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UNIVERSAL BOUNDS FOR EIGENVALUES OF THE POLYHARMONIC OPERATORS

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Abstract. We study eigenvalues of polyharmonic operators on compact Riemannian manifolds with boundary (possibly empty). In particular, we prove a universal inequality for the eigenvalues of the polyharmonic operators on compact domains in a Euclidean space. This inequality controls the kth eigenvalue by the lower eigenvalues, independently of the particular geometry of the domain. Our inequality is sharper than the known Payne-Pólya-Weinberg type inequality and also covers the important Yang inequality on eigenvalues of the Dirichlet Laplacian. We also prove universal inequalities for the lower order eigenvalues of the polyharmonic operator on compact domains in a Euclidean space which in the case of the biharmonic operator and the buckling problem strengthen the estimates obtained by Ashbaugh. Finally, we prove universal inequalities for eigenvalues of polyharmonic operators of any order on compact domains in the sphere.

1. Introduction

Let Ω be a connected bounded domain with smooth boundary in an n (≥ 2)-dimensional Euclidean space $\mathbb{R}^n$ and let ν be the outward unit normal vector field of $\partial\Omega$. Denote by Δ the Laplace operator on $\mathbb{R}^n$ and let l be a positive integer. Solutions of $\Delta u = 0$ on a domain $\Omega \subset \mathbb{R}^n$ are of course the classical harmonic functions which describe the equilibrium position of an elastic homogeneous membrane. Solutions of $\Delta^2 u = 0$ are called biharmonic, and they model equilibria of homogeneous plates. Similarly, solutions of $\Delta^l u = 0,\ l \in \mathbb{N}$, are called polyharmonic.

One then naturally considers the eigenvalue problem

$$\begin{aligned} &(-\Delta)^l u = \lambda u \quad \text{in } \ \Omega, \\ &u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0. \end{aligned} \tag{1.1}$$

Let

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

denote the successive eigenvalues, where each eigenvalue is repeated according to its multiplicity.

The case $l = 1$ has of course been well-studied, beginning with the work of Weyl [We] and Courant-Hilbert [CH]. But also for $l \geq 2$, polyharmonic functions have

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interesting applications in physics. For example, the Airy function in mechanics is a bi-harmonic function. More generally, a clamped plate in equilibrium is a solution of the bi-harmonic problem

$$(-\Delta)^2 v = 0 \quad \text{in} \quad \Omega,$$
$$v|_{\partial\Omega} = \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

in a two-dimensional Ω. An oscillating clamped plate then satisfies

$$(-\Delta)^2 v + v_{tt} = 0 \quad \text{in} \quad \Omega,\ t \geq 0,$$
$$v|_{\partial\Omega} = \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0,$$

and a separation of variables $v(x, y, t) = u(x, y)g(t)$ leads to the eigenvalue problem

$$(-\Delta)^2 u = \lambda u \quad \text{in} \quad \Omega,$$
$$u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0.$$

This problem has already been studied by Courant [Co]. He derived the Weyl type law

$$\lambda_k \sim \left(\frac{4\pi k}{area(\Omega)}\right)^2.$$

In this paper, we investigate the eigenvalues of the problem (1.1) for general l. We are interested in so-called universal properties, that is, properties that do not depend on the specific domain Ω, but only on its dimension n. These universal properties then take the form of relations between different eigenvalues. Naturally, the first eigenvalue λ_1 plays a distinguished role. Since this eigenvalue can often be estimated in terms of the geometry of Ω, one can then also derive geometric estimates for higher eigenvalues from such universal bounds, but this is not explored in the present paper.

Let us now put our results into the context of those known for $l = 1$. Payne, Pólya and Weinberger proved in [PPW1] and [PPW2] that

$$\frac{\lambda_2}{\lambda_1} \leq 3 \quad \text{for} \quad \Omega \subset \mathbb{R}^2$$

and conjectured that

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{\text{disk}}$$

with equality if and only if Ω is a disk. For $n \geq 2$, the analogous statements are

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n} \quad \text{for} \quad \Omega \subset \mathbb{R}^n,$$

and the *PPW conjecture*

$$\frac{\lambda_2}{\lambda_1} \leq \left.\frac{\lambda_2}{\lambda_1}\right|_{n-\text{ball}},$$

with equality if and only if Ω is an n-ball. This important *PPW conjecture* was solved by Ashbaugh and Benguria in [AB1], [AB2], [AB3]. In [PPW2], Payne,

Pólya and Weinberger also proved the bound

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \cdots, \tag{1.2}$$

for $\Omega \subset \mathbb{R}^2$. This result easily extends to $\Omega \subset \mathbb{R}^n$ as

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \cdots . \tag{1.3}$$

Much interesting work has been done in generalizing (1.3), e.g., in [A1], [A2], [AH], [CY1], [Ha], [HM1], [HM2], [HP], [HS], [HY], [LeP], [Y]. Here we mention two results in this direction. In 1980, Hile and Protter proved [HP]

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad \text{for } k = 1, 2, \cdots . \tag{1.4}$$

In 1991, Yang [Y] proved the following much stronger inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n}\right) \lambda_i \right) \leq 0, \quad \text{for } \ k = 1, 2, \cdots . \tag{1.5}$$

The inequality (1.5), as observed by Yang himself, and as later proved, e.g., in [A1], [A2], [AH], is the strongest of the classical inequalities that are derived following the scheme devised by Payne-Pólya-Weinberger. Yang's inequality provided a marked improvement for eigenvalues of large index. Recently, some Yang type inequalities on eigenvalues of the problem (1.1) for the case $l > 1$ have been proved in [CY2], [CY3], [WX1], [WX2] and [WC]. We remark that there is an error in the line below (3.1) of [WX1] where boundary terms are dropped from an integration by parts, without a reason that these terms should vanish.

For general l, the Payne-Pólya-Weinberg type inequality reads (cf. [CQ], [H]):

$$\lambda_{k+1} \leq \lambda_k + \frac{4l(n + 2l - 2)}{n^2 k^2} \left(\sum_{i=1}^{k} \lambda_i^{1/l} \right) \left(\sum_{i=1}^{k} \lambda_i^{(l-1)/l} \right). \tag{1.6}$$

In this paper, we obtain a universal inequality of Yang type for the eigenvalues of the problem (1.1) for any l. Indeed, we consider the more general eigenvalue problem:

$$\begin{aligned} &(-\Delta)^l u = \lambda u \quad \text{in} \ \ M, \\ &u|_{\partial M} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial M} = \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial M} = 0, \end{aligned} \tag{1.7}$$

where M is a compact Riemannian manifold with boundary (possibly empty), Δ is the Laplacian operator on M (for general results for the case $l = 1$, see e.g. [Ch]). We will prove a general inequality for the eigenvalues of the problem (1.7) (see Theorem 2.1). By using this inequality, we show that when M is a bounded connected domain in $\mathbb{R}^n$ with smooth boundary, then the eigenvalues of the problem

(1.1) satisfy (see Theorem 3.1):

(1.8)

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \left(\frac{4l(n+2l-2)}{n^2}\right)^{1/2}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\lambda_i^{(l-1)/l}\right)^{1/2}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i^{1/l}\right)^{1/2}.$$

When $l=1$, (1.8) is just Yang's inequality (1.5). As a consequence of (1.8), we have the following two estimates for the $(k+1)$-th eigenvalue in terms of the first k-eigenvalues of the problem (1.1):

(1.9)

$$\lambda_{k+1} \leq \left\{\left(\frac{2l(n+2l-2)}{k^2n^2}\right)^2\left(\sum_{i=1}^{k}\lambda_i^{(l-1)/l}\right)^2\left(\sum_{i=1}^{k}\lambda_i^{1/l}\right)^2-\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right\}^{1/2} + \frac{1}{k}\sum_{i=1}^{k}\lambda_i+\frac{2l(n+2l-2)}{k^2n^2}\left(\sum_{i=1}^{k}\lambda_i^{(l-1)/l}\right)\left(\sum_{i=1}^{k}\lambda_i^{1/l}\right),$$

(1.10)

$$\lambda_{k+1} \leq \left\{\left(\frac{2l(n+2l-2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2-\left(1+\frac{4l(n+2l-2)}{n^2}\right)\frac{1}{k}\sum_{j=1}^{k}\left(\lambda_j-\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right\}^{1/2} + \left(1+\frac{2l(n+2l-2)}{n^2}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i.$$

Notice that (1.9) is much stronger than (1.6).

In [AB4], Ashbaugh and Benguria showed that when $l=1$, the first $n+1$ eigenvalues of the problem (1.1) satisfy the inequality

$$\lambda_2+\lambda_3+\cdots+\lambda_{n+1}\leq(n+4)\lambda_1. \tag{1.11}$$

Ashbaugh showed in [A1] that when $l=2$,

$$\lambda_2+\lambda_3+\cdots+\lambda_{n+1}\leq(n+24)\lambda_1. \tag{1.12}$$

In this paper, we prove a similar inequality for any l which covers the inequality (1.11) when $l=1$ and improves (1.12) when $l=2$ (cf. Theorem 4.1). The reason why the dimension n comes in here is that coordinate functions in n-dimensional Euclidean space when used as test functions yield useful inequalities.

Consider now the so-called buckling problem:

$$\Delta^2u=-\Lambda\Delta u \quad \text{in } \Omega, \qquad u|_{\partial\Omega}=\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0, \tag{1.13}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$.

Let

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots$$

denote the successive eigenvalues for (1.13). Payne, Pólya and Weinberger [PPW2] proved

$$\Lambda_2/\Lambda_1 < 3 \quad \text{for} \quad \Omega \subset \mathbb{R}^2.$$

For $\Omega \subset \mathbb{R}^n$ this reads

$$\Lambda_2/\Lambda_1 < 1 + 4/n.$$

Subsequently, Hile and Yeh [HY] obtained the improved bound

$$\frac{\Lambda_2}{\Lambda_1} \leq \frac{n^2 + 8n + 20}{(n+2)^2} \qquad \text{for} \quad \Omega \subset \mathbb{R}^n.$$

Ashbaugh [A1] proved:

$$\sum_{i=1}^{n} \Lambda_{i+1} \leq (n+4)\Lambda_1. \tag{1.14}$$

Cheng and Yang [CY2] obtained:

$$\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n+2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i. \tag{1.15}$$

In this paper, we will prove the following inequality which strengthens (1.14) (cf. Theorem 4.2):

$$\sum_{i=1}^{n} \Lambda_{i+1} + \frac{4(\Lambda_2 - \Lambda_1)}{n+4} \leq (n+4)\Lambda_1. \tag{1.16}$$

We will also show that the first $n+1$ eigenvalues of the following more general problem,

$$\begin{aligned} &(-\Delta)^l u = -\Lambda \Delta u \quad \text{in} \quad \Omega, \\ &u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0, \end{aligned} \tag{1.17}$$

where $l \geq 2$ is a fixed integer, satisfy (cf. Theorem 4.3):

$$\sum_{k=1}^{n} \frac{k}{2l+k} (\Lambda_{n+2-k} - \Lambda_1) < 4(l-1)\Lambda_1. \tag{1.18}$$

In the final part of this paper, we will prove universal inequalities for eigenvalues of the polyharmonic operator of any order on compact domains with boundary in a unit sphere. For similar inequalities for eigenvalues of the Laplacian on compact domains in a sphere, we refer to [CY1], [AH] and the references therein.

2. General inequalities for eigenvalues of the harmonic operator of any order on Riemannian manifolds

In this section, we prove some general inequalities for eigenvalues of the polyharmonic operators on compact Riemannian manifolds.

Theorem 2.1. *Let $(M, \langle\ ,\ \rangle)$ be an n-dimensional compact connected Riemannian manifold with boundary ∂M (possibly empty) and let ν be the outward unit normal vector field of ∂M. Let l be a positive integer and denote by Δ the Laplacian operator of M. Consider the eigenvalue problem*

$$\begin{aligned}(-\Delta)^l u &= \lambda u \quad in \ \ M,\\ u|_{\partial M} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial M} &= \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial M} = 0.\end{aligned} \tag{2.1}$$

Let $\lambda_i,\ i = 1, \cdots,$ be the i-th eigenvalue of the problem (2.1) and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$\begin{aligned}&(-\Delta)^l u_i = \lambda_i u_i \quad in\ M,\\ &u_i|_{\partial M} = \left.\frac{\partial u_i}{\partial \nu}\right|_{\partial M} = \cdots = \left.\frac{\partial^{l-1} u_i}{\partial \nu^{l-1}}\right|_{\partial M} = 0,\\ &\int_M u_i u_j = \delta_{ij}, \quad for\ any\ \ i,\ j = 1, 2, \cdots.\end{aligned}$$

Then for any function $h \in C^{l+2}(M) \cap C^{l+1}(\partial M)$ and any positive integer k, we have

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_M h u_i \left((-\Delta)^l (h u_i) - \lambda_i h u_i\right)\\ &\qquad \le \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) || \left((-\Delta)^l (h u_i) - \lambda_i h u_i\right) ||^2,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_M \left(-h u_i^2 \Delta h - 2 h u_i \langle \nabla h, \nabla u_i\rangle\right)\\ &\qquad \le \delta \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_M h u_i \left((-\Delta)^l (h u_i) - \lambda_i h u_i)\right)\\ &\qquad + \sum_{i=1}^{k} \frac{(\lambda_{k+1} - \lambda_i)}{\delta} \left|\left|\langle \nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right|\right|^2\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_M \left(-h u_i^2 \Delta h - 2 h u_i \langle \nabla h, \nabla u_i\rangle\right)\\ &\le \delta \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) || \left((-\Delta)^l (h u_i) - \lambda_i h u_i)\right) ||^2\\ &\quad + \frac{1}{\delta} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \left|\left|\langle \nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right|\right|^2,\end{aligned} \tag{2.4}$$

where δ is any positive constant, $||g||^2 = \int_M g^2$.

Proof of Theorem 2.1. The inequality (2.2) follows from Theorem 2.1 in [AH]. In fact, by taking $N = 1,\ B_1 = hId,\ A = (-\Delta)^l$ in Theorem 2.1 of [AH], one easily

gets that

$$\rho_i \equiv \langle [A, B_1]u_i, B_1 u_i\rangle = \int_M hu_i \left((-\Delta)^l (hu_i) - \lambda_i hu_i\right)$$

and

$$\Lambda_i \equiv ||[A, B_1]u_i||^2 = ||\left((-\Delta)^l (hu_i) - \lambda_i hu_i\right)||^2,$$

which, by using (2.5) in [AH], gives (2.2).

We use some similar calculations as in [CY4] to prove (2.3). For $i = 1, \cdots, k$, consider the functions $\phi_i : M \to \mathbb{R}$ given by

$$\phi_i = hu_i - \sum_{j=1}^{k} r_{ij} u_j, \tag{2.5}$$

where

$$r_{ij} = \int_M hu_i u_j. \tag{2.6}$$

Since

$$\phi_i|_{\partial M} = \left.\frac{\partial \phi_i}{\partial \nu}\right|_{\partial M} = \cdots = \left.\frac{\partial^{l-1}\phi_i}{\partial \nu^{l-1}}\right|_{\partial M} = 0$$

and

$$\int_M u_j \phi_i = 0, \quad \forall\ i, j = 1, \cdots, k,$$

it follows from the Rayleigh-Ritz inequality that

$$\begin{aligned} \lambda_{k+1} \int_M \phi_i^2 &\leq \int_M \phi_i (-\Delta)^l \phi_i \\ &= \lambda_i ||\phi_i||^2 + \int_M \phi_i \left((-\Delta)^l \phi_i - \lambda_i hu_i\right) \\ &= \lambda_i ||\phi_i||^2 + \int_M \phi_i \left((-\Delta)^l (hu_i) - \lambda_i hu_i\right) \\ &= \lambda_i ||\phi_i||^2 + \int_M hu_i \left((-\Delta)^l (hu_i) - \lambda_i hu_i\right) - \sum_{j=1}^{k} r_{ij} s_{ij}, \end{aligned} \tag{2.7}$$

where

$$s_{ij} = \int_M \left((-\Delta)^l (hu_i) - \lambda_i hu_i\right) u_j.$$

Notice that if $u \in C^{l+2}(M) \cap C^{l+1}(\partial M)$ satisfies

$$u|_{\partial M} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial M} = \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial M} = 0, \tag{2.8}$$

then

$$\begin{aligned} u|_{\partial M} &= \nabla u|_{\partial M} = \Delta u|_{\partial M} = \nabla(\Delta u)|_{\partial M} = \cdots = \Delta^{m-1} u\big|_{\partial M} \\ &= \nabla(\Delta^{m-1} u)\big|_{\partial M} = 0, \qquad \text{when} \quad l = 2m \end{aligned} \tag{2.9}$$

and

(2.10)
$$\begin{aligned} u|_{\partial M} = \nabla u|_{\partial M} = \Delta u|_{\partial M} = \nabla(\Delta u)|_{\partial M} = \cdots = \Delta^{m-1}u\big|_{\partial M} = \nabla(\Delta^{m-1}u)\big|_{\partial M} \\ = \Delta^m u|_{\partial M} = 0, \qquad \text{when} \quad l = 2m+1. \end{aligned}$$

Observe that both u_j and hu_i satisfy the boundary condition (2.8) and so they satisfy (2.9) when $l = 2m$ and (2.10) when $l = 2m+1$. Thus we can use integration by parts to conclude that

$$\int_M u_j(-\Delta)^l(hu_i) = \int_M hu_i(-\Delta)^l(u_j) = \lambda_j r_{ij},$$

which gives

$$s_{ij} = (\lambda_j - \lambda_i)r_{ij}. \tag{2.11}$$

Set

$$p_i(h) = (-\Delta)^l(hu_i) - \lambda_i hu_i;$$

then we have from (2.7) and (2.11) that

$$(\lambda_{k+1} - \lambda_i)||\phi_i||^2 \leq \int_M \phi_i p_i(h) = \int_M hu_i p_i(h) + \sum_{j=1}^{k}(\lambda_i - \lambda_j)r_{ij}^2. \tag{2.12}$$

Set

$$t_{ij} = \int_M u_j \left(\langle \nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right); \tag{2.13}$$

then $t_{ij} + t_{ji} = 0$ and

$$\int_M (-2)\phi_i \left(\langle \nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right) = w_i + 2\sum_{j=1}^{k} r_{ij}t_{ij}, \tag{2.14}$$

where

$$w_i = \int_M (-hu_i^2 \Delta h - 2hu_i\langle \nabla h, \nabla u_i\rangle). \tag{2.15}$$

Multiplying (2.14) by $(\lambda_{k+1}-\lambda_i)^2$ and using the Schwarz inequality and (2.12), we get

$$
\begin{aligned}
(2.16)\quad & (\lambda_{k+1}-\lambda_i)^2\left(w_i+2\sum_{j=1}^k r_{ij}t_{ij}\right)\\
= \;& (\lambda_{k+1}-\lambda_i)^2\int_M(-2)\phi_i\left(\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)-\sum_{j=1}^k t_{ij}u_j\right)\\
\leq \;& \delta(\lambda_{k+1}-\lambda_i)^3||\phi_i||^2+\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\int_M\left|\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}-\sum_{j=1}^k t_{ij}u_j\right|^2\\
= \;& \delta(\lambda_{k+1}-\lambda_i)^3||\phi_i||^2+\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\left|\left|\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right|\right|^2-\sum_{j=1}^k t_{ij}^2\right)\\
\leq \;& \delta(\lambda_{k+1}-\lambda_i)^2\left(\int_M hu_ip_i(h)+\sum_{j=1}^k(\lambda_i-\lambda_j)r_{ij}^2\right)\\
& +\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left(\left|\left|\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right|\right|^2-\sum_{j=1}^k t_{ij}^2\right).
\end{aligned}
$$

Summing over i and noticing that $r_{ij}=r_{ji},\ t_{ij}=-t_{ji}$, we infer

$$
\begin{aligned}
& \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2w_i-2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)r_{ij}t_{ij}\\
\leq \;& \delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M hu_ip_i(h)+\sum_{i=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\delta}\left|\left|\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right|\right|^2\\
& -\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)\delta(\lambda_i-\lambda_j)^2r_{ij}^2-\sum_{i,j=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\delta}t_{ij}^2.
\end{aligned}
$$

Hence (2.3) is true. Substituting (2.2) into (2.3), one gets (2.4).

We end this section by listing some lemmas which are needed in the next sections.

Lemma 2.1. *Let u_i and $\lambda_i, i=1,2,\cdots$, be as in Theorem 2.1. Then*

$$
(2.17)\qquad 0\leq\int_M u_i(-\Delta)^ku_i\leq\lambda_i^{k/l},\quad k=1,\cdots,l-1.
$$

Proof. When $k\in\{1,\cdots,l-1\}$ is even, we have

$$
\int_M u_i(-\Delta)^ku_i=\int_M u_i\Delta^ku_i=\int_M\left(\Delta^{k/2}u_i\right)^2\geq 0.
$$

On the other hand, if $k \in \{1, \cdots, l-1\}$ is odd,

$$\begin{aligned}\int_M u_i(-\Delta)^k u_i &= -\int_M u_i \Delta^k u_i \\ &= -\int_M \Delta^{(k-1)/2} u_i \Delta \left(\Delta^{(k-1)/2} u_i\right) \\ &= \int_M \left|\nabla \left(\Delta^{(k-1)/2} u_i\right)\right|^2 \\ &\geq 0.\end{aligned}$$

Thus the inequality at the left hand side of (2.17) holds.

We claim that for any $k = 1, \cdots, l-1$,

$$\left(\int_M u_i(-\Delta)^k u_i\right)^{k+1} \leq \left(\int_M u_i(-\Delta)^{k+1} u_i\right)^k. \tag{2.18}$$

Since

$$\left(\int_M u_i \Delta u_i\right)^2 \leq \int_M u_i^2 \int_M (\Delta u_i)^2 = \int_M u_i \Delta^2 u_i,$$

we know that (2.18) holds when $k = 1$.

Suppose that (2.18) holds for $k-1$, that is,

$$\left(\int_M u_i(-\Delta)^{k-1} u_i\right)^k \leq \left(\int_M u_i(-\Delta)^k u_i\right)^{k-1}. \tag{2.19}$$

When k is even, we have

(2.20)

$$\begin{aligned}\int_M u_i(-\Delta)^k u_i &= \int_M \Delta^{k/2-1} u_i \Delta \left(\Delta^{k/2} u_i\right) \\ &= -\int_M \left\langle \nabla \left(\Delta^{k/2-1} u_i\right), \nabla \left(\Delta^{k/2} u_i\right)\right\rangle \\ &\leq \left(\int_M \left|\nabla \left(\Delta^{k/2-1} u_i\right)\right|^2\right)^{1/2} \left(\int_M \left|\nabla \left(\Delta^{k/2} u_i\right)\right|^2\right)^{1/2} \\ &= \left(-\int_M \Delta^{k/2-1} u_i \Delta^{k/2} u_i\right)^{1/2} \left(-\int_M \Delta^{k/2} u_i \Delta^{k/2+1} u_i\right)^{1/2} \\ &= \left(\int_M u_i(-\Delta)^{k-1} u_i\right)^{1/2} \left(\int_M u_i(-\Delta)^{k+1} u_i\right)^{1/2}.\end{aligned}$$

On the other hand, when k is odd,

$$\begin{aligned}\int_M u_i(-\Delta)^k u_i &= \int_M (-\Delta)^{(k-1)/2} u_i (-\Delta)^{(k+1)/2} u_i \\ &\leq \left(\int_M \left((-\Delta)^{(k-1)/2} u_i\right)^2\right)^{1/2} \left(\int_M \left((-\Delta)^{(k+1)/2} u_i\right)^2\right)^{1/2} \\ &= \left(\int_M u_i(-\Delta)^{k-1} u_i\right)^{1/2} \left(\int_M u_i(-\Delta)^{k+1} u_i\right)^{1/2}.\end{aligned} \tag{2.21}$$

Thus we always have

$$\int_M u_i(-\Delta)^k u_i \leq \left(\int_M u_i(-\Delta)^{k-1}u_i\right)^{1/2}\left(\int_M u_i(-\Delta)^{k+1}u_i\right)^{1/2}. \tag{2.22}$$

Substituting (2.19) into (2.22), we know that (2.18) is true for k. Using (2.18) repeatedly, we get

$$\int_M u_i(-\Delta)^k u_i \leq \left(\int_M u_i(-\Delta)^{k+1}u_i\right)^{k/(k+1)} \leq \cdots \leq \left(\int_M u_i(-\Delta)^l u_i\right)^{k/l} = \lambda_i^{k/l}. \tag{2.23}$$

This shows that the inequality at the right hand side of (2.17) also holds.

Lemma 2.2. *Let*

$$C = \left\{ z = (z_1, \cdots, z_n) \in \mathbb{R}^n \mid z_i \geq 0,\ i = 1, \cdots, n, \sum_{j=1}^n z_j = 1 \right\}.$$

Consider the function $f : C \to R$ defined by

$$f((z_1, \cdots, z_n)) = \sum_{i=1}^n \frac{z_i^2}{1+4z_i}. \tag{2.24}$$

Then

$$\min_{z\in C} f(z) = f\left(\left(\frac{1}{n}, \cdots, \frac{1}{n}\right)\right) = \frac{1}{n+4}. \tag{2.25}$$

Proof of Lemma 2.2. We minimize the function

$$\sum_{i=1}^n \frac{z_i^2}{1+4z_i}$$

with the constraint

$$\sum_{j=1}^n z_j = 1,\ z_i \geq 0,\ i = 1, \cdots, n.$$

By means of the method of the Lagrange multiplier, we consider the following function:

$$g = \sum_{i=1}^n \frac{z_i^2}{1+4z_i} + \lambda\left(\sum_{j=1}^n z_j - 1\right),$$

where λ is the Lagrange multiplier. The minimum point of $\sum_{i=1}^n z_i^2/(1+4z_i)$ is a critical point of g. Taking the derivative of g with respect to z_i, we have

$$\frac{2z_i(1+4z_i) - 4z_i^2}{(1+4z_i)^2} + \lambda = 0.$$

Multiplying the above equaltion by $(1+4z_i)^2$ and simplifying, we get

$$(16\lambda + 4)z_i^2 + (8\lambda + 2)z_i + \lambda = 0.$$

Hence at most two of the z_i's are distinct from each other at a critical point of g. Assume without loss of generality that $z_1 = z_2 = \cdots = z_p = s,\ z_{p+1} = \cdots = z_{p+q} = t$ with $p + q = n$. Then $ps + qt = 1$ and so we have

(2.26)
$$\begin{aligned}
\sum_{i=1}^{n} \frac{z_i^2}{(1+4z_i)} &= \frac{ps^2}{1+4s} + \frac{qt^2}{1+4t} \\
&= \frac{ps^2 + qt^2 + 4st(ps+qt)}{(1+4s)(1+4t)} \\
&= \frac{ps^2 + qt^2 + 4st}{(1+4s)(1+4t)} \\
&= \frac{(p+q+4)(ps^2 + qt^2 + 4st(ps+qt))}{(n+4)(1+4s)(1+4t)} \\
&= \frac{p^2s^2 + q^2t^2 + pq(s^2+t^2) + 4ps^2 + 4qt^2 + 4(p+q)st + 16st}{(n+4)(1+4s)(1+4t)} \\
&= \frac{1 - 2pqst + pq(s^2+t^2) + 4ps^2 + 4qt^2 + 4(p+q)st + 16st}{(n+4)(1+4s)(1+4t)} \\
&\geq \frac{1 + 4ps^2 + 4qt^2 + 4(p+q)st + 16st}{(n+4)(1+4s)(1+4t)} \\
&= \frac{1 + 4(ps+qt)(s+t) + 16st}{(n+4)(1+4s)(1+4t)} = \frac{1}{n+4}.
\end{aligned}$$

This completes the proof of Lemma 2.2.

Lemma 2.3. *Let $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ be three sequences of nonnegative real numbers with $\{a_i\}$ decreasing and $\{b_i\}$ and $\{c_i\}_{i=1}^m$ increasing. Then the following inequality holds:*

$$\left(\sum_{i=1}^{m} a_i^2 b_i\right)\left(\sum_{i=1}^{m} a_i c_i\right) \leq \left(\sum_{i=1}^{m} a_i^2\right)\left(\sum_{i=1}^{m} a_i b_i c_i\right). \tag{2.27}$$

Proof. When $m = 1$, (2.27) holds trivially. Suppose that (2.27) holds when $m = k$, that is,

$$\left(\sum_{i=1}^{k} a_i^2 b_i\right)\left(\sum_{i=1}^{k} a_i c_i\right) \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} a_i b_i c_i\right). \tag{2.28}$$

Then when $m = k+1$, we have from (2.28) that

$$
\begin{aligned}
(2.29)\quad & \left(\sum_{i=1}^{k+1} a_i^2\right)\left(\sum_{i=1}^{k+1} a_i b_i c_i\right) - \left(\sum_{i=1}^{k+1} a_i^2 b_i\right)\left(\sum_{i=1}^{k+1} a_i c_i\right) \\
= \; & \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} a_i b_i c_i\right) - \left(\sum_{i=1}^{k} a_i^2 b_i\right)\left(\sum_{i=1}^{k} a_i c_i\right) + a_{k+1}^2 \sum_{i=1}^{k} a_i b_i c_i \\
& - a_{k+1}^2 b_{k+1} \sum_{i=1}^{k} a_i c_i + a_{k+1} b_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 - a_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 b_i \\
\geq \; & a_{k+1}^2 \sum_{i=1}^{k} a_i b_i c_i - a_{k+1}^2 b_{k+1} \sum_{i=1}^{k} a_i c_i + a_{k+1} b_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 - a_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 b_i \\
= \; & -a_{k+1}^2 \sum_{i=1}^{k} (b_{k+1} - b_i) a_i c_i + a_{k+1} c_{k+1} \sum_{i=1}^{k} a_i^2 (b_{k+1} - b_i) \\
= \; & \sum_{i=1}^{k} a_{k+1} a_i (b_{k+1} - b_i)(c_{k+1} a_i - a_{k+1} c_i) \\
\geq \; & 0,
\end{aligned}
$$

where in the last inequality we have used the fact that

$$a_{k+1} a_i (b_{k+1} - b_i)(c_{k+1} a_i - a_{k+1} c_i) \geq 0, \quad i = 1, \cdots, k.$$

Thus (2.27) holds for $m = k+1$. This completes the proof of Lemma 2.3.

The following result is the so-called *Reverse Chebyshev Inequality* (cf. [HLP]).

Lemma 2.4. *Suppose $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$ are two real sequences with $\{a_i\}$ increasing and $\{b_i\}$ decreasing. Then the following inequality holds:*

$$\sum_{i=1}^{m} a_i b_i \leq \frac{1}{m}\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{i=1}^{m} b_i\right). \tag{2.30}$$

The following lemma can also be found in [HLP].

Lemma 2.5. *Let $\{c_k\}_{k=1}^l$ and $\{d_k\}_{k=1}^l$ be two increasing real sequences. Then for any permutation $\{i_1, \cdots, i_l\}$ of $\{1, \cdots, l\}$, we have*

$$\sum_{k=1}^{l} c_k d_{i_k} \geq c_1 d_l + c_2 d_{l-1} + \cdots + c_l d_1.$$

Remark. Lemma 2.4 also admits a probabilistic interpretation. We may assume that the a_i and b_i are nonnegative and satisfy $\sum_{i=1}^m a_i = 1 = \sum_{i=1}^m b_i$ so that we can interpret them as the probabilities for observing i under the laws a or b, resp. One then needs to prove that $\sum_{i=1}^m a_i b_i \leq \frac{1}{m}$. When the b_i are all the same, that is, $= \frac{1}{m}$, the inequality is obviously an equality, and when b_i is decreasing instead of being constant, the right hand side stays the same, but the left hand side can only become smaller, because then higher weights are placed on those i with smaller a_i. Thus, the inequality follows. In fact, Lemma 2.5 above admits a similar interpretation.

3. Universal inequalities for eigenvalues of the polyharmonic operators on compact domains in $\mathbb{R}^n$

In this section, we will prove universal bounds on eigenvalues of the polyharmonic operator on bounded domains in a Euclidean space by using Theorem 2.1.

Theorem 3.1. *Let Ω be a connected bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$ and let Δ be the Laplacian of $\mathbb{R}^n$. Denote by λ_i the i-th eigenvalue of the eigenvalue problem:*

$$\begin{aligned}&(-\Delta)^l u = \lambda u \quad in \;\; \Omega,\\ &u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0.\end{aligned}$$

Then we have

(3.1)

$$\begin{aligned}&\sum_{i=1}^{k} (\lambda_{k+1}-\lambda_i)^2\\ &\le \left(\frac{4l(n+2l-2)}{n^2}\right)^{1/2} \left(\sum_{i=1}^{k} (\lambda_{k+1}-\lambda_i)^2 \lambda_i^{(l-1)/l}\right)^{1/2} \left(\sum_{i=1}^{k} (\lambda_{k+1}-\lambda_i)\, \lambda_i^{1/l}\right)^{1/2}.\end{aligned}$$

Corollary 3.1. *Under the same assumptions as in Theorem 3.1, we have*

$$\begin{aligned}\lambda_{k+1} \le \frac{1}{k}\sum_{i=1}^{k}\lambda_i &+ \frac{2l(n+2l-2)}{k^2n^2}\left(\sum_{i=1}^{k}\lambda_i^{(l-1)/l}\right)\left(\sum_{i=1}^{k}\lambda_i^{1/l}\right)\\ &+\left\{\left(\frac{2l(n+2l-2)}{k^2n^2}\right)^2\left(\sum_{i=1}^{k}\lambda_i^{(l-1)/l}\right)^2\left(\sum_{i=1}^{k}\lambda_i^{1/l}\right)^2\right.\\ &\left.-\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\lambda_{k+1} \le \left(1+\frac{2l(n+2l-2)}{n^2}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i &+ \left\{\left(\frac{2l(n+2l-2)}{n^2}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2\right.\\ &\left.-\left(1+\frac{4l(n+2l-2)}{n^2}\right)\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i-\frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right\}^{1/2}.\end{aligned} \tag{3.3}$$

Proof of Theorem 3.1. Let $x_1, x_2, \cdots, x_n$ be the standard Euclidean coordinate functions of $\mathbb{R}^n$. Let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i of the problem (3.1), $i = 1, \cdots$; then

$$\Delta x_\alpha = 0,\ \nabla x_\alpha = (\underbrace{0, \cdots, 0, 1}_{\alpha}, 0, \cdots, 0), \quad \alpha = 1, 2, \cdots, n, \tag{3.4}$$

which implies that

$$(3.5)\qquad \begin{aligned}(-\Delta)^l(x_\alpha u_i) &= x_\alpha(-\Delta)^l u_i + 2l(-1)^l\langle\nabla x_\alpha, \nabla(\Delta^{l-1}u_i)\rangle\\ &= \lambda_i x_\alpha u_i + 2l(-1)^l\langle\nabla x_\alpha, \nabla(\Delta^{l-1}u_i)\rangle.\end{aligned}$$

Taking $h = x_\alpha$ in (2.3), we infer for any $\delta > 0$ that

$$\begin{aligned}&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega(-2x_\alpha u_i\langle\nabla x_\alpha, \nabla u_i\rangle)\\ \le\ &\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\int_\Omega 2l(-1)^l x_\alpha u_i\langle\nabla x_\alpha, \nabla(\Delta^{l-1}u_i)\rangle\\ &+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\,||\langle\nabla x_\alpha, \nabla u_i\rangle||^2.\end{aligned}$$

Summing over α, we have

$$(3.6)\qquad \begin{aligned}&\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^n\int_\Omega(-2x_\alpha u_i\langle\nabla x_\alpha, \nabla u_i\rangle)\\ \le\ &2l\delta\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^n\int_\Omega(-1)^l x_\alpha u_i\langle\nabla x_\alpha, \nabla(\Delta^{l-1}u_i)\rangle\\ &+\frac{1}{\delta}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\sum_{\alpha=1}^n||\langle\nabla x_\alpha, \nabla u_i\rangle||^2.\end{aligned}$$

Observe that

$$(3.7)\qquad \sum_{\alpha=1}^n|\nabla x_\alpha|^2 = n,\quad \sum_{\alpha=1}^n\langle\nabla x_\alpha, \nabla u_i\rangle^2 = |\nabla u_i|^2.$$

Hence

$$(3.8)\qquad \sum_{\alpha=1}^n\int_\Omega(-2x_\alpha u_i\langle\nabla x_\alpha, \nabla u_i\rangle) = \frac{1}{2}\sum_{\alpha=1}^n\int_\Omega u_i^2\Delta x_\alpha^2 = \int_\Omega u_i^2\sum_{\alpha=1}^n|\nabla x_\alpha|^2 = n.$$

From (2.17), we infer

$$(3.9)\qquad \sum_{\alpha=1}^n||\langle\nabla x_\alpha, \nabla u_i\rangle||^2 = \int_\Omega|\nabla u_i|^2 = \int_\Omega(-u_i\Delta u_i) \le \lambda_i^{1/l}.$$

Since

$$\Delta^{l-1}(x_\alpha u_i) = 2(l-1)\langle\nabla x_\alpha, \nabla(\Delta^{l-2}u_i)\rangle + x_\alpha\Delta^{l-1}u_i,$$

we have

(3.10)

$$\begin{aligned}\int_\Omega x_\alpha u_i\langle\nabla x_\alpha, \nabla(\Delta^{l-1}u_i)\rangle &= \int_\Omega x_\alpha u_i\Delta^{l-1}\langle\nabla x_\alpha, \nabla u_i\rangle\\ &= \int_\Omega\Delta^{l-1}(x_\alpha u_i)\langle\nabla x_\alpha, \nabla u_i\rangle\\ &= \int_\Omega\big(2(l-1)\langle\nabla x_\alpha, \nabla(\Delta^{l-2}u_i)\rangle + x_\alpha\Delta^{l-1}u_i\big)\langle\nabla x_\alpha, \nabla u_i\rangle.\end{aligned}$$

On the other hand,

$$\int_\Omega x_\alpha u_i \langle \nabla x_\alpha, \nabla(\Delta^{l-1} u_i)\rangle = -\int_\Omega \Delta^{l-1} u_i \ \mathrm{div}(x_\alpha u_i \nabla x_\alpha) = -\int_\Omega \Delta^{l-1} u_i (|\nabla x_\alpha|^2 u_i + x_\alpha \langle \nabla x_\alpha, \nabla u_i\rangle), \tag{3.11}$$

where $\mathrm{div}(X)$ denotes the divergence of X. Combining (3.10) and (3.11), we obtain

$$\begin{aligned} &\int_\Omega x_\alpha u_i \langle \nabla x_\alpha, \nabla(\Delta^{l-1} u_i)\rangle \\ = &\int_M \left\{ (l-1)\langle \nabla x_\alpha, \nabla(\Delta^{l-2} u_i)\rangle \langle \nabla x_\alpha, \nabla u_i\rangle - \frac{1}{2}\Delta^{l-1} u_i |\nabla x_\alpha|^2 u_i \right\}. \end{aligned} \tag{3.12}$$

It then follows from (2.17), (3.7), (3.12) and

$$\sum_{\alpha=1}^n \langle \nabla x_\alpha, \nabla(\Delta^{l-2} u_i)\rangle \langle \nabla x_\alpha, \nabla u_i\rangle = \langle \nabla u_i, \nabla(\Delta^{l-2} u_i)\rangle$$

that

$$\begin{aligned} &\sum_{\alpha=1}^n \int_\Omega (-1)^l x_\alpha u_i \langle \nabla x_\alpha, \nabla(\Delta^{l-1} u_i)\rangle \\ = &\sum_{\alpha=1}^n \int_\Omega (-1)^l \left\{ (l-1)\langle \nabla x_\alpha, \nabla(\Delta^{l-2} u_i)\rangle \langle \nabla x_\alpha, \nabla u_i\rangle - \frac{1}{2}\Delta^{l-1} u_i |\nabla x_\alpha|^2 u_i \right\} \\ = &\int_\Omega (-1)^l \left\{ (l-1)\langle \nabla(\Delta^{l-2} u_i), \nabla u_i\rangle - \frac{n}{2} u_i \Delta^{l-1} u_i \right\} \\ = &\left(l-1+\frac{n}{2}\right) \int_\Omega u_i (-\Delta)^{l-1} u_i \\ \le &\left(l-1+\frac{n}{2}\right) \lambda_i^{(l-1)/l}. \end{aligned} \tag{3.13}$$

Substituting (3.8), (3.9) and (3.13) into (3.6), one gets

$$\begin{aligned} &n \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \\ \le &\ l(n+2l-2)\delta \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \lambda_i^{(l-1)/l} + \frac{1}{\delta} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i^{1/l}. \end{aligned} \tag{3.14}$$

Taking

$$\delta = \left\{ \frac{\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i^{1/l}}{l(n+2l-2) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \lambda_i^{(l-1)/l}} \right\}^{1/2},$$

we get (3.1).

In the proof of Corollary 3.1 we will use the reverse-Chebyshev inequality which was used earlier for similar purposes in [A1] and [AH].

Proof of Corollary 3.1. It follows from (2.30) that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i^{1/l} \le \frac{1}{k} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right) \left(\sum_{i=1}^k \lambda_i^{1/l} \right) \tag{3.15}$$

and

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\,\lambda_i^{(l-1)/l} \le \frac{1}{k}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right)\left(\sum_{i=1}^{k}\lambda_i^{(l-1)/l}\right). \tag{3.16}$$

Introducing (3.15) and (3.16) into (3.1), we infer

$$\begin{aligned}&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\\ \le\ & \frac{4l(n+2l-2)}{k^2n^2}\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\right)\left(\sum_{i=1}^{k}\lambda_i^{(l-1)/l}\right)\left(\sum_{i=1}^{k}\lambda_i^{1/l}\right).\end{aligned}$$

Solving this quadratic polynomial about λ_{k+1}, one gets (3.2).

From (2.27), we have

$$\begin{aligned}&\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\,\lambda_i^{(l-1)/l}\right)\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\,\lambda_i^{1/l}\right)\\ \le\ & \left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right)\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\,\lambda_i\right).\end{aligned}$$

It then follows from (3.1) that

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{4l(n+2l-2)}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\,\lambda_i, \tag{3.17}$$

which implies (3.3).

4. Universal inequalities for lower order eigenvalues of the polyharmonic operators on compact domains in $\mathbb{R}^n$

In [AB4], Ashbaugh and Benguria showed that when $l=1$, the first $n+1$ eigenvalues of the problem (1.1) satisfy the inequality $\lambda_2+\lambda_3+\cdots+\lambda_{n+1}\le(n+4)\lambda_1$. Also, Ashbaugh showed in [A1] that when $l=2$, $\lambda_2+\lambda_3+\cdots+\lambda_{n+1}\le(n+24)\lambda_1$. The following result generalizes the estimate by Ashbaugh and Benguria to any l and strengthens the above Ashbaugh's inequality.

Theorem 4.1. *Under the same assumptions as in Theorem 3.1, we have*

$$\sum_{i=2}^{n+1}\lambda_i+\sum_{i=1}^{n-1}\frac{2(l-1)i}{2l+i-1}(\lambda_{n+1-i}-\lambda_1)\le(n+4l(2l-1))\lambda_1. \tag{4.1}$$

Proof of Theorem 3.2. As in the proof of Theorem 3.1, we let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i of the problem (3.1), $i=1,\cdots$. We first *claim* that there exists a Cartesian coordinate system $(x_1,...,x_n)$ of $\mathbb{R}^n$ so that the following orthogonality conditions are satisfied:

$$\int_\Omega x_iu_1u_j=0 \quad \text{for} \quad 1\le j\le i\le n. \tag{4.2}$$

Indeed, by choosing the origin properly, we can assume that there exist Cartesian coordinates $(y_1,...,y_n)$ of R^n such that

$$\int_\Omega y_iu_1^2=0 \quad \text{for} \quad i=1,...,n. \tag{4.3}$$

Consider the matrix A defined by

$$A = \begin{bmatrix} \int_\Omega y_1 u_1 u_2 & \int_\Omega y_1 u_1 u_3 & \cdots & \int_\Omega y_1 u_1 u_{n+1} \\ \int_\Omega y_2 u_1 u_2 & \int_\Omega y_2 u_1 u_2 & \cdots & \int_\Omega y_2 u_1 u_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \int_\Omega y_n u_1 u_2 & \int_\Omega y_n u_1 u_3 & \cdots & \int_\Omega y_n u_1 u_{n+1} \end{bmatrix}.$$

From the orthogonalization of Gram-Schmidt (QR-factorization theorem), we know that A can be written as

$$B = TA,$$

where $T = (t_{ij})$ is an orthogonal $n \times n$ matrix and B is an upper triangular matrix. Hence, we have, for any k and j with $k > j$,

$$\sum_{l=1}^{n} t_{kl} \int_\Omega y_l u_1 u_{j+1} = 0.$$

Defining new coordinate functions x_k, by $x_k = \sum_{j=1}^{n} t_{kj} y_j$, one has, for any $i, j = 1, 2, ..., n$, satisfying $i > j$,

$$\int_\Omega x_i u_1 u_{j+1} = 0. \tag{4.4}$$

Combining (4.3) and (4.4), we know that our *claim* is true.

Since (4.2) holds, for each $i = 1, ..., n$, we get from the Rayleigh-Ritz inequality, (3.5) and (3.12) that

(4.5)

$$\begin{aligned} \lambda_{i+1} &\leq \frac{\int_\Omega x_i u_1 (-\Delta)^l (x_i u_1)}{\int_\Omega x_i^2 u_1^2} \\ &= \lambda_1 + \frac{2l \int_\Omega (-1)^l x_i u_1 \langle \nabla x_i, \nabla(\Delta^{l-1} u_1)\rangle}{\int_\Omega x_i^2 u_1^2} \\ &= \lambda_1 + \frac{2l \int_\Omega (-1)^l \left\{ (l-1)\langle \nabla x_i, \nabla(\Delta^{l-2} u_1)\rangle \langle \nabla x_i, \nabla u_1\rangle - \frac{1}{2} u_1 \Delta^{l-1} u_1 \right\}}{\int_\Omega x_i^2 u_1^2}. \end{aligned}$$

Since

$$1 = \int_\Omega u_1^2 = -2 \int_\Omega x_i u_1 \langle \nabla x_i, \nabla u_1 \rangle,$$

we have by squaring both sides and using the Cauchy-Schwarz inequality that

$$1 \leq 4 \int_\Omega \langle \nabla x_i, \nabla u_1 \rangle^2 \int_\Omega x_i^2 u_1^2,$$

which gives

$$\frac{1}{\int_\Omega x_i^2 u_1^2} \leq 4 \int_\Omega \langle \nabla x_i, \nabla u_1 \rangle^2. \tag{4.6}$$

Substituting (4.6) into (4.5) yields

(4.7)

$$\begin{aligned} \lambda_{i+1} - \lambda_1 &\leq \left(2l \int_\Omega (-1)^l \left\{ (l-1)\langle \nabla x_i, \nabla(\Delta^{l-2} u_1)\rangle \langle \nabla x_i, \nabla u_1\rangle - \frac{1}{2} u_1 \Delta^{l-1} u_1 \right\} \right) \\ &\times \left(4 \int_\Omega \langle \nabla x_i, \nabla u_1 \rangle^2 \right). \end{aligned}$$

Set

(4.8)
$$a=\int_\Omega u_1(-\Delta)^{l-1}u_1,\quad a_i=\int_\Omega(-1)^l\langle\nabla x_i,\nabla(\Delta^{l-2}u_1)\rangle\langle\nabla x_i,\nabla u_1\rangle,\quad i=1,\cdots,n;$$

then

$$\sum_{i=1}^n a_i=a. \tag{4.9}$$

Taking a permutation $\{i_1,\cdots,i_n\}$ of $\{1,\cdots,n\}$ so that

$$a_{i_n}\le a_{i_{n-1}}\le\cdots\le a_{i_1}, \tag{4.10}$$

it then follows from (4.9) and (4.10) that

$$a_{i_k}\le\frac{1}{k}a,\ k=1,\cdots,n. \tag{4.11}$$

Substituting (4.11) into (4.7), we get

$$\lambda_{i_k+1}-\lambda_1\le 8l\left(\frac{(l-1)}{k}a+\frac{a}{2}\right)\int_\Omega\langle\nabla x_{i_k},\nabla u_1\rangle^2. \tag{4.12}$$

Multiplying (4.12) by $\frac{(2l-1)k}{2(l-1)+k}$ and simplifying, one has

$$\begin{aligned}\left(1+\frac{2(l-1)(k-1)}{2(l-1)+k}\right)(\lambda_{i_k+1}-\lambda_1)&=\frac{(2l-1)k}{2(l-1)+k}(\lambda_{i_k+1}-\lambda_1)\\&\le 4l(2l-1)a\int_\Omega\langle\nabla x_{i_k},\nabla u_1\rangle^2.\end{aligned} \tag{4.13}$$

Summing over k, we have

$$\begin{aligned}4l(2l-1)a\int_\Omega|\nabla u_1|^2&\ge\sum_{k=1}^n\left(1+\frac{2(l-1)(k-1)}{2(l-1)+k}\right)(\lambda_{i_k+1}-\lambda_1)\\&=\sum_{k=1}^n(\lambda_k-\lambda_1)+\sum_{k=2}^n\frac{2(l-1)(k-1)}{2(l-1)+k}(\lambda_{i_k+1}-\lambda_1).\end{aligned} \tag{4.14}$$

Observe that $\{i_1,\cdots,i_n\}$ is a permutation of $\{1,\cdots,n\}$. We *claim* that there is a permutation $\{q_2,q_3,\cdots,q_n\}$ of $\{1,...,n-1\}$ such that

$$\sum_{k=2}^n\frac{2(l-1)(k-1)}{2(l-1)+k}(\lambda_{i_k+1}-\lambda_1)\ge\sum_{k=2}^n\frac{2(l-1)(k-1)}{2(l-1)+k}(\lambda_{q_k+1}-\lambda_1). \tag{4.15}$$

In fact, if $i_1=n$, then $\{i_2,\cdots,i_n\}$ is a permutation of $\{1,\cdots,n-1\}$ and there is nothing to prove. On the other hand, if $i_1=m\in\{1,2,\cdots,n-1\}$, then $\{i_2,i_3,\cdots,i_n\}=\{1,2,\cdots,m-1,m+1,\cdots,n\}$ and so there is a $j\in\{2,\cdots,n\}$ such that $i_j=n$, which implies that

$$\{i_2,i_3,\cdots,i_{j-1},i_{j+1},\cdots,i_n\}=\{1,2,\cdots,m-1,m+1,\cdots,n-1\} \tag{4.16}$$

and

$$\begin{aligned}
(4.17)\qquad & \sum_{k=2}^{n}\frac{2(l-1)(k-1)}{2(l-1)+k}\left(\lambda_{i_k+1}-\lambda_1\right)\\
= & \sum_{k=2,k\neq j}^{n}\frac{2(l-1)(k-1)}{2(l-1)+k}\left(\lambda_{i_k+1}-\lambda_1\right)+\frac{2(l-1)(j-1)}{2(l-1)+j}\left(\lambda_{n+1}-\lambda_1\right)\\
\geq & \sum_{k=2,k\neq j}^{n}\frac{2(l-1)(k-1)}{2(l-1)+k}\left(\lambda_{i_k+1}-\lambda_1\right)+\frac{2(l-1)(j-1)}{2(l-1)+j}\left(\lambda_{m+1}-\lambda_1\right).
\end{aligned}$$

From (4.16), we know that $\{i_2, i_3, \cdots, i_{j-1}, m, i_{j+1}, \cdots, i_n\}$ is a permutation of $\{1, 2, \cdots, m-1, m, m+1, \cdots, n-1\}$. Set $i_2 = q_2, \cdots, i_{j-1} = q_{j-1}, m = q_j, i_{j+1} = q_{j+1}, \cdots, i_n = q_n$; then $\{q_2, q_3, \cdots, q_n\}$ is a permutation of $\{1, ..., n-1\}$ and we can rewrite (4.17) as (4.15). Thus our *claim* is true.

Since $\left\{\frac{2(l-1)(k-1)}{2(l-1)+k}\right\}_{k=2}^{n}$ and $\{\lambda_k-\lambda_1\}_{k=2}^{n}$ are two increasing sequences and $\{q_1+1, q_2+1, \cdots, q_n+1\}$ is a permutation of $\{2, \cdots, n\}$, we conclude from Lemma 2.5 that

$$(4.18)\qquad \sum_{k=2}^{n}\frac{2(l-1)(k-1)}{2(l-1)+k}\left(\lambda_{q_k+1}-\lambda_1\right)\geq\sum_{k=2}^{n}\frac{2(l-1)(k-1)}{2(l-1)+k}(\lambda_{n-k+2}-\lambda_1).$$

Thus we have from (2.17), (4.14) and (4.18) that

$$\begin{aligned}
(4.19)\qquad & \sum_{k=1}^{n}\lambda_{i+1}+\sum_{k=2}^{n}\frac{2(l-1)(k-1)}{2(l-1)+k}(\lambda_{n-k+2}-\lambda_1)\\
\leq\ & n\lambda_1+4l(2l-1)a\int_\Omega|\nabla u_1|^2\\
=\ & n\lambda_1+4l(2l-1)\left(\int_\Omega u_1(-\Delta)^{l-1}u_1\right)\left(\int_\Omega u_1(-\Delta u_1)\right)\\
\leq\ & n\lambda_1+4l(2l-1)\lambda_1^{(l-1)/l}\cdot\lambda_1^{1/l}=(n+4l(2l-1))\lambda_1.
\end{aligned}$$

This is just the inequality (4.1). The proof of Theorem 3.2 is completed.

Our next result is to prove the inequality (1.16) as mentioned in the introduction.

Theorem 4.2. *Let Ω be a connected bounded domain with smooth boundary in $\mathbb{R}^n$. Denote by ν the outward unit normal vector field of $\partial\Omega$ and let $\Lambda_i,\ i=1,\cdots,n+1$, be the first $n+1$ eigenvalues of the following buckling problem:*

$$(4.20)\qquad \Delta^2 u=-\Lambda\Delta u\quad\text{in}\ \ \Omega,\qquad u|_{\partial\Omega}=\left.\frac{\partial u}{\partial\nu}\right|_{\partial\Omega}=0.$$

Then,

$$(4.21)\qquad \sum_{i=1}^{n}\Lambda_{i+1}+\frac{4(\Lambda_2-\Lambda_1)}{n+4}\leq(n+4)\Lambda_1.$$

Proof of Theorem 4.2. Let us denote by u_i the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i of the buckling problem (4.20), $i=1,\cdots$. That is,

we have

$$\Delta^2 u_i = -\Lambda_i \Delta u \quad \text{in} \quad \Omega, \qquad u_i|_{\partial\Omega} = \left.\frac{\partial u_i}{\partial \nu}\right|_{\partial\Omega} = 0, \tag{4.22}$$

$$\int_\Omega \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}, \quad \forall \quad i,j = 1, \cdots. \tag{4.23}$$

Using similar discussions as in the proof of Theorem 4.1, we can find a set of Cartesian coordinates $(x_1, ..., x_n)$ of $\mathbb{R}^n$ so that the following orthogonality conditions are satisfied:

$$\int_\Omega \langle \nabla(x_i u_1), \nabla u_j \rangle = -\int_\Omega x_i u_1 \Delta u_j = 0 \quad \text{for} \quad 1 \le j \le i \le n. \tag{4.24}$$

Now we start with the well-known Rayleigh-Ritz inequality

$$\Lambda_{i+1} \le \frac{\int_\Omega \phi \Delta^2 \phi}{\int_\Omega |\nabla \phi|^2}, \tag{4.25}$$

which is satisfied by any sufficiently smooth function ϕ such that

$$\phi = \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

$$\int_\Omega \langle \nabla \phi, \nabla u_j \rangle = 0, \ j = 1, \cdots, i.$$

Setting $u = u_1$, we choose as our trial function

$$\phi = x_i u,$$

which clearly satisfies the above boundary condition, and by (4.24) the orthogonality condition also. Thus we have

$$\Lambda_{i+1} \int_\Omega |\nabla(x_i u)|^2 \le \int_\Omega x_i u \Delta^2 (x_i u), \quad i = 1, \cdots, n. \tag{4.26}$$

Let us calculate

$$\int_\Omega |\nabla(x_i u)|^2 = \int_\Omega x_i^2 |\nabla u|^2 + 2\int_\Omega x_i u u_{x_i} + \int_\Omega u^2 = \int_\Omega x_i^2 |\nabla u|^2, \tag{4.27}$$

where $u_{x_i} = \langle \nabla x_i, \nabla u \rangle = \frac{\partial u}{\partial x_i}$. As for the right hand side of (4.26), we have

$$\begin{aligned} \int_\Omega x_i u \Delta^2 (x_i u) &= \int_\Omega x_i u \left(x_i \Delta^2 u + 4 \Delta u_{x_i} \right) \\ &= -\Lambda_1 \int_\Omega x_i^2 u \Delta u + 4 \int_\Omega x_i u \Delta u_{x_i}. \end{aligned} \tag{4.28}$$

By integration by parts, one gets

$$\begin{aligned} \int_\Omega x_i^2 u \Delta u &= -\int_\Omega \langle \nabla(x_i^2 u), \nabla u \rangle \\ &= -\int_\Omega x_i^2 |\nabla u|^2 - 2\int_\Omega x_i u u_{x_i} \\ &= -\int_\Omega x_i^2 |\nabla u|^2 + \int_\Omega u^2. \end{aligned} \tag{4.29}$$

Thus,

$$\int_\Omega x_i u \Delta^2 (x_i u) = \Lambda_1 \int_\Omega x_i^2 |\nabla u|^2 - \Lambda_1 \int_\Omega u^2 + 4 \int_\Omega x_i u \Delta u_{x_i}. \tag{4.30}$$

Substituting (4.27) and (4.30) into (4.26) and dividing both sides by $\int_\Omega x_i^2|\nabla u|^2$, we get

$$\Lambda_{i+1}-\Lambda_1 \le \frac{-\Lambda_1\int_\Omega u^2+4\int_\Omega x_i u\Delta u_{x_i}}{\int_\Omega x_i^2|\nabla u|^2}. \tag{4.31}$$

We have

$$\begin{aligned}\int_\Omega x_i u\Delta u_{x_i} &= -\int_\Omega \langle\nabla(x_i u),\nabla u_{x_i}\rangle\\ &= -\int_\Omega (x_i\langle\nabla u,\nabla u_{x_i}\rangle+uu_{x_ix_i})\\ &= \frac{1}{2}\int_\Omega|\nabla u|^2+\int_\Omega u_{x_i}^2\\ &= \frac{1}{2}+\int_\Omega u_{x_i}^2.\end{aligned} \tag{4.32}$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned}1 &= \int_\Omega(-u\Delta u)=\left(\int_\Omega(-u\Delta u)\right)^2\le\left(\int_\Omega u^2\right)\left(\int_\Omega(\Delta u)^2\right)\\ &= \left(\int_\Omega u^2\right)\left(\int_\Omega u\Delta^2 u\right)=\left(\int_\Omega u^2\right)\Lambda_1\int_\Omega|\nabla u|^2=\Lambda_1\int_\Omega u^2.\end{aligned} \tag{4.33}$$

Substituting (4.32) and (4.33) into (4.31), we get

$$\Lambda_{i+1}-\Lambda_1\le\frac{1+4\int_\Omega u_{x_i}^2}{\int_\Omega x_i^2|\nabla u|^2}. \tag{4.34}$$

Since

$$\begin{aligned}\left(\int_\Omega\langle\nabla(x_iu),\nabla u_{x_i}\rangle\right)^2 &\le \left(\int_\Omega|\nabla(x_iu)|^2\right)\left(\int_\Omega|\nabla u_{x_i}|^2\right)\\ &= \left(\int_\Omega x_i^2|\nabla u|^2\right)\left(\int_\Omega|\nabla u_{x_i}|^2\right),\end{aligned} \tag{4.35}$$

it follows that

$$\frac{\left(\int_\Omega\langle\nabla(x_iu),\nabla u_{x_i}\rangle\right)^2}{\int_\Omega x_i^2|\nabla u|^2}\le\int_\Omega|\nabla u_{x_i}|^2. \tag{4.36}$$

Combining (4.32) and (4.36), one gets

$$\frac{1+4\int_\Omega u_{x_i}^2+4\left(\int_\Omega u_{x_i}^2\right)^2}{\int_\Omega x_i^2|\nabla u|^2}\le 4\int_\Omega|\nabla u_{x_i}|^2. \tag{4.37}$$

Set

$$b_i=\int_\Omega u_{x_i}^2 \quad\text{and}\quad \epsilon_i=\frac{\left(\int_\Omega u_{x_i}^2\right)^2}{1+4\int_\Omega u_{x_i}^2}. \tag{4.38}$$

It then follows from (4.34) and (4.37) that

$$(1+4\epsilon_i)(\Lambda_{i+1}-\Lambda_1)\le 4\int_\Omega|\nabla u_{x_i}|^2. \tag{4.39}$$

Since $u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0$, we know that $u_{x_i}|_{\partial\Omega} = 0$, which implies from the divergence theorem that

$$\int_\Omega |\nabla u_{x_i}|^2 = -\int_\Omega u_{x_i}\Delta u_{x_i} = -\int_\Omega u_{x_i}(\Delta u)_{x_i}.$$

Let $e_i = (\underbrace{0,\cdots,0,1}_{i},0,\cdots,0)$ and consider the vector field $X = u(\Delta u)_{x_i}e_i$. We infer from the divergence theorem and $X|_{\partial\Omega} = 0$ that

$$0 = \int_\Omega \mathrm{div} X = \int_\Omega (u_{x_i}(\Delta u)_{x_i} + u(\Delta u)_{x_ix_i}).$$

Hence, we have

$$\int_\Omega |\nabla u_{x_i}|^2 = \int_\Omega u(\Delta u)_{x_ix_i}. \tag{4.40}$$

Substituting (4.40) into (4.39) and summing on i from 1 to n, we have

$$\begin{aligned} \sum_{i=1}^n (1+4\epsilon_i)(\Lambda_{i+1}-\Lambda_1) &\le 4\int_\Omega u\left(\sum_{i=1}^n (\Delta u)_{x_ix_i}\right) \\ &= -4\Lambda_1\int_\Omega u\Delta u \\ &= 4\Lambda_1\int_\Omega |\nabla u|^2 = 4\Lambda_1. \end{aligned} \tag{4.41}$$

Now we want to estimate the left hand side of the above inequality. We have

$$\sum_{i=1}^n (1+4\epsilon_i)(\Lambda_{i+1}-\Lambda_1) \ \ge\ \sum_{i=1}^n (\Lambda_{i+1}-\Lambda_1) + 4(\Lambda_2-\Lambda_1)\sum_{i=1}^n \epsilon_i. \tag{4.42}$$

From the definition, we know that

$$\sum_{i=1}^n b_i = \int_\Omega |\nabla u|^2 = 1. \tag{4.43}$$

Thus we deduce from Lemma 2.2 that

$$\sum_{i=1}^n \epsilon_i = f((b_1,\cdots,b_n)) \ge \frac{1}{n+4}. \tag{4.44}$$

Combining (4.41), (4.42) and (4.44), we get

$$\sum_{i=2}^{n+1}\Lambda_i + \frac{4(\Lambda_2-\Lambda_1)}{n+4} \le (n+4)\Lambda_1. \tag{4.45}$$

This completes the proof of Theorem 4.2.

The final result of this section is to prove the inequality (1.18). That is, we have

Theorem 4.3. *Let $l \ge 2$ be a positive integer and let Ω be a connected bounded domain with smooth boundary in $\mathbb{R}^n$. Consider the eigenvalue problem*

$$\begin{aligned} &(-\Delta)^l u = -\Lambda\Delta u \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\bigg|_{\partial\Omega} = \cdots = \frac{\partial^{l-1}u}{\partial\nu^{l-1}}\bigg|_{\partial\Omega} = 0. \end{aligned} \tag{4.46}$$

Let

$$0 < \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_{n+1}$$

denote the first $n+1$ eigenvalues of the above problem. Then we have

$$\sum_{k=1}^{n} \frac{k}{2l+k}(\Lambda_{n+2-k} - \Lambda_1) < 4(l-1)\Lambda_1. \tag{4.47}$$

Proof of Theorem 4.3. Let v_i be the i-th orthonormal eigenfunction of the problem (4.46) corresponding to the eigenvalue Λ_i, $i = 1, 2, \cdots$, that is, v_i satisfies

$$\begin{aligned} &(-\Delta)^l v_i = -\Lambda_i \Delta v_i \quad \text{in} \quad \Omega, \\ &v_i|_{\partial\Omega} = \left.\frac{\partial v_i}{\partial \nu}\right|_{\partial\Omega} = 0 = \cdots = \left.\frac{\partial^{l-1} v_i}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0, \\ &\int_\Omega \langle \nabla v_i, \nabla v_j \rangle = \delta_{ij}, \quad \forall\ i, j. \end{aligned}$$

As in the proof of Theorem 4.1, we take a set of Cartesian coordinates $(x_1, ..., x_n)$ of $\mathbb{R}^n$ so that the following orthogonality conditions are satisfied:

$$\int_\Omega \langle \nabla(x_i v_1), \nabla v_j \rangle = -\int_\Omega x_i v_1 \Delta v_j = 0 \quad \text{for} \quad 1 \le j \le i \le n. \tag{4.48}$$

The Rayleigh-Ritz inequality now states that

$$\Lambda_{i+1} \le \frac{\int_\Omega \phi(-\Delta)^l \phi}{\int_\Omega |\nabla \phi|^2}, \tag{4.49}$$

which is satisfied by any sufficiently smooth function ϕ such that

$$\begin{aligned} &\phi = \frac{\partial \phi}{\partial \nu} = \cdots = \frac{\partial^{l-1}\phi}{\partial \nu^{l-1}} = 0 \ \text{ on } \ \partial\Omega, \\ &\int_\Omega \langle \nabla \phi, \nabla v_j \rangle = 0, \ j = 1, \cdots, i. \end{aligned}$$

Setting $v = v_1$, we choose as our trial function $\phi = x_i v$, which clearly satisfies the above boundary condition, and by (4.48) the orthogonality condition also. Then we have

$$\Lambda_{i+1} \le \frac{\int_\Omega x_i v(-\Delta)^l (x_i v)}{\int_\Omega |\nabla(x_i v)|^2}, \ i = 1, \cdots, n. \tag{4.50}$$

Since $\Delta x_i = 0, i = 1, 2, \cdots, n$, we have

$$\begin{aligned} (-\Delta)^l (x_i v) &= x_i(-\Delta)^l v + 2l(-1)^l (\Delta^{l-1} v)_{x_i} \\ &= \Lambda_1 x_i(-\Delta v) + 2l(-1)^l (\Delta^{l-1} v)_{x_i}. \end{aligned} \tag{4.51}$$

As calculated in (4.27) and (4.29), we have

$$\int_\Omega |\nabla(x_i v)|^2 = \int_\Omega x_i^2 |\nabla v|^2, \ \int_\Omega x_i^2 v \Delta v = -\int_\Omega x_i^2 |\nabla v|^2 + \int_\Omega v^2. \tag{4.52}$$

Substituting (4.51) and (4.52) into (4.50), we get

$$\Lambda_{i+1} - \Lambda_1 \le \frac{-\Lambda_1 \int_\Omega v^2 + 2l(-1)^l \int_\Omega x_i v (\Delta^{l-1} v)_{x_i}}{\int_\Omega x_i^2 |\nabla v|^2}. \tag{4.53}$$

Since

$$\Delta^{l-1}(x_i v) = 2(l-1)(\Delta^{l-2} v)_{x_i} + x_i \Delta^{l-1} v,$$

we have

$$\begin{aligned}\int_\Omega x_i v(\Delta^{l-1}v)_{x_i} &= \int_\Omega x_i v\Delta^{l-1}v_{x_i}\\ &= \int_\Omega \Delta^{l-1}(x_i v)v_{x_i}\\ &= \int_\Omega (2(l-1)(\Delta^{l-2}v)_{x_i} + x_i\Delta^{l-1}v)v_{x_i}\\ &= \int_\Omega (-2(l-1)\Delta^{l-2}vv_{x_ix_i} + x_i\Delta^{l-1}vv_{x_i}).\end{aligned} \tag{4.54}$$

On the other hand, we have from the divergence theorem that

$$\int_\Omega x_i v(\Delta^{l-1}v)_{x_i} = -\int_\Omega \Delta^{l-1}v(v + x_i v_{x_i}). \tag{4.55}$$

Combining (4.54) and (4.55), we obtain

$$\begin{aligned}\int_\Omega x_i v(\Delta^{l-1}v)_{x_i} &= \int_\Omega \left((l-1)(\Delta^{l-2}v)_{x_i}v_{x_i} - \frac{1}{2}v\Delta^{l-1}v\right)\\ &= \int_\Omega \left(-(l-1)(\Delta^{l-2}v)_{x_ix_i}v - \frac{1}{2}v\Delta^{l-1}v\right).\end{aligned} \tag{4.56}$$

Substituting (4.56) into (4.53), we infer

$$\Lambda_{i+1} - \Lambda_1 \le \frac{-\Lambda_1\int_\Omega v^2 + l\int_\Omega(-2(l-1)((-\Delta)^{l-2}v)_{x_ix_i}v + v(-\Delta)^{l-1}v)}{\int_\Omega x_i^2|\nabla v|^2}. \tag{4.57}$$

By using the same arguments as in the proof of (4.37), one deduces

$$\frac{1 + 4\int_\Omega v_{x_i}^2 + 4\left(\int_\Omega v_{x_i}^2\right)^2}{\int_\Omega x_i^2|\nabla v|^2} \le 4\int_\Omega |\nabla v_{x_i}|^2,$$

which gives

$$\frac{1}{\int_\Omega x_i^2|\nabla v|^2} \le 4\int_\Omega |\nabla v_{x_i}|^2 \tag{4.58}$$

and since $\sum_{i=1}^n \int_\Omega v_{x_i}^2 = \int_\Omega |\nabla v|^2 = 1$, we know that there exists at least one $i \in \{1, \cdots, n\}$ such that (4.58) is a strict inequality. Set

$$a = \int_\Omega v(-\Delta)^{l-1}v, \quad a_i = \int_\Omega(-((-\Delta)^{l-2}v)_{x_ix_i}v), \ i = 1, \cdots, n; \tag{4.59}$$

then

$$\sum_{i=1}^n a_i = a. \tag{4.60}$$

Introducing (4.59) into (4.57), we have

$$\Lambda_{i+1} - \Lambda_1 \le 4\left(-\Lambda_1\int_\Omega v^2 + l(2(l-1)a_i + a)\right)\int_\Omega |\nabla v_{x_i}|^2, \tag{4.61}$$

and there exists at least $i \in \{1, \cdots, n\}$ such that (4.61) is a strict inequality. Take a permutation $\{i_1, \cdots, i_n\}$ of $\{1, \cdots, n\}$ so that

$$a_{i_n} \le a_{i_{n-1}} \le \cdots \le a_{i_1}. \tag{4.62}$$

It then follows from (4.60) and (4.62) that

$$a_{i_k} \leq \frac{1}{k} a, \ k = 1, \cdots, n. \tag{4.63}$$

Substituting (4.63) into (4.61), we get

$$\begin{aligned} \Lambda_{i_k+1} - \Lambda_1 &\leq 4\left(-\Lambda_1 \int_\Omega v^2 + l(2(l-1)a_{i_k} + a)\right) \int_\Omega |\nabla v_{x_{i_k}}|^2 \\ &\leq 4\left(-\Lambda_1 \int_\Omega v^2 + l\left(\frac{2(l-1)}{k} + 1\right) a\right) \int_\Omega |\nabla v_{x_{i_k}}|^2 \end{aligned} \tag{4.64}$$

and for some $k \in \{1, \cdots, n\}$, (4.64) is a strict inequality. Before we can finish the proof of Theorem 4.3, let us prove the following inequalities:

$$\int_\Omega v(-\Delta)^{l-1} v \leq \Lambda_1^{(l-2)/(l-1)}, \tag{4.65}$$

$$\int_\Omega v\Delta^2 v \leq \Lambda_1^{1/(l-1)}, \tag{4.66}$$

$$\int_\Omega v^2 \geq \Lambda_1^{-1/(l-1)}. \tag{4.67}$$

First observe as in the proof of Lemma 2.1 that for any $k = 1, \cdots, l-1$,

$$\int_\Omega v(-\Delta)^k v \geq 0.$$

When $l = 2$, (4.65) and (4.66) hold obviously and in this case we have from the Schwarz inequality that

$$1 = \int_\Omega (-v\Delta v) = \left(\int_\Omega (-v\Delta v)\right)^2 \leq \left(\int_\Omega v^2\right)\left(\int_\Omega (\Delta v)^2\right) = \Lambda_1 \int_\Omega v^2.$$

Hence (4.67) holds when $l = 2$.

Assume now that $l > 2$. We claim that for any $k = 2, \cdots, l-1$,

$$\left(\int_\Omega v(-\Delta)^k v\right)^k \leq \left(\int_\Omega v(-\Delta)^{k+1} v\right)^{k-1}. \tag{4.68}$$

Since

$$\int_\Omega v\Delta^2 v = \int_\Omega \Delta v \Delta v = -\int_\Omega \nabla\Delta v \nabla v,$$

we have from the Schwarz inequality that

$$\left(\int_\Omega v\Delta^2 v\right)^2 \leq \left(\int_\Omega |\nabla\Delta v|^2\right)\left(\int_\Omega |\nabla v|^2\right) = -\int_\Omega \Delta v \Delta^2 v = \int_\Omega v(-\Delta^3 v). \tag{4.69}$$

Hence (4.68) holds when $k = 2$. Suppose that (4.68) holds for $k-1$, that is,

$$\left(\int_\Omega v(-\Delta)^{k-1} v\right)^{k-1} \leq \left(\int_\Omega v(-\Delta)^k v\right)^{k-2}. \tag{4.70}$$

As in the proof of (2.26), we have

$$\int_\Omega v(-\Delta)^k v \le \left(\int_\Omega v(-\Delta)^{k-1}v\right)^{1/2}\left(\int_\Omega v(-\Delta)^{k+1}v\right)^{1/2}. \tag{4.71}$$

Substituting (4.70) into (4.71), we know that (4.68) is true for k. Using (4.68) repeatedly, we get

$$\begin{aligned}\int_\Omega v(-\Delta)^k v &\le \left(\int_\Omega v(-\Delta)^{k+1}v\right)^{(k-1)/k} \le \cdots \\ &\le \left(\int_\Omega v(-\Delta)^{l}v\right)^{(k-1)/(l-1)} = \Lambda_1^{(k-1)/(l-1)}.\end{aligned}$$

Taking $k=2$ and $k=l-1$ in the above inequality, respectively, one gets (4.65) and (4.66). On the other hand, we have from the Schwarz inequality and (4.66) that

$$\begin{aligned}1 &= \int_\Omega(-v\Delta v) = \left(\int_\Omega(-v\Delta v)\right)^2 \le \left(\int_\Omega v^2\right)\left(\int_\Omega(\Delta v)^2\right) \\ &= \left(\int_\Omega v^2\right)\left(\int_\Omega v\Delta^2 v\right) \le \Lambda_1^{1/(l-1)}\left(\int_\Omega v^2\right).\end{aligned} \tag{4.72}$$

This proves (4.67). Now we continue with the proof of Theorem 4.3. Substituting (4.65) and (4.67) into (4.64) and multiplying both sides by $\frac{k}{k+2l}$, we get

$$\frac{k}{k+2l}(\Lambda_{i_k+1}-\Lambda_1) \le 4(l-1)\Lambda_1^{(l-2)/(l-1)}\int_\Omega |\nabla v_{x_{i_k}}|^2 \tag{4.73}$$

and for some $k\in\{1,\cdots,n\}$, the above inequality is a strict inequality. Thus by summing on k and using $\int_\Omega v\Delta^2 v \le \Lambda_1^{1/(l-1)}$, one gets

$$\begin{aligned}\sum_{k=1}^n \frac{k}{2l+k}(\Lambda_{i_k+1}-\Lambda_1) &< 4(l-1)\Lambda_1^{(l-2)/(l-1)}\sum_{k=1}^n\int_\Omega|\nabla v_{x_{i_k}}|^2 \\ &= 4(l-1)\Lambda_1^{(l-2)/(l-1)}\sum_{k=1}^n\int_\Omega|\nabla v_{x_k}|^2 \\ &= 4(l-1)\Lambda_1^{(l-2)/(l-1)}\int_\Omega v\Delta^2 v \le 4(l-1)\Lambda_1.\end{aligned} \tag{4.74}$$

Since $\left\{\frac{k}{2l+k}\right\}_{k=1}^n$ and $\{\Lambda_{k+1}-\Lambda_1\}_{k=1}^n$ are two increasing sequences, we have from Lemma 2.5 that

$$\sum_{k=1}^n \frac{k}{2l+k}(\Lambda_{i_k+1}-\Lambda_1) \ge \sum_{k=1}^n \frac{k}{2l+k}(\Lambda_{n+2-k}-\Lambda_1). \tag{4.75}$$

Substituting (4.75) into (4.74), we get (4.47). This completes the proof of Theorem 4.3.

5. Eigenvalues of the polyharmonic operators on compact domains in a unit sphere

In this section, we will prove universal inequalities for eigenvalues of the polyharmonic operators on compact connected domains in a unit n-sphere S^n. Let l be a positive integer and for $p = 0, 1, 2, ...$, define the polynomials $F_p(t)$ inductively by

$$\begin{aligned} &F_0(t) = 1, \quad F_1(t) = t - n, \\ &F_p(t) = (2t-2)F_{p-1}(t) - (t^2 + 2t - n(n-2))F_{p-2}(t), \quad p = 2, \cdots . \end{aligned} \tag{5.1}$$

Set

$$F_l(t) = t^l + a_{l-1}t^{l-1} + \cdots + a_1 t + a_0. \tag{5.2}$$

Theorem 5.1. *Let λ_i be the i-th eigenvalue of the following eigenvalue problem:*

$$\begin{aligned} &(-\Delta)^l u = \lambda u \quad in \ \ \Omega, \\ &u|_{\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \cdots = \left.\frac{\partial^{l-1} u}{\partial \nu^{l-1}}\right|_{\partial\Omega} = 0, \end{aligned}$$

where Ω is a compact connected domain in a unit n-sphere S^n. Then we have

$$\begin{aligned} &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq \ & \frac{1}{n} \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \left(|a_{l-1}| \lambda_i^{(l-1)/l} + \cdots + |a_1| \lambda_i^{1/l} + |a_0| \right) \right\}^{1/2} \\ & \times \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(n^2 + 4\lambda_i^{1/2} \right) \right\}^{1/2}. \end{aligned} \tag{5.3}$$

Corollary 5.1. *Under the same assumptions as in Theorem 5.1, we have*

$$\begin{aligned} \lambda_{k+1} \leq \ & \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{1}{2n^2k^2} \left(\sum_{i=1}^{k} \left(|a_{l-1}| \lambda_i^{(l-1)/l} + \cdots + |a_1| \lambda_i^{1/l} + |a_0| \right) \right) \\ & \times \left(kn^2 + 4 \sum_{i=1}^{k} \lambda_i^{1/l} \right) \\ & + \left\{ \frac{1}{4n^4k^4} \left(\sum_{i=1}^{k} \left(|a_{l-1}| \lambda_i^{(l-1)l} + \cdots + |a_1| \lambda_i^{1/l} + |a_0| \right) \right)^2 \left(kn^2 + 4 \sum_{i=1}^{k} \lambda_i^{1/l} \right)^2 \right. \\ & \left. - \frac{1}{k} \sum_{i=1}^{k} \left(\lambda_i - \frac{1}{k} \sum_{j=1}^{k} \lambda_j \right)^2 \right\}^{1/2} \end{aligned} \tag{5.4}$$

and

$$\lambda_{k+1} \leq U_{k+1} + \sqrt{U_{k+1}^2 - V_{k+1}}, \tag{5.5}$$

where

(5.6)
$$U_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{1}{2n^2k}\sum_{i=1}^{k}\left(|a_{l-1}|\lambda_i^{(l-1)/l} + \cdots + |a_1|\lambda_i^{1/l} + |a_0|\right)\left(n^2 + 4\lambda_i^{1/l}\right)$$

and

(5.7)
$$V_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\lambda_i^2 + \frac{1}{n^2k}\sum_{i=1}^{k}\lambda_i\left(|a_{l-1}|\lambda_i^{(l-1)/l} + \cdots + |a_1|\lambda_i^{1/l} + |a_0|\right)\left(n^2 + 4\lambda_i^{1/l}\right).$$

Proof of Theorem 5.1. Let $x_1, x_2, \cdots, x_{n+1}$ be the standard coordinate functions of the Euclidean space $\mathbb{R}^{n+1}$; then

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{\alpha=1}^{n+1} x_\alpha^2 = 1\}.$$

It is well known that

$$\Delta x_\alpha = -n x_\alpha, \quad \alpha = 1, \cdots, n+1. \tag{5.8}$$

Let u_i be the i-th orthonormal eigenfunction corresponding to the eigenvalue λ_i, $i = 1, 2, \cdots$. For any $\delta > 0$, by taking $h = x_\alpha$ in (2.4), we have

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \int_\Omega (-x_\alpha u_i^2 \Delta x_\alpha - 2x_\alpha u_i \langle \nabla x_\alpha, \nabla u_i\rangle) \\
&\qquad \le \delta \sum_{i=1}^{k+1}(\lambda_{k+1} - \lambda_i)^2 \int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i) - \lambda_i x_\alpha u_i) \\
&\qquad\quad + \frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left|\left|\langle \nabla x_\alpha, \nabla u_i\rangle + \frac{u_i \Delta x_\alpha}{2}\right|\right|^2.
\end{aligned}$$

Taking the sum on α from 1 to $n+1$, we get

(5.9)
$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \sum_{\alpha=1}^{n+1}\int_\Omega \left(-x_\alpha u_i^2 \Delta x_\alpha - 2x_\alpha u_i \langle \nabla x_\alpha, \nabla u_i\rangle\right) \\
&\qquad \le \delta \sum_{i=1}^{k+1}(\lambda_{k+1} - \lambda_i)^2 \sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i) - \lambda_i x_\alpha u_i) \\
&\qquad\quad + \frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\sum_{\alpha=1}^{n+1}\left|\left|\langle \nabla x_\alpha, \nabla u_i\rangle + \frac{u_i \Delta x_\alpha}{2}\right|\right|^2.
\end{aligned}$$

Using $\sum_{\alpha=1}^{n+1} x_\alpha^2 = 1$, (2.17) and (5.8), we infer

$$\sum_{\alpha=1}^{n+1} \int_\Omega \left(-x_\alpha u_i^2 \Delta x_\alpha - 2x_\alpha u_i \langle \nabla x_\alpha, \nabla u_i\rangle\right) \tag{5.10}$$
$$= \int_\Omega \left(\left(\sum_{\alpha=1}^{n+1} x_\alpha^2\right) n u_i^2 - u_i \left\langle \nabla \left(\sum_{\alpha=1}^{n+1} x_\alpha^2\right), \nabla u_i \right\rangle\right) = \int_\Omega n u_i^2 = n,$$

$$\sum_{\alpha=1}^{n+1} \left|\left|\langle \nabla x_\alpha, \nabla u_i\rangle + \frac{u_i \Delta x_\alpha}{2}\right|\right|^2 \tag{5.11}$$
$$= \int_\Omega \sum_{\alpha=1}^{n+1} \left(\langle \nabla x_\alpha, \nabla u_i\rangle^2 - n\langle \nabla x_\alpha, \nabla u_i\rangle u_i x_\alpha + \frac{n^2 u_i^2 x_\alpha^2}{4}\right)$$
$$= \frac{n^2}{4} + \int_\Omega |\nabla u_i|^2$$
$$= \frac{n^2}{4} + \int_\Omega u_i(-\Delta u_i)$$
$$\leq \frac{n^2}{4} + \lambda_i^{1/l}.$$

For any smooth function f on Ω, we have from the Bochner formula that

$$\begin{aligned} \frac{1}{2}\Delta|\nabla f|^2 &= |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \mathrm{Ric}(\nabla f, \nabla f) \\ &= |\nabla^2 f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + (n-1)|\nabla f|^2, \end{aligned} \tag{5.12}$$

where Ric is the Ricci tensor of S^n. Thus for any smooth function g on Ω,

$$\frac{1}{2}\Delta|\nabla g|^2 = |\nabla^2 g|^2 + \langle \nabla g, \nabla(\Delta g)\rangle + (n-1)|\nabla g|^2 \tag{5.13}$$

and

$$\frac{1}{2}\Delta|\nabla(f+g)|^2 = |\nabla^2(f+g)|^2 + \langle \nabla(f+g), \nabla(\Delta(f+g))\rangle + (n-1)|\nabla(f+g)|^2. \tag{5.14}$$

Subtracting the sum of (5.12) and (5.13) from (5.14), we get

$$\Delta\langle \nabla f, \nabla g\rangle = 2\langle \nabla^2 f,\ \nabla^2 g\rangle + \langle \nabla f, \nabla(\Delta g)\rangle + \langle \nabla g, \nabla(\Delta f)\rangle + 2(n-1)\langle \nabla f,\ \nabla g\rangle, \tag{5.15}$$

where

$$\langle \nabla^2 f,\ \nabla^2 g\rangle = \sum_{s,t=1}^{n} \nabla^2 f(e_s, e_t)\nabla^2 g(e_s, e_t),$$

$e_1, \cdots, e_n$ being orthonormal vector fields locally defined on Ω. Since

$$\nabla^2 x_\alpha = -x_\alpha \langle\ ,\ \rangle,$$

we infer from (5.15) by taking $f = x_\alpha$ that

$$\begin{aligned} \Delta\langle \nabla x_\alpha, \nabla g\rangle &= -2x_\alpha \Delta g + \langle \nabla x_\alpha, \nabla(\Delta g)\rangle + (n-2)\langle \nabla x_\alpha,\ \nabla g\rangle \\ &= -2x_\alpha \Delta g + \langle \nabla x_\alpha, \nabla((\Delta + (n-2))g)\rangle. \end{aligned} \tag{5.16}$$

For each $q = 0, 1, \cdots$, thanks to (5.8) and (5.16), there are polynomials B_q and C_q of degrees less than or equal to q such that

$$\Delta^q(x_\alpha g) = x_\alpha B_q(\Delta)g + 2\langle \nabla x_\alpha, \nabla(C_q(\Delta)g)\rangle. \tag{5.17}$$

It is obvious that

$$B_0 = 1, \quad B_1 = t - n, \quad C_0 = 0, \quad C_1 = 1. \tag{5.18}$$

It follows from (5.8), (5.16) and (5.17) that

$$\begin{aligned} \Delta^q(x_\alpha g) &= \Delta(\Delta^{q-1}(x_\alpha g)) \\ &= \Delta(x_\alpha B_{q-1}(\Delta)g + 2\langle \nabla x_\alpha, \nabla(C_{q-1}(\Delta)g)\rangle) \\ &= x_\alpha((\Delta - n)B_{q-1}(\Delta) - 4\Delta C_{q-1}(\Delta))g \\ &\qquad + 2\langle \nabla x_\alpha, \nabla((B_{q-1}(\Delta) + (\Delta + (n-2))C_{q-1}(\Delta))g)\rangle. \end{aligned} \tag{5.19}$$

Thus, for any $q = 2, \cdots$, we have

$$B_q(\Delta) = (\Delta - n)B_{q-1}(\Delta) - 4\Delta C_{q-1}(\Delta), \tag{5.20}$$

$$C_q(\Delta) = B_{q-1}(\Delta) + (\Delta + (n-2))C_{q-1}(\Delta). \tag{5.21}$$

Consequently, we have

$$\begin{aligned} B_q(\Delta) &= (2\Delta - 2)B_{q-1}(\Delta) - (\Delta + n - 2)B_{q-1}(\Delta) - 4\Delta C_{q-1}(\Delta) \\ &= (2\Delta - 2)B_{q-1}(\Delta) - (\Delta + n - 2)((\Delta - n)B_{q-2}(\Delta) - 4\Delta C_{q-2}(\Delta)) \\ &\qquad - 4\Delta C_{q-1}(\Delta) \\ &= (2\Delta - 2)B_{q-1}(\Delta) - (\Delta^2 + 2\Delta - n(n-2))B_{q-2}(\Delta) \\ &\quad + 4\Delta[B_{q-2}(\Delta) + (\Delta + n - 2)C_{q-2}(\Delta) - C_{q-1}(\Delta)] \\ &= (2\Delta - 2)B_{q-1}(\Delta) - (\Delta^2 + 2\Delta - n(n-2))B_{q-2}(\Delta), \quad q = 2, \cdots. \end{aligned} \tag{5.22}$$

Since (5.18) and (5.22) hold, we know that $B_q = F_q, \ \forall q = 0, 1, \cdots$. It follows from (5.17) and the divergence theorem that

$$\begin{aligned} &\int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i) - \lambda_i x_\alpha u_i) \\ =& \int_\Omega x_\alpha u_i \left((-1)^l (x_\alpha B_l(\Delta)u_i + 2\langle \nabla x_\alpha, \nabla(C_l(\Delta)u_i)\rangle) - \lambda_i x_\alpha u_i\right) \\ =& \int_\Omega x_\alpha u_i \left((-1)^l \left(x_\alpha(\Delta^l + a_{l-1}\Delta^{l-1} + \cdots + a_0)u_i \right.\right. \\ &\qquad \left.\left. + 2\langle \nabla x_\alpha, \nabla(C_l(\Delta)u_i)\rangle\right) - \lambda_i x_\alpha u_i\right) \\ =& \int_\Omega (-1)^l x_\alpha u_i \left(x_\alpha(a_{l-1}\Delta^{l-1} + \cdots + a_0)u_i + 2\langle \nabla x_\alpha, \nabla(C_l(\Delta)u_i)\rangle\right). \end{aligned} \tag{5.23}$$

Summing on α, one has from $\nabla\left(\sum_{\alpha=1}^{n+1} x_\alpha^2\right) = 0$ and (2.17) that

$$\begin{aligned} &\int_\Omega x_\alpha u_i((-\Delta)^l(x_\alpha u_i) - \lambda_i x_\alpha u_i) \\ =& \int_\Omega u_i(-1)^l(a_{l-1}\Delta^{l-1} + \cdots + a_0)u_i \\ \le& |a_{l-1}| \int_\Omega u_i(-\Delta)^{l-1}u_i + \cdots + |a_1| \int_\Omega u_i(-\Delta)u_i + |a_0| \int_\Omega u_i^2 \\ \le& |a_{l-1}|\lambda_i^{(l-1)/l} + \cdots + |a_1|\lambda_i^{1/l} + |a_0|. \end{aligned} \tag{5.24}$$

Substituting (5.10), (5.11) and (5.24) into (5.9), we infer

$$\begin{aligned} n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \quad &\leq \quad \delta\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|) \\ &\quad +\frac{1}{\delta}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/l}+\frac{n^2}{4}\right). \end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_i^{1/l}+\frac{n^2}{4}\right)}{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|)}\right\}^{1/2}, \tag{5.25}$$

we get (5.1). This completes the proof of Theorem 5.1.

Proof of Corollary 5.1. From Lemma 2.4, we have

$$\begin{aligned} &\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|\right) \\ \leq\ &\frac{1}{k}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right\}\left\{\sum_{i=1}^{k}\left(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|\right)\right\} \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(n^2+4\lambda_i^{1/l}\right) &\leq \frac{1}{k}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\right\}\left\{\sum_{i=1}^{k}\left(n^2+4\lambda_i^{1/l}\right)\right\} \\ &= \frac{1}{k}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\right\}\left\{kn^2+4\sum_{i=1}^{k}\lambda_i^{1/l}\right\}. \end{aligned} \tag{5.27}$$

Substituting (5.26) and (5.27) into (5.3), we get

$$\begin{aligned} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{1}{n^2k^2}&\left\{\sum_{i=1}^{k}\left(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|\right)\right\} \\ &\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\right\}\left\{kn^2+4\sum_{i=1}^{k}\lambda_i^{1/l}\right\}. \end{aligned} \tag{5.28}$$

Solving this quadratic polynomial of λ_{k+1}, we have (5.4).

On the other hand, one gets by using (2.27) that

$$\begin{aligned} &\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\left(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|\right)\right\} \\ &\quad\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(n^2+4\lambda_i^{1/2}\right)\right\} \leq \left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right\} \\ &\quad\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|\right)\left(n^2+4\lambda_i^{1/2}\right)\right\}, \end{aligned} \tag{5.29}$$

which, combining with (5.3), gives

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{1}{n^2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(|a_{l-1}|\lambda_i^{(l-1)/l}+\cdots+|a_1|\lambda_i^{1/l}+|a_0|\right)\left(n^2+4\lambda_i^{1/l}\right). \tag{5.30}$$

Hence

$$\lambda_{k+1} \leq U_{k+1}+\sqrt{U_{k+1}^2-V_{k+1}}, \tag{5.31}$$

where U_{k+1} and V_{k+1} are given by (5.6) and (5.7), respectively. Thus (5.5) holds.

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