain different conditions from (1.4) (which can work for any Blaschke sequence) on the angular distribution of a Blaschke sequence $\{z_k\}$ to be a $D_s$-zero set, $0 < s < 1$.

Proposition 1. Suppose that $0 < s < 1$ and $\{z_k\} \subset \mathbb{D}$. If there exists $r_0 \in (0, 1)$ such that
\[
M(\{z_k\}) \overset{\text{def}}{=} \inf_{r_0 \leq |z| < 1} \sum_k \frac{(1 - |z_k|^2)(1 - |z|^2)^s}{|1 - \overline{z}z|^2} > 0,
\]
then $\{z_k\}$ is not a $D_s$-zero set.

Proof. Suppose that $\{z_k\}$ is a $D_s$-zero set and satisfies (6.1). Then, there exists $F \in D_s$ which vanishes uniquely on $\{z_k\}$, so $F = f \cdot B$, where $f \in D_s$ and $B$ is the Blaschke product with zeros $\{z_k\}$. Thus, Lemma 6 and (6.1) imply that
\[
\infty > \sum_k (1 - |z_k|^2) \int_D \frac{|f(z)|^2|B_k(z)|^2}{|1 - \overline{z}z|^2} \frac{dA(z)}{(1 - |z|^2)^{1-s}} \\
\geq \int_D |f(z)|^2|B(z)|^2 \left( \sum_k \frac{(1 - |z_k|^2)(1 - |z|^2)^s}{|1 - \overline{z}z|^2} \right) \frac{dA(z)}{(1 - |z|^2)} \\
\geq M(\{z_k\}) \int_D |F(z)|^2 \frac{dA(z)}{(1 - |z|^2)};
\]
consequently $F \equiv 0$. This finishes the proof.

This result allows us to make constructions of Blaschke sequences which are not $D_s$-zero sets.

Corollary 2. For $0 < s < 1$, set
\[
z_{k,j}^{(s)} \overset{\text{def}}{=} \left(1 - 2^{-1-s} \right) \exp \left( \frac{2\pi j}{2^{k-1}} \right), \quad k = 0, 1, 2, \ldots, \\
\quad j = 0, 1, \ldots, 2^k - 1.
\]
The sequence $\{z_{k,j}^{(s)}\}$ is not a $D_s$-zero set.

Proof. There is $\beta = \beta(s) > 0$ such that for each $z \in \mathbb{D}$ we can find a pair $(k(z), j(z))$ with $1 - |z| > 1 - |z_{k(z),j(z)}^{(s)}|$, and
\[
|1 - \overline{z_{k(z),j(z)}^{(s)} z}|^2 \leq \beta(1 - |z|^2)^{1+s}.
\]
Therefore
\[ \sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} \frac{(1 - |z_{k,j}|^2)(1 - |z|^2)^s}{|1 - \overline{z}_{k,j}z|^2} \geq \frac{(1 - |z_{k(j,z)}|^2)(1 - |z|^2)^s}{|1 - \overline{z}_{k,j}z|^2} \geq C\beta^{-1}, \]
so, by Proposition 2 \( \{z^{(s)}\} \) is not a \( D_s \)-zero set. \( \square \)

6.2. M"obius invariant spaces generated by \( D_s \). The space \( \mathcal{Q}_s \), \( 0 \leq s < \infty \), is the M"obius invariant space generated by \( D_s \), that is, \( f \in \mathcal{Q}_s \) if
\[ \sup_{a \in \mathbb{D}} \| f \circ \varphi_a - f(n) \|_{D_s}^2 < \infty. \]

It is known that \( \mathcal{Q}_1 \) coincides with \( BMOA \). However, if \( 0 < s < 1 \), \( \mathcal{Q}_s \) is a proper subspace of \( BMOA \) and has many interesting properties (see the detailed monograph \([42]\)).

As usual, for a space of analytic functions \( X \), we shall write \( M(X) \) for the algebra of (pointwise) multipliers of \( X \), that is,
\[ M(X) \overset{\text{def}}{=} \{ g \in H(\mathbb{D}) : gf \in X \text{ for all } f \in X \}. \]

**Theorem 7.** Suppose that \( 0 < s \leq 1 \). Then \( \mathcal{D}_s, \mathcal{Q}_s, \mathcal{Q}_s \cap H^\infty \) and \( M(\mathcal{D}_s) \) have the same zero sets.

**Proof.** If \( s = 1 \), the result is well known because \( \mathcal{D}_1 = H^2, M(H^2) = H^\infty \) and \( \mathcal{Q}_1 = BMOA \). If \( 0 < s < 1 \), by \([26]\) Corollary 13 the zeros sets of \( \mathcal{D}_s \) and \( M(\mathcal{D}_s) \) coincide, so the result follows from the chain of embeddings (see \([4]\) Lemma 5.1))
\[ M(\mathcal{D}_s) \subset \mathcal{Q}_s \cap H^\infty \subset \mathcal{Q}_s \subset \mathcal{D}_s. \]
This finishes the proof. \( \square \)

Since from different values of \( s \in (0,1) \), the \( \mathcal{D}_s \)-zero sets are not the same, we obtain directly the following result.

**Corollary 3.** Suppose that \( 0 \leq s < p < 1 \). Then there exists \( Z \subset \mathbb{D} \), which is a \( \mathcal{Q}_p \)-zero set but not a \( \mathcal{Q}_s \)-zero set.

A stronger result, in the following sense, can be proved. A sequence \( \{z_n\} \) interpolating for \( \mathcal{Q}_p \cap H^\infty \), \( 0 < p < 1 \), if for each bounded sequence \( \{w_k\} \) of complex numbers, there exists \( f \in \mathcal{Q}_p \cap H^\infty \) such that \( f(z_k) = w_k \) for all \( k \). A characterization of these sequences in terms of \( p \)-Carleson measures is given in \([30]\).

It is clear that each interpolating sequence for \( \mathcal{Q}_p \cap H^\infty \) is a \( \mathcal{D}_p \)-zero set.

**Theorem 8.** Suppose that \( 0 < s < p < 1 \). Then, there exists \( Z = \{z_n\}_{n=0}^{\infty} \subset \mathbb{D} \) which is an interpolating sequence for \( \mathcal{Q}_p \cap H^\infty \) and such that it is not a \( \mathcal{D}_s \)-zero set.

**Proof.** Set
\[ z_n = \left( 1 - \frac{1}{n^{1/s}} \right) e^{i\theta_n}, \quad n = 2, 3, \ldots, \]
where
\[ \theta_n = \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{2n}, \quad n = 2, 3, \ldots. \]

The proof of \([29]\) Theorem 5.10 \] gives that \( \{z_n\} \) is not a \( \mathcal{D}_p \)-zero set. Moreover, borrowing the argument of the proof of \([32]\) Theorem 2, we have that \( \{z_n\} \) is
Corollary 4. Suppose that $0 < s < p < 1$. Then

$$M(Q_p, Q_s) \overset{\text{def}}{=} \{ g \in H(D) : gf \in Q_s \text{ for all } f \in Q_p \} = \{0\}.$$  

Proof. Suppose that $M(Q_p, Q_s) \neq \{0\}$. Let $g \in M(Q_p, Q_s)$, $g \neq 0$ and denote by $W$ its zero set. By Corollary 3 there exists $f \in Q_p$, $f \neq 0$, whose sequence of zeros $Z$ is not a $Q_s$-zero set. It is clear that $Z \cup W$ is the zero set of $fg \in Q_s$, and since $g \in Q_s$, $W$ satisfies the Blaschke condition. Now, taking $B$ to be the Blaschke product with zeros $W$ and bearing in mind that $Q_s$ has the $f$-property (see Corollary 1 of [14] or Corollary 5.4.1 of [42]), we obtain that $\frac{f}{g} \in Q_s$, whose zero set is $Z$. This finishes the proof.  

7. Further remarks

We would like to emphasize that conditions $(ii)$ and $(iii)$ of Theorem 1 are equivalent when $\{z_n\}$ is a finite union of separated Blaschke sequences. So, it seems natural to ask whether or not for finite unions of separated Blaschke sequences, condition $(ii)$ implies that $\{z_n\}$ is a $D_s$-zero set. Although we are not able to answer this question, if the function $g$ has some additional regularity properties, one can prove that condition $(ii)$ implies that $\{z_n\}$ is a $D_s$-zero set, as the following result shows.

Proposition 2. Let $\{z_n\} \subset D$ be a Blaschke sequence, $0 < s < 1$ and $\alpha > \frac{1-s}{2}$. If there exists a function $g \in \Lambda_\alpha$ such that

$$\sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty,$$

then $\{z_n\}$ is a $D_s$-zero set.

Proof. Let $B$ be the Blaschke product with zeros $\{z_n\}$. We shall prove that $gB \in D_s$. Using the fact that $g \in \Lambda_\alpha$, and [43] Lemma 4.2.2, one has

$$\sum_n (1 - |z_n|^2) \int_D |g(z) - g(z_n)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2} \, dA(z)$$

$$\leq C \sum_n (1 - |z_n|^2) \int_D \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2 - 2^n} \, dA(z)$$

$$\leq C \sum_n (1 - |z_n|^2) < \infty. \tag{7.1}$$

Also, by our assumption and [43] Lemma 4.2.2,

$$\sum_n (1 - |z_n|^2)|g(z_n)|^2 \int_D \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2} \, dA(z)$$

$$\leq C \sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty. \tag{7.2}$$
Now, since $\Lambda_\alpha \subset D_s$ for $\alpha > \frac{1-s^2}{2}$, it follows easily from (7.1) and (7.2) that
\[
\|gB\|_{D_s}^2 \leq C\|g\|_{D_s}^2 + C\int_D |(gB')(z)|^2 (1-|z|^2)^s \, dA(z) < \infty.
\]

In view of all this, we state the following related problem.

**Problem.** For $0 < s < 1$, describe those separated Blaschke sequences $\{z_n\} \subset \mathbb{D}$ such that there is $g \in D_s$, $g \neq 0$, with
\[
\sum_n |g(z_n)|^2 (1-|z_n|^2)^s < \infty.
\]

Another interesting problem is to find sufficient conditions in order for a sequence $\{z_n\}$ to be a zero set for the analytic Besov space $B_p$, $1 < p < \infty$ (see [43, Chapter 5]). Since the point evaluations are bounded linear functionals in $B_p$, there are reproducing kernels $k_z \in B_{p'}$, where $p'$ is the conjugate exponent of $p$. Also, it is well known that
\[
\|k_z\|_{B_{p'}}^{-p} \approx \left(\log \frac{1}{1-|z|}\right)^{-(p-1)}.
\]
So, bearing in mind (1.1), it seems natural to ask the following.

**Question.** Let $1 < p < \infty$, and let $\{z_n\} \subset \mathbb{D}$ such that
\[
\sum_n \left(\log \frac{1}{1-|z_n|^2}\right)^{-(p-1)} < \infty.
\]
Is the sequence $\{z_n\}$ a $B_{p'}$-zero set?

In order to answer that question, it seems that a more constructive proof of the case $p = 2$ (the Shapiro-Shields result [39]) must be given, not relying so heavily on Hilbert space techniques.

**References**


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Departament de Matemàtica Aplicada i Analisi, Universitat de Barcelona, 08007 Barcelona, Spain
E-mail address: jordi.pau@ub.edu

Departament de Anàlisis Matemàtico, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain
E-mail address: japelaez@uma.es