TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 5, May 2011, Pages 2715–2732
S 0002-9947(2010)05253-5
Article electronically published on November 30, 2010

LINEAR INEQUALITIES FOR ENUMERATING CHAINS IN PARTIALLY ORDERED SETS

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Abstract. We characterize the linear inequalities satisfied by flag f-vectors of all finite bounded posets. We do the same for semipure posets. In particular, the closed convex cone generated by flag f-vectors of bounded posets of fixed rank is shown to be simplicial, and the closed cone generated by flag f-vectors of semipure posets of fixed rank is shown to be polyhedral. The extreme rays of both of these cones are described explicitly in terms of quasisymmetric functions. The extreme rays of the first cone are then used to define a new basis for the algebra of quasisymmetric functions. This basis has nonnegative structure constants, for which a combinatorial interpretation involving lattice paths is given.

1. Introduction

The flag f-vector of a poset records the numbers of chains in the poset that pass through prescribed sets of ranks. In this paper we adapt techniques of Billera and Hetyei [3] to give a characterization of the linear inequalities satisfied by flag f-vectors of all (finite) bounded posets. We also obtain a characterization of the linear inequalities satisfied by flag f-vectors of all semipure posets. As an application we give a combinatorial interpretation of the structure constants of a new basis in the algebra of quasisymmetric functions.

Let P be a finite bounded poset, meaning P has a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$. We will assume that $\hat{0} \neq \hat{1}$. The *rank* of an element $p \in P$, denoted $\mathrm{rk}(p)$, is the length of the longest chain from $\hat{0}$ up to p. We call $\mathrm{rk}(\hat{1})$ the rank of P. If P is of rank $n+1$ and S is a subset of $[n] = \{1, 2, \ldots, n\}$, let $f_S(P)$ denote the number of chains $\hat{0} < p_1 < \cdots < p_k < \hat{1}$ in P such that $\{\mathrm{rk}(p_1), \ldots, \mathrm{rk}(p_k)\} = S$. The function from $2^{[n]}$ to $\mathbb{N}$ given by $S \mapsto f_S(P)$ will be called the *flag f-vector* of P. We say that P is *graded* (or *pure*) if every maximal chain in P has the same length, and *semipure* (following [10, 11]) if every interval $[p, q]$ in P such that $q \neq \hat{1}$ is graded. The main result of [3] gives necessary and sufficient conditions on a sequence of real numbers $a_S, S \subseteq [n]$, under which

$$\sum_{S \subseteq [n]} a_S \cdot f_S(P) \geq 0$$

Received by the editors April 10, 2009 and, in revised form, October 19, 2009.
2010 *Mathematics Subject Classification.* Primary 06A07, 05E99, 16T30.
Key words and phrases. Flag f-vector, bounded poset, semipure poset, quasisymmetric function.

holds for every graded poset P of rank $n+1$. This paper develops analogous results in the broader context of bounded posets.

Flag f-vectors have been studied extensively for various special classes of posets (for example, face lattices of convex polytopes [2]). In addition to recording essential enumerative information about chains, flag f-vectors can also encode more subtle topological or representation-theoretic information about a poset [7, 19]. In fact, their definition extends naturally to colored simplicial complexes, and in this context, flag f-vectors can be characterized both numerically [14] and in terms of, say, homology of completely balanced Cohen-Macaulay complexes [8, 18]. More recently, the development of a theory of shellability for nonpure posets [10, 11] and the companion theory of sequentially Cohen-Macaulay complexes [20] has drawn attention to various classes of nonpure posets and complexes (see, for example, [12, 15, 17, 23]), although little consideration has been given to properties of flag f-vectors of arbitrary bounded posets.

To describe our characterization of linear inequalities satisfied by flag f-vectors of bounded posets, let us first make a simple observation: In a bounded poset, any chain whose minimum element has rank $k > 1$ can be extended to a longer chain by adjoining a new element of arbitrary rank smaller than k. Hence for any bounded poset P of rank $n+1$,

$$f_S(P) - f_{S-\{\min S\}}(P) \geq 0 \quad \text{for every nonempty subset } S \subseteq [n], \tag{1.1}$$

where $\min S$ denotes the smallest element in S. Our first main result, Theorem 2.1, implies that these inequalities, along with the trivial inequality $f_\emptyset(P) \geq 0$, are essentially the only ones satisfied by flag f-vectors of arbitrary bounded posets of rank $n+1$; all other linear inequalities can be obtained as nonnegative linear combinations of these.

Turning to semipure posets, in Theorem 3.2 we characterize the linear inequalities satisfied by flag f-vectors of all semipure posets of fixed rank. Our characterization is given in terms of blockers of families of intervals and is quite similar to the corresponding result for graded posets [3, Theorem 2.1]. In a sense, the inequalities for semipure posets can be thought of as "lifts" (a term that is made precise in the proof of Proposition 4.4) of inequalities for graded posets of lower ranks.

In Section 4 we refine Theorem 3.2 by explicitly determining the extreme rays of the closure of the cone generated by flag f-vectors of all semipure posets (Proposition 4.4). The results are stated in terms of quasisymmetric functions, a convenient language for handling generating functions of flag f-vectors. In addition, we observe that the extreme rays of the cone of flag f-vectors of bounded posets of rank $n+1$ form a basis for the vector space of quasisymmetric functions of degree $n+1$, and in Section 5 we explore an algebraic consequence of this observation; namely, we exploit the fact that these extreme rays represent "limit posets" to give a lattice-path interpretation of the structure constants for this basis.

In the Appendix we prove that Ehrenborg's F-quasisymmetric function [13] extends to a Hopf algebra homomorphism from a certain graded Hopf algebra that includes bounded posets to the Hopf algebra of quasisymmetric functions. This fact, which generalizes a result of Ehrenborg [13, Proposition 4.4], is needed in our discussion in Section 5.

Throughout this paper, if n and m are integers, then we let $[n, m] = \{i \in \mathbb{Z} \mid n \leq i \leq m\}$. Usually we will write $[n]$ for $[1, n]$.

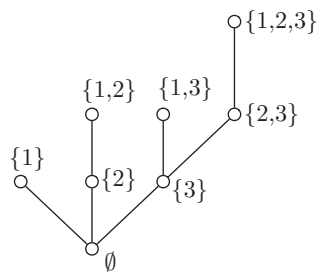

FIGURE 1. The partial order $\preccurlyeq$ on subsets of $\{1, 2, 3\}$.

For the remainder of this section we review some material that is needed for the rest of the paper.

The dual vector space of chain operators. Taking the perspective of [3, 6], we view f_S as a *chain operator* on bounded posets and write f_S^{n+1} to mean the operator that assigns a bounded poset P to $f_S(P)$ if $\mathrm{rk}(P) = n + 1$, and to 0 otherwise. It follows from [6, Proposition 1.1] that the operators f_S^{n+1}, $S \subseteq [n]$, are linearly independent, and hence they span a 2^n-dimensional vector space over $\mathbb{R}$, denoted by $\mathcal{A}_{n+1}$. When n is understood from the context we will drop the superscripts and write f_S for f_S^{n+1}.

It is sometimes convenient to think of $\mathcal{A}_{n+1}$ as the dual of the vector space spanned by the flag f-vectors of all bounded posets of rank $n+1$. If we let $\mathit{flag}(P)$ denote the flag f-vector of P and define $f_S(\mathit{flag}(P)) = f_S(P)$, then this interpretation of $\mathcal{A}_{n+1}$ makes sense. This perspective is perhaps most helpful when discussing extreme rays and facets of certain cones associated with collections of posets. More specifically, if $\mathcal{P}$ is some collection of bounded posets of rank $n + 1$ and $\mathcal{C} \subseteq \mathbb{R}^{2^n}$ is the closure of the convex cone generated by the flag f-vectors of all posets in $\mathcal{P}$, then the dual cone $\mathcal{C}^* := \{\varphi \in \mathcal{A}_{n+1} \mid \forall v \in \mathcal{C}, \varphi(v) \geq 0\}$ is precisely the set of all nonnegative linear forms on $\mathcal{P}$; that is, we have

$$\mathcal{C}^* = \{\varphi \in \mathcal{A}_{n+1} \mid \varphi(P) \geq 0 \text{ for all } P \in \mathcal{P}\}.$$

In particular, if $\mathcal{C}$ is polyhedral, then a nonnegative linear form φ is an extreme ray of $\mathcal{C}^*$ if and only if the inequality $\varphi \geq 0$ (short for $\{v \mid \varphi(v) \geq 0\}$) determines a facet of $\mathcal{C}$.

2. Inequalities for bounded posets

Throughout this section let n be a fixed nonnegative integer.

We define a partial order $\preccurlyeq$ on the set of subsets of $[n]$ by setting $\emptyset \preccurlyeq S$ for all $S \subseteq [n]$, and $T \preccurlyeq Q \cup T$ for all $T \subseteq [n]$, $T \neq \emptyset$, and $Q \subseteq [1, \min T - 1]$. For example, when $n = 6$ we have $\{3, 4, 6\} \preccurlyeq S$ if and only if $S = \{3, 4, 6\} \cup T$ where $T \subseteq \{1, 2\}$. See Figure 1 for the entire poset in the case $n = 3$.

For every nonempty subset $S \subseteq [n]$, we define the linear form $e_S \in \mathcal{A}_{n+1}$ by

$$e_S = f_S - f_{S-\{\min S\}} \quad \text{and set} \quad e_\emptyset = f_\emptyset. \tag{2.1}$$

We will write e_S^{n+1} when we wish to emphasize the degree of the linear form. Since the chain operators f_S are linearly independent, it is clear that the linear forms e_S are as well.

The main result of this section says that the 2^n linear forms e_S are the extreme rays of the cone of nonnegative linear forms (i.e., $\{\varphi \in \mathcal{A}_{n+1} \mid P \text{ bounded} \implies \varphi(P) \geq 0\}$), and it gives a simple way to determine whether a linear form $\sum_{S\subseteq[n]} a_S \cdot f_S$ is in this cone.

Theorem 2.1. *Let $a_S \in \mathbb{R}$ for every $S \subseteq [n]$. The following are equivalent:*

(1) *For every bounded poset P of rank $n+1$,*
$$\sum_{S\subseteq[n]} a_S \cdot f_S(P) \geq 0.$$

(2) *For every $T \subseteq [n]$,*
$$\sum_{S\succcurlyeq T} a_S \geq 0.$$

(3) *The linear form $\sum_{S\subseteq[n]} a_S \cdot f_S$ is in the convex cone generated by $\{e_S \mid S \subseteq [n]\}$.*

Example 2.2. When $n = 2$, this theorem implies that $a\, f_\emptyset(P) + b\, f_{\{1\}}(P) + c\, f_{\{2\}}(P) + d\, f_{\{1,2\}}(P) \geq 0$ for every bounded poset P of rank 3 if and only if the following four inequalities hold:

$$\begin{array}{ccccccccc} a & + & b & + & c & + & d & \geq & 0 \\ & & b & & & & & \geq & 0 \\ & & & & c & + & d & \geq & 0 \\ & & & & & & d & \geq & 0 \end{array}$$

or equivalently, the linear form $a\, f_\emptyset + b\, f_{\{1\}} + c\, f_{\{2\}} + d\, f_{\{1,2\}} \in \mathcal{A}_3$ is a conic combination of $f_\emptyset$, $f_{\{1\}} - f_\emptyset$, $f_{\{2\}} - f_\emptyset$, and $f_{\{1,2\}} - f_{\{2\}}$.

Before proving Theorem 2.1, we need to construct sequences of posets with suitable limit flag f-vectors, as in [3].

Definition 2.3. Let $T = \{t_1 < t_2 < \cdots < t_k\}$ be a subset of $[n]$. For each $N > 0$, define the poset $P(n, T, N)$ as follows. The elements of $P(n, T, N)$ are all pairs $(a, b) \in [0, n+1] \times [1, N]$ such that $b = 1$ whenever $a \notin T$. The partial order on P is defined by letting $(a, b) < (c, d)$ if $a < c$ and either $b = 1$ or $a, c \in T$. See Figure 2 for an example when $n = 5$, $T = \{2, 3, 5\}$ and $N = 3$.

The poset $P(n, T, N)$ has rank $n+1$, a unique minimum $\hat{0} = (0, 1)$ and a unique maximum $\hat{1} = (n+1, 1)$. Moreover, for any $(a, b) \in P(n, T, N)$ we have $\mathrm{rk}(a, b) = a$. If $T \neq \emptyset$, the parameter N may be thought of as the "width" of $P(n, T, N)$ in the sense that there are N elements of rank i for any $i \in T$ (but only one element of rank i if $i \notin T$). If $T = \emptyset$, then $P(n, T, N)$ is a single chain of $n+1$ elements for every N.

Proposition 2.4. *For every $S, T \subseteq [n]$, we have*

$$\lim_{N\to\infty} \frac{1}{N^{|T|}} f_S(P(n, T, N)) = \begin{cases} 1 & \text{if } S \succcurlyeq T, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

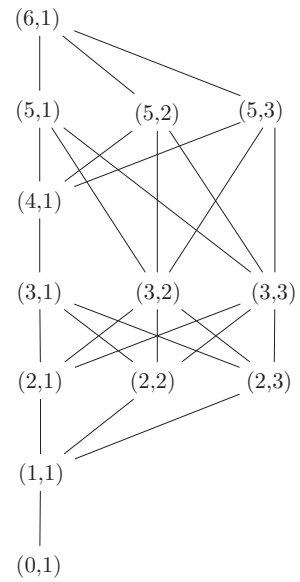

FIGURE 2. The poset $P(5, \{2, 3, 5\}, 3)$.

Proof. This follows directly from a result proved later in the paper, namely Proposition 5.1, as we discuss in Remark 5.2. □

For each $T \subseteq [n]$, the vector $(\lim_{N\to\infty} \frac{1}{N^{|T|}} f_S(P(n,T,N)) : S \subseteq [n])$ will be called a *limit flag f-vector*. Limit flag f-vectors have a particularly simple form when expressed in terms of the e_S. To be precise, a direct consequence of Proposition 2.4 is that

$$\lim_{N\to\infty} \frac{1}{N^{|T|}} e_S(P(n,T,N)) = \delta_{S,T}, \tag{2.3}$$

where $\delta_{S,T}$ is the Kronecker delta function.

Now we determine the extreme rays and facets of the closed cone of flag f-vectors of bounded posets. The following is basically the dual version of Theorem 2.1.

Theorem 2.5. *The closure of the convex cone generated by the flag f-vectors of all bounded posets of rank $n+1$ is a 2^n-dimensional simplicial cone. Its extreme rays are the 2^n limit flag f-vectors, and its facet inequalities are given by $e_S \geq 0$, for $S \subseteq [n]$.*

Proof. It follows from (1.1) that the simplicial cone determined by the 2^n inequalities $e_S \geq 0$ contains the flag f-vector of every bounded poset of rank $n+1$, so in particular it contains all 2^n limit flag f-vectors. On the other hand, by (2.3) every limit flag f-vector lies on all but one of the 2^n hyperplanes $e_S = 0$, $S \subseteq [n]$, and therefore spans an extreme ray of the simplicial cone. Hence this simplicial cone is precisely the cone of flag f-vectors described in the theorem. □

Proof of Theorem 2.1. By Proposition 2.4,

$$\sum_{S \succcurlyeq T} a_S = \lim_{N\to\infty} \frac{1}{N^{|T|}} \cdot \sum_{S\subseteq[n]} a_S \cdot f_S(P(n,T,N)). \tag{2.4}$$

If Condition (1) holds, then the right side of (2.4) is nonnegative, hence so is the left side, and Condition (2) follows. If Condition (2) holds, then by (2.4) the linear form $\sum_{S\subseteq[n]} a_S \cdot f_S$ is nonnegative when applied to every limit flag f-vector. It follows from Theorem 2.5 that this linear form is nonnegative on all bounded posets and is therefore a nonnegative linear combination of the extremal linear forms e_S, $S \subseteq [n]$. Hence we have proved that (2) implies (3). Finally, the assertion that (3) implies (1) follows from the fact that $e_S^{n+1}(P) \geq 0$ for every bounded poset P of rank $n+1$, as noted in (1.1). □

3. Inequalities for semipure posets

Let n be a nonnegative integer. A family of nonempty intervals in $[n]$ is called an *interval system in* $[n]$. Let $X \subseteq [n]$ and let $\mathcal{I}$ be an interval system in $[n]$. The *blocker of* $\mathcal{I}$ *in* X is defined to be the family of sets

$$\mathcal{B}_X(\mathcal{I}) = \{S \subseteq X \mid S \cap I \neq \emptyset \text{ for all } I \in \mathcal{I}\}.$$

In particular, $\mathcal{B}_X(\emptyset)$ consists of all subsets of X. For example, if $n = 4$, $X = \{1,2,3,4\}$, and $\mathcal{I} = \{[1,1],[2,4],[3,4]\}$, then $\mathcal{B}_X(\mathcal{I}) = \{\{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}, \{1,3\}, \{1,3,4\}, \{1,4\}\}$. For any interval system $\mathcal{I}$, let $\hat{\mathcal{I}}$ denote the set of minimal intervals in $\mathcal{I}$ with respect to inclusion. Thus $\hat{\mathcal{I}}$ is an antichain interval system. Continuing our previous example, we have $\hat{\mathcal{I}} = \{[1,1],[3,4]\}$. It is clear that, in general, $\mathcal{B}_X(\mathcal{I}) = \mathcal{B}_X(\hat{\mathcal{I}})$. For this reason it usually suffices to assume that $\mathcal{I}$ is an antichain interval system. Additional facts about blockers appear in [3].

Recall the following theorem of Billera and Hetyei [3].

Theorem 3.1. *Let $a_S \in \mathbb{R}$ for every $S \subseteq [n]$. Then $\sum_{S\subseteq[n]} a_S \cdot f_S(P) \geq 0$ for every graded poset P of rank $n+1$ if and only if*

$$\sum_{S\in\mathcal{B}_{[n]}(\mathcal{I})} a_S \geq 0 \tag{3.1}$$

for all antichain interval systems $\mathcal{I}$ in $[n]$.

The main result in this section is a version of Theorem 3.1 for semipure posets.

Theorem 3.2. *Let $a_S \in \mathbb{R}$ for every $S \subseteq [n]$. Then $\sum_{S\subseteq[n]} a_S \cdot f_S(P) \geq 0$ for every semipure poset P of rank $n+1$ if and only if*

(1) $\displaystyle\sum_{S\subseteq[n]} a_S \geq 0$

and

(2) $\displaystyle\sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}} \geq 0$ *for all $m \in [n]$ and antichain interval systems $\mathcal{I}$ in $[m-1]$.*

Example 3.3. In [3, Example 3.4] the inequality $f_{\{1,3\}}(P) - f_{\{1\}}(P) + f_{\{2\}}(P) - f_{\{3\}}(P) \geq 0$ was shown to hold for every *graded* poset P of rank 4. This inequality is no longer valid if P is allowed to range over semipure posets of rank 4. We can see this by applying Theorem 3.2 with $n = 3$, $a_{\{1,3\}} = 1, a_{\{1\}} = -1, a_{\{2\}} = 1, a_{\{3\}} = -1$, and $a_S = 0$ for all other $S \subseteq [3]$, and noting that when $m = 1$ and $\mathcal{I} = \emptyset$, we have $\sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}} = a_{\{1\}} = -1$; hence Condition (2) fails.

Example 3.4. By applying Theorem 3.2 with $n = 3$, $a_{\{1,2,3\}} = 1, a_{\{1,3\}} = -1, a_{\{1,2\}} = 1, a_{\{2\}} = -1$, and $a_S = 0$ for all other $S \subseteq [3]$, we deduce that for every semipure poset P of rank 4,

$$f_{\{1,2,3\}}(P) - f_{\{1,3\}}(P) + f_{\{1,2\}}(P) - f_{\{2\}}(P) \geq 0. \tag{3.2}$$

Condition (1) holds since $\sum_{S\subseteq[3]} a_S = 0$, and the inequalities in Condition (2) are verified in Table 1.

TABLE 1. Verification of Condition (2) in Theorem 3.2 for the expression $f_{\{1,2,3\}}(P) - f_{\{1,3\}}(P) + f_{\{1,2\}}(P) - f_{\{2\}}(P)$.

m	$\mathcal{I}$	$\sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}}$	Value of $\sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}}$ when $\sum_{S\subseteq[3]} a_S \cdot f_S(P) =$ $f_{\{1,2,3\}}(P) - f_{\{1,3\}}(P) + f_{\{1,2\}}(P) - f_{\{2\}}(P)$
1	$\emptyset$	$a_{\{1\}}$	0
2	$\emptyset$	$a_{\{2\}} + a_{\{1,2\}}$	0
	$\{[1,1]\}$	$a_{\{1,2\}}$	1
3	$\emptyset$	$a_{\{3\}} + a_{\{1,3\}} + a_{\{2,3\}} + a_{\{1,2,3\}}$	0
	$\{[1,1]\}$	$a_{\{1,3\}} + a_{\{1,2,3\}}$	0
	$\{[1,1],[2,2]\}$	$a_{\{1,2,3\}}$	1
	$\{[1,2]\}$	$a_{\{1,3\}} + a_{\{2,3\}} + a_{\{1,2,3\}}$	0
	$\{[2,2]\}$	$a_{\{2,3\}} + a_{\{1,2,3\}}$	1

The remainder of this section is devoted to proving Theorem 3.2. As before, we will prove the necessity of Conditions (1) and (2) in Theorem 3.2 by considering flag f-vectors of appropriate sequences of posets. First recall the following definition from [3].

Definition 3.5. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ be an interval system in $[n]$. For each $N > 0$, define $P(n, \mathcal{I}, N)$ to be the poset whose elements are lists of the form $(i; p_1, p_2, \ldots, p_k)$, where $i \in [0, n+1]$ and for every $j \in [1, k]$,

$$p_j \in \begin{cases} [1, N] & \text{if } i \in I_j, \\ \{*\} & \text{otherwise,} \end{cases}$$

where $*$ is a symbol different from every integer. Set $(i; p_1, p_2, \ldots, p_k) < (i'; p'_1, p'_2, \ldots, p'_k)$ if $i < j$ and for every $j \in [1, k]$, either $p_j = p'_j$ or $* \in \{p_j, p'_j\}$.

The poset $P(n, \mathcal{I}, N)$ is graded of rank $n+1$, with minimum $\hat{0} = (0; *, \ldots, *)$ and maximum $\hat{1} = (n+1; *, \ldots, *)$. See Figure 3(a). We will build on posets of the form $P(n, \mathcal{I}, N)$ to obtain a suitable family of semipure posets. Recall [21] that if P and Q are posets on disjoint sets, then their ordinal sum, denoted $P \oplus Q$, is the poset on $P \cup Q$ with $x \leq y$ in $P \oplus Q$ if and only if $x \leq y$ in P, $x \leq y$ in Q, or $x \in P$ and $y \in Q$; their disjoint union, denoted $P + Q$, is the poset on $P \cup Q$ such that $x \leq y$ in P if and only if $x \leq y$ in P or $x \leq y$ in Q. Let $\mathbf{n}$ denote the poset $[n]$ with the usual total order.

For positive integers n, m, N with $m \in [n]$, and an interval system $\mathcal{I}$ in $[m-1]$, let $k = n - m + 1$ and define the poset $Q(n, m, \mathcal{I}, N)$ by

$$Q(n, m, \mathcal{I}, N) = (P(m-1, \mathcal{I}, N) - \{\hat{1}\}) \oplus (\mathbf{k} + \underbrace{\mathbf{1} + \cdots + \mathbf{1}}_{N-1 \text{ terms}}) \oplus \mathbf{1}.$$

Note that $Q(n, m, \mathcal{I}, N)$ is a semipure poset of rank $n+1$. See Figure 3.

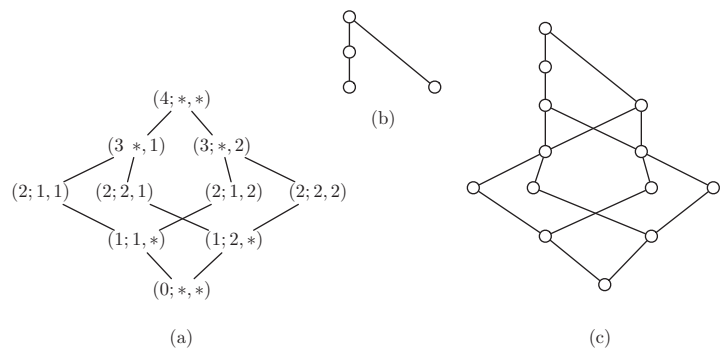

FIGURE 3. (a) $P(3, \{[1,2],[2,3]\}, 2)$; (b) $(\mathbf{2} + \mathbf{1}) \oplus \mathbf{1}$; (c) $Q(5, 4, \{[1,2],[2,3]\}, 2)$.

Proposition 3.6. *Let $m \in [n]$ and let $\mathcal{I}$ be an antichain interval system in $[m-1]$. For every $S \subseteq [n]$, we have*

(3.3)
$$\lim_{N\to\infty} \frac{1}{N^{|\mathcal{I}|+1}} f_S(Q(n, m, \mathcal{I}, N)) = \begin{cases} 1 & \text{if } S = T \cup \{m\} \text{ for some } T \in \mathcal{B}_{[m-1]}(\mathcal{I}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $T \subseteq [m-1]$. It follows immediately from the definitions that

$$f_T(Q(n, m, \mathcal{I}, N)) = f_T(P(m-1, \mathcal{I}, N)). \tag{3.4}$$

Moreover, since every interval in $Q(n, m, \mathcal{I}, N)$ of the form $[\hat{0}, q]$, where $\mathrm{rk}(q) = m$, is isomorphic to the poset $P(m-1, \mathcal{I}, N)$, and there are N elements of rank m in $Q(n, m, \mathcal{I}, N)$, we have

$$f_{T\cup\{m\}}(Q(n, m, \mathcal{I}, N)) = N \cdot f_T(P(m-1, \mathcal{I}, N)). \tag{3.5}$$

Plugging the formula $f_T(P(m-1, \mathcal{I}, N)) = N^{|\{I \in \mathcal{I} | S \cap I \neq \emptyset\}|}$ (see [3, Proposition 2.5]) into (3.4) and (3.5) and taking the limit, we deduce that (3.3) holds if $S \subseteq [m]$.

Suppose that $S \not\subseteq [m]$, in which case $m < n$. Since the elements in $Q(n, m, \mathcal{I}, N)$ from rank $m+1$ to n form a chain, there is no loss of generality if we assume that $S = T \cup \{s\}$ for some $T \subseteq [m]$ and $s \in [m+1, n]$. If $m \notin T$, then we have $f_S(Q(n, m, \mathcal{I}, N)) = f_T(P(m-1, \mathcal{I}, N)) \leq N^{|\mathcal{I}|}$, and (3.3) follows. If $m \in T$, then by definition of $Q(n, m, \mathcal{I}, N)$, there is only one element $q \in Q(n, m, \mathcal{I}, N)$ such that $\mathrm{rk}(q) = m$ and $q < s$, and so $f_S(Q(n, m, \mathcal{I}, N)) \leq f_{S-\{m,s\}}(Q(n, m, \mathcal{I}, N)) = f_{T-\{m\}}(P(m-1, \mathcal{I}, N)) \leq N^{|\mathcal{I}|}$, and (3.3) follows. □

Now we are ready to prove the necessity of Conditions (1) and (2) in Theorem 3.2.

Proposition 3.7. *Let $a_S \in \mathbb{R}$ for every $S \subseteq [n]$. Suppose that $\sum_{S\subseteq[n]} a_S \cdot f_S(P) \geq 0$ for every semipure poset P of rank $n+1$. Then Conditions (1) and (2) in Theorem 3.2 hold.*

Proof. If P is a chain of rank $n+1$, then $f_S(P) = 1$ for all $S \subseteq [n]$, and so $\sum_{S\subseteq[n]} a_S \geq 0$, which proves Condition (1). For any $m \in [n]$ and antichain interval system $\mathcal{I}$ in $[m-1]$, we have

$$\sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}} = \lim_{N\to\infty} \frac{1}{N^{|\mathcal{I}|+1}} \sum_{S\subseteq[n]} a_S \cdot f_S(Q(n,m,\mathcal{I},N)) \geq 0.$$

This establishes Condition (2). □

Finally, we establish the sufficiency of Conditions (1) and (2).

Proposition 3.8. *Let $a_S \in \mathbb{R}$ for every $S \subseteq [n]$. Suppose that Conditions (1) and (2) in Theorem 3.2 hold. Then $\sum_{S\subseteq[n]} a_S \cdot f_S(P) \geq 0$ for every bounded poset P of rank $n+1$.*

Proof. Let P be a bounded poset of rank $n+1$. Given $m \in [n]$ and $t \in P$, let $P_m = \{p \in P \mid \mathrm{rk}(p) = m\}$ and $P_{\leq t}$ be the induced subposet $\{p \in P \mid p \leq t\}$. For each $m \in [n]$, we fix an arbitrary element t_m of rank m. We have

$$\begin{aligned}\sum_{S\subseteq[n]} a_S f_S(P) &= a_\emptyset + \sum_{m=1}^{n} \sum_{S\subseteq[m-1]} a_{S\cup\{m\}} f_{S\cup\{m\}}(P) \\ &= a_\emptyset + \sum_{m=1}^{n} \sum_{t\in P_m} \sum_{S\subseteq[m-1]} a_{S\cup\{m\}} f_S(P_{\leq t}) \\ &= a_\emptyset + \sum_{m=1}^{n} \sum_{S\subseteq[m-1]} a_{S\cup\{m\}} f_S(P_{\leq t_m}) \\ &\quad + \sum_{m=1}^{n} \sum_{t\in P_m-\{t_m\}} \sum_{S\subseteq[m-1]} a_{S\cup\{m\}} f_S(P_{\leq t}).\end{aligned} \tag{3.6}$$

For any $m \in [n]$ and $t \in P_m$, the poset $P_{\leq t}$ is graded. Then by [3, Proposition 2.8] we get $\sum_{S\subseteq[m-1]} a_{S\cup\{m\}} f_S(P_{\leq t}) \geq 0$ for every $m \in [n]$ and $t \in P_m$. Thus it suffices to show that the sum of the first two terms in (3.6) is nonnegative.

In the proof of [3, Proposition 2.8] it was shown that there is a way of associating the maximal chains $C_1, C_2, \ldots$ of any graded poset Q of rank m with interval systems $\mathcal{I}_1, \mathcal{I}_2, \ldots$ in $[m-1]$ so that one of the $\mathcal{I}_i$'s is empty, say $\mathcal{I}_1 = \emptyset$, and

$$\sum_{S\subseteq[m-1]} a_S f_S(Q) = \sum_{S\subseteq[m-1]} a_S + \sum_{i\geq 2} \sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I}_i)} a_S.$$

It follows that

$$\begin{aligned}&a_\emptyset + \sum_{m=1}^{n} \sum_{S\subseteq[m-1]} a_{S\cup\{m\}} f_S(P_{\leq t_m}) \\ &\qquad = a_\emptyset + \sum_{m=1}^{n} \left(\sum_{S\subseteq[m-1]} a_{S\cup\{m\}} + \sum_{\mathcal{I}} \sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}} \right) \\ &\qquad = \sum_{S\subseteq[n]} a_S + \sum_{m=1}^{n} \sum_{\mathcal{I}} \sum_{S\in\mathcal{B}_{[m-1]}(\mathcal{I})} a_{S\cup\{m\}} \geq 0.\end{aligned} \tag{3.7}$$

This completes the proof. □

4. Extreme rays of cones of flag f-vectors

In this section we describe the extreme rays of the three cones generated by flag f-vectors of bounded posets, graded posets, and semipure posets, using the language of quasisymmetric functions. We begin with a brief review of quasisymmetric functions. The reader is referred to [22, §7.19] for additional details and references.

4.1. **Quasisymmetric functions.** Let $x_1, x_2, \ldots$ be commuting variables. A *quasisymmetric function* (over $\mathbb{R}$) is a formal power series $F \in \mathbb{R}[[x_1, x_2, \ldots]]$ of finite degree such that for any sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of positive integers, the coefficients of the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ and $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ in F are the same for any $i_1 < \cdots < i_k$. The vector space $\mathcal{Q}$ of all quasisymmetric functions forms a graded algebra under power series multiplication.

We now describe an important basis for $\mathcal{Q}$. Let $n \geq 0$ and $S = \{s_1 < \cdots < s_k\} \subseteq [n]$. If we set $s_0 = 0$ and $s_{k+1} = n+1$, then the *monomial quasisymmetric function* M_S is defined by

$$M_S = \sum_{i_1 < i_2 < \cdots < i_{k+1}} x_{i_1}^{s_1 - s_0} \, x_{i_2}^{s_2 - s_1} \cdots x_{i_{k+1}}^{s_{k+1} - s_k}.$$

Note that M_S is homogeneous of degree $n+1$. When we wish to make the degree explicit, we will write M_S^{n+1}. The set $\{M_S \mid S \subseteq [n]\}$ is a basis of the vector space of homogeneous quasisymmetric functions of degree $n+1$, denoted by $\mathcal{Q}_{n+1}$. We set $M_\emptyset^0 = 1$ and $\mathcal{Q}_0 = \mathbb{R}$.

A useful way to encode the flag f-vector of a bounded poset P is through its *F-quasisymmetric function*, defined by

$$F(P) = \sum_{S \subseteq [n]} f_S(P) \cdot M_S. \tag{4.1}$$

This generating function was introduced by Ehrenborg [13], who proved that $F(\cdot)$ is a homomorphism of Hopf algebras from the reduced incidence Hopf algebra of graded posets to $\mathcal{Q}$. In Appendix A we show how Ehrenborg's result can be extended to bounded posets.

4.2. **Extreme rays of three cones.** In the following, if $X \subseteq \mathcal{Q}_{n+1}$, then we let $\operatorname{cone} X$ denote the closure of the convex cone in $\mathcal{Q}_{n+1}$ generated by X. We will describe the extreme rays of the following cones:

$$\begin{aligned}
\mathcal{C}_{n+1}^{\mathrm{BD}} &= \operatorname{cone}\{F(P) \in \mathcal{Q} \mid P \text{ is a bounded poset of rank } n+1\},\\
\mathcal{C}_{n+1}^{\mathrm{SP}} &= \operatorname{cone}\{F(P) \in \mathcal{Q} \mid P \text{ is a semipure poset of rank } n+1\},\\
\mathcal{C}_{n+1}^{\mathrm{GR}} &= \operatorname{cone}\{F(P) \in \mathcal{Q} \mid P \text{ is a graded poset of rank } n+1\}.
\end{aligned}$$

The extreme rays of $\mathcal{C}_{n+1}^{\mathrm{BD}}$ are given in Theorem 2.5, and the extreme rays of $\mathcal{C}_{n+1}^{\mathrm{GR}}$ are known from [3]; here we are just translating those results into the language of quasisymmetric functions. Some additional effort is needed to describe the extreme rays of $\mathcal{C}_{n+1}^{\mathrm{SP}}$.

For every $n \geq 0$ and $S \subseteq [n]$, define $E_S \in \mathcal{Q}_{n+1}$ by

$$E_S = \lim_{N \to \infty} \frac{1}{N^{|S|}} F(P(n, S, N)). \tag{4.2}$$

We sometimes write E_S^{n+1} to emphasize the degree, and we set $E_\emptyset^0 = 1$.

The following result is a direct consequence of Theorem 2.5.

Proposition 4.1. *The extreme rays of $\mathcal{C}_{n+1}^{\mathrm{BD}}$ are precisely the quasisymmetric functions E_S where $S \subseteq [n]$.*

Let us note the following change-of-basis formula:

Proposition 4.2. *Let $S \subseteq [n]$. If $S = \emptyset$, then set $r = n+1$, and otherwise set $r = \min S$. Then*

$$E_S = \sum_{S \preccurlyeq T \subseteq [n]} M_T, \tag{4.3}$$

$$M_S = E_S - \sum_{i=1}^{r-1} E_{S \cup \{i\}}. \tag{4.4}$$

Consequently, $\{E_S \mid S \subseteq [n]\}$ is a basis for $\mathcal{Q}_{n+1}$.

Proof. Formula (4.3) follows immediately from Proposition 2.4. To obtain (4.4), apply Möbius inversion to get $M_S = \sum_{T \succcurlyeq S} \mu(S,T) E_T$, where μ is the Möbius function of the set of subsets of $[n]$ partially ordered by $\preccurlyeq$. If $T \succcurlyeq S$, then $T = \{t_1, \ldots, t_j\} \cup S$ for some $\{t_1 < \cdots < t_j\} \subseteq [r-1]$, so the interval $[S,T]$ is linearly ordered, with $S \prec \{t_j\} \cup S \prec \{t_{j-1}, t_j\} \cup S \prec \cdots \prec \{t_1, \ldots, t_j\} \cup S = T$. Thus, by the well-known formula for the Möbius function of a linearly ordered set (e.g., [21, Example 3.8.1]), we have

$$\mu(S,T) = \begin{cases} 1 & \text{if } T = S, \\ -1 & \text{if } T = \{t\} \cup S \text{ for some } t \in [r-1], \\ 0 & \text{otherwise.} \end{cases}$$

Formula (4.4) follows. □

Next, recall Billera and Hetyei's result on the extreme rays of $\mathcal{C}_{n+1}^{\mathrm{GR}}$. Given a nonnegative integer n, an interval system $\mathcal{I}$ in $[n]$, and a subset X of $[n]$, the corresponding *interval quasisymmetric function* $F_{X,\mathcal{I}} \in \mathcal{Q}_{n+1}$ is defined by

$$F_{X,\mathcal{I}} = \sum_{S \in \mathcal{B}_X(\mathcal{I})} M_S. \tag{4.5}$$

We write $F_{X,\mathcal{I}}^{n+1}$ when we need to make the degree explicit. The following is essentially Proposition 3.1 and Corollary 3.2 of [3] (cf. [5, Proposition 1.5]).

Proposition 4.3. *The extreme rays of $\mathcal{C}_{n+1}^{\mathrm{GR}}$ are precisely the interval quasisymmetric functions $F_{[n],\mathcal{I}}$ such that $\mathcal{I}$ is an antichain interval system in $[n]$. The number of extreme rays is the Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$.*

Finally, we describe the extreme rays of $\mathcal{C}_{n+1}^{\mathrm{SP}}$. Let $\mathcal{E}_{n+1}$ be the subset of $\mathcal{Q}_{n+1}$ consisting of $F_{[n],\emptyset}$ together with every $F_{[m-1],\mathcal{I} \cup \{\{m\}\}}$ such that $m \in [n]$ and $\mathcal{I}$ is an antichain interval system in $[m-1]$.

Proposition 4.4. *The extreme rays of $\mathcal{C}_{n+1}^{\mathrm{SP}}$ are precisely the quasisymmetric functions in $\mathcal{E}_{n+1}$. The number of extreme rays is $|\mathcal{E}_{n+1}| = \sum_{m=0}^{n} \frac{1}{m+1}\binom{2m}{m}$.*

Proof. It follows from Theorem 3.2 that the set $\mathcal{E}_{n+1}$ generates $\mathcal{C}^{\mathrm{SP}}_{n+1}$. It remains to show that these generators are conically independent. First, notice that $F^{n+1}_{[n],\emptyset}$ is the only member of $\mathcal{E}_{n+1}$ whose expansion in the monomial basis involves $M_\emptyset$, so it cannot be a conic combination of the others. Next, it is clear by definition that each $F^{n+1}_{[m-1],\mathcal{I}\cup\{\{m\}\}}$ cannot be written as a nontrivial conic combination involving terms of the form $F^{n+1}_{[m'-1],\mathcal{I}\cup\{\{m'\}\}}$ where $m' \neq m$. Thus it suffices to show that for every $m \in [n]$, the members of the form $F^{n+1}_{[m-1],\mathcal{I}\cup\{\{m\}\}}$ are conically independent. For each m, consider the linear lifting operator from $\mathcal{Q}_m$ to $\mathcal{Q}_{n+1}$ that takes M^m_S to $M^{n+1}_{S\cup\{m\}}$. Since $F^m_{[m-1],\mathcal{I}}$ lifts to $F^{n+1}_{[m-1],\mathcal{I}\cup\{\{m\}\}}$, and since quasisymmetric functions of the form $F^m_{[m-1],\mathcal{I}}$ are extreme rays of $\mathcal{C}^{\mathrm{GR}}_m$, their lifts to $\mathcal{Q}_{n+1}$ must be conically independent.

Having described $\mathcal{E}_{n+1}$ in terms of lifting extreme rays, the cardinality of $\mathcal{E}_{n+1}$ can be computed by summing the cardinalities of the sets of extreme rays of $\mathcal{C}^{\mathrm{GR}}_m$ for $m = 1, \ldots, n$, plus 1 for the extra element $F^{n+1}_{[n],\emptyset}$. Those cardinalities are Catalan numbers, as noted in Proposition 4.3. □

5. Multiplication of extreme rays

In this section we derive a combinatorial rule for multiplication of E-basis elements. Let $n, m \geq 0$, $S \subseteq [n-1]$, and $T \subseteq [m-1]$. By Proposition A.1, and by the defining formula (4.2),

$$\begin{aligned}
E^n_S \cdot E^m_T &= \left(\lim_{N\to\infty} \frac{1}{N^{|S|}} F(P(n-1,S,N))\right)\left(\lim_{M\to\infty} \frac{1}{M^{|T|}} F(P(m-1,T,M))\right) \\
&= \lim_{N\to\infty} \frac{1}{N^{|S|+|T|}} F(P(n-1,S,N))F(P(m-1,T,N)) \\
&= \lim_{N\to\infty} \frac{1}{N^{|S|+|T|}} F(P(n-1,S,N)\times P(m-1,T,N)) \\
&= \lim_{N\to\infty} \frac{1}{N^{|S|+|T|}} \sum_{R\subseteq[n+m-1]} f_R(P(n-1,S,N)\times P(m-1,T,N)) M^{n+m}_R. \qquad (5.1)
\end{aligned}$$

The manipulation of limits in the second equality is justified because the product of any two quasisymmetric functions $\sum_S a_S M_S \cdot \sum_S b_S M_S$ has the form $\sum_S c_S M_S$, where each c_S is a polynomial in the a_T's and b_T's. The last expression, being the limit of generating functions of a sequence of bounded posets, must have a nonnegative expansion in the E-basis. Our goal is to give an explicit combinatorial interpretation of these coefficients.

It will be convenient to work with a refinement of the flag f-vector of the Cartesian product of two posets. In the following proposition, we will assume that A is a sequence of the form $0 = a_0 \leq a_1 \leq \cdots \leq a_k = n$, B is a sequence of the form $0 = b_0 \leq b_1 \leq \cdots \leq b_k = m$ having the same length as A, and $a_i < a_{i+1}$ or $b_i < b_{i+1}$ for every $i \in [0, k-1]$. For every bounded poset P of rank n and Q of rank m, let

$$\begin{aligned}
f_{A,B}(P\times Q) &= \text{number of chains } \hat{0} = (p_0,q_0) < (p_1,q_1) < \cdots < (p_k,q_k) = \hat{1} \\
&\qquad \text{in } P\times Q \text{ such that } \mathrm{rk}(p_i) = a_i \text{ and } \mathrm{rk}(q_i) = b_i \text{ for all } i \in [0,k].
\end{aligned}$$

Set $\underline{A} = \{a_0, a_1, \ldots, a_k\} - \{0, n\} \subseteq [n-1]$ and $\underline{B} = \{b_0, b_1, \ldots, b_k\} - \{0, m\} \subseteq [m-1]$.

Proposition 5.1. *Let $n, m > 0$ and let A and B be as before. For any $S \subseteq [n-1]$ and $T \subseteq [m-1]$, we have*

(5.2)
$$\lim_{N\to\infty} \frac{1}{N^{|S|+|T|}} f_{A,B}(P(n-1,S,N)\times P(m-1,T,N)) = \begin{cases} 1 & \text{if } \underline{A} \succcurlyeq S \text{ and } \underline{B} \succcurlyeq T, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Observe that $f_{A,B}(P(n-1,S,N) \times P(m-1,T,N))$ is the number of sequences (α_0,β_0), (α_1,β_1), $\ldots$, (α_k,β_k), with $\alpha_i, \beta_i \in [N]$ for all i, such that $((a_0,\alpha_0),(b_0,\beta_0)) < \cdots < ((a_k,\alpha_k),(b_k,\beta_k))$ is a chain in $P(n-1,S,N) \times P(m-1,T,N)$. Suppose that we have $S = \{s_1 < \cdots < s_\ell\}$. Let $s_{\ell+1} = n$. In what follows, besides using closed intervals $[u,v] = \{u, u+1, \ldots, v\}$, we will be working with half-open intervals $[u,v) = \{u, u+1, \ldots, v-1\}$ and open intervals $(u,v) = \{u+1, u+2, \ldots, v-1\}$. By definition of $P(n-1,S,N)$, $\alpha_i = 1$ whenever $a_i \notin S$ or whenever there is some $s_j \in S$ such that $a_i = s_j$ and $\underline{A} \cap (s_j, s_{j+1}) \neq \emptyset$. Thus, the only indices i for which α_i could possibly take on the full range of values in $[N]$ are those in the set $I = \{i \in [0,k] \mid \exists s_j \in S, a_i = s_j \text{ and } \underline{A} \cap (s_j, s_{j+1}) = \emptyset\}$. Since $\{a_i \mid i \in I\} = \{s_j \in S \mid \underline{A} \cap [s_j, s_{j+1}) = \{s_j\}\}$, it follows that there are at most $N^{|\{s_j \in S \mid \underline{A} \cap [s_j, s_{j+1}) = \{s_j\}\}|} \leq N^{|S|}$ possibilities for $(\alpha_1, \ldots, \alpha_k)$, and the upper bound $N^{|S|}$ is attainable only if $\{s_j \in S \mid \underline{A} \cap [s_j, s_{j+1}) = \{s_j\}\} = S$, or equivalently, $\underline{A} \succcurlyeq S$. A similar analysis for the possible number of sequences $(\beta_1, \ldots, \beta_k)$ yields the upper bound $N^{|T|}$, which is attainable only if $\underline{B} \succcurlyeq T$. It follows that $\lim_{N\to\infty} f_{A,B}(P(n-1,S,N) \times P(m-1,T,N))/N^{|S|+|T|} = 0$ if $\underline{A} \not\succcurlyeq S$ or $\underline{B} \not\succcurlyeq T$.

Now suppose that $\underline{A} \succcurlyeq S$ and $\underline{B} \succcurlyeq T$. Let $(\alpha_0,\beta_0), \ldots, (\alpha_k,\beta_k)$ be any sequence with $\alpha_i, \beta_i \in [N]$ for all i, subject to the following restrictions: if $a_i \notin S$, then $\alpha_i = 1$; if $a_i = a_{i+1}$, then $\alpha_i = \alpha_{i+1}$; if $b_i \notin T$, then $\beta_i = 1$; if $b_i = b_{i+1}$, then $\beta_i = \beta_{i+1}$. There are precisely $N^{|S|+|T|}$ such sequences. We claim that $((a_0,\alpha_0),(b_0,\beta_0)) < \cdots < ((a_k,\alpha_k),(b_k,\beta_k))$ is a chain in $P(n-1,S,N) \times P(m-1,T,N)$. From this it follows that $\lim_{N\to\infty} f_{A,B}(P(n-1,S,N) \times P(m-1,T,N))/N^{|S|+|T|} = 1$.

To prove the claim, fix $i \in [0,k-1]$. We must show that (1): $(a_i,\alpha_i) \leq (a_{i+1},\alpha_{i+1})$, and (2): $(b_i,\beta_i) \leq (b_{i+1},\beta_{i+1})$. If $a_i = a_{i+1}$, then $\alpha_i = \alpha_{i+1}$ and so (1) holds. If $a_i \notin S$, then $\alpha_i = 1$, so (1) holds regardless of the value of α_{i+1}. Suppose that $a_i \in S$ and $a_i < a_{i+1}$. Since $\underline{A} \succcurlyeq S$, we have $a_{i+1} = n$ or $a_{i+1} \in S$. If $a_{i+1} = n$, then (a_{i+1},α_{i+1}) is the maximal element of $P(n-1,N,S)$ and so (1) holds. If $a_{i+1} \in S$, then, since both a_i and a_{i+1} are in S, (1) holds regardless of the values of α_i and α_{i+1}. The completes the proof of (1). The proof of (2) is similar. □

Remark 5.2. The limit $\lim_{N\to\infty} \frac{1}{N^{|S|}} f_T(P(n,S,N))$ can be computed from Proposition 5.1 as follows. Given a strictly increasing sequence A of the form $0 < a_1 < \cdots < a_k = n$, let B be the sequence $0 = b_0 = b_1 = \cdots = b_{k-1} < b_k = 1$. Then $f_{\underline{A}}(P(n-1,S,N)) = f_{A,B}(P(n-1,S,N) \times P(1,\emptyset,N))$, and (5.2) reduces to (2.2).

In the following theorem, we will think of $[0,n] \times [0,m]$ as a poset with the usual product order: $(a,b) \leq (a',b')$ if and only if $a \leq a'$ and $b \leq b'$. For a chain $C = \{(0,0) = (a_0,b_0) < (a_1,b_1) < \cdots < (a_j,b_j) = (n,m)\}$ in $[0,n] \times [0,m]$, let $\mathrm{set}(C) = \{a_1 + b_1, \ldots, a_{j-1} + b_{j-1}\} \subset [n+m-1]$.

Theorem 5.3. *Let $n, m > 0$, $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n-1]$ and $T = \{t_1 < t_2 < \cdots < t_\ell\} \subseteq [m-1]$. Then*

$$E_S^n \cdot E_T^m = \sum_C E_{\mathrm{set}(C)}^{n+m}, \tag{5.3}$$

where the sum is over all chains $C = \{(0,0) = (a_0, b_0) < (a_1, b_1) < \cdots < (a_j, b_j) = (n, m)\}$ in $[0, n] \times [0, m]$ such that $\{a_0, a_1, \ldots, a_j\} - \{0, n\} \succcurlyeq S$, $\{b_0, b_1, \ldots, b_j\} - \{0, m\} \succcurlyeq T$, $a_1 = s_1$ whenever $b_1 = 0$, and $b_1 = t_1$ whenever $a_1 = a_2$.

Proof. Let $\mathcal{C}$ denote the set of all chains $\{(0,0) = (a_0, b_0) < (a_1, b_1) < \cdots < (a_j, b_j) = (n, m)\}$ in $[0, n] \times [0, m]$ such that $\{a_0, a_1, \ldots, a_j\} - \{0, n\} \succcurlyeq S$ and $\{b_0, b_1, \ldots, b_j\} - \{0, m\} \succcurlyeq T$. It follows from (5.1) and Proposition 5.1 that

$$E_S \cdot E_T = \sum_{C \in \mathcal{C}} M_{\mathrm{set}(C)}. \tag{5.4}$$

Fix $C = \{(0,0) = (a_0, b_0) < \cdots < (a_j, b_j) = (n, m)\}$. For $C' \in \mathcal{C}$, we set $C \preccurlyeq^* C'$ if $C \subseteq C' \subseteq C \cup \{(a, 0) \mid 0 \le a \le a_1\} \cup \{(a_1, b) \mid 0 \le b \le b_1\}$. We see that C is minimal in $\mathcal{C}$ with respect to $\preccurlyeq^*$ if and only if one of the following conditions holds: $a_1 = s_1$ whenever $b_1 = 0$, and $b_1 = t_1$ whenever $a_1 = a_2$. Moreover, there is a unique minimal chain $C' \in \mathcal{C}$ such that $C' \preccurlyeq^* C$. This minimal chain is obtained from C by deleting every $(a, b) \in C$ such that $(0,0) < (a, b) < (s_1, 0)$ or such that there is some $(a, b') \in \mathcal{C}$ satisfying $(a, b) < (a, b') \le (a, t_1)$ and there is no $(a'', b'') \in \mathcal{C}$ satisfying $(0, 1) \le (a'', b'') < (a, b)$. Thus,

$$\sum_{C \in \mathcal{C}} M_{\mathrm{set}(C)} = \sum_{C \text{ minimal in } \mathcal{C}} \sum_{C \preccurlyeq^* C'} M_{\mathrm{set}(C')}.$$

Fixing $C \in \mathcal{C}$ as before, the correspondence $C' \to \mathrm{set}(C')$ is a bijection from $\{C' \in \mathcal{C} \mid C \preccurlyeq^* C'\}$ to $\{R \subseteq [n+m-1] \mid \mathrm{set}(C) \preccurlyeq R\}$. Thus,

$$\sum_{C' \in \mathcal{C} : C \preccurlyeq^* C'} M_{\mathrm{set}(C')} = \sum_{\mathrm{set}(C) \preccurlyeq R \subseteq [n+m-1]} M_R = E_{\mathrm{set}(C)},$$

and the theorem follows. □

When computing $E_S \cdot E_T$ using (5.3), it is helpful to visualize chains in $[0, n] \times [0, m]$ as lattice paths in an $n \times m$ grid. The sum in (5.3) is over all lattice paths (chains) $(0,0) = (a_0, b_0) \to (a_1, b_1) \to \cdots \to (a_j, b_j) = (n, m)$ that have

- at least one vertex on every vertical line segment $\{s_1\} \times [0, m]$, $\{s_2\} \times [0, m]$, $\ldots$, $\{s_k\} \times [0, m]$;
- at least one vertex on every horizontal line segment $[0, n] \times \{t_1\}$, $[0, n] \times \{t_2\}$, $\ldots$, $[0, n] \times \{t_\ell\}$;
- no vertices in the regions $[s_1 + 1, s_2 - 1] \times [0, m], [s_2 + 1, s_3 - 1] \times [0, m]$, $\ldots$, $[s_k + 1, n - 1] \times [0, m]$;
- no vertices in the regions $[0, n] \times [t_1 + 1, t_2 - 1], [0, n] \times [t_2 + 1, t_3 - 1]$, $\ldots$, $[0, n] \times [t_\ell + 1, m - 1]$;
- $b_1 \neq 0$ unless $a_1 = s_1$;
- $a_1 \neq a_2$ unless $b_1 = t_1$.

See Figures 4 and 5.

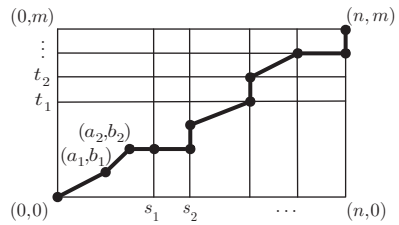

FIGURE 4. A typical chain in the multiplication formula for $E_S \cdot E_T$, where $S = \{s_1 < \cdots < s_k\} \subseteq [n-1]$ and $T = \{t_1 < \cdots < t_\ell\} \subseteq [m-1]$.

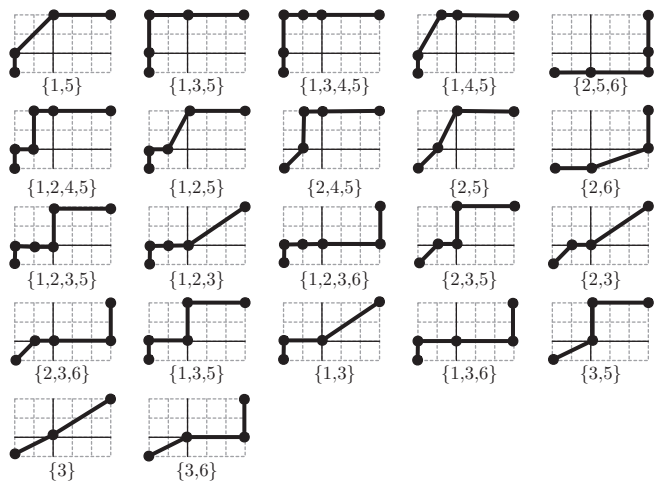

FIGURE 5. Lattice paths for the product $E^5_{\{2\}} \cdot E^3_{\{1\}} = E^8_{\{1,5\}} + E^8_{\{1,3,5\}} + E^8_{\{1,3,4,5\}} + E^8_{\{1,4,5\}} + E^8_{\{2,5,6\}} + E^8_{\{1,2,4,5\}} + E^8_{\{1,2,5\}} + E^8_{\{2,4,5\}} + E^8_{\{2,5\}} + E^8_{\{2,6\}} + E^8_{\{1,2,3,5\}} + E^8_{\{1,2,3\}} + E^8_{\{1,2,3,6\}} + E^8_{\{2,3,5\}} + E^8_{\{2,3\}} + E^8_{\{2,3,6\}} + E^8_{\{1,3,5\}} + E^8_{\{1,3\}} + E^8_{\{1,3,6\}} + E^8_{\{3,5\}} + E^8_{\{3\}} + E^8_{\{3,6\}}$.

6. Concluding remarks

In this paper we have determined the extreme rays of the cones $\mathcal{C}_n^{\mathrm{BD}}$ and $\mathcal{C}_n^{\mathrm{SP}}$. The extreme rays of $\mathcal{C}_n^{\mathrm{GR}}$ are given in [3, Proposition 4.1]. Since $\mathcal{C}_n^{\mathrm{BD}}$ is simplicial, its facets can be described easily, as we do in Theorem 2.5. However, the problem of characterizing the facets of the other two cones is not trivial and is still open. Partial results on the facets of $\mathcal{C}_n^{\mathrm{GR}}$ are given in [3].

Another cone that has been carefully studied is the one generated by flag f-vectors of planar graded posets. In describing this cone, Billera and Hetyei [4] introduced a "planar flag h-vector" and proved that the corresponding linear forms are the extreme rays of the cone of nonnegative forms on planar graded posets. This is analogous to the relation between the "flag e-vector" used in our paper and the cone of nonnegative linear forms on bounded posets. Moreover, when expressing entries in the ordinary or planar flag h-vectors as well as the flag e-vector in terms of flag f-numbers, all of the coefficients are 0, 1 or -1. In each case the coefficient

pattern can be explained by the fact that we are doing Möbius inversion on some distributive or join-distributive lattice; in the case of the flag e-vector the lattice is a chain, as discussed in our proof of Proposition 4.2. It turns out that both the ordinary and planar flag h-vectors fit neatly into Hetyei's general theory of G-shellability of balanced simplicial complexes [15], which simultaneously generalizes the notion of CL-shellability [9] as well as the notion of planar decomposition developed in [4]. Again Möbius inversion on join-distributive lattices is key to describing the relation between the flag f-vector and Hetyei's "G-generalized flag h-vector". Even though our results on bounded posets seem not to fit into Hetyei's theory, the similarities between our flag e-vector and the planar and ordinary flag h-vectors suggests the possibility of a common generalization of CL-shellability and planar decomposition that includes bounded posets and is different from the approach in [15]. (We thank the referee for this suggestion.)

Appendix A. Generalizing the F-quasisymmetric function to bounded posets

In [13, Proposition 4.4], Ehrenborg proves that the linear map from the reduced incidence Hopf algebra of graded posets to the Hopf algebra of quasisymmetric functions that takes P to $F(P)$ is a homomorphism of graded Hopf algebras. Here we extend this result to the context of bounded posets.

Let P be a bounded poset. A map λ from the set of cover relations (i.e., edges in the Hasse diagram) of P to the positive integers will be called *rank-compatible* if the sum of the labels along every maximal chain in P is the same. The pair (P, λ) will be called an RC-poset. For example, the labeling that takes each cover relation $p < q$ to $\mathrm{rk}(p) - \mathrm{rk}(q)$ is rank-compatible and will be called the *canonical labeling*. We define the *rank* of an RC-poset (P, λ) to be the sum of the edge-labels of any maximal chain in P. It will also be convenient to call the one-element poset together with the empty map an RC-poset of rank 0. Two RC-posets are said to be isomorphic if there is an isomorphism of the underlying posets that preserves the edge-labeling.

For each integer $n \geq 0$, let $\mathcal{P}_n$ be the vector space with basis consisting of the isomorphism classes of RC-posets (P, λ) of rank n. Let $\mathcal{P}$ be the graded vector space $\mathcal{P} = \bigoplus_{n=0}^{\infty} \mathcal{P}_n$. We now define a product and coproduct on $\mathcal{P}$ as follows. We will define these operations on actual RC-posets rather than isomorphism classes, but it is easy to see that these operations induce well-defined operations on isomorphism classes. Let (P, λ) and (Q, α) be RC-posets. Define their product to be

$$(P, \lambda) * (Q, \alpha) = (P \times Q, \lambda * \alpha),$$

where $P \times Q$ is the Cartesian product of P with Q, and $\lambda * \alpha$ is the labeling that takes each cover relation $(p, q) < (p', q')$ to $\lambda(p < p')$ if $p < p'$ and $q = q'$, and to $\alpha(q < q')$ if $p = p'$ and $q < q'$. Then $(P, \lambda) * (Q, \alpha)$ is an RC-poset. For the coproduct, note that if $[p, q]$ is an interval in P, and if $\lambda_{[p,q]}$ denotes the restriction of λ to the cover relations in $[p, q]$, then $([p, q], \lambda_{[p,q]})$ is an RC-poset. Thus we may define a coproduct Δ on $\mathcal{P}$ by

$$\Delta((P, \lambda)) = \sum_{p \in P} ([\hat{0}, p], \lambda_{[\hat{0},p]}) \otimes ([p, \hat{1}], \lambda_{[p,\hat{1}]}).$$

It is routine to verify that the graded vector space $\mathcal{P}$ together with the product $*$ and coproduct Δ is a graded, connected bialgebra, and hence a graded Hopf algebra. See, e.g., [16, 13] for a more thorough discussion of these ideas.

Observe that the subspace of $\mathcal{P}$ spanned by RC-posets (P, λ) such that P is graded and λ is the canonical labeling is isomorphic to the reduced incidence Hopf algebra of graded posets. We may now extend Ehrenborg's map to a linear map $F : \mathcal{P} \to \mathcal{Q}$ by defining, for every RC-poset (P, λ) such that $|P| \geq 1$,

$$F((P,\lambda)) = \sum_{\hat{0}=p_0<\cdots<p_k=\hat{1}} M_{\{\lambda(p_0,p_1),\lambda(p_0,p_1)+\lambda(p_1,p_2),\ldots,\lambda(p_0,p_1)+\cdots+\lambda(p_{k-2},p_{k-1})\}},$$

where the sum is over all chains in P, and where $\lambda(p_i, p_{i+1})$ stands for the rank of the RC-poset $([p_i, p_{i+1}], \lambda_{[p_i,p_{i+1}]})$. If λ is the canonical labeling, then $F((P, \lambda))$ agrees with our definition of $F(P)$ in (4.1). If in addition P is graded, then $F((P, \lambda))$ is just Ehrenborg's F-quasisymmetric function of P.

Proposition A.1. *The map $F : \mathcal{P} \to \mathcal{Q}$ is a homomorphism of graded Hopf algebras.*

Proof. Consider the linear functional $\zeta : \mathcal{P} \to \mathbb{R}$ given by $\zeta((P, \lambda)) = 1$ for every RC-poset (P, λ). Clearly ζ is a character (i.e., a homomorphism of algebras), so by the universal property of combinatorial Hopf algebras [1, Theorem 4.1], ζ induces a Hopf algebra homomorphism from $\mathcal{P}$ to $\mathcal{Q}$. Moreover, by [1, Formula (4.2)] this induced map is precisely F. □

Acknowledgements

We thank Gábor Hetyei for inspiring discussions and the anonymous referee for several valuable comments and suggestions, including a simplification of the proof of Theorem 2.1.

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