ON THE COMPLEXITY
OF THE UNIFORM HOMEOMORPHISM RELATION
BETWEEN SEPARABLE BANACH SPACES

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Abstract. We investigate the uniform homeomorphism relation between separable Banach spaces and the related relation of local equivalence. We completely characterize the descriptive complexity of local equivalence in the Borel reducibility hierarchy. This also provides a lower bound for the complexity of the uniform homeomorphism.

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1. Introduction

Recently, there has been a growing interest in understanding the complexity of common analytic equivalence relations between separable Banach spaces via the notion of Borel reducibility in descriptive set theory (see [Bos] [FG] [FLR] [FR1] [FR2] [Mc]). In general, the notion of Borel reducibility yields a hierarchy (though not linear) among equivalence relations in terms of their relative complexity.

The most important relations between separable Banach spaces include the isometry, the isomorphism, the equivalence of bases, and in nonlinear theory, Lipschitz...
and uniform homeomorphisms. The exact complexity of the first four relations has been completely determined by recent work in the field. Using earlier work of Weaver [W], Melleray [Me] proved that the isometry between separable Banach spaces is a universal orbit equivalence relation. Rosendal [Ro] studied the equivalence of bases and showed that it is a complete $K_\sigma$ equivalence relation. Using the work of Argyros and Dodos [AD] on amalgamations of Banach spaces, Ferenczi, Louveau, and Rosendal [FLR] recently showed that the isomorphism, the (complemented) biembeddability and the Lipschitz equivalence between separable Banach spaces, as well as the permutative equivalence of Schauder bases, are complete analytic equivalence relations. The Borel reducibility among these equivalence relations, as well as some other equivalence relations we will be dealing with in this paper, is illustrated in Figure 1 (see Section 2 below for the definitions of the equivalence relations). Note that in particular the complete analytic equivalence relation $E_{\Sigma^1_1}$ is the most complex one in the Borel reducibility hierarchy of all analytic equivalence relations.

![Figure 1](https://example.com/f1.png)

**Figure 1.** Equivalence relations characterizing the complexity of classification problems for Banach spaces.

The main problem left open now is to determine the exact complexity of the uniform homeomorphism between separable Banach spaces (see Problem 23, [FLR]). Recall that Banach spaces $X$ and $Y$ are uniformly homeomorphic if there exists a uniformly continuous bijection $\phi : X \rightarrow Y$ such that $\phi^{-1}$ is also uniformly continuous. The understanding of the uniform homeomorphism relation between Banach spaces, also known as the uniform classification of Banach spaces, is in fact one of the main programs in the nonlinear theory of Banach spaces.

Compared to the linear theory, results about the uniform classification are significantly harder to prove, and their proofs often use a combination of metric, geometric, and topological arguments (for a good survey of methods and results, see Chapter 10, [BL]). Moreover, previous efforts have been mostly focused on the
uniform classification of classical Banach spaces. For instance, it is well known that for \(1 \leq p \neq q < \infty\), \(\ell_p\) and \(\ell_q\) are not uniformly homeomorphic (due to Ribe [Ri]). Also, for \(p \neq 2\), \(\ell_p\) and \(L_p\) are not uniformly homeomorphic (due to Bourgain [Bon] for \(1 \leq p < 2\) and Gorelik [Go] for \(2 < p < \infty\)). In fact, it turns out that for \(1 < p < \infty\), the uniform structure of \(\ell_p\) completely determines its linear structure (a result due to Johnson, Lindenstrauss, and Schechtman [JLS]). This also generalizes to certain finite sums of \(\ell_p\) spaces.

From the point of view of descriptive set theory, all previously known results on the uniform classification give some lower bound estimates for its complexity. However, these lower bounds are no more complex than \(\text{id}(2^{\omega})\), the least complicated bound in Figure 1. In contrast to this, it is conceivable that the uniform classification is as complex as Lipschitz and isomorphic classifications, that is, it is \(E_{\Sigma^1_1}\). Thus there is a huge gap between what was conjectured and what could be verified.

In this paper, we give a slightly improved lower bound for the complexity of the uniform classification. We show that the complete \(K_\sigma\) equivalence relation (represented by the equivalence relation \(\ell_\infty\), to be defined in Section 2) is Borel reducible to the uniform homeomorphism relation between separable Banach spaces. As shown in Figure 1 this in particular implies that the uniform homeomorphism relation is at least as complex as the equivalence of bases.

The study of the uniform classification is essentially devoted to understanding what aspects of the linear structure of Banach spaces are invariant under uniform homeomorphisms. As an important example, a fundamental theorem of Ribe [Ri] asserts that the local structure of finite-dimensional subspaces is such an invariant. The proof of our main theorem is a straightforward application of this theorem of Ribe. Moreover, we will isolate a concept of local equivalence between separable Banach spaces in Section 3 and prove in Section 5 that it is Borel bireducible with \(\ell_\infty\). This means that the lower bound we have reached for the complexity of the uniform classification is the best possible with the consideration of local structures.

We can now state the main theorems of this paper.

**Theorem 1.1.** There exists a Borel family \(\{S_{\bar{x}} : \bar{x} \in \mathbb{R}^\omega\}\) of separable Banach spaces such that the following are equivalent for any \(\bar{x}, \bar{y} \in \mathbb{R}^\omega\):

(a) \(\bar{x} - \bar{y} \in \ell_\infty\);
(b) \(S_{\bar{x}}\) and \(S_{\bar{y}}\) are uniformly homeomorphic;
(c) \(S_{\bar{x}}\) and \(S_{\bar{y}}\) are isomorphic;
(d) \(S_{\bar{x}}\) and \(S_{\bar{y}}\) are locally equivalent.

**Theorem 1.2.** The local equivalence between separable Banach spaces is Borel bireducible with \(\ell_\infty\).

Of course in general the local structure is not sufficient to determine the uniform structure (for instance, as the results of Bourgain and Gorelik mentioned above show). It is anticipated that the complexity of the uniform classification is much more complex than \(\ell_\infty\). To verify this it would be enough to show that the equivalence relation \(E_\omega^0\) is Borel reducible to the uniform homeomorphism relation. As Figure 1 suggests, \(E_\omega^0\) is in some sense the least complex equivalence relation not Borel reducible to \(\ell_\infty\).

In Section 6 we generalize the construction in the proof of Theorem 1.1 and consider a variety of classes of separable Banach spaces with a similar construction.
scheme. The uniform homeomorphism relations for these classes are all no more complex than $\ell_\infty$. We then determine exactly what kind of complexity the uniform homeomorphism relations on these classes can achieve. It turns out that they can only be $\ell_\infty$, $E_1$, $E_0$, or smooth.

Our constructions in Sections 4 and 6 will yield only classes of separable Banach spaces for which the uniform homeomorphism and the isomorphism relations coincide. In general it is well known that the uniform homeomorphism is a genuinely coarser equivalence relation than the isomorphism (see, for example, Section 10.4, [BL]). Therefore it is of interest to study the question of how many different isomorphism classes a single uniform homeomorphic class can contain. In Section 7 we prove the following related result.

**Theorem 1.3.** There exists a Borel class $C$ of mutually uniformly homeomorphic separable Banach spaces such that the equality relation of countable sets of real numbers, denoted $=^+$, is Borel reducible to the isomorphism relation on $C$.

The rest of the paper is organized as follows. In Section 2 we define all benchmark equivalence relations relevant to our discussions in this paper and review the Borel reducibility theory of equivalence relations on Polish spaces. In Section 3 we explain how to apply the framework of the descriptive set theory of equivalence relations to the uniform classification of separable Banach spaces. We also define the notion of local equivalence and show that it is $\Sigma_0^3$ in two different codings of separable Banach spaces. In Section 4 we give the construction of the family in Theorem 1.1 and prove some basic properties. In Section 5 we give the proofs of both Theorems 1.1 and 1.2. We also generalize the $\ell_\infty$ equivalence relation and prove a dichotomy theorem characterizing its possible complexity. In Section 6 the construction of Section 4 is generalized and the possible complexity of the uniform homeomorphism relations for the resulting classes is completely determined. Finally in Section 7 we prove Theorem 1.3.

### 2. Preliminaries on the Borel reducibility hierarchy

In this section we review the Borel reducibility hierarchy of analytic equivalence relations for the convenience of the reader. We give the definitions of all equivalence relations mentioned in Figure 1 and recall their characteristic properties. The reader can find more details in the references provided below, or see [Ga2].

Descriptive set theory studies definable sets and relations on Polish spaces. Recall that a *Polish* space is a separable and completely metrizable topological space. Examples of Polish spaces include $\omega = \mathbb{N}$ with the discrete topology, $\mathbb{R}$ with the usual topology, $\mathbb{R}^\omega$ with the product topology, and the Cantor space $2^\omega = \{0,1\}^\omega$. The simplest examples of definable subsets of a Polish space are the Borel sets. Recall that the collection of all *Borel* sets is the smallest $\sigma$-algebra containing all open sets. Thus all Borel subsets of a Polish space can also be arranged in a hierarchy according to their *descriptive complexity*. In this hierarchy the simplest ones are open sets and closed sets. On the next level we have the $F_\alpha$ sets and $G_\delta$ sets, which are respectively countable unions of closed sets and countable intersections of open sets. To continue, we call a set $\Sigma_0^\alpha$ if it is a countable union of $G_\delta$ sets, and $\Pi_0^\alpha$ if it is a countable intersection of $F_\alpha$ sets. In general, one can define the classes $\Sigma_0^\alpha$, $\Pi_0^\alpha$ in the same fashion for all countable ordinals $\alpha$. However, in this paper we will not deal with any set beyond $\Sigma_3^0$ and $\Pi_3^0$. 

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It is well known that \( \Sigma^0_3 \) and \( \Pi^0_3 \) are distinct classes. To prove that a given subset \( A \) of a Polish space \( X \) is not \( \Sigma^0_3 \), the usual method is to try to show that \( A \) is \( \Pi^0_3 \)-hard, that is, given any \( \Pi^0_3 \) subset \( B \) of \( 2^\omega \), there is a continuous function \( f : 2^\omega \rightarrow X \) such that \( B = f^{-1}(A) \). If \( A \) is \( \Pi^0_3 \) and \( \Pi^0_3 \)-hard, then it is said to be \( \Pi^0_3 \)-complete. For more on this topic, see Section 22, [K].

The Borel structure thus given by a Polish topology is known as a standard Borel structure. A Borel space (that is, a space with a distinguished \( \sigma \)-algebra of subsets) is called a standard Borel space if its Borel structure is standard, that is, induced by an underlying Polish topology. A function \( f \) between Polish spaces (or standard Borel spaces) is Borel if \( f^{-1}(A) \) is Borel for any Borel set \( A \). Any two uncountable standard Borel spaces are Borel isomorphic to each other.

Other than the examples of Polish spaces mentioned above, we recall another well-known example of a standard Borel space, the Effros Borel space. Let \( X \) be a Polish space and \( F(X) \) be the hyperspace of all closed subsets of \( X \). The Effros Borel structure is the Borel structure on \( F(X) \) generated by basic Borel sets of the form

\[
\{ F \in F(X) : F \cap U \neq \emptyset \}
\]

for some open subset \( U \) of \( X \). A Polish topology generating the Effros Borel structure was discovered by Beer [B]. We also recall its definition. Let \( d \) be a compatible complete metric on \( X \). For any \( x \in X \) and \( F \in F(X) \), define

\[
d(x, F) = \inf\{ d(x, y) : y \in F \}.
\]

Now consider the topology generated by all subbasic open sets of the form

\[
\{ F \in F(X) : d(x, F) < a \} \text{ or } \{ F \in F(X) : d(x, F) > a \}
\]

for some \( x \in X \) and \( a \in \mathbb{R} \). This topology is known as the Wijsman topology on \( F(X) \); Hess [He] observed that it generates the Effros Borel structure, and Beer [B] proved that it is Polish.

The next level of definable subsets beyond Borel subsets of a Polish space consists of analytic ones. Recall that a subset of a Polish space is analytic (or \( \Sigma^1_1 \)) if it is the continuous image of a Polish space. It is well known that every Borel set is analytic. For more information on Polish spaces, Borel and analytic subsets, and Borel functions, see [K].

Let \( X \) be a Polish space and \( E \) an equivalence relation on \( X \). We say that \( E \) is analytic if \( E \) is an analytic subset of \( X \times X \). Similarly we also speak of Borel equivalence relations, or even \( F_\sigma \), \( G_\delta \), \( \Sigma^1_1 \), \( \Pi^0_3 \) equivalence relations, respectively.

The notion of Borel reducibility defined below is fundamental in the theory of equivalence relations as it explores the relative structural complexity of equivalence relations. Let \( X, Y \) be Polish spaces and \( E, F \) be equivalence relations on \( X, Y \), respectively. We say that \( E \) is Borel reducible to \( F \), and denote it by \( E \leq_B F \), if there is a Borel function \( f : X \rightarrow Y \) such that for all \( x_1, x_2 \in X \),

\[
x_1 E x_2 \iff f(x_1) F f(x_2).
\]

If \( E \leq_B F \), then intuitively \( E \) is no more complex than \( F \), since any complete invariants for the \( F \)-equivalence classes can be composed with \( f \) to obtain complete invariants for the \( E \)-equivalence classes. In the case of both \( E \leq_B F \) and \( F \leq_B E \), then we denote \( E \sim_B F \) and say that \( E \) and \( F \) are Borel bireducible. If \( E \sim_B F \), then intuitively \( E \) and \( F \) have the same complexity.

Next we define the benchmark equivalence relations in Figure [11]
(1) The equivalence relation \( id(2^\omega) \) is the identity (or equality) relation on the Cantor space \( 2^\omega \), that is, \((x, y) \in id(2^\omega) \) iff \( x = y \). Since all uncountable standard Borel spaces are Borel isomorphic to each other, this relation is Borel bireducible with the identity relation on any uncountable Polish (or standard Borel) space. An equivalence relation that is Borel reducible to \( id(2^\omega) \) is said to be smooth or concretely classifiable, since it is possible to assign a concrete real number as a complete invariant for each of its equivalence classes.

(2) The equivalence relation \( E_0 \) is the eventual agreement relation on \( 2^\omega \). In symbols, for \( x = (x_n), y = (y_n) \in 2^\omega \),
\[
x E_0 y \iff \exists m \in \omega \ \forall n \geq m \ x_n = y_n.
\]
In the Borel reducibility hierarchy for Borel equivalence relations, \( E_0 \) is the minimum one beyond \( id(2^\omega) \) [HaKL].

(3) The equivalence relation \( E_1 \) is the eventual agreement relation for countable sequences of real numbers. In symbols, for \( \vec{x} = (x_n), \vec{y} = (y_n) \in \mathbb{R}^\omega \),
\[
\vec{x} E_1 \vec{y} \iff \exists m \in \omega \ \forall n \geq m \ x_n = y_n.
\]
It is easy to see that \( E_0 \leq_B E_1 \). In the definition of \( E_1 \) the space \( \mathbb{R} \) can be replaced by the Cantor space \( 2^\omega \) or the Baire space \( \omega^\omega \) without affecting the complexity of the resulting equivalence relation, since \( \mathbb{R} \) is Borel isomorphic to any uncountable Polish space. We will use the alternate versions of the definition in this paper without further elaboration. In the Borel reducibility hierarchy, \( E_1 \) is an immediate successor of \( E_0 \) ([KL]), that is, if \( E \leq_B E_1 \), then \( E \) is Borel bireducible with either \( E_1 \) or \( E_0 \), or else \( E \) is smooth.

(4) For \( 1 \leq p \leq \infty \) the equivalence relation \( E_\ell_p \) is defined on \( \mathbb{R}^\omega \) as follows: for \( \vec{x} = (x_n), \vec{y} = (y_n) \in \mathbb{R}^\omega \),
\[
\vec{x} E_\ell_p \vec{y} \iff \vec{x} - \vec{y} \in \ell_p.
\]
When there is no danger of confusion we simply use \( \ell_p \) to denote the equivalence relation \( E_\ell_p \). Dougherty and Hjorth [DH] showed that for \( 1 \leq p \leq q < \infty \), \( \ell_p \leq_B \ell_q \). The first author [Ga1] extended this to include the case \( q = \infty \). It is also known that \( E_1 \leq_B \ell_\infty \) [Ga1] and \( E_1 \not\leq_B \ell_p \) for \( p < \infty \) [KL].

The equivalence relation \( \ell_\infty \) is perhaps the most important equivalence relation for this paper. Rosendal [Ro2] showed that it is a complete \( K_\sigma \) equivalence relation, that is, for any equivalence relation \( E \) on a Polish space \( X \), if \( E \) is \( K_\sigma \) (that is, a countable union of compact subsets of \( X^2 \)), then \( E \leq_B \ell_\infty \). In particular, if every \( E \)-equivalence class is countable, then \( E \leq_B \ell_\infty \).

(5) An equivalence relation is called countable if every one of its equivalence classes is countable. Among all countable Borel equivalence relations there exists a maximum one in terms of Borel reducibility [DJK]. We denote any such equivalence relation by \( E_\infty \). By the remark above, \( E_\infty \leq \ell_\infty \). An equivalence relation \( E \) is essentially countable if \( E \leq_B E_\infty \).

(6) The equivalence relation \( =^+ \) codes the equality relation for countable sets of real numbers. In symbols, for \( \vec{x} = (x_n), \vec{y} = (y_n) \in \mathbb{R}^\omega \),
\[
\vec{x} =^+ \vec{y} \iff \{x_n : n \in \omega\} = \{y_n : n \in \omega\}.
\]
It is an easy consequence of a classical theorem of Luzin and Novikov (see Theorem 18.10, [K]) in descriptive set theory that \( E_\omega \leq B =^+ \). It is also known that \( \ell_\infty \not\leq_B =^+ \) (by results of Kechris and Louveau [KL]) and \( =^+ \not\leq_B \ell_\infty \) (see below).

(7) The equivalence relation \( E_0^\omega \) is defined on \((2^\omega)^\omega\) as follows: for \( \vec{x} = (x_n), \vec{y} = (y_n) \in (2^\omega)^\omega\),

\[
\vec{x} E_0^\omega \vec{y} \iff \forall n \in \omega \ x_n E_0 y_n.
\]

\( E_0^\omega \) has been studied explicitly or implicitly in, for example, [Sa], [HK], and [HjKL]. Note that it is a \( \Pi^0_3 \) equivalence relation. Results of Solecki [Sa] imply that \( E_0^\omega \) is not essentially countable, that is, \( E_0^\omega \not\leq_B E_{\infty} \). Further results of Hjorth, Kechris, and Louveau [HK], [HjKL] imply that \( E_0^\omega \) is not Borel reducible to any \( \Sigma^0_3 \) equivalence relations. Thus in particular \( E_0^\omega \not\leq_B \ell_\infty \). It is a somewhat elusive task to trace the references for this result; for the convenience of the reader we will give a direct proof of it later in this section.

The importance of \( E_0^\omega \) lies in both the fact that it is combinatorially easy to analyze and the speculation that it seems to be the simplest (or even minimum) equivalence relation not reducible to \( \ell_\infty \). For instance, it is relatively simple to show that \( E_0^\omega \leq B =^+ \) (we will give a proof later in this section); therefore it follows immediately that \( =^+ \not\leq_B \ell_\infty \). Also, when we need to consider equivalence relations which seem to be more complex than \( \ell_\infty \), the reducibility of \( E_0^\omega \) to them gives good test questions.

(8) The equivalence relation \( E_G^\infty \) is the universal orbit equivalence relation induced by Borel actions of Polish groups. We shall not deal with general orbit equivalence relations in this paper. Therefore we will omit the details of the definition of \( E_G^\infty \). The interested reader can find more information in [BK], [GK], or [Me].

(9) Among all analytic equivalence relations on Polish spaces, there is a universal one, that is, every other analytic equivalence relation is Borel reducible to it. Following [LR] we denote it by \( E_{\Sigma^1_1} \). As we mentioned in the Introduction this equivalence relation plays an important role when natural equivalence relations between separable Banach spaces are considered. However, results in this paper only involve equivalence relations much less complex than \( E_{\Sigma^1_1} \).

In the rest of the paper we will consider more equivalence relations, but most of them will be closely related to \( \ell_\infty \).

In the remainder of this section we give some proofs of facts related to \( E_0^\omega \) (see (7) above) for the convenience of the reader. We fix some notation to be used in these proofs, as well as in the rest of the paper. First, we fix once and for all a computable bijection \( \langle \cdot, \cdot \rangle : \omega \times \omega \rightarrow \omega \). Next, we let \( \omega^{<\omega} \) denote the countable set of all finite sequences of natural numbers. For \( s \in \omega^{<\omega} \), we fix \( |s| \) denote the length of \( s \), that is, if \( s = (s_1, \ldots, s_n) \), then \( |s| = n \). The empty sequence is denoted \( \emptyset \), and we set \(|\emptyset| = 0\). If \( s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_m) \in \omega^{<\omega} \), then we let

\[
s * t = \begin{cases} (s_1, \ldots, s_n, t_{n+1}, \ldots, t_m), & \text{if } m > n, \\ s, & \text{if } m \leq n. \end{cases}
\]

Then \(|s * t| = \max\{|s|, |t|\}\), and \( s * t \) is obtained by replacing the first \(|s| \) many elements of \( t \) by \( s \). This definition also makes sense when \( t \in \omega^{<\omega} \) is replaced by
an element of \( \omega^{\omega} \). For \( s, t \) as above we also let

\[ s \oplus t = (s_1, t_1, s_2, t_2, \ldots). \]

Then \( |s \oplus t| = |s| + |t| \). This definition makes sense when both \( s \) and \( t \) are replaced by elements of \( \omega^{\omega} \). For \( x, y \in \omega^{\omega} \), \( x \oplus y \) is obtained from shuffling the elements of \( x \) and \( y \) into a single sequence.

**Lemma 2.1.** \( E_0^\omega \subseteq B = \ast^+ \).

**Proof.** Let \( s_0^0, s_1^1, \ldots \) be an enumeration of \( \omega^{<\omega} \). Fix some \( \bar{x} = (z_n) \in (2^{\omega})^\omega \) such that for all \( i \neq j \in \omega \), \( (z_i, z_j) \notin E_0 \). For \( \bar{x} = (x_n) \in (2^{\omega})^\omega \), let \( f(x) = (y_n) \), where for \( n = (i, j) \),

\[ y_n = z_i \oplus (s^i \cdot x_j). \]

It is easy to verify that \( f \) is a Borel (in fact continuous) reduction from \( E_0^\omega \) to \( \ast^+ \). \( \square \)

**Lemma 2.2.** \( E_0^\omega \) is not Borel reducible to any \( \Sigma_3^0 \) equivalence relation.

**Proof.** Suppose \( X \) is a Polish space, \( E \) a \( \Sigma_3^0 \) equivalence relation on \( X \), and \( f : (2^{\omega})^\omega \to X \) a Borel function such that for all \( \bar{x}, \bar{y} \in (2^{\omega})^\omega \),

\[ \bar{x} E_0^\omega \bar{y} \iff f(x) E f(y). \]

Since \( f \) is Borel, and hence Baire measurable, there is a comeager set \( C \subseteq (2^{\omega})^\omega \) such that \( f \mid C \) is continuous. We may assume \( C \) is a \( G_\delta \) set. We may now compute \( E_0^\omega \cap (C \times C) \) to be \( \Sigma_4^0 \), namely,

\[ (x, y) \in E_0^\omega \cap (C \times C) \iff x, y \in C \wedge (f(x), f(y)) \in E. \]

Since \( f \mid C \) is continuous and \( E \in \Sigma_3^0 \), this shows \( E_0^\omega \cap (C \times C) \) to be \( \Sigma_4^0 \). To get a contradiction it thus suffices to prove the following claim.

**Claim.** For every comeager set \( C \subseteq (2^{\omega})^\omega \), \( E_0^\omega \cap (C \times C) \) is \( \Pi_3^0 \)-complete.

**Proof.** Let \( B = \{ x \in 2^{\omega} : \forall i \in \omega \exists j \in \omega \forall k \geq j \ x((i, k)) = 1 \} \). Then \( B \) is clearly \( \Pi_3^0 \). We first show that \( B \) is \( \Pi_3^0 \)-complete. For this, let \( A \subseteq 2^{\omega} \) be \( \Pi_3^0 \), say \( A = \bigcap_i \bigcup_j \bigcap_k A_{i,j,k} \), where each \( A_{i,j,k} \) is clopen. Define \( \rho : 2^{\omega} \to 2^{\omega} \) as follows. For each \( i, k \in \omega \), let \( a_{x,i,k} \in \omega \) be the least integer \( j \leq k \), if one exists, such that \( x \in A_{i,j,k} \) for all \( k' \leq k \). Let \( \rho(x)(i, k) = 1 \) iff \( a_{x,i,k} \) and \( a_{x,i,k-1} \) are both defined and are equal. Otherwise set \( \rho(x)(i, k) = 0 \). The map \( \rho \) is continuous from \( 2^{\omega} \) to \( 2^{\omega} \), and \( x \in A \) iff \( \rho(x) \in B \). Thus, \( B \) is \( \Pi_3^0 \)-complete.

Note that \( (2^{\omega})^\omega \) is homeomorphic to \( 2^{\omega \times \omega} \). For notational simplicity we work with \( 2^{\omega \times \omega} \) below, and identify it with \( (2^{\omega})^\omega \). If \( s \in 2^{n \times m} \) for some \( n, m \in \omega \), then the basic clopen set determined by \( s \), denoted by \( N_s \), is the set \( \{ x \in 2^{\omega \times \omega} : \forall i < n, j < m \ x(i, j) = s(i, j) \} \). Write \( C = \bigcap_n D_n \), where each \( D_n \) is open dense in \( 2^{\omega \times \omega} \).

We next define two continuous functions \( \pi_1, \pi_2 : 2^{\omega} \to 2^{\omega \times \omega} \) so that

\[ x \in B \iff (\pi_1(x), \pi_2(x)) \in E_0^\omega \cap (C \times C). \]

For each sequence \( s \in 2^n \) we will define values \( \pi_1(s), \pi_2(s) \in 2^{p(n) \times p(n)} \) for some \( p = p(n) \) which depends only on \( n \). We will then take, for \( x \in 2^{\omega} \), \( \pi_1(x) = \bigcup_n \pi_1(x \upharpoonright n) \) and likewise for \( \pi_2(x) \).

Suppose inductively that for some \( n \in \omega \) and every sequence \( s \in 2^n \) we have defined \( \pi_1(s), \pi_2(s) \in 2^{p \times p} \) for some \( p = p(n) \in \omega \) which depends only on \( n \).
Suppose also that \( N_{\pi_1(s)} \cap N_{\pi_2(s)} \subseteq D_n \) for each \( s \in 2^n \). Let \( n + 1 = \langle i, k \rangle \). For each \( s' \in 2^{n+1} \) extending \( s \in 2^n \), extend \( t_1 := \pi_1(s) \) and \( t_2 := \pi_2(s) \) to \( t_1', t_2' \) by letting \( t'_1(i, p(n) + k) = t'_2(i, p(n) + k) = 1 \) if \( s(n) = 1 \), and otherwise setting \( t'_1(i, p(n) + k) = 0 \), \( t'_2(i, p(n) + k) = 1 \). Extend \( t_1', t_2' \) to \( t_1'', t_2'' \) in \( 2^{q_n \times q_n} \), where \( q_n = p(n) + n \), by setting all other undefined values to 0. Note that all of the \( t_1'', t_2'' \) are elements of \( 2^{q_n \times q_n} \). Let \( p(n+1) \) be large enough so that there is a finite function \( h_{n+1} : (p(n+1) \times p(n+1)) \to \{0, 1\} \) such that for all of the \( t_1'', t_2'' \) we have that \( u_1 = t_1'' \cup h_{n+1} + u_2 = t_2'' \cup h_{n+1} \) determine basic open sets with \( N_u \subseteq D_{n+1} \). We can achieve this in \( 2 \cdot 2^{n+1} \) steps, using the fact that \( D_{n+1} \) is dense open. Set \( \pi_1(s') = u_1, \pi_2(s') = u_2 \). Note that for any \( s_1, s_2 \in 2^{n+1} \) and \( a, b \in \{1, 2\} \), \( \pi_a(s_1), \pi_b(s_2) \) differ in at most one point of \( (p(n+1) \times p(n+1)) \to \{0, 1\} \). Thus \( (\pi_1(x), \pi_2(x)) \not\in E^\omega_0 \).

Conversely, if \( x \notin B \), then for some \( i \), there are infinitely many \( j \) such that \( x(i, j) = 0 \). Fix such an \( i \). For each \( j \) with \( x(i, j) = 0 \) let \( n = \langle i, k \rangle \), and we have that \( \pi_1(x)(n) \) and \( \pi_2(x)(n) \) disagree at \( (i, p(n) + j) \). This implies that \( -\pi_1(x) E^\omega_0 \pi_2(x) \).

This completes the proof of Lemma 2.2.

3. Codings of separable Banach spaces and the local equivalence

To apply the descriptive set-theoretic framework to the study of equivalence relations on separable Banach spaces, the collection of separable Banach spaces must be viewed as a Polish space.

One way to do this is to use the well-known theorem of Banach and Mazur that \( C[0, 1] \) is a universal separable Banach space, that is, every separable Banach space is linearly isometric to a (necessarily closed) subspace of \( C[0, 1] \). The collection of all separable Banach spaces is then viewed as a subspace of the hyperspace \( F(C[0, 1]) \) with the Wijsman topology (see Section 2). Let

\[ \mathcal{B} = \{ F \in F(C[0, 1]) : F \text{ is a linear subspace of } C[0, 1] \}. \]

We check below that \( \mathcal{B} \) is a Polish subspace.

**Lemma 3.1.** \( \mathcal{B} \) is a \( G_\delta \) subspace of \( F(C[0, 1]) \), hence Polish.

**Proof.** Fix a countable dense \( D \subseteq C[0, 1] \). Let \( d \) be the metric on \( C[0, 1] \) given by its norm. We claim that for any \( F \in F(C[0, 1]) \), \( F \in \mathcal{B} \) if and only if

\[ \forall p, q, a, b \in \mathbb{Q} \, \forall x, y \in D \, [d(x, F) < a \land d(y, F) < b \implies d(px + qy, F) < |pa| + |qb|]. \]

If \( F \) is a linear subspace of \( C[0, 1] \) the demonstrated condition clearly holds. Conversely, suppose the condition holds. Since \( F \) is closed, it suffices to show that for all \( u, v \in F \) and \( p, q \in \mathbb{Q} \), \( pu + qv \in F \). For this take two sequences \( x_n, y_n \in D \) such that \( d(x_n, u), d(y_n, v) < 2^{-n} \). Then \( d(px_n + qy_n, F) < (|p| + |q|)2^{-n} \) by the assumption, and \( d(px_n + qy_n, pu + qv) < (|p| + |q|)2^{-n} \). Thus \( d(pu + qv, F) < (|p| + |q|)2^{-n+1} \).
Since \( n \) is arbitrary, we have that \( d(\psi u + qv, F) = 0 \) and \( \psi u + qv \in F \). The claim implies immediately that \( \mathfrak{B} \) is \( G_\delta \) in the Wijsman topology of \( F(C[0, 1]) \). \( \square \)

In discussing Banach spaces with a distinguished Schauder basis another representation is often used. A fundamental result of Pełczyński [P] says that there is a universal basis \( U = (e_i) \), that is, for every separable Banach space with a basis \((X, B)\), where \( B = (x_i) \), there is a one-to-one map \( f: B \to U \) which extends to a linear isomorphism from \( X \) to the space spanned by \( f(B) \). In this manner, the collection of separable Banach spaces with a basis can be identified with the Cantor space \( 2^\omega \). For \( x \in 2^\omega \), let \( X_x \) be the separable Banach space with a basis coded by \( x \). The space of all Banach spaces with infinite bases corresponds to \( [\omega]^\omega \), the set of all infinite subsets of \( 2^\omega \), which is a \( G_\delta \) subset of \( 2^\omega \) and a Polish space in its own right.

In practice it is often easier to work with the following direct coding for Banach spaces with infinite bases. Fix once and for all for the rest of the paper an enumeration \((s^n)\) of \( \mathbb{Q}^{<\omega} \), the set of all finite sequences of rational numbers. To any separable Banach space with an infinite basis \((Y, (y_i))\), we associate a sequence of real numbers \((r_n) \in \mathbb{R}^\omega \), where

\[
r_n = \left\| \sum_{i=1}^k a^n_i y_i \right\|_Y
\]

if \( s^n = (a^n_1, \ldots, a^n_k) \). Recall that a normalized (i.e., all \( y_i \) have norm 1) basis \((y_i)\) is called monotone if the projections onto initial segments of \( (y_i) \) have norm 1. There is no loss of generality in restricting to monotone bases, since for every normalized basis we can take an equivalent norm for which the basis is monotone. Let \( \mathfrak{B}_b \subseteq \mathbb{R}^\omega \) be the set of all possible sequences \((r_n)\) associated with Banach spaces with a monotone basis.

Again we check below that \( \mathfrak{B}_b \) is a Polish space. We henceforth use the phrase “Banach space with basis” to denote a pair \((X, B)\), where \( X \) is a Banach space, and \( B \) is a basis. It is these objects that are coded by the reals in \( \mathfrak{B}_b \).

**Lemma 3.2.** \( \mathfrak{B}_b \) is a closed subspace of \( \mathbb{R}^\omega \), hence is Polish.

**Proof.** For notational convenience in this proof we identify \( s^n = (a^n_1, \ldots, a^n_k) \) with the infinite sequence \((a^n_1, \ldots, a^n_k, 0, 0, \ldots)\). Then it makes sense to speak of \( s^n + s^m \in \mathbb{Q}^{<\omega} \) for any \( n, m \in \omega \), and \( ps^n \in \mathbb{Q} \) for any \( p \in \mathbb{Q} \) and \( n \in \omega \). Now for any \((r_n) \in \mathbb{R}^\omega \), \((r_n) \in \mathfrak{B}_b \) if all of the following hold:

(i) if \( s^n = (0, \ldots, 0, 1, x, 0, \ldots, 0) \), then \( r_n = 1 \);

(ii) if \( s^m \) coincides with an initial segment of \( s^n \), then \( r_m \leq r_n \);

(iii) if \( s^n + s^m = s^l \), then \( r_l \leq r_n + r_m \);

(iv) for any \( p \in \mathbb{Q} \), if \( s^n = ps^n \), then \( r_m = |p|r_n \).

The conditions listed are closed for \((r_n) \in \mathbb{R}^\omega \). Note that (i) and (ii) imply that the basis is monotone, which implies that any nonzero linear combination of the \( y_i \) has positive norm. \( \square \)

Given any \((r_n) \in \mathfrak{B}_b \), by the proof of Lemma 3.2 we can associate a Banach space with an infinite basis whose norm function is approximated by the sequence \((r_n)\). In this manner each element of \( \mathfrak{B}_b \) codes a Banach space with basis. For \( y \in \mathfrak{B}_b \), let \( Y_y \) be the space coded by \( y \).
We remark that the two codings for Banach spaces with bases are equivalent in the following precise sense. It is easy to see that there is a continuous function \( \varphi : [\omega]^\omega \to \mathfrak{B}_b \) such that for all \( x \in [\omega]^\omega \), \( X_x \) is linearly isometric to \( Y_{\varphi(x)} \). Conversely, by the proof of Pelczyński’s theorem \([P]\) there is also a Borel function \( \psi : \mathfrak{B}_b \to [\omega]^\omega \) such that for all \( y \in \mathfrak{B}_b \), \( Y_y \) is linearly isometric to \( X_{\psi(y)} \).

As for the relationship between codings using elements of \( \mathfrak{B} \) versus those of \( \mathfrak{B}_b \), we denote by \( \mathfrak{B}_\beta \) the subspace of \( \mathfrak{B} \) consisting of all linear subspaces of \( C[0,1] \) admitting bases. It follows immediately from the proof of the Banach-Mazur theorem that there is a Borel function \( \Phi : \mathfrak{B}_b \to \mathfrak{B}_\beta \subseteq \mathfrak{B} \) such that for all \( y \in \mathfrak{B}_b \), \( Y_y \) is linearly isometric to \( \Phi(y) \). Intuitively, in defining \( \Phi(y) \) one omits the given basis and obtains an isomorphic (in fact isometric) copy of the space as a subspace of \( C[0,1] \). It is easy to see that \( \mathfrak{B}_\beta \) coincides with the isomorphic saturation of \( \Phi(\mathfrak{B}_b) \), denoted \([\Phi(\mathfrak{B}_b)]\), which is also the same as the isometric saturation of \( \Phi(\mathfrak{B}_b) \). Obviously both \( \Phi(\mathfrak{B}_b) \) and \( \mathfrak{B}_\beta = [\Phi(\mathfrak{B}_b)] \) are analytic subsets of \( \mathfrak{B} \). However, it is not known whether either of them is Borel.

Rosendal has pointed out that the function \( \Phi \) can be improved to be injective, that is, there is a Borel function \( \Psi : \mathfrak{B}_b \to [0,1] \) to be a Borel injection and \( \varphi : C[0,1] \oplus_{\infty} C[0,1] \to C[0,1] \) a linear isometric embedding. For any \( y \in \mathfrak{B}_b \), let

\[
\Psi(y) = \{ \varphi(v, \lambda(y)v) : v \in C[0,1] \}.
\]

Then \( \Psi(y) \) and \( \Phi(y) \) are linearly isometric, and \( \Psi \) is obviously injective because of the injectivity of \( \varphi \) and \( \psi \). It follows that \( \Psi(\mathfrak{B}_b) \) is Borel. Note that \( \mathfrak{B}_\beta = [\Psi(\mathfrak{B}_b)] = [\Phi(\mathfrak{B}_b)] \), and the question about its Borelness remains unresolved.

Next we turn to equivalence relations between separable Banach spaces.

We remark first that the uniform homeomorphism relation is analytic as an equivalence relation on either \( \mathfrak{B} \) or \( \mathfrak{B}_b \). This was noted in \([FLR]\), and in fact it is proved there that the uniform homeomorphism relation on all Polish metric spaces is complete analytic. For the convenience of the reader we recall the following argument. Let \( \approx \) denote the uniform homeomorphism relation on \( \mathfrak{B} \). Then for \( X, Y \in \mathfrak{B} \), \( X \approx Y \) iff there exist \( (x_n), (y_n) \in C[0,1]^\omega \) such that

1. \( x_n \in X \) and \( y_n \in Y \) for all \( n \in \omega \);
2. the sets \( D_X := \{ x_n : n \in \omega \} \) and \( D_Y := \{ y_n : n \in \omega \} \) are dense in \( X \) and \( Y \), respectively;
3. the map \( f : D_X \to D_Y \) with \( f(x_n) = y_n \) for all \( n \in \omega \) is a uniformly continuous bijection, with \( f^{-1} \) also uniformly continuous.

One direction of the equivalence is clear. For the other direction, we note that the uniform homeomorphism \( f \) defined on a dense set \( D_X \) can be uniquely extended to a necessarily uniform homeomorphism of the entire space, since Cauchy sequences in \( D_X \) will correspond to Cauchy sequences in \( D_Y \) by the uniform continuity of \( f \) and \( f^{-1} \). Now the conditions (a) through (c) are all Borel conditions for \( X, Y, (x_n) \), and \( (y_n) \). Hence \( \approx \) is analytic. It also follows immediately that the uniform homeomorphism relation on \( \mathfrak{B}_b \) is analytic via the pullback of the Borel function \( \Phi \) defined above.

In the remainder of this section we define a notion of local equivalence inspired by Ribe’s theorem \([Ri]\) and study its basic properties. In doing this we recall some concepts and results from Banach space theory. All other unexplained terms and facts can be found in \([BL]\) or \([T]\).
Recall that, for linearly isomorphic Banach spaces $X$ and $Y$, the Banach-Mazur distance between $X$ and $Y$ is defined as
\[ d(X, Y) := \inf \{ \|T\|\|T^{-1}\| : T : X \to Y \text{ is an isomorphism} \}. \]

The following theorem is a fundamental result about uniform homeomorphisms.

**Theorem 3.3 (Ribe [Ri]).** If $X$ and $Y$ are uniformly homeomorphic Banach spaces, then there exists a constant $C > 0$ such that for every finite-dimensional subspace $E$ of $X$ there exists a finite-dimensional subspace $F$ of $Y$ such that $d(E, F) \leq C$, and vice versa.

This motivates the following concept.

**Definition 3.4.** Let $X$ and $Y$ be Banach spaces. We say that $X$ and $Y$ are locally equivalent, denoted by $X \equiv_L Y$, if there exists a constant $C > 0$ such that for every finite-dimensional subspace $E$ of $X$ there exists a finite-dimensional subspace $F$ of $Y$ such that $d(E, F) \leq C$, and vice versa.

Here we refer to the structure of finite-dimensional subspaces of a Banach space as its local structure. In the literature the local equivalence between $X$ and $Y$ is sometimes informally referred to as $X$ and $Y$ having the same finite-dimensional subspaces. Ribe’s theorem states that uniformly homeomorphic spaces are locally equivalent. The converse is not true. For instance, as we mentioned in the Introduction, $\ell_p$ is not uniformly homeomorphic to $L_p$ for $1 \leq p < \infty$, $p \neq 2$; however, they are locally equivalent.

In the following we compute the descriptive complexity of the local equivalence as an equivalence relation on either the Polish space $\mathcal{B}$ of all separable Banach spaces or the Polish space $\mathcal{B}_b$ of all separable Banach spaces with basis.

**Lemma 3.5.** Local equivalence is a $\Sigma^0_3$ equivalence relation on either $\mathcal{B}$ or $\mathcal{B}_b$.

**Proof.** First consider $\equiv_L$ as an equivalence relation on $\mathcal{B}_b$. Let $(X, (e_i))$ and $(Y, (f_i))$ be the Banach spaces with basis coded as elements of $\mathcal{B}_b$ by $x, y \in \mathbb{R}^\omega$. Note that every finite-dimensional subspace of $(X, (e_i))$ can be approximated by a space with a (finite) basis consisting of finite rational linear combinations of the $e_i$. We use the enumeration $\{s^n\}$ of $\mathbb{Q}^{\leq \omega}$ in the definition of $\mathcal{B}_b$. For $s^n = (a^n_1, \ldots, a^n_k) \in \mathbb{Q}^{\leq \omega}$, let $s^n(X) = \sum_{i=1}^k a^n_i e_i$. For $\bar{n} = n_1, \ldots, n_M \in \omega$, let $X_{\bar{n}}$ be the $\leq N$-dimensional subspace of $X$ with basis $s^{n_1}(X), \ldots, s^{n_M}(X)$. Similarly we define $s^m(Y)$ and $Y_{\bar{m}}$ for $\bar{m} = m_1, \ldots, m_M \in \omega$.

Let $I$ be the set of $\bar{n} = (n_1, \ldots, n_N) \in \mathbb{N}^{\leq \omega}$ such that the vectors $s^{n_1}, \ldots, s^{n_N}$ are linearly independent. Note that if $X \in \mathcal{B}_b$ is coded by $x$, and $\bar{n} \in I$, then the vectors $s^{n_1}(X), \ldots, s^{n_N}(X)$ are linearly independent in the space $X$ (since $x$ codes a monotone basis for $X$).

With this notation we have that $X \equiv_L Y$ iff
\[ \exists M \geq 1 \forall N \geq 1 \forall (n_1, \ldots, n_N) \in I \exists (m_1, \ldots, m_N) \in I \quad d(X_{n_1, \ldots, n_N}, Y_{m_1, \ldots, m_N}) < M \]
and vice versa. It suffices to show that for fixed $\bar{n}$ and $\bar{m}$ that the relation on $\mathcal{B}_b$ given by
\[ U(x, y) \iff d(X_{\bar{n}}, Y_{\bar{m}}) < M \]
is open, where $X^x$ and $Y^y$ denote the spaces coded by $x$ and $y$. Fix $x, y$ with $U(x, y)$. Let $T : X^x \rightarrow Y^y$ be a linear isomorphism with $\|T\|\|T^{-1}\| < M$. Let
\( x_1, \ldots, x_N \) denote \( s^{n_1}(X), \ldots, s^{n_N}(X) \), and let \( y_1 = T(x_1), \ldots, y_N = T(x_N) \). For \( x' \in \mathcal{B}_0 \), let \( x'_1, \ldots, x'_N \) denote \( s^{n_1}(X'), \ldots, s^{n_N}(X') \), where \( X' \) is the space coded by \( x' \). Let \( T': X'_N \to Y \) be the linear map defined by \( T'(x'_1) = y_1, \ldots, T'(x'_N) = y_N \).

It suffices by symmetry to show that for any \( \epsilon > 0 \) there is an open \( V \subseteq \mathbb{R}^\omega \) containing \( x \) such that for all \( x' \in \mathcal{B}_0 \cap V \) we have \( \|T(x') - T(x)\| < \epsilon \). Let \( \rho > 0 \) be such that \( \rho < \min \{\|x_i\|\} \leq \max \{\|x_i\|\} < \frac{1}{\rho} \). Let \( 0 < \eta < \inf \{\|\alpha x_1 + \cdots + \alpha_N x_N\|_X : \alpha \in S_N\} \), where \( S_N = \{\alpha : \sum \alpha_i^2 = 1\} \). By definition of the product topology on \( \mathbb{R}^\omega \), there is a clearly an open set \( V_1 \) about \( x \) such that for \( x' \in V_1 \) we have \( \rho < \min \{\|x_i'\|\} \leq \max \{\|x_i'\|\} < \frac{1}{\rho} \). It thus suffices to show that for all \( \epsilon > 0 \) there is a neighborhood \( V \subseteq V_1 \) of \( x \) such that for all \( x' \in V \) we have that \( \|alpha_1 x_1 + \cdots + alpha_N x_N\|_X - \|alpha'_1 x'_1 + \cdots + alpha_N x'_N\|_X < \epsilon \left(\frac{\sqrt{n_1 \rho}}{\mathcal{N}_{n_1 \rho}}\right) \), for all \( \alpha \in S_N \).

For then, letting \( v = alpha_1 x_1 + \cdots + alpha_N x_N \) and \( v' = alpha'_1 x'_1 + \cdots + alpha_N x'_N \) we have (noting \( T(v) = T(v') \)) and assuming \( \epsilon < \frac{\sqrt{\rho}}{2} \):

\[
\frac{\|T(v)\| - \|T(v')\|}{\|v\| - \|v'\|} = \frac{\|T(v)\|}{\|v\|} \frac{\|T(v')\|}{\|v'\|} (\|v\| - \|v'\|) \leq \frac{\|T\|N}{2} \frac{2}{\eta^2} (\|v\| - \|v'\|) \leq \epsilon.
\]

Let \( \mathfrak{N} \subseteq S_N \cap \mathcal{Q}^N \) be such that for all \( \alpha \in S_N \), there is a \( \bar{q} \in \mathfrak{N} \) such that \( |\alpha_i - q_i| < \frac{\sqrt{\rho}}{3N} \) for all \( i \), where \( \delta = \frac{\sqrt{\rho}}{2N^2} \) (the rational points on \( S_N \) are dense. Given \( \alpha \in S_N \), let \( \bar{q} \in \mathfrak{N} \) be such that \( |\alpha_i - q_i| < \frac{\sqrt{\rho}}{3N} \) for all \( 1 \leq i \leq N \). We have that \( \|\sum \alpha_i x_i\| - \|\sum q_i x_i\| < \left(\frac{\sqrt{\rho}}{3N}\right) \max \{\|x_i\|\} \leq \frac{\epsilon}{2} \), with a similar estimate for \( x' \). Since the \( q_i \) and \( s^{n_i} \) are rational, if \( V \) is a small enough neighborhood of \( x \) and \( x' \in V \) we will have \( \|\sum q_i x_i\| - \|\sum q_i x_i\| < \frac{\epsilon}{4} \). Thus, \( \|\sum \alpha_i x_i\| - \|\sum \alpha_i x_i\| < \epsilon \).

Next consider \( \equiv_x \) as an equivalence relation on \( \mathcal{B} \). Fix a countable dense \( D \subseteq C[0,1] \). For this part of the proof let \( d \) be the metric on \( C[0,1] \) given by the norm. Let \( Q \) be the set of all quadruples \((s,t,n,q)\) such that

1. \( s, t \in D^\omega \) (the set of all finite sequences of elements of \( D \)), \( n \in \omega \), \( q \in \mathfrak{N} \);
2. there is some \( k \in \omega \) such that \( s, t \in D^k \), that is, \( |s| = |t| = k \);
3. if \( s = (s_1, \ldots, s_k) \), \( t = (t_1, \ldots, t_k) \), then for any \( x_1, \ldots, x_k, y_1, \ldots, y_k \in C[0,1] \) such that \( d(s_i, x_i), d(t_i, y_i) < 2^{-n} \) for \( 1 \leq i \leq k \), letting \( T \) be the linear map from \( \text{span}(x_1, \ldots, x_k) \) to \( \text{span}(y_1, \ldots, y_k) \) with \( T(x_i) = y_i \) for \( 1 \leq i \leq k \), we have that \( \|T\|/\|T^{-1}\| < q \).

Note that if \( x_1, \ldots, x_k, y_1, \ldots, y_k \in C[0,1] \) and the linear map \( T : \text{span}(x_1, \ldots, x_k) \to \text{span}(y_1, \ldots, y_k) \) sending \( x_i \) to \( y_i \) satisfies \( \|T\|/\|T^{-1}\| < C \) for some \( C > 0 \), then there is a quadruple \((s,t,n,q)\) in \( Q \) such that \( q < C \) and for all \( 1 \leq i \leq k \), \( d(s_i, x_i), d(t_i, y_i) < 2^{-n} \).

We claim that for any \( X, Y \in F(C[0,1]) \), \( X \equiv_L Y \) if

\[
\exists C \in Q \forall k \in \omega \forall x_1, \ldots, z_k \in D \forall x \in \mathfrak{N} \{ \forall 1 \leq i \leq k \ d(z_i, X) < \epsilon \implies \exists (s, t, n, q) \in Q \ q < C \land 2^{-n} < \epsilon \}
\]

To prove the claim, first suppose \( X \equiv_L Y \), and let \( C > 0 \) be a witness. Suppose \( z_1, \ldots, z_k \in D \) and \( \epsilon \) are given, and for \( 1 \leq i \leq k \), let \( x_i \in X \) be such that \( d(x_i, z_i) < \epsilon \). By the local equivalence between \( X \) and \( Y \) there are \( y_1, \ldots, y_k \in Y \) such that the linear map \( T : \text{span}(x_1, \ldots, x_k) \to \text{span}(y_1, \ldots, y_k) \) sending \( x_i \) to \( y_i \) satisfies that \( \|T\|/\|T^{-1}\| < C \). We have a quadruple \((s,t,n,q)\) in \( Q \) such that \( q < C \), \( d(s_i, x_i), d(t_i, y_i) < 2^{-n} \) for all \( 1 \leq i \leq k \). Moreover, we may choose \( n \) to be large enough such that \( 2^{-n} < \epsilon \) and \( d(s_i, z_i) < \epsilon \). This verifies the displayed property.
For the other implication, let $C$ be as in the displayed property. Let $x_1, \ldots, x_k \in X$ be given. Let $\epsilon > 0$ be sufficiently small compared with $k$. Let $z_1, \ldots, z_k \in D$ with $d(z_i, x_i) < \epsilon$ for all $1 \leq i \leq k$. Then the displayed property gives a quadruple $(s, t, n, q) \in Q$. Thus $q < C$, $2^{-n} < \epsilon$, and for $1 \leq i \leq k$, $d(s_i, z_i) < \epsilon$, $d(t_i, Y) < 2^{-n}$, $d(s_i, X) < 2^{-n}$. In particular there are $y_1, \ldots, y_k \in Y$ such that $d(t_i, y_i) < 2^{-n}$ for all $1 \leq i \leq k$, and by the definition of $Q$, the linear map $T : \text{span}(s_1, \ldots, s_k) \to \text{span}(y_1, \ldots, y_k)$ sending $s_i$ to $y_i$ satisfies that $\|T\|\|T^{-1}\| < q$. Since $d(s_i, x_i) < 2\epsilon$, and $\epsilon$ is sufficiently small, the map $S : \text{span}(x_1, \ldots, x_k) \to \text{span}(y_1, \ldots, y_k)$ sending $x_i$ to $y_i$ satisfies that $\|S\|\|S^{-1}\| < C + 1$.

The displayed property is apparently $^0 \Sigma_3$ in the Wijsman topology on $F(C[0, 1])$. It follows that $\equiv_L$ is $^0 \Sigma_3$ on $\mathcal{B}$.

We can also consider the local equivalence on the space $2^\omega$ of codes for Banach spaces with basis via the Pełczyński universal basis. Recall that there is a continuous function $\varphi : [\omega]^\omega \to \mathcal{B}_b$ such that for any $x \in 2^\omega$, $X_x$ is linearly isometric to $Y_{\varphi(x)}$. Via this map the local equivalence on $[\omega]^\omega$ is continuously reduced to $\equiv_L$ on $\mathcal{B}_b$. It follows that the local equivalence on $2^\omega$ is also $^0 \Sigma_3$.

Now it follows from Lemma [2.2] that $E_0^\omega$ is not Borel reducible to $\equiv_L$ on either $\mathcal{B}$ or $\mathcal{B}_b$, and by Lemma [2.1] $=^*$ is not Borel reducible also to either of them. In Section 5 we will prove that in fact $\equiv_L$ (on either $\mathcal{B}$ or $\mathcal{B}_b$) is Borel bireducible to $\ell_\infty$, thus completely determining its complexity in the Borel reducibility hierarchy.

4. The uniform homeomorphism on a class of Banach spaces

In this section we construct a class of Banach spaces and completely characterize its uniform homeomorphism relation. In the construction and proofs we use a few well-known results in Banach space theory. Our standard reference for undefined terms and unexplained results is [1].

Recall that two given bases $(x_i)$ and $(y_i)$ of Banach spaces are said to be $C$-equivalent for $C > 0$ if there exist positive constants $A, B$ with $AB \leq C$ such that for all scalar sequences $(a_i)$,

$$\frac{1}{A} \left\| \sum_i a_i x_i \right\| \leq \left\| \sum_i a_i y_i \right\| \leq B \left\| \sum_i a_i x_i \right\|.$$  

We will make use of the following important notion in the study of the local structures of Banach spaces. Let $X$ be a Banach space and let $1 \leq p \leq 2$. The type $p$ constant $T_{p,n}(X)$ of $X$ over $n$ vectors is the smallest positive number such that for arbitrary $n$ vectors $x_1, \ldots, x_n \in X$,

$$\left( \text{Ave}_{\epsilon_i = \pm 1} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|^2 \right)^{1/2} \leq T_{p,n}(X) \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p}.$$  

$X$ is said to have type $p$ if $T_p(X) = \sup_n T_{p,n}(X) < \infty$. For $\ell_q^\omega$ spaces, type $p$ constants can be easily computed and they satisfy the following estimates:

$$T_p(\ell_q^\omega) \leq \left( n \max(0, 1/q - 1/p) \right) \leq T_p(\ell_q^\omega) \leq c q^{1/2} n \max(0, 1/q - 1/p) \quad \text{for } 1 \leq q < \infty,$$

where $c$ is a universal constant. Moreover, the proof of the lower estimate in (4.2) also shows that for $n \leq k$,

$$T_{p,n}(\ell_q^k) \leq c q^{1/2} n \max(0, 1/q - 1/p).$$
Note that type is a linear notion; in particular, if $T : Y \to X$ is a linear embedding, then $T_{p,n}(Y) \leq \|T\|\|T^{-1}\|T_{p,n}(X)$.

For a sequence $\bar{p} = (p_i) \in (1, 2)^\omega$, by $S_{\bar{p}}$ we will denote the $\ell_2$-direct sum of finite-dimensional $\ell_{p_i}^n$ spaces for a fixed sequence of increasing dimensions $(n_i)$. That is,

$$S_{\bar{p}} := \left( \bigoplus_{i=1}^{\infty} \ell_{p_i}^n \right)_2.$$

The next theorem singles out a class of such spaces on which isomorphism, local equivalence and uniform homeomorphism relations all coincide.

**Theorem 4.1.** Let $I_i = [l_i, r_i]$ be a sequence of successive intervals in $(1, 2)$. Then there exists $(n_i) \in \omega^\omega$ such that for $p_i = (p_i)$ and $q_i = (q_i)$ with each $p_i, q_i \in I_i$, we have that $S_{p_i}$ is uniformly homeomorphic to $S_{q_i}$ if and only if there exists a constant $C \geq 1$ such that

$$|\frac{1}{n_i} - \frac{1}{r_i}| \leq C$$

for all $i \in \omega$.

**Proof.** For any sequence of dimensions $(n_i)$, if $n_i^{1/r_i} \leq C$ for some $C < \infty$, then $d(\ell_{p_i}^n, \ell_{q_i}^n) \leq C$ for all $i \in \omega$. In fact, in this case the unit vector bases are $C$-equivalent. From this it easily follows that $S_{p_i}$ and $S_q$ are C-isomorphic, and in particular, they are uniformly homeomorphic.

Let $I_i = [l_i, r_i]$ be a sequence of given intervals in $(1, 2)$. Pick a sequence $(n_i)$ of natural numbers such that

$$(4.4) \quad \sup n_i^{1/r_i} = \infty \text{ and } n_i^{1/r_i+1} \geq n_i^{1/l_i}, \quad i = 1, 2, 3, \ldots.$$ 

Suppose without loss of generality that $p_i < q_i \in [l_i, r_i]$ with $\sup_i n_i^{1/p_i} = \infty$. By Ribe’s Theorem [23] it is sufficient to show that the spaces in the sequence $(\ell_{p_i}^n)$ do not linearly embed in $S_{\bar{q}}$ with a uniform constant.

Fix $i_0 \in \omega$. Put $n = n_{i_0}$, $p = p_{i_0}$ and let $T : \ell_{p}^n \to S_{\bar{q}}$ be a linear embedding. Since

$$T_{2,n}(\ell_{p}^n) \leq \|T\|\|T^{-1}\|T_{2,n}(S_{\bar{q}}),$$

and $T_{2,n}(\ell_{p}^n) \geq n^{1/p-1/2}$, we need an upper estimate for $T_{2,n}(S_{\bar{q}})$.

Let $I_{i_0}$ be such that $p, q = q_{i_0} \in I_{i_0}$. That is, $q_{i_0-1} < p < q_{i_0} = q$. Let $x^1, \ldots, x^n \in S_{\bar{q}}$. Then, writing each $x^j$ as $\sum_{i=1}^{\infty} x_{i}^j$, where $x_{i}^j \in \ell_{p_i}^n$, we have

$$\begin{align*}
\text{Ave}_{\epsilon_j = \pm 1} \left( \sum_{j=1}^{n} \varepsilon_j x^j \right)^2 & = \text{Ave}_{\epsilon_j = \pm 1} \left( \sum_{j=1}^{n} \varepsilon_j \left( \sum_{i=1}^{\infty} x_{i}^j \right)^2 \right) = \text{Ave}_{\epsilon_j = \pm 1} \left( \sum_{j=1}^{n} \sum_{i=1}^{\infty} \varepsilon_j x_{i}^j \right)^2 \\
& \leq \sum_{i < i_0} \text{Ave}_{\epsilon_j = \pm 1} \left( \sum_{j=1}^{n} \varepsilon_j x_{i}^j \right)^2 + \sum_{i \geq i_0} \text{Ave}_{\epsilon_j = \pm 1} \left( \sum_{j=1}^{n} \varepsilon_j x_{i}^j \right)^2 \\
& \leq \sum_{i < i_0} T_{2, n}(\ell_{p_i}^n) \sum_{j=1}^{n} \left\| x_{j}^i \right\|^2 + \sum_{i \geq i_0} T_{2, n}(\ell_{p_i}^n) \sum_{j=1}^{n} \left\| x_{j}^i \right\|^2.
\end{align*}$$

Using the estimates (4.2) for $i < i_0$ and (4.3) for $i \geq i_0$ sums, the last inequality is less than or equal to

$$\begin{align*}
\sum_{i < i_0} c_i q_i n_i^{2/q_i-1} \sum_{j=1}^{n} \left\| x_{j}^i \right\|^2 + \sum_{i \geq i_0} c_i q_i n_i^{2/q_i-1} \sum_{j=1}^{n} \left\| x_{j}^i \right\|^2,
\end{align*}$$

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which is, by (4.4), less than

\[ 2e^2n^{2/q-1} \sum_{j=1}^{n} \sum_{i=1}^{\infty} \|x_i^{j}\|_{q_i}^2 = 2e^2n^{2/q-1} \sum_{j=1}^{n} \|x_j\|_{S_{q_j}}^2. \]

Thus, we have shown that \( T_{2,n}(S_{q}) \leq 2\sqrt{2cn^{1/q-1/2}}. \) It follows that

\[ \|T\|\|T^{-1}\| \geq \frac{n^{1/p-1/2}}{2\sqrt{2cn^{1/q-1/2}}} = \frac{n^{1/p-1/q}}{\sqrt{2c}}. \]

\[ \Box \]

5. The Complexity of the Uniform Homeomorphism and the Local Equivalence

In this section we prove the main theorems of our paper. In doing this we also define some natural equivalence relations and characterize their complexity. Some of the equivalence relations to be defined in this section have already been considered in [Ro]. For instance, Lemma 5.1, Definition 5.7, and the beginning of Theorem 5.8 can be found in [Ro]. For the sake of completeness we give all definitions and proofs in a self-contained manner.

For notational clarity we use the following convention in this section. Let \( X \) be a set. We use \( \vec{x} \) to denote an element of \( X^\omega \), the set of all infinite sequences of elements of \( X \). The coordinates of \( \vec{x} \) will be denoted by \( x(n) \) for \( n \in \omega \). Thus \( \vec{x} = (x(n)) = (x(0), x(1), \ldots) \). This is slightly different from previous sections, but it provides the most convenience for the arguments of this section.

Recall that the equivalence relation \( E_{\ell_\infty} \) (simply \( \ell_\infty \) when there is no danger of confusion) is the equivalence relation on \( \mathbb{R}^\omega \) defined by

\[ \vec{x} E_{\ell_\infty} \vec{y} \iff \exists C \forall n |x(n) - y(n)| < C \]

for \( \vec{x}, \vec{y} \in \mathbb{R}^\omega \). We consider the following variation. Let \( B \) be the set of all infinite increasing sequences of positive real numbers without an upper bound. For any \( \vec{b} = (b(0), b(1), \ldots) \in B \), we denote by \( E^\vec{b}_{\ell_\infty} \) the equivalence relation \( E_{\ell_\infty} \) restricted to the set \( \prod_{n \in \omega} [0, b(n)] \).

Lemma 5.1. For any \( \vec{b} \in B, E_{\ell_\infty} \leq_E E^\vec{b}_{\ell_\infty} \).

Proof. For each \( n \in \omega \), let \( \rho_n \) be a linear map from \([−b(n), b(n)] \) onto \([0, b(n)] \). Define \( \pi: \mathbb{R}^\omega \rightarrow \prod_{n \in \omega}[0, b(n)] \) by

\[ \pi(\vec{x})(\langle i, j \rangle) = \begin{cases} \rho_i(x(i)), & \text{if } x(i) \in [−b(j), b(j)], \\ 0, & \text{if } x(i) < −b(j), \\ b(j), & \text{if } x(i) > b(j), \end{cases} \]

for all \( i, j \in \omega \). Clearly \( \pi(\vec{x}) \in \prod_{n \in \omega}[0, b(n)] \). Note that if \( \pi(\vec{x}_1) = \vec{y}_1, \pi(\vec{x}_2) = \vec{y}_2 \), then

\[ |y_1(n) - y_2(n)| \leq |x_1(i) - x_2(i)| \]

for all \( n = \langle i, j \rangle \in \omega \). Thus \( \vec{x}_1 E_{\ell_\infty} \vec{x}_2 \) implies \( \pi(\vec{x}_1) E^\vec{b}_{\ell_\infty} \pi(\vec{x}_2) \). Suppose \( \vec{x}_1 \) is not \( E_{\ell_\infty} \)-equivalent to \( \vec{x}_2 \). Then for any \( C > 0 \), there is an \( i \in \omega \) such that \( |x_1(i) - x_2(i)| > C \). Let \( j \) be a large enough integer such that \( b(j) > \max\{|x_1(i), |x_2(i)|\} \). Let \( n = \langle i, j \rangle \). Then \( y_1(n) = \rho_j(x_1(i)) \) and \( y_2(n) = \rho_j(x_2(i)) \), and so \( |y_1(n) - y_2(n)| | > C/2 \). So, \( \pi(\vec{x}_1) \) is not \( E^\vec{b}_{\ell_\infty} \)-equivalent to \( \pi(\vec{x}_1) \). This shows that \( \pi \) is a reduction from \( E_{\ell_\infty} \) to \( E^\vec{b}_{\ell_\infty} \). It is clear that \( \pi \) is a Borel function. \( \Box \)
Theorem 5.2. The equivalence relation $\ell_\infty$ is Borel reducible to the uniform homeomorphism relation on either $\mathcal{B}_b$ or $\mathcal{B}$.

Proof. By Lemma 5.1 it suffices to define a Borel reduction from $E^5_{\ell_\infty}$ (for some $\bar{b} \in B$) to the uniform homeomorphism relation for Banach spaces with a basis. To construct the Banach spaces we use the proof of Theorem 4.1. For this fix $\bar{b} \in B \cap \omega^\omega$ with $b(0) > 0$. For all $i \in \omega$ put

$$\delta_i = \frac{1}{b(i) 2^i} \quad \text{and} \quad n_i = 2^{\delta_i} = 2^{b(i)2^i}.$$ 

Let $I_i$ be a sequence of successive intervals in $(1, 2)$ with $|I_i| = 2^{-i-1}$. Assume $I_i = [l_i, r_i]$. Since $n_i^2 \leq n_{i+1}$, equation (4.4) is satisfied. The sequences $(n_i)$ and $(I_i)$ will be used as in the proof Theorem 4.1. Let $\sigma_i$ be the affine bijection between $[0, b(i)]$ and $I_i$. For $\bar{x} \in \prod_{n \in \omega}[0, b(n)]$, define $\rho(\bar{x}) \in \mathbb{R}^\omega$ by $\rho(\bar{x})(i) = \sigma_i(x(i))$. Finally, define $\pi(\bar{x}) = S_{\rho(\bar{x})}$. So, for all $\bar{x} \in \prod_{n \in \omega}[0, b(n)]$, $\pi(\bar{x})$ is a separable Banach space with a basis.

We show that $\pi$ is the desired reduction. It is straightforward to check that $\pi$ is Borel as a map from $\prod_{n \in \omega}[0, b(n)]$ to $\mathcal{B}_b$. Granting that $\pi$ is a reduction, then composed with the Borel map $\Phi: \mathcal{B}_b \to \mathcal{B}$, it would be a reduction to the uniform homeomorphism relation on $\mathcal{B}$.

To verify that $\pi$ is a reduction, consider $\bar{x}_1, \bar{x}_2 \in \prod_{n \in \omega}[0, b(n)]$. From Theorem 4.1 we have that $S_{\rho(\bar{x}_1)}$ and $S_{\rho(\bar{x}_2)}$ are uniformly homeomorphic iff

$$\exists C > 0 \ \forall i \in \omega \quad n_i \left| \frac{1}{\rho(\bar{x}_1)(i)} - \frac{1}{\rho(\bar{x}_2)(i)} \right| < C.$$ 

By taking the logarithm we get that this is equivalent to

$$\exists D > 0 \ \forall i \in \omega \quad \left| \frac{\log(n_i)}{\rho(\bar{x}_1)(i)} - \frac{\log(n_i)}{\rho(\bar{x}_2)(i)} \right| < D.$$ 

Using the definition of $\rho$ we get that the inequality is equivalent to

$$\frac{\log(n_i)}{\sigma_i(\bar{x}_1(i)) - \sigma_i(\bar{x}_2(i))} \frac{1}{\sigma_i(\bar{x}_1(i)) \cdot \sigma_i(\bar{x}_2(i))} < D.$$ 

Since $\sigma_i(\bar{x}_1(i)) \in (1, 2)$ and likewise for $\bar{x}_2$, the statement is thus equivalent to

$$\exists D > 0 \ \forall i \in \omega \quad \log(n_i) |\sigma_i(\bar{x}_1(i)) - \sigma_i(\bar{x}_2(i))| < D.$$ 

By the linearity of $\sigma_i$, we in fact have

$$|\sigma_i(\bar{x}_1(i)) - \sigma_i(\bar{x}_2(i))| = \frac{|x_1(i) - x_2(i)|}{b(i) \cdot 2^{i+1}}.$$ 

Finally, our choice of $(n_i)$ guarantees that

$$\frac{1}{2} \leq \frac{|x_1(i) - x_2(i)|}{b(i) \cdot 2^i} \leq 2.$$ 

Therefore, the statement is eventually equivalent to $\exists D > 0 \ \forall i \in \omega |\bar{x}_1(i) - \bar{x}_2(i)| < D$, that is, $\bar{x}_1 E^5_{\ell_\infty} \bar{x}_2$. □

Theorem 4.1 is now a direct corollary of the above proof. In particular we have the following corollary.
Theorem 5.3. The equivalence relation $\ell_\infty$ is Borel reducible to the local equivalence on either $\mathcal{B}_b$ or $\mathcal{B}$, that is, $\ell_\infty \leq_B \equiv_L$.

This gives half of Theorem 1.2. Next we prove Theorem 1.2 by showing the reverse reduction. We will use the following concept and lemma.

Definition 5.4. For $X = (X, d)$ a Polish metric space, let $F_X$ be the equivalence relation on $X^\omega$ defined by
\[ x F_X y \iff \exists C > 0 \left[ \forall i j d(x(i), y(j)) < C \land \forall i \exists j d(y(i), x(j)) < C \right]. \]

Lemma 5.5. For every Polish metric space $(X, d)$, $F_X \leq_B \ell_\infty$.

Proof. Fix a 1-net $R = \{r_0, r_1, \ldots \}$ in $X$. We define $\pi : X^\omega \to \mathbb{R}^\omega$ by
\[ \pi(x)(i) = d(r_i, \{x(0), x(1), \ldots \}). \]
It is easy to check that $\pi$ is a Borel function. We verify that it is a reduction from $F_X$ to $E_{\ell_\infty}$. Suppose $x F_X y$, and let $C > 0$ be a witness. For any $z \in X$, if $\delta = d(z, \{x(0), x(1), \ldots \})$, then $d(z, \{y(0), y(1), \ldots \}) \leq \delta + C$. So, $|d(z, \{x(0), x(1), \ldots \}) - d(z, \{y(0), y(1), \ldots \})| \leq C$. Thus, $\pi(x) \leq_B \pi(y)$.

Conversely, suppose $x$ is not $F_X$-equivalent to $y$. Let $C > 0$ be arbitrary. Then there is a $k$ such that $d(x(k), \{y(0), y(1), \ldots \}) > C$ or $d(y(k), \{x(0), x(1), \ldots \}) > C$. Without loss of generality, assume the former. Let $i$ be such that $d(x(k), r_i) < 1$. Then $\pi(x)(i) < 1$, but $\pi(y)(i) > C - 1$. So $\pi(x) \not\leq_B \pi(y)$. □

Theorem 5.6. The local equivalence on either $\mathcal{B}$ or $\mathcal{B}_b$ is Borel reducible to the equivalence relation $\ell_\infty$, that is, $\equiv_L \leq_B \ell_\infty$.

Proof. Let $F$ be the collection of finite-dimensional Banach spaces (presented with bases). The following distance function is a separable metric on $F$. If two spaces $(X, (x_1, \ldots, x_n))$ and $(Y, (y_1, \ldots, y_n))$ are both $n$-dimensional, let $\rho_n(X, Y) = \max\{\log(||T||), \log(||T^{-1}||)\}$, where $T : X \to Y$ is the linear isomorphism sending $x_i$ to $y_i$. By truncating we may assume each $\rho_n \leq n$, and we may then put together the $\rho_n$ to obtain a metric on $F$ (if $\dim(X) \neq \dim(Y)$, we set $\rho(X, Y) = \dim(X) + \dim(Y)$). Let $(F, \rho)$ be the Polish space obtained by completing $\rho$.

First consider $\equiv_L$ as an equivalence relation on $\mathcal{B}_b$. Given a separable Banach space $X$ with basis $(x_i)$, we define $\pi(X) \in \mathcal{F}^\omega$ as follows. Let $\chi_1, \chi_2, \ldots$ enumerate $(\mathbb{Q}^{<\omega})^{<\omega}$. Suppose $\chi_i = (q_{i1}, \ldots, q_{ik})$. Then let $\pi(X)(i)$ be the $k$-dimensional subspace of $X$ with basis $(e_1, \ldots, e_k)$, where $e_i = \sum q_{ij}x_j$ for all $1 \leq i \leq k$. It is routine to check that $\pi$ is a Borel function. Note that every finite-dimensional subspace of $X$ is approximated arbitrarily closely in the $\rho$ metric by a term of the sequence $\pi(X)$. It is then straightforward from the definition of the local equivalence that $X \equiv_L Y$ if $\pi(X) F_{\mathcal{F}} \pi(Y)$. Thus, $\pi$ is a reduction from $\equiv_L$ to $E_{\mathcal{F}}$. We are done by Lemma 5.5.

We modify the above argument to work for $\equiv_L$ as an equivalence relation on $\mathcal{B}$. Let $d$ be the metric on $C[0,1]$ given by the norm. Let $D \subseteq C[0,1]$ be countable dense. Fix an enumeration of $D^{<\omega} \times \omega$ as $\langle s_0, n_0 \rangle, \langle s_1, n_1 \rangle, \ldots$. Fix a Borel function $\sigma : F(C[0,1]) \to C[0,1]$ such that $\sigma(F) \in F$ for all nonempty $F \in F(C[0,1])$ (see Theorem 12.13, [K]). For $x \in C[0,1]$, $F \in F(C[0,1])$, and $n \in \omega$, let
\[
\sigma_n(x, F) = \left\{ \begin{array}{ll}
\sigma(F \cap \{u \in C[0,1] : d(x, u) < \frac{1}{n} \}) & \text{if } F \cap \{u \in C[0,1] : d(x, u) < \frac{1}{n} \} \neq \emptyset,
\sigma(F), & \text{otherwise.}
\end{array} \right.
\]
Given a separable Banach space $X \in \mathcal{B}$, let $\pi(X)(i) \in \tilde{F}$ code the finite-dimensional subspace of $X$ with basis $(\sigma_n(s_i(0), X), \ldots, \sigma_n(s_i(|s_i| - 1), X))$. Since $D$ is dense, every finite-dimensional subspace of $X$ is approximated arbitrarily closely by spaces of the form $\pi(X)(i)$. Thus, $\pi$ is a Borel reduction from $\equiv_L$ to $E_{\tilde{F}}$. □

Theorem 5.2 is immediate from Theorems 5.3 and 5.6. To summarize, we have shown that the local equivalence between separable Banach spaces has the same complexity as $\ell_\infty$, and the uniform homeomorphism relation is at least as complex as $\ell_\infty$. Thus we have obtained the sharpest result possible for the uniform classification by considering the local structures of Banach spaces alone.

The equivalence relation $F_X$ we used in the above proof is sort of a generalization of the $\ell_\infty$ equivalence relation on the space of countable subsets of a Polish metric space equipped with the Hausdorff metric. In the remainder of this section we consider a full generalization of $\ell_\infty$ to arbitrary Polish metric spaces and characterize its complexity.

**Definition 5.7.** Let $X = (X, d)$ be a Polish metric space. The equivalence relation $E_{\ell_\infty(X)}$, or simply $\ell_\infty(X)$, on $X^\omega$ is defined as

$$\bar{x}E_{\ell_\infty(X)}\bar{y} \iff \exists C > 0 \ \forall i \in \omega \ \ d(x(i), y(i)) < C.$$ 

We have the following dichotomy for the complexity of $\ell_\infty(X)$ in the Borel reducibility hierarchy for any Polish metric space $X$.

**Theorem 5.8.** Let $X = (X, d)$ be a Polish metric space with $d$ unbounded. Then $\ell_\infty(X)$ is Borel bireducible with either $\ell_\infty$ or $E_1$.

**Proof.** We first reduce $\ell_\infty(X)$ to $\ell_\infty$. Fix a countable 1-net $R = \{r_0, r_1, \ldots\}$ in $X$, that is, $R \subseteq X$ with $d(r_i, r_j) > 1$ for all $i \neq j$, and $\forall x \in X \ \exists i \in \omega \ \ d(x, r_i) \leq 1$. This can be done in any separable metric space. Define $\pi : X^\omega \to \mathbb{R}^\omega$ by

$$\pi(\bar{x})(i, j) = d(x(i), r_j)$$

for any $i, j \in \omega$. Then $\pi$ is continuous, in particular Borel. Let $\bar{x}, \bar{y} \in X^\omega$. If $\bar{x}E_{\ell_\infty(X)}\bar{y}$, then let $C > 0$ be such that $d(x(i), y(i)) < C$ for all $i \in \omega$. Then it follows that for any $j \in \omega$, $|d(x(i), r_j) − d(y(i), r_j)| < C$. So $\pi(\bar{x})E_{\ell_\infty}\pi(\bar{y})$. Conversely, if $\pi(\bar{x})E_{\ell_\infty}\pi(\bar{y})$, then for some $C > 0$ we have that $\forall i \in \omega \ \forall j \ |d(x(i), r_j) − d(y(i), r_j)| < C$. It we take $j$ so that $d(x(i), r_j) \leq 1$, then this implies that $d(x(i), y(i)) < C + 2$, and so $\bar{x}E_{\ell_\infty(X)}\bar{y}$.

Next we reduce $E_1$ to $\ell_\infty(X)$. Since $\mathbb{R}$ is Borel isomorphic to both $2^\omega$ and $\omega^\omega$, we may work with $E_1$ defined on either $(2^\omega)^\omega$ or $(\omega^\omega)^\omega$, whichever is more convenient.

Fix a sequence $(z_n) \in X^\omega$ with $\lim_n d(z_0, z_n) = \infty$. Define $\tau : (2^\omega)^\omega \to X^\omega$ by

$$\tau(\bar{x})(i, j) = \begin{cases} z_i, & \text{if } x_i(j) = 1, \\ z_0, & \text{otherwise,} \end{cases}$$

for all $i, j \in \omega$. Again $\tau$ is continuous, hence Borel. Given $\bar{x}, \bar{x}' \in (2^\omega)^\omega$ and $n \in \omega$, we have that $x_k = x'_k$ for all $k \geq n$ if $\tau(\bar{x}) = \tau(\bar{x}')$ and $\tau(\bar{x}) = \tau(\bar{x}')$ only disagree where they take values in $\{z_0, \ldots, z_{n−1}\}$. This implies that $\bar{x}E_1\bar{x}'$ if $\tau(\bar{x})E_{\ell_\infty(X)}\tau(\bar{x}')$.

We have shown in this section that $E_1 \leq_B \ell_\infty(X) \leq_B \ell_\infty$. If $Y$ is a 1-net in $X$, then clearly $\ell_\infty(X)$ is bireducible with $\ell_\infty(Y)$. So, without loss of generality we may assume that $X$ is countable. For every positive $C$, let $\sim_C$ be the equivalence relation on $X$ which is the transitive closure of the relation $\{(x, y) : d(x, y) < C\}$. We call the $\sim_C$-equivalence classes the $C$-components of $X$. We now consider two cases.
Case I. For all $C$ there is a bound $K_C$ on the diameter of the $C$-components.

In this case we reduce $\ell_\infty(X)$ to $E_1$. For each positive integer $n$, let $A^n_0, A^n_1, \ldots$ enumerate (with repetition) the $n$-components of $X$. Given $\vec{x} = (x_0, x_1, \ldots) \in X^\omega$, define $\vec{y} = \pi(\vec{x}) \in (\omega^n)^\omega$ by $y_n(m) = j$ iff $x_m \in A^n_j$. Suppose first that $\vec{x} E_{\ell_\infty(X)} \vec{x}$, say $\forall n \ d(x_n, x'_n) \leq N$. Then, for all $n \geq N$, we have that for all $m$, $x_m$ and $x'_m$ lie in the same $n$-component, since any two points in two distinct $n$-components have $d$ distance greater than $n$. This shows that $y_n = y'_n$ for all $n \geq N$, and so $\vec{y} E_1 \vec{y}'$. Conversely, suppose $\forall n \geq N$, $y_n = y'_n$. So, for all $n \geq N$ and all $m$, $x_m$ and $x'_m$ lie in the same $n$-component. In particular, $x_m$ and $x'_m$ lie in the same $N$-component for all $m$, and so $d(x_m, x'_m) \leq K_m$ for all $m$, that is, $\vec{x} E_{\ell_\infty(X)} \vec{x}'$.

Case II. For some $C$ and every $K$, there is a $C$-component of diameter greater than $K$.

In this case we reduce $\ell_\infty$ to $\ell_\infty(X)$. Fix $C$ as in the case hypothesis. By a $C$-path we mean a finite sequence of points $y_0, y_1, \ldots, y_n$ from $X$ such that $d(y_i, y_{i+1}) < C$ for all $i$. Note that all the points of a $C$-path lie in the same $C$-component of $X$. Let $p_0, p_1, \ldots$ enumerate all of the $C$-paths in $X$. If $p = (y_0, \ldots, y_n)$ is a $C$-path and $\ell \in \omega$, let $p(\ell) = y_i$ if $i \leq n$ and otherwise let $p(\ell) = y_n$. Clearly $d(p(\ell), p(j)) \leq C|\ell - j|$ for any $C$-path $p$ and any $\ell, j \in \omega$. Given $x \in (\omega^n)^\omega$, define $\vec{y} = \pi(x) \in X^\omega$ by $y_{1,i} = p_i(x(j))$. If $\forall m \ |x(m) - x'(m)| \leq N$, then $\forall m \ d(y_m, y'_m) \leq CN$ from the above observation. Suppose then that $x$ is not $E_{\ell_\infty(X)}$-equivalent to $x'$. Given $k$, let $A \subseteq X$ be a $C$-component with diameter greater than $k$. Let $z, w \in A$ with $d(z, w) > k$. Let $p$ be a $C$-path from $z$ to $w$. Say $p = (z = z_0, z_1, \ldots, w = z_n)$. Let $i$ be such that $|x(i) - x'(i)| > n$. From $p$ we can easily obtain a $C$-path $q$ such that $q(x(i)) = z_0$ and $q(x'(i)) = z_n$ (have the path $q$ start at $z_0$, remain at $z_0$ for an appropriate number of steps, then follow $p$, and then remain at $z_n$). Say $q = q_j$. Then $y_{1, j} = q(x(i)) = z_0$ and $y'_{1, j} = q(x'(i)) = z_n$. Thus $d(y_{1, j}, y'_{1, j}) \geq k$. Since this is true for all $k$, we have that $\vec{y}$ is not $E_{\ell_\infty(X)}$-equivalent to $\vec{y}'$.

6. Some special classes of separable Banach spaces

In this section we generalize the construction in Section §I to obtain some classes of separable Banach spaces. For each of these classes it turns out that the isomorphism, the uniform homeomorphism, and the local equivalence relations on it coincide. We also obtain some characterizations for the possible complexity of these equivalence relations.

We will use the following equivalence relation on $2^\omega$ and a characterization of its possible complexity.

**Definition 6.1.** For any sequence $\vec{t} = (t_i) \in 2^\omega$ with $t_i \geq 0$ for all $i$, let $E_{\vec{t}}$ be the equivalence relation on $2^\omega$ defined by

$$x E_{\vec{t}} y \iff \sup_i |x(i) - y(i)| < \infty.$$ 

**Theorem 6.2.** For any $\vec{t} \in 2^\omega$ with $t_i \geq 0$ for all $i$, $E_{\vec{t}}$ is either smooth, Borel bireducible with $E_0$, or Borel bireducible with $E_1$.

**Proof.** If $(t_i)$ is bounded, then $E_{\vec{t}}$ is trivial, and in particular smooth. So we assume $\vec{t}$ is unbounded. We inductively define a finite or infinite sequence $n_0 < n_1 < \ldots$ of natural numbers as follows. Let $n_0$ be the least $n \in \omega$, if one exists, such that...
\{(i: t_i \leq n) \} \) is infinite. Suppose \( n_k \) is defined; then let \( n_{k+1} \) be the least \( n > n_k \), if one exists, such that \( \{i: n_k < t_i \leq n\} \) is infinite. If \( n_k \) is defined, we also let \( A_k = \{i: n_{k-1} < t_i \leq n_k\} \).

First assume that \( n_k \) is defined for all \( k \in \omega \). Thus, \( A_k \) is defined for all \( k \) and the \( A_k \) form a partition of \( \omega \). Note that each \( A_k \) is infinite by definition. Let \( e'_k, i \in \omega \), enumerate \( A_k \). Define \( f: 2^\omega \to (2^\omega)^\omega \) by \( f(x)_k(i) = x(e'_k). \) Clearly \( x_1 E_{\ell^2} x_2 \) iff the sequences of reals coded by \( \{e'_k\} \) is, the proof of Theorem 6.5 below.

Easily \( h \) is a Borel reduction of \( E_{\ell^2} \) to \( E_0 \). Thus, \( f \) is a Borel reduction of \( E_{\ell^2} \) to \( E_1 \). In fact, \( f \) is a bijection between \( 2^\omega \) and \( (2^\omega)^\omega \), so its inverse gives a reduction from \( E_1 \) to \( E_{\ell^2} \).

Finally, suppose that \( n_0 < \cdots < n_\ell \) are defined, while \( n_{\ell+1} \) is not. Since \( (t_i) \) is unbounded, we must have that \( \omega - \bigcup_{k \leq \ell} A_k \) is infinite. Let \( e'_{\ell+1}, i \in \omega \), enumerate \( A_k \) for \( k \leq \ell \), and let \( e'_{\ell+1}, i \in \omega \), enumerate \( \omega - \bigcup_{k \leq \ell} A_k \). Define \( g: 2^\omega \to 2^\omega \) by \( g(x)(i) = x(e'_{\ell+1}) \). Clearly \( f \) is a Borel reduction of \( E_{\ell^2} \) to \( E_0 \). For the other direction, define \( h: 2^\omega \to 2^\omega \) by

\[
h(y)(j) = \begin{cases} y(i), & \text{if } j = e'_{\ell+1}, \\ 0, & \text{otherwise}. \end{cases}
\]

Easily \( h \) is a reduction of \( E_0 \) to \( E_{\ell^2} \). \( \square \)

The above proof can be simplified in view of known facts about \( E_1 \) and \( E_0 \) (see Section 2). In fact, if \( E \preceq_B E_1 \), then \( E \) is either smooth or Borel bireducible with either \( E_0 \) or \( E_1 \) by the dichotomy theorems of \([\text{HaK}] \) and \([\text{KL}] \). However, we gave the full proof here since it is self-contained and gives some information about the combinatorial structure of the equivalence relation \( E_{\ell^2} \). This will happen again for the proof of Theorem 6.5 below.

As in Section 4 we consider sequences \( \vec{p} = (p_i), \vec{q} = (q_i) \in \mathbb{R}^\omega \), and \( \vec{n} = (n_i) \in \omega^\omega \) such that

\[
1 < p_i < q_i < p_{i+1} < 2, \ n_i > 0 \quad \text{and} \quad \frac{n_{i-1}^{q_i}}{n_{i+1}^{p_i}} > \frac{n_i^{q_i}}{n_i^{p_i}}. \tag{6.1}
\]

Let \( B_{\vec{p},\vec{q},\vec{n}} \) be the collection of Banach spaces of the form

\[
X = \left( \bigoplus_{i=0}^{\infty} \ell_{p_i}^{(p_i)} \right)_2,
\]

where \( r_i \in \{p_i, q_i\} \). \( B_{\vec{p},\vec{q},\vec{n}} \) can be viewed as a closed subspace of \( \mathfrak{B}_b \). To see this, first code the elements of \( B_{\vec{p},\vec{q},\vec{n}} \) by elements of \( 2^\omega \) in the natural manner (i.e., \( x(i) \) determines whether to use \( \ell_p \) or \( \ell_q \)). By using a fixed bijection between \( \omega \times \omega \) and \( \omega \), we fix an order of enumeration of the basis elements for all the spaces in \( B_{\vec{p},\vec{q},\vec{n}} \).

This induces a map \( f \) from \( 2^\omega \) to \( \mathfrak{B}_b \subseteq \mathbb{R}^\omega \), which is easily seen to be continuous. Then \( f(2^\omega) \) is a closed subset of \( \mathfrak{B}_b \) which represents the set of spaces in \( B_{\vec{p},\vec{q},\vec{n}} \).

Clearly each \( B_{\vec{p},\vec{q},\vec{n}} \) contains continuum many elements.

**Theorem 6.3.** For any \( \vec{p}, \vec{q}, \vec{n} \) satisfying (6.1) above, the uniform homeomorphism relation on \( B_{\vec{p},\vec{q},\vec{n}} \) is either smooth, Borel bireducible to \( E_0 \), or Borel bireducible to \( E_1 \).
Proof. Consider the sequence of numbers \( t_i = \frac{1}{\sqrt{n}} + \frac{1}{n} \). First suppose that the sequence \((t_i)\) is bounded. In this case, all of the spaces \( \mathcal{B}_{\mathcal{F},\mathcal{A}} \) are isomorphic, so the uniform homeomorphism relation on \( \mathcal{B}_{\mathcal{F},\mathcal{A}} \) is trivial.

Suppose next that \((t_i)\) is unbounded. For each \( X \in \mathcal{B}_{\mathcal{F},\mathcal{A}} \), let \( z(X) = 2^\omega \) be the real \( z \) such that \( z(i) = 0 \) if \( X \) involves \( \ell_n^\mathcal{F} \), and \( z(i) = 1 \) if \( X \) involves \( \ell_n^\mathcal{A} \).

From the proof of Theorem 6.3, we have that for \( X, Y \in \mathcal{B}_{\mathcal{F},\mathcal{A}} \), \( X \) is uniformly homeomorphic to \( Y \) iff \( \sup(t_i | z(X)(i) - z(Y)(i)) < \infty \), that is, \( z(X) E_I z(Y) \). Therefore we are done by Theorem 6.2. \( \square \)

We now extend Theorem 6.3 to some even larger classes of separable Banach spaces. Again we define and study some new equivalence relations.

Definition 6.4. For any sequence \( \mathcal{B} = (B_i) \), where each \( B_i \) is a finite subset of \( \mathbb{R} \), let \( E_\mathcal{B} \) denote the equivalence relation \( \ell_\infty \) restricted on \( \prod_{i \in \omega} B_i \).

For \( \mathcal{B} = (B_i) \) as in the above definition, let \( b_i = \sup\{|a| : a \in B_i\} \). Then \( E_\mathcal{B} \) is also \( E_\infty \) restricted to \( \prod_{i \in \omega} B_i \). Thus \( E_\mathcal{B} \leq B E_\infty^\mathcal{F} \leq B \ell_\infty^\mathcal{F} \leq B \ell_\infty \).

Theorem 6.5. Let \( \mathcal{B} = (B_i) \), where each \( B_i \) is a finite subset of \( \mathbb{R} \). Then \( E_\mathcal{B} \) is either smooth, Borel bireducible with \( E_0 \), Borel bireducible with \( E_1 \), or Borel bireducible with \( \ell_\infty \).

Proof. By translating each \( B_i \) we may assume that each \( B_i \) consists of nonnegative real numbers and contains 0 as its least element. Let \( b_i = \max B_i \). If \( (b_i) \) is bounded, then \( E \) is a trivial equivalence relation, and so is smooth. So, we assume \((b_i)\) is unbounded. Also, we may assume that \( B_i \subseteq \omega \), for we may replace \( B_i \) by \( \{|a| : a \in B_i\} \).

For \( i, n \in \omega \), let \( F_n^i \) denote the finite equivalence relation on \( B_i \) given by the transitive closure of the relation

\[
x R_n y \iff |x - y| \leq n.
\]

For each \( i, n, \) let \( a_n^i(0), \ldots, a_n^i(k) \) enumerate the \( F_n^n \) classes of \( B_i \) in increasing order (i.e., \( \max(a_n^i(l)) < \min(a_n^i(l+1)) \)). Here \( k = k(i, n) \) depends on \( i \) and \( n \).

First consider the case where for some \( n \), there is no bound on the size of the \( F_n^n \) equivalence classes. That is, \( \forall b \exists i \exists |a_n^i(l)| > b \). Fix such an \( n \). Let \( i_0 < i_1 < \ldots \) be a subsequence and \( l_0, l_1, \ldots \) a sequence such that \( |a_n^i(l)| > m \). We know that \( E_{\ell_\infty} \leq E_{C_i} \), where \( C_i = \{0, 1, \ldots, i\} \). So, it suffices to show in this case that \( E_{C_i} \leq E_{\mathcal{B}} \), as it then gives that \( E_{\mathcal{B}} \) is Borel bireducible with \( \ell_\infty \). Let \( Z = \prod_i C_i \) and \( X = \prod_i B_i \). Define \( \pi: Z \rightarrow X \) by

\[
\pi(z)(i) = \begin{cases} 0, & \text{if } i \notin \{i_0, i_1, \ldots\}, \\
\text{the } z(m)\text{-th element of } a_n^i(l_m), & \text{if } i = i_m.
\end{cases}
\]

Then for all \( x, y \in Z \) we have \( |x(m) - y(m)| \leq |\pi(x)(i_m) - \pi(y)(i_m)| \leq n|x(m) - y(m)| \). It follows that \( \pi \) is a Borel reduction from \( E_{C_i} \) to \( E_{\mathcal{B}} \).

Next consider the case where for each \( n \) there is a bound \( K_n \) on the size of the \( F_n^n \) equivalence classes, that is, \( \forall i \forall l |a_n^i(l)| < K_n \). We first show in this case that \( E_{\mathcal{B}} \leq B E_1 \). Define a map \( \tau \) from \( X = \prod_i B_i \) to \( (\omega^\omega)^\omega \) as follows. For \( x \in X \) let \( \tau(x)(n) \in \omega^\omega = y_n \) be the real such that \( y_n(i) = \text{the unique } l \text{ such that } x(i) \in a_n^i(l) \).

Consider \( x, y \in X \). If \( x E_{\mathcal{B}} y \), then for some \( C > 0 \) we have \( |x(i) - y(i)| < C \) for all \( i \). Then for each \( i \) we must have that \( x(i), y(i) \) lie in the same class of \( F_n^n \), since any two points in distinct \( F_n^n \) classes are at least \( n \) apart.
This shows that \( \tau(x)(m) = \tau(y)(m) \) for all \( m \geq n \). That is, \( \tau(x)E_1(\omega)\tau(y) \), where \( E_1(\omega) \) refers to the \( E_1(\omega) \) (eventual agreement) relation on \( \langle \omega^\omega \rangle \). Conversely, suppose \( \tau(x)E_1(\omega)\tau(y) \). Fix \( n \) so that for all \( m \geq n \), \( \tau(x)(m) = \tau(y)(m) \). Then for all \( i \) we have that \( |x(i) - y(i)| \leq nK_n \), since any two points in the same \( F_n^i \) equivalence class are at most \( nK_n \) apart. Thus, \( xE_\beta y \). Thus, \( \tau \) is a Borel reduction of \( E_\beta \) to \( E_1(\omega) \). However, it is easy to see that \( E_1(\omega) \leq_B E_1 \), so \( E_\beta \leq_B E_1 \) in this case.

We next consider subcases. First assume that for every \( n \) there is a \( D_n \) such that for infinitely many \( i \) we have that \( g_n^i < D_n \), where \( g_n^i \) is the minimum distance between distinct \( F_n^i \) classes (and \( = 0 \) if there is only one \( F_n^i \) class). Note that \( g_n^i > n \) unless \( B_i \) is a single \( F_n^i \). We may therefore get a sequence \( k_0 < k_1 < \cdots \) such that for all \( n \), there are infinitely many \( i \) such that \( k_n < g_n^i \leq k_n + 1 \). We may then find pairwise disjoint infinite sets \( A_n \subseteq \{ i : k_n < g_n^i < k_n + 1 \} \). For each \( n \) and each \( i \in A_n \), let \( l = l(i, n) \) and \( l' = l'(i, n) \) be such that the distance between \( a_n^i(l) \) and \( a_n^i(l') \) is between \( k_n \) and \( k_n + 1 \). Define \( \varphi : \langle 2^n \rangle^\omega \rightarrow X \) as follows. Given \( y = (y_0, y_1, \ldots) \in \langle 2^n \rangle^\omega \), let \( \varphi(y) = x \in X \), where \( x(i) = 0 \) if \( i \notin \bigcup \limits \limits_n A_n \), and for \( i \in A_n \), say if \( i \) is the \( j \)-th element of \( A_n \), then \( x(i) \) is the least element of \( a_n^i(l) \) if \( y_n(j) = 0 \) and the least element of \( a_n^i(l') \) if \( y_n(j) = 1 \). Note that for any \( x, x' \) in the range of \( \varphi \), we always have that for all \( i \in A_n \) that \( |x(i) - x'(i)| \leq k_n + 1 + nK_n \). Also, if \( y_n \neq y_n' \), then for some \( i \in A_n \) we have that \( |x(i) - x'(i)| \geq k_n \). It follows that \( \varphi \) is a Borel reduction of \( E_1 \) to \( E_\beta \). Thus, \( E_\beta \) is Borel bireducible with \( E_1 \).

Finally assume that for some \( n \) we have that
\[
\forall C > 0 \exists i_C \forall i \geq i_C \ g_n^i > C.
\]
Define \( \psi : X \rightarrow \omega^\omega \) by \( \psi(x)(i) = \) the unique \( l \) such that \( x(i) \in a_n^i(l) \). If \( xE_\beta y \), then we must have \( \psi(x)E_0 \psi(y) \) as \( g_n^i \) tends to infinity with \( i \). Conversely, if \( \psi(x)E_0 \psi(y) \), then \( xE_\beta y \) since \( |x(i) - y(i)| \leq nK_n \) for all \( i \). So, \( E_\beta \leq E_0 \) in this case. It is easy to embed \( E_0 \) into \( E_\beta \) using the fact that the \( b_i \) are unbounded. So, \( E_\beta \) is bireducible with \( E_0 \) in this case.

To define our generalized classes of Banach spaces, we again fix a sequence of successive intervals \( I_i = [l_i, r_i] \) with \( l_{i+1} > r_i \), and integers \( \bar{n} = (n_i) \) as in Theorem 4.1. Once again, we assume that
\[
\frac{1}{r_{i+1} - l_i} \geq n_i^\frac{1}{\bar{n}}.
\]
For each \( i \), let \( S_i \subseteq [l_i, r_i] \) be a finite set.

**Definition 6.6.** For \( I_i, n_i \), and \( S_i \) as above, let \( B_{S_i, \bar{n}} \) be the collection of separable Banach spaces of the form \( X = \left( \sum_{i=1}^\infty \ell_{r_i} \right)^\omega \), where \( r_i \in S_i \). Let \( E_{S_i, \bar{n}} \) denote the uniform homeomorphism relation on the collection \( B_{S_i, \bar{n}} \).

We note that \( B_{S_i, \bar{n}} \) can be regarded as a closed subspace of \( 2^\omega \). Alternatively, we may regard \( B_{S_i, \bar{n}} \) as the space \( \prod \limits_i S_i \) (\( S_i \) having the discrete topology) which is homeomorphic to \( 2^\omega \). These two topologies give the same Borel structure on \( B_{S_i, \bar{n}} \).

**Theorem 6.7.** For any \( I, \bar{I}, \bar{n} \) as above, \( E_{S_i, \bar{n}} \) is either smooth, Borel bireducible with \( E_0 \), Borel bireducible with \( E_1 \), or Borel bireducible with \( \ell_\infty \).

**Proof.** Suppose \( X, Y \in B_{S_i, \bar{n}} \), say \( X \) corresponds to the sequence \( (p_i) \) (where \( p_i \in S_i \)), and \( Y \) corresponds to \( (q_i) \). Again the proof of Theorem 4.1 shows that \( XE_{S_i, \bar{n}}Y \) if \( \left( n_i^\frac{1}{\bar{n}}, \frac{1}{\bar{n}} \right) \) is bounded. Define \( \pi : B_{S_i, \bar{n}} \rightarrow \mathbb{R}^\omega \) as follows. If \( X \) corresponds to
the sequence \((p_i)\), then let \(\pi(X)(i) = \frac{1}{p_i} \log(n_i)\). Note that all of the \(\pi(X)(i)\) take values in the finite set \(B_i := \{\frac{1}{p_i} \log(n_i) : p_i \in S_i\}\). We then have that \(XE_{\tilde{S}, \tilde{u}} Y\) iff \(\pi(X)E_{\tilde{S}, \tilde{u}} \pi(Y)\). Moreover, \(\pi\) is a bijection between \(B_{\tilde{S}, \tilde{u}}\) and \(\prod B_i\). Thus \(E_{\tilde{S}, \tilde{u}}\) is Borel bijective with \(E_{\tilde{S}}\). We are done by Theorem 6.5.

7. Nonisomorphic uniformly homeomorphic Banach spaces

Fix a countable dense set \(D \subseteq (2, 3)\). If \(A\) is a countable subset of \((2, 3) \setminus D\), then we associate to \(A\) the separable Banach space

\[
X_A = \left( \sum_{p \in D} \ell_p \right)_{c_0} \oplus \left( \sum_{q \in A} \ell_q \right)_{c_0}.
\]

Since \((2, 3) \setminus D\) is Borel bijective with \(\mathbb{R}\), to prove Theorem 1.3 it suffices to show that \(X_A\) is uniformly homeomorphic to \(X_B\) for any countable \(A, B \subseteq (2, 3) \setminus D\), and \(X_A\) is not isomorphic to \(X_B\) for \(A \neq B\). The following well-known lemma, for which we sketch a proof for convenience, verifies the second requirement.

**Lemma 7.1.** If \(A \subseteq (2, 3)\) is countable and \(q \notin A\), then \(\ell_q\) is not isomorphic to a subspace of \(\left( \sum_{p \in A} \ell_p \right)_{c_0}\).

**Proof.** Let \(A = \{p_n : n \in \mathbb{N}\}\). For \(k \in \mathbb{N}\), denote by \(P_k\) the natural projection onto \(\left( \sum_{n=1}^{k} \ell_{p_n} \right)_{c_0}\). Suppose there exists an \(X \subseteq \left( \sum_{n=1}^{\infty} \ell_{p_n} \right)_{c_0}\), which is isomorphic to \(\ell_q\). Consider two mutually exclusive cases.

(i) Suppose that there exist \(\varepsilon > 0\) and \(k \in \mathbb{N}\) such that for all \(x \in X\), \(\|P_k x\| \geq \varepsilon \|x\|\). Put \(X' = P_k(X)\). Then \(T : X \to X'\) defined by \(Tx = P_k(x)\), for all \(x \in X\), is an isomorphism with \(\|T^{-1}\| \leq 1/\varepsilon\). That is, \(\ell_q\) is isomorphic to a subspace of the finite direct sum \(\left( \sum_{n=1}^{k} \ell_{p_n} \right)_{c_0}\), which is impossible unless \(q = p_n\) for some \(1 \leq n \leq k\).

(ii) Suppose that for all \(\varepsilon > 0\) and all \(k \in \mathbb{N}\), there exists a normalized \(x \in X\) with \(\|P_k x\| < \varepsilon \|x\|\). Let \((\varepsilon_i) \setminus 0\) such that \(\sum_i \varepsilon_i < 1/4\). Construct inductively a sequence of normalized \((x_i) \in X\) and \(0 < k_1 < k_2 < k_3 \ldots\) such that \(\|x_i - P_{k_i} x_i\| \leq \varepsilon_i\) and \(\|P_{k_i} x_{i+1}\| < \varepsilon_{i+1}\), and put \(x'_i = (P_{k_i} - P_{k_{i-1}}) x_i\). Then \((x'_i)_{i=1}^{\infty}\) is a sequence of disjointly supported vectors, thus equivalent to the unit vector basis of \(c_0\). Since \(\sum_{i=1}^{\infty} \|x_i - x'_i\| \leq \sum_i \varepsilon_i + \varepsilon_{i+1} - \varepsilon_i < 1/2\), this implies that \((x_i) \subseteq X\) is equivalent to the \(c_0\) basis, a contradiction.

**Theorem 7.2.** Let \(D \subseteq (2, 3)\) be dense and \(A \subseteq (2, 3) \setminus D\) be countable. Then \(X = \left( \sum_{p \in D} \ell_p \right)_{c_0}\) is uniformly homeomorphic to \(X_A = \left( \sum_{p \in D} \ell_p \right)_{c_0} \oplus \left( \sum_{q \in A} \ell_q \right)_{c_0}\).

**Remark 7.3.** The proof is a slight generalization of Theorem 10.28 in [BL]. The idea of the proof is due to Ribe [Ri2] who proved it in a special case. This was later extended by Aharoni and Lindenstrauss [AL] to a more general setting (see Ben for a nice exposition). We will reproduce the main steps of the proof, following [BL] with the necessary modifications, and also present an additional step (Lemma 7.4) clarifying an obscure point there.

**Proof.** Recall that for \(x = (x_i) \in \ell_p\), the Mazur map \(\varphi_{p,q} : \ell_p \to \ell_q\) is defined by

\[
\varphi_{p,q}(x) = \|x\|^{1 - \frac{q}{p}} (\text{sign}(x_i)|x_i|^{\frac{q}{p}})_{i=1}^{\infty}.
\]
\( \varphi \) is positively homogeneous and, for each \( K > 0 \), it is a uniform homeomorphism of the \( K \)-ball in \( \ell_p \) onto the \( K \)-ball in \( \ell_q \). Moreover, for every \( M \), the family \( \{ \varphi_{p,q} : 1 \leq p, q \leq M \} \) is a family of equi-uniform homeomorphisms where each \( \varphi_{p,q} \) is restricted to the ball of radius \( \exp(1/|p - q|) \) (see Proposition 9.2, [BL]).

Let \( \{ q_j \} \) be an enumeration of \( A \). Since \( D \) is dense, there exist disjoint infinite subsets \( I_j = \{ (j, n) : n \in \mathbb{N} \} \subset \mathbb{N}, j = 1, 2, \ldots \), such that \( p_{(j,n)} \to q_j \) for each \( j \). To simplify the notation, we write \( \varphi_{j,n} \) for the Mazur map \( \varphi_{p_{(j,n)},q_j} : \ell_{p_{(j,n)}} \to \ell_{q_j} \). By passing to subsequences of the \( I_j \)'s, if necessary, we can and will assume that the family \( \{ \varphi_{j,n}, (\varphi_{j,n})^{-1} : j, n \in \mathbb{N} \} \) of maps where each is restricted to the \( 2^n \)-balls of its domain is equi-uniformly continuous.

In the next step we solely work on copies of the \( \ell_{q_j} \)'s. The goal is to construct continuous paths of homeomorphisms between two particular invertible operators \( S_0^j \) and \( S_1^j \) described below in a ‘uniform’ manner. For a fixed \( j \), this follows from the fact that the general linear group of invertible operators on \( \ell_q \) is contractible (cf., e.g., [MI]). Since we require the paths to be independent of the \( j \)'s, we give them explicitly.

**Lemma 7.4.** There exists a continuous path \( \tau \to V_\tau \), \( 0 \leq \tau \leq 1/2 \), of invertible operators on \( \ell_q \) such that \( V_0 \) is the identity and \( V_{1/2} = (u, w, v) \), for all \( (u, v, w) \in \ell_q \). Moreover, \( \| V_\tau \| \leq 2 \), and the path is independent of \( 1 \leq q < \infty \) in the sense that the matrix representation of \( V_\tau \) with respect to the decomposition \( \ell_q \) does not depend on \( q \).

**Proof.** Consider an isomorphism \( D : \ell_q \to \ell_q \). \( D \) induces an isomorphism \( \ell_q \to \ell_q \) mapping \( (u, v, w) \) to \((v, w, (Du)_{11}, (Du)_{12}, \ldots) \). Regarding the latter as a sequence of scalars and composing with obvious isometries, the operator \( V_{1/2} = J \oplus I \oplus \ldots \), where \( I \) is the \( 2 \times 2 \) identity matrix and \( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). For \( 0 \leq \tau \leq 1/4 \), consider the path \( V_\tau \) of invertible operators defined by \( V_\tau = A_\tau \oplus A_\tau \oplus A_\tau \oplus \ldots \) where \( A_\tau = \begin{pmatrix} B_\tau & C_\tau \\ -C_\tau & B_\tau \end{pmatrix} \), and

\[
B_\tau = \begin{pmatrix} \cos^2 2\pi \tau & \sin^2 2\pi \tau \\ \sin^2 2\pi \tau & \cos^2 2\pi \tau \end{pmatrix}, \quad C_\tau = \begin{pmatrix} -\cos 2\pi \tau & \sin 2\pi \tau & -\cos 2\pi \tau & \sin 2\pi \tau \\ \sin 2\pi \tau & -\cos 2\pi \tau & -\sin 2\pi \tau & \cos 2\pi \tau \\ -\cos 2\pi \tau & \sin 2\pi \tau & \cos 2\pi \tau & -\sin 2\pi \tau \\ -\sin 2\pi \tau & -\cos 2\pi \tau & \sin 2\pi \tau & \cos 2\pi \tau \end{pmatrix}.
\]

Thus, the path connects the identity to \( V_{1/4} = J \oplus J \oplus \ldots \).

For \( 1/4 \leq \tau \leq 1/2 \), we continue the path by \( V_\tau = J \oplus A_\tau \oplus A_\tau \oplus \ldots \). Thus \( V_{1/2} = J \oplus I \oplus I \oplus \ldots \), as desired. Note that since \( A_{1/4} = J \oplus J \), two definitions of \( V_{1/4} \) coincide and therefore it is well defined. Clearly, the paths are independent of \( 1 \leq q < \infty \), and an easy computation shows that \( \| V_\tau \| \leq 2 \) for all \( 0 \leq \tau \leq 1/2 \).

Now for each \( j \) let \( T_j : (\ell_{q_j} \oplus \ell_{q_j})_{q_j} \to \ell_{q_j} \) be a linear isometry, and consider the following isometries from \( (\ell_{q_j} \oplus \ell_{q_j} \oplus \ell_{q_j})_{q_j} \) onto \( (\ell_{q_j} \oplus \ell_{q_j})_{q_j} \), induced by the \( T_j \)'s:

\[
S_0^j(u, v, w) = (T_j(u, v), w) \quad \text{and} \quad S_1^j(u, v, w) = (v, T_j(u, w)) \quad \text{for} \quad (u, v, w) \in (\ell_{q_j} \oplus \ell_{q_j} \oplus \ell_{q_j})_{q_j}.
\]

**Lemma 7.5.** For all \( j \) and \( 0 \leq \tau \leq 1 \), there is a homogeneous norm-preserving homeomorphism \( h_\tau^j : (\ell_{q_j} \oplus \ell_{q_j} \oplus \ell_{q_j})_{q_j} \to (\ell_{q_j} \oplus \ell_{q_j})_{q_j} \) such that \( h_0^j = S_0^j \) and...
that the desired norm-preserving maps. Note that the inverses of the normalized maps
is defined twice. However, the two defi-
ing its to the operator \((u, v) \rightarrow (v, u)\). To get the path \(U_j\), start with the
isomorphism \(E_j^1 : (\ell_q_j \oplus \ell_q_j)_{q_j} \rightarrow (\ell_q_j \oplus \ell_q_j + \ldots)_{q_j}\) defined by \(E_j^1(u, v) = ((D^j u)_1, (D^j u)_2, (D^j v)_2, \ldots)\), where \(D_j^j : \ell_q_j \rightarrow \ell_q_j(\ell_q_j)\) is an isomorphism.

Then put \(U_j = (E_j^1)^{-1}V_j E_j^1\), where \(V_j = V_j^{(2t-1)/4}\). Note that the norm of the
\(S_j^j\)’s and their inverses are uniformly bounded, and it is clear from the formulas
that the \(S_j^j\)’s are Lipschitz in \(\tau\). Finally, putting \(h_j^j(x) = \|x\|S_j^j(x)/\|S_j^j(x)\vertical yields
the desired norm-preserving maps. Note that the inverses of the normalized maps
have the same form, that is, \((h_j^j)^{-1}(y) = \|y\|(S_j^j)^{-1}(y)/\|(S_j^j)^{-1}(y)\|\). \(\square\)

The desired uniform homeomorphism from \(X_A\) onto \(X\) will be defined by \(\phi(x) = g_{t|x|}(x)\), where \(g_{t}(x) = \|x\|\tilde{g}_t(x)/\|\tilde{g}_t(x)\|, t > 0\) and \(\tilde{g}_t\) is defined below.

Every \(x \in X_A\) has a unique representation of the form \(\sum_j u_j + \sum_n x_{j,n}\), where \(u_j \in \ell_{q_j}, q_j \in A\) and \(x_{j,n} \in \ell_{p_j}, p_j \in D\). For notational convenience we split the
second sum and write this as

\[
x = \sum_{j=1}^{\infty} (u_j + \sum_n x_{j,n}) + \sum_{i \in I_0} x_i,
\]

where \(x_{j,n} \in \ell_{p_{j,n}}\) and \(I_0\) is the set of indices which do not belong to any \(I_j\)’s,
\(j = 1, 2, \ldots\).

Using the Mazur maps we define \(\psi_{j,n} : (\ell_{p_{j,n}} \oplus \ell_{p_{j,n+1}})_{c_0} \rightarrow (\ell_{q_j} \oplus \ell_{q_j})_{q_j}\) by

\[
\psi_{j,n}(x_{j,n}, x_{j,n+1}) = (\varphi_{j,n}(x_{j,n}), \varphi_{j,n+1}(x_{j,n+1})).
\]

Then the \(\psi_{j,n}\)’s are equi-uniform homeomorphisms between \(2^n\)-balls of the domain
and \(2^{1/q_j}2^n\)-balls of the range. Let \(\omega(\varepsilon)\) denote the common bound of the moduli of
continuity of \(\varphi_{j,n}\) and \(\psi_{j,n}\) and of their inverses.

For \(n = 0, 1, \ldots\) put \(\alpha_n = 2^n - 1\). For \(\alpha_n \leq t \leq \alpha_{n+1}\) define

\[
\tilde{g}_t(x) = \sum_{j=1}^{\infty} \left[ \psi_{j,n}^{-1}(h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_{j,n}, x_{j,n+1}))) \right] + \sum_{i \neq n+1} x_{j,i} + \sum_{i \in I_0} x_i.
\]

Here for each \(j\), the map only replaces three coordinates in the block \((u_j, \ldots, x_{j,n},
x_{j,n+1}, \ldots)\) by \((\ldots, y_{j,n}, y_{j,n+1}, \ldots)\), where \((y_{j,n}, y_{j,n+1}) = \psi_{j,n}^{-1}(h_{2^{-n}(t-\alpha_n)}(u_j,\psi_{j,n}(x_{j,n}, x_{j,n+1})))\). Note that for \(t = \alpha_n\), \(\tilde{g}_t\) is defined twice. However, the two definitions using \(h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n-1}(x_{j,n-1}, x_{j,n}))\) and the one using \(h_{\alpha_n}(u_j, \psi_{j,n}(x_{j,n}, x_{j,n+1}))\) coincide; therefore it is well-defined.

It is clear from the definition that the \(\tilde{g}_t\)’s are homogeneous. It remains to check that \(\{\tilde{g}_t : 0 \leq t < \infty\}\) is a family of equi-uniform homeomorphisms. This will imply
the same for the norm-preserving \(g_j\)’s (see the remarks before Theorem 10.28, BL).

Let \(x = \sum_{j=1}^{\infty} (u_j + \sum_n x_{j,n}) + \sum_{i \in I_0} x_i\) and
\(y = \sum_{j=1}^{\infty} (v_j + \sum_n y_{j,n}) + \sum_{i \in I_0} y_i\) be in \(X_A\) such that \(\|x\|, \|y\| \leq \alpha_{n+1}\) and \(\|x - y\| < 1\), and let \(\alpha_n \leq t, s \leq \alpha_{n+1}\).
Then $\|g_t(x) - g_s(y)\|$ is bounded by

$$
\left\| \sum_j \left[ \psi_{j,n}^{-1} \left( h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_j,n, x_j,n+1)) \right) \right. \\
- \left. \left. \psi_{j,n}^{-1} \left( h_{2^{-n}(s-\alpha_n)}(v_j, \psi_{j,n}(y_j,n, y_j,n+1)) \right) \right] \right\|_{c_0} \\
+ \left\| \sum_j \sum_{i \neq n,n+1} (x_{j,i} - y_{j,i}) + \sum_{i \in I_0} (x_i - y_i) \right\|_{c_0}.
$$

The second term is bounded by $\|x - y\|$. By Lemma 7.5 for all $j$,

$$
\left\| \psi_{j,n}^{-1} \left( h_{2^{-n}(t-\alpha_n)}(u_j, \psi_{j,n}(x_j,n, x_j,n+1)) \right) - \psi_{j,n}^{-1} \left( h_{2^{-n}(s-\alpha_n)}(v_j, \psi_{j,n}(y_j,n, y_j,n+1)) \right) \right\|
$$

is bounded by

$$
\omega \left( \left\{ \|u_j - v_j\| + \omega(\|x_{j,n} - y_{j,n}\| + \|x_{j,n+1} - y_{j,n+1}\|) \right\} + \alpha_{n+1} |2^{-n}s - 2^{-n}t| \right),
$$

where the constant $K$ and the function $\omega$ are independent of $j$. Since $\varepsilon \leq C\omega(\varepsilon)$ for some constant $C$ and all $\varepsilon \leq 1$, and since the estimates are independent of the $j$'s, it follows that there exist constants $L_1$ and $L_2$ such that

$$
\|g_t(x) - g_s(y)\| \leq L_1 \omega(L_2 \omega(\|x - y\|) + |s - t|).
$$

The same estimates hold for the inverses as well. \hfill \Box

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**References**


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