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UNIQUENESS OF FINITE TOTAL CURVATURES AND THE STRUCTURE OF RADIAL SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS

JANN-LONG CHERN, ZHI-YOU CHEN, AND YONG-LI TANG

ABSTRACT. In this article, we are concerned with the semilinear elliptic equation

$$\Delta u + K(|x|)|u|^{p-1}u = 0 \quad \text{in } \mathbf{R}^n \setminus \{\mathbf{0}\},$$

where $n > 2$, $p > 1$, and $K(|x|) > 0$ in $\mathbf{R}^n$. The correspondence between the initial values of regularly positive radial solutions of the above equation and the associated finite total curvatures will be derived. In addition, we also conduct the zeros of radial solutions in terms of the initial data under specific conditions on K and p. Furthermore, based on the Pohozaev identity and openness for the regions of initial data corresponding to certain types of solutions, we obtain the whole structure of radial solutions depending on various situations.

1. Introduction and main results

The nonlinear elliptic equation

$$\Delta u(x) + K(|x|)|u(x)|^{p-1}u(x) = 0 \quad \text{in } \mathbf{R}^n \setminus \{\mathbf{0}\}, \tag{1.1}$$

where $n > 2$ and $p > 1$, arises from various topics such as astrophysics, combustion, and differential geometry. To cite an instance, let (M, g) be a Riemannian manifold of dimension n with $n \geq 3$, and let K be a prescribed function on M. In order to find a new metric g_1 on M with K as its scalar curvature, which is conformal to the original one g (that is, $g_1 = \psi g$ for some function $\psi > 0$ on M), we write $\psi = u^{\frac{4}{n-2}}$. Then this problem is equivalent to finding positive solutions of the equation

$$\frac{4(n-1)}{n-2}\Delta_g u - ku + Ku^{\frac{n+2}{n-2}} = 0,$$

where Δ_g and k are the Laplacian and scalar curvature in the g metric, respectively. If $M = \mathbf{R}^n$ and $g = (\delta_{ij})$, then $k = 0$, and hence the above equation can be reduced to (1.1) with $p = \frac{n+2}{n-2}$ after a suitable scaling and sign changing of K. In the above example, we call (1.1) the scalar curvature equation.

In this paper, we are concerned with the radial solutions u for (1.1), i.e., $u = u(r)$, $r = |x|$. Then (1.1) is reduced to the ordinary differential equation

$$u''(r) + \frac{n-1}{r}u'(r) + K(r)|u(r)|^{p-1}u(r) = 0, \quad r > 0.$$

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First, we consider the following initial value problem:

$$\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) + K(r)|u(r)|^{p-1}u(r) = 0, \quad r > 0, \\ u(0) = \alpha,\ u'(0) = 0, \end{cases} \tag{1.2}$$

where $K(r)$ satisfies the following conditions:

$$\frac{rK'(r)}{K(r)} \text{ is nonincreasing on } (0,\infty), \tag{1.3}$$

$$\begin{cases} K(r) \in C^1(0,\infty), \\ K(r) > 0 \text{ on } (0,\infty), \\ rK(r) \in L^1(0,1), \\ r^{-(3+2\lambda)}K(r) \in L^1(1,\infty), \end{cases} \tag{1.4}$$

with

$$\lambda = \frac{(n-2)p-(n+2)}{2}$$

and $\alpha \in \mathbf{R}$ is a given initial value. We know that (1.2) possesses a unique solution on $[0,\infty)$, and we denote it by $u(r;\alpha)$.

One of the major purposes in this article is conducting the uniqueness of the finite total curvatures, which is described in the following context, associated with solutions of (1.2) with respect to the initial values. To achieve our goal, set

$$\ell = \lim_{r\to\infty} \frac{rK'(r)}{K(r)},$$
$$\sigma = \lim_{r\to 0} \frac{rK'(r)}{K(r)}$$

and assume that σ is finite throughout this paper. In addition, we consider the following assumptions involving $K(r)$ in different situations:

$$-\infty \le \ell < \lambda < \sigma < +\infty, \tag{1.5}$$

$$\frac{rK'(r)}{K(r)} \le \lambda (\not\equiv \lambda), \tag{1.6}$$

$$\frac{rK'(r)}{K(r)} \ge \lambda (\not\equiv \lambda), \tag{1.7}$$

$$\frac{rK'(r)}{K(r)} \equiv \lambda \tag{1.8}$$

for $r > 0$.

Remark 1.1. For any $K(r)$ satisfying (1.3), all possible situations of $K(r)$ are the cases mentioned in (1.5)-(1.8) above.

Let $u(r;\alpha)$ be a positive solution of (1.2) on $[0,\infty)$. We define

$$\mathcal{T}(\alpha) = \int_0^\infty r^{n-1}K(r)u^p(r;\alpha)\, dr. \tag{1.9}$$

Generally, we call $\mathcal{T}(\alpha)$ the total curvature associated with the solution $u(r;\alpha)$ for (1.2).

Many research efforts have been dedicated to the structure of solutions of (1.1). In [13], some uniqueness and nonuniqueness results of positive radial solutions have been made for the case where (1.8) is satisfied. Moreover, by considering the growth rate of K different from (1.3), [7] studied the weighted p-Laplacian equation. In a more recent work, [6] dealt with the positive radial solutions of (1.1) when the monotonicity assumption (1.3) fails. See also, for example, [9], [11], [15], [17], [18], and the references therein.

By means of Theorem 1 in [16], we see that (1.2) possesses a unique positive solution with finite total curvature if (1.5) holds. Some interesting questions about the total curvature arise: Under what conditions on K does (1.2) possess more than one positive solution with finite total curvature? Furthermore, are the finite total curvatures uniquely determined by $u(r;\alpha)$ and hence by α? On the other hand, [10] shows that the solutions of (1.2) may separate from each other entirely or meet infinitely many times depending on various situations for K and p. We wonder whether or not only two situations stated above may occur for solutions of (1.2). To answer these questions partially, we establish our first main consequence in the following.

Theorem 1.1. *Suppose that* (1.8) *holds and* $p > 3$*. Then the following assertions are true.*

(a) *The solution* $u(r;\alpha)$ *of* (1.2) *is positive on* $[0,\infty)$ *and* $\mathcal{T}(\alpha)$ *is finite for all* $\alpha > 0$*, where* $\mathcal{T}(\alpha)$ *is defined in* (1.9).

(b) $\mathcal{T}(\alpha) \in C^1(0,\infty)$*. Furthermore, for any* $T > 0$*, there exists a unique positive solution* $u(r)$ *of* (1.2) *satisfying*

$$\int_0^\infty r^{n-1}K(r)u^p(r)\ dr = T.$$

(c) *Any two distinct solutions of* (1.2) *intersect exactly once on* $[0,\infty)$*. Furthermore, if* $R(\alpha)$ *is the unique intersection point of* $u(r;\alpha)$ *and* $u(r;\alpha_0)$ *for* $\alpha > \alpha_0 > 0$*, then* $R(\alpha)$ *is strictly decreasing on* (α_0,∞) *and*

$$\lim_{\alpha\to\infty} R(\alpha) = 0.$$

The concept of the linearization (see, e.g., [1], [8]) plays a significant role in deriving many qualitative properties of solutions. We will apply the arguments involving the linearized equations which were used in [3] and [4] to verify Theorem 1.1 in Section 2.

For accomplishing the second part of our results, we introduce two initial value problems as follows:

$$\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) + K(r)|u(r)|^{p-1}u(r) = 0, \quad r > 0, \\ u(1) = \theta,\ u'(1) = \eta \end{cases} \tag{1.10}$$

and

$$\begin{cases} u''(r) + \dfrac{n-1}{r}u'(r) + K(r)|u(r)|^{p-1}u(r) = 0, \quad r > 0, \\ \lim\limits_{r\to\infty} r^{n-2}u(r) = \beta, \end{cases} \tag{1.11}$$

where $K(r)$ satisfies (1.3) and (1.4), and $\theta,\eta,\beta \in \mathbf{R}$ are given initial data. Conventionally, we denote the solutions of (1.10) and (1.11) by $u(r;\theta,\eta)$ and $u(r;\beta)$, respectively, if there is no confusion. Obviously, both $u(r;\alpha)$, which is a solution of (1.2), and $u(r;\beta)$ are solutions of (1.10).

Moreover, we define the following curves on the (θ, η)-plane as

$$\gamma_1^{\pm}(\pm\alpha) = (u(1;\pm\alpha), u'(1;\pm\alpha)) \quad (u(r;\pm\alpha) \text{ are solutions of } (1.2)), \tag{1.12}$$

$$\gamma_2^{\pm}(\pm\beta) = (u(1;\pm\beta), u'(1;\pm\beta)) \quad (u(r;\pm\beta) \text{ are solutions of } (1.11)) \tag{1.13}$$

for $\alpha, \beta > 0$. We note that all curves defined in (1.12) and (1.13) are smooth by (1.4), and we let $\Gamma_1^{\pm}$ and $\Gamma_2^{\pm}$ be the ranges of $\gamma_1^{\pm}(\pm\alpha)$ and $\gamma_2^{\pm}(\pm\beta)$ over $(0,\infty)$, respectively.

Remark 1.2. It is known that, by virtue of Theorem 2.2 in [2] , γ_1^+ and γ_2^+ emanate from the origin tangentially to the θ^+-axis and the half line $(n-2)\theta + \eta$ with $\theta > 0$, respectively, and go through the fourth quadrant in the (θ, η)-plane.

One of our aims in this paper is to conduct the number of zeros of solutions for (1.10) in terms of the initial data in the (θ, η)-plane, which can be classified into certain regions characterized by $\gamma_1^{\pm}$ and $\gamma_2^{\pm}$ under specific conditions on $K(r)$ and p. Furthermore, based on the Pohozaev identity and openness for the regions of initial data corresponding to certain types of solutions, which is known as the shooting method (see, e.g., [5]), we also obtain the whole structure of solutions of (1.10) depending on various situations as mentioned in (1.5)-(1.8).

In addition, according to the behaviors at the origin and infinity, we introduce various types of solution $u(r)$ to (1.10) as follows:

- *Type* R^+-$*$ (*resp.*, R^--$*$)*:* $u(r)$ is regular at $r = 0$, i.e., $u(r)$ converges to a positive (resp., negative) constant as $r \to 0$.
- *Type* $S^{\pm}$-$*$*:* $u(r)$ is singular at $r = 0$, i.e., $u(r) \to \pm\infty$ as $r \to 0$.
- *Type* O-$*$*:* $u(r)$ has infinitely many zeros near $r = 0$.
- *Type* $*$-R^+ (*resp.*, $*$-R^-)*:* $r^{n-2}u(r)$ converges to a positive (resp., negative) constant as $r \to \infty$.
- *Type* $*$-$S^{\pm}$*:* $r^{n-2}u(r) \to \pm\infty$ as $r \to \infty$.
- *Type* $*$-O*:* $u(r)$ has infinitely many zeros near $r = \infty$.

To discuss the zeros of solutions in the following context, we put an additional symbol over the hyphen in the notation of various types to indicate the total number of zeros on $(0,\infty)$. For instance, we say that $u(r)$ is a solution of Type $\mathrm{R}^+ \overset{k}{-} *$, which means that it is of Type R^+-$*$ and has exactly k zeros on $(0,\infty)$.

By virtue of Yanagida [16], it is well known that the numbers of zeros and the behaviors at infinity for solutions of (1.2) can be clarified under certain conditions related to $K(r)$, as mentioned below.

Theorem A. *Suppose that* (1.5) *holds. Then the following assertions concerning the solution* $u(r;\alpha)$ *of* (1.2) *are true.*

(i) *There exist* $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \infty$ *with* $\lim_{k\to\infty} \alpha_k = \infty$ *such that* $u(r;\alpha)$ *is of Type* $R^+ \overset{2k}{-} R^+$ (*resp., Type* $R^+ \overset{2k+1}{-} R^-$) *if and only if* $\alpha = \alpha_{2k+1}$ (*resp.,* $\alpha = \alpha_{2k+2}$)*, and* $u(r;\alpha)$ *is of Type* $R^+ \overset{2k}{-} S^+$ (*resp., Type* $R^+ \overset{2k+1}{-} S^-$) *for* $\alpha \in (\alpha_{2k}, \alpha_{2k+1})$ (*resp.,* $\alpha \in (\alpha_{2k+1}, \alpha_{2k+2})$)*, where* $k \in \mathbf{N} \cup \{0\}$.

(ii) *Let* $\beta_k = \lim_{r\to\infty} r^{n-2}|u(r;\alpha_k)|$. *Then* $\{\beta_k\}$ *is a monotone increasing positive sequence with* $\lim_{k\to\infty} \beta_k = \infty$.

Moreover, [16] also provided the following facts under other conditions on $K(r)$.

Theorem B. (i) *If* (1.6) *holds, then the solution* $u(r;\alpha)$ *of* (1.2) *is of Type* $R^+ \overset{0}{-} S^+$ *for all* $\alpha > 0$.

(ii) *If* (1.7) *holds, then the solution* $u(r;\alpha)$ *of* (1.2) *is of Type* R^+-O *for all* $\alpha > 0$.
(iii) *If* (1.8) *holds, then the solution* $u(r;\alpha)$ *of* (1.2) *is of Type* $R^+ \stackrel{0}{—} R^+$ *for all* $\alpha > 0$.

Now, we state our second main assertion as follows.

Theorem 1.2. *Suppose that* (1.5) *holds and* $p \geq \frac{2+\sigma}{n-2}$. *Then the following assertions involving the solution* $u(r;\theta,\eta)$ *of* (1.10) *in the* (θ,η)*-plane are true.*

(a) *In the* (θ,η)*-plane,* $u(r;\theta,\eta)$ *is of Type* S^+-$*$ *(resp., Type* S^--$*$*) if* (θ,η) *lies on the region* Ω_1^+ *(resp.,* Ω_1^-*) bounded by* $\Gamma_1^\pm$ *and* $\{\mathbf{0}\}$*, and containing* $\{(0,\eta) : -\eta_1 < \eta < 0$ *(resp.,* $0 < \eta < \eta_1)\}$ *for some* $\eta_1 > 0$.
(b) *In the* (θ,η)*-plane,* $u(r;\theta,\eta)$ *is of Type* $*$-S^+ *(resp., Type* $*$-S^-*) if* (θ,η) *lies on the region* Ω_2^+ *(resp.,* Ω_2^-*) bounded by* $\Gamma_2^\pm$ *and* $\{\mathbf{0}\}$*, and containing* $\{(0,\eta) : 0 < \eta < \eta_2)$ *(resp.,* $-\eta_2 < \eta < 0)\}$ *for some* $\eta_2 > 0$.
(c) *Let* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *be described as in Theorem A. Then, for* $k \in \mathbf{N} \cup \{0\}$,

$$\begin{cases} \gamma_1^\pm(\pm\alpha_{2k+1}) = \gamma_2^\pm(\pm\beta_{2k+1}), \\ \gamma_1^\pm(\pm\alpha_{2k+2}) = \gamma_2^\mp(\mp\beta_{2k+2}) \end{cases}$$

and

$$\begin{cases} u(r;\gamma_1^\pm(\pm\alpha_{2k+1})) \text{ are of Type } R^\pm \stackrel{2k}{—} R^\pm, \\ u(r;\gamma_1^\pm(\pm\alpha_{2k+2})) \text{ are of Type } R^\pm \stackrel{2k+1}{—} R^\mp. \end{cases}$$

Furthermore, $u(r;\gamma_1^+(\alpha))$ *is of Type* $R^+ \stackrel{2k}{—} S^+$ *(resp., Type* $R^+ \stackrel{2k+1}{—} S^-$*) for* $\alpha \in (\alpha_{2k}, \alpha_{2k+1})$ *(resp.,* $\alpha \in (\alpha_{2k+1}, \alpha_{2k+2})$*).*

The types of solutions associated with initial data on Γ_1^- *and* $\Gamma_2^\pm$ *can be characterized by means of* (a)-(c).
(d) *Let* E_1 *be a connected component of* Ω_1^+ *and suppose the intersection of* E_1 *with each quadrant in* (θ,η)*-plane is also connected (including the empty set). Then the solutions* $u(r;\theta,\eta)$ *have the same number of zeros on* $(0,1]$ *for* $(\theta,\eta) \in E_1$ *with* $\theta < 0$*, as they do for* $(\theta,\eta) \in E_1$ *with* $\theta > 0$*. Moreover, if the intersection of the interior of* E_1 *and* η^+*-axis (resp.,* η^-*-axis) is nonempty, then* $u(r;\theta_1,\eta_1)$ *possesses exactly one more zero on* $(0,1]$ *than* $u(r;\theta_2,\eta_2)$ *for* $(\theta_i,\eta_i) \in E_1$ *with* $\eta_i > 0$ *(resp.,* $\eta_i < 0$*)* $(i = 1,2)$ *and* $\theta_2 < 0 \leq \theta_1$ *(resp.,* $\theta_1 \leq 0 < \theta_2$*). The situation for* Ω_1^- *is the same as that for* Ω_1^+.
(e) *Let* E_2 *be a connected component of* Ω_2^+ *and suppose the intersection of* E_2 *with each quadrant in the* (θ,η)*-plane is also connected (including the empty set). Then the solutions* $u(r;\theta,\eta)$ *have the same number of zeros on* $[1,\infty)$ *for* $(\theta,\eta) \in E_2$ *with* $\theta < 0$*, as they do for* $(\theta,\eta) \in E_2$ *with* $\theta > 0$*. Moreover, if the intersection of the interior of* E_2 *and* η^+*-axis (resp.,* η^-*-axis) is nonempty, then* $u(r;\theta_1,\eta_1)$ *possesses exactly one more zero on* $[1,\infty)$ *than* $u(r;\theta_2,\eta_2)$ *for* $(\theta_i,\eta_i) \in E_2$ *with* $\eta_i > 0$ *(resp.,* $\eta_i < 0$*)* $(i = 1,2)$ *and* $\theta_1 \leq 0 < \theta_2$ *(resp.,* $\theta_2 < 0 \leq \theta_1$*). The situation for* Ω_2^- *is the same as that for* Ω_2^+.

According to Remark 1.2 and Theorem 1.2, we obtain the whole solution structure of (1.10), as pictured in Figure 1, for the case of (1.5) and $p \geq \frac{2+\sigma}{n-2}$.

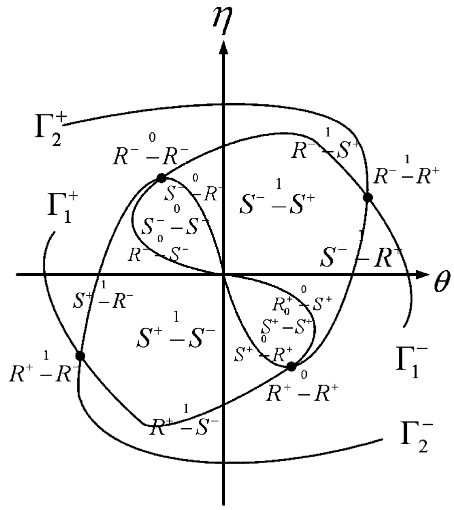

FIGURE 1. $p \geq \frac{2+\sigma}{n-2}$ and (1.5) holds

For $\gamma \in \mathbf{R}$ and $n > 10 + 4\gamma$, we define

$$p_{\pm}(\gamma, n) = \frac{(n-2)^2 - 2(\gamma+2)(n+\gamma) \pm 2(\gamma+2)\sqrt{(n+\gamma)^2 - (n-2)^2}}{(n-2)(n-10-4\gamma)}. \tag{1.14}$$

The following three theorems are related to the structures of solutions of (1.10) under various conditions (1.6)-(1.8) on $K(r)$.

Theorem 1.3. *Suppose that* (1.6) *holds and* $p \geq \frac{2+\sigma}{n-2}$*. Then the following assertions involving the solution* $u(r;\theta,\eta)$ *of* (1.10) *in the* (θ,η)*-plane are true.*

(a) $\Gamma_1^i \cap \Gamma_2^j = \emptyset$*, where* $i, j = \pm$*, and* $\gamma_2^{\pm}(\pm\beta)$ *have no limit points in the* (θ,η)*-plane as* $\beta \to \infty$*.*

(b) *Suppose that* p_+ *is defined in* (1.14)*. If* $n > 10 + 4\sigma$ *and* $p > p_+(\sigma, n) > \frac{n+2+2\sigma}{n-2}$*, then*

$$\lim_{\alpha \to \infty} \gamma_1^{\pm}(\pm\alpha) = \boldsymbol{p}^{\pm} \text{ exist,}$$

and $u(r;\boldsymbol{p}^{\pm})$ *are the unique solutions of Type* $S^{\pm}$*-*$S^{\pm}$*.*

(c) $u(r;\theta,\eta)$ *is of Type O-*$*$ *for* $(\theta,\eta) \notin \overline{\Gamma_1^+} \cup \overline{\Gamma_1^-}$ *and is of Type* $*$*-*S^+ *(resp.,* $*$*-*S^-*) if* $(\theta.\eta)$ *lies on the region bounded by* $\Gamma_2^{\pm}$ *and* $\{\mathbf{0}\}$*, and containing* $\{(\theta,0) : 0 < \theta < \varepsilon\}$ *(resp.,* $\{(\theta,0) : -\varepsilon < \theta < 0\}$*) for some* $\varepsilon > 0$*.*

(d) $u(r;\theta,\eta)$ *has no zeros on* $(0,\infty)$ *unless it is oscillatory.*

Following Theorem 1.3, we depict the structure of solutions of (1.10) in Figure 2 for the case of (1.6), $n > 10 + 4\sigma$ and $p > \max\left\{\frac{2+\sigma}{n-2}, p_+\right\}$.

Remark 1.3. The total curvature associated with the solution $u(r;\alpha)$ of (1.2) with (1.6) is infinite for all $\alpha > 0$.

Theorem 1.4. *Suppose that* (1.7) *holds and* $p \geq \frac{2+\sigma}{n-2}$*. Then the following assertions involving the solution* $u(r;\theta,\eta)$ *of* (1.10) *in the* (θ,η)*-plane are true.*

(a) $\Gamma_1^i \cap \Gamma_2^j = \emptyset$*, where* $i, j = \pm$*, and* $\gamma_1^{\pm}(\pm\beta)$ *have no limit points in* (θ,η)*-plane as* $\beta \to \infty$*.*

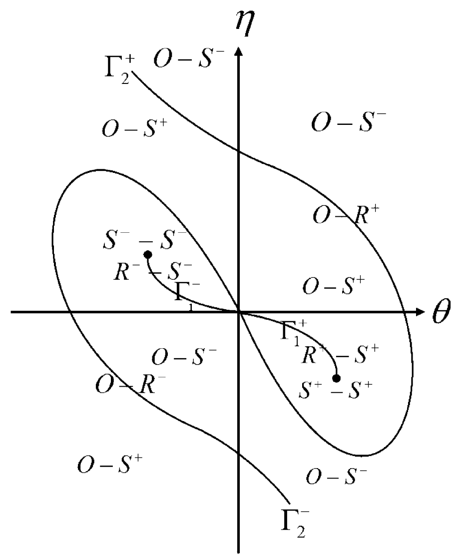

FIGURE 2. $n > 10 + 4\sigma$, $p > \max\left\{\frac{2+\sigma}{n-2}, p_+\right\}$, and (1.6) holds

(b) *Suppose that* p_- *is defined in* (1.14). *If* $n > 10 + 4(2\lambda - \ell)$ *and* $1 < p < p_-(2\lambda - \ell, n) < \frac{n+2+2\ell}{n-2}$, *then*

$$\lim_{\alpha\to\infty} \gamma_2^{\pm}(\pm\beta) = \boldsymbol{q}^{\pm} \text{ exist,}$$

and $u(r; \boldsymbol{q}^{\pm})$ *are the unique solutions of Type* $S^{\pm}$-$S^{\pm}$.

(c) $u(r; \theta, \eta)$ *is of Type* $*$-O *for* $(\theta, \eta) \notin \overline{\Gamma_2^+} \cup \overline{\Gamma_2^-}$ *and is of Type* S^+-$*$ (*resp.,* S^--$*$) *if* $(\theta.\eta)$ *lies on the region bounded by* $\Gamma_1^{\pm}$ *and* $\{\mathbf{0}\}$, *and containing* $\{(0, \eta) : -\varepsilon < \eta < 0\}$ (*resp.,* $\{(0, \eta) : 0 < \eta < \varepsilon\}$) *for some* $\varepsilon > 0$.

(d) $u(r; \theta, \eta)$ *has no zeros on* $(0, \infty)$ *unless it is oscillatory.*

As shown in Figure 3, we attain the solution structure for (1.10) in the case of (1.7), $n > 10 + 4(2\lambda - \ell)$, and $\max\left\{\frac{2+\sigma}{n-2}, 1\right\} < p < p_- < \frac{n+2+2\ell}{n-2}$ by virtue of the above theorem.

Remark 1.4. In addition to including the conclusions of Theorem 2 in [16] (Theorem B), we also establish, as depicted in Theorems 1.3 and 1.4 above, that the solutions of (1.2) and (1.11) will converge towards uniquely certain types of solutions as the initial data tend to infinity (or minus infinity) under further conditions of p.

Theorem 1.5. *Suppose that* (1.8) *holds and* $p \geq \frac{2+\sigma}{n-2}$. *Then the following assertions involving the solution* $u(r; \theta, \eta)$ *of* (1.10) *in the* (θ, η)*-plane are true.*

(a) $\Gamma_1^+ = \Gamma_2^+ \equiv \Gamma^+$ *and* $\Gamma_1^- = \Gamma_2^- \equiv \Gamma^-$. *Furthermore, both* Γ^+ *and* Γ^- *eventually return to the origin.*

(b) $u(r; \theta, \eta)$ *is of Type* $R^{\pm}$-$R^{\pm}$ *for* $(\theta, \eta) \in \Gamma^{\pm}$ *and is of Type* $S^{\pm}$-$S^{\pm}$ *if* (θ, η) *lies on the bounded regions* $F^{\pm}$ *bounded by* $\Gamma^{\pm}$ *and* $\{\mathbf{0}\}$. *Moreover,* $u(r; \theta, \eta)$ *is of Type O-O if* $(\theta, \eta) \notin \overline{F^+} \cup \overline{F^-}$.

(c) $u(r; \theta, \eta)$ *has no zeros on* $(0, \infty)$ *unless it is oscillatory.*

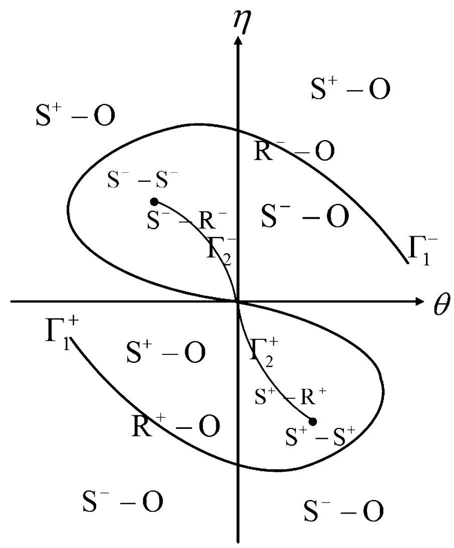

FIGURE 3. $n > 10 + 4(2\lambda - \ell)$, $\max\left\{\frac{2+\sigma}{n-2}, 1\right\} < p < p_- < \frac{n+2+2\ell}{n-2}$, and (1.7) holds

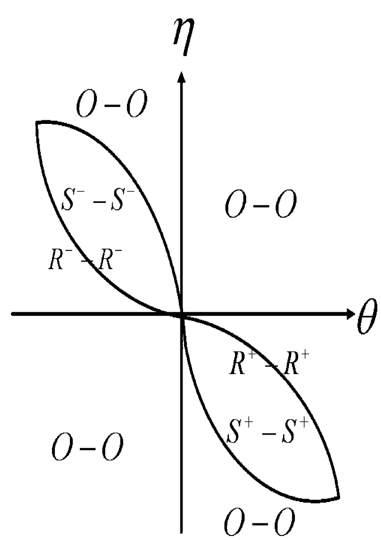

FIGURE 4. $p \geq \frac{2+\sigma}{n-2}$ and (1.8) holds

We illustrate the structure of solutions based on Theorem 1.5 in Figure 4.

Finally, we conclude this section with two specific examples related to (1.10) as follows.

Example 1.1. Suppose $K_1(r) = \frac{1}{1+r^2}$ on $(0, \infty)$. Then

(a) if $p > p_+(0, n) > \frac{n+2}{n-2}$, the structure of solutions for (1.10) with $K = K_1$ is the same as that depicted in Figure 2 based on Theorem 1.3;
(b) if $\frac{n}{n-2} < p < \frac{n+2}{n-2}$, the structure of solutions for (1.10) with $K = K_1$ is the same as that depicted in Figure 1 based on Theorem 1.2.

We remark that equation (1.10) with $K = K_1$ and $n = 3$ in the above example is well known as the Matukuma equation (see, e.g., [12], [14], [19]), which was introduced by Matukuma as a mathematical model for a global cluster of stars.

Example 1.2. Suppose $K_2(r) = r^\gamma$ on $(0, \infty)$. Then

(a) if $-2 < \gamma \leq \lambda$ and $p > p_+(\gamma, n)$, the structure of solutions for (1.10) with $K = K_2$ is the same as that illustrated in Figure 2 based on Theorem 1.3;
(b) if $\gamma \geq \lambda$ and $\frac{n+2+\gamma}{n-2} < p < p_-(2\lambda - \gamma, n)$, the structure of solutions for (1.10) with $K = K_2$ is the same as that illustrated in Figure 3 based on Theorem 1.4;
(c) if $\gamma = \lambda$, the structure of solutions for (1.10) with $K = K_2$ is the same as that illustrated in Figure 4 based on Theorem 1.5.

This article is organized as follows. In the next section, the complete verification of Theorem 1.1 will be given. In Section 3, we make preparations for the demonstrations of our remaining main conclusions. We introduce the Pohozaev identity and derive the properties of openness to conduct various types of solutions for (1.10). Moreover, we also attain the monotonicity of the numbers of zeros for solutions with respect to the η-axis in the (θ, η)-plane. In fact, these preliminary works are the essential elements to realize the structure of solutions. Finally, we present the complete proofs of Theorems 1.2-1.5 in Section 4.

2. Proof of Theorem 1.1

Throughout this section, we assume that (1.8) holds in (1.2), i.e., $\frac{rK'(r)}{K(r)} \equiv \lambda$ for $r > 0$. Let $u(r; \alpha)$ be a positive solution of (1.2) and $\phi(r) \equiv \phi(r; \alpha) = \frac{\partial u}{\partial \alpha}(r; \alpha)$. Then ϕ verifies the following linearized equation:

$$\begin{cases} \phi'' + \dfrac{n-1}{r}\phi' + pK(r)u^{p-1}\phi = 0, \quad r > 0, \\ \phi(0) = 1, \ \ \phi'(0) = 0. \end{cases} \tag{2.1}$$

In order to prove Theorem 1.1, we need several lemmas stated as follows.

Remark 2.1. In Theorem 1.1, the proof of (a) and the existence of a solution for the prescribed total curvature in (b) follow the fact of (b) in Theorem 1.5, which will be confirmed in Section 4. Therefore, we have that $r^{n-2}u(r; \alpha)$ tends to some positive constant (depending on α) as $r \to \infty$ for all $\alpha > 0$.

Lemma 2.1. *Suppose $u(r) \equiv u(r; \alpha)$ is a positive solution of* (1.2)*; then $\frac{ru'}{u}$ is decreasing on* $[0, \infty)$.

Proof. For any $c \in \mathbf{R}$, let $V_c(r) = V_c(r; \alpha) = ru'(r) + cu(r)$. Then V_c satisfies

$$\begin{cases} V_c'' + \dfrac{n-1}{r}V_c' + pK(r)u^{p-1}V_c = K(r)f(c; u), \quad r > 0, \\ V_c(0; \alpha) = c\alpha, \ \ V_c'(0; \alpha) = 0, \end{cases} \tag{2.2}$$

where

$$f(c;u) = [(p-1)c - (2+\lambda)]u^p.$$

From (1.2) and (2.2), we obtain that for $r \geq 0$,

$$\begin{aligned} r^{n-1}u^2(r)\left(\frac{ru'}{u}\right)' &= -r^{n-1}(V_0u' - uV_0') \\ &= -\int_0^r s^{n-1}(V_0\Delta u - u\Delta V_0)\,ds \\ &= -\int_0^r [(p-1)su' + (2+\lambda)u]s^{n-1}K(s)u^p(s)\,ds \\ &= \frac{1-p}{p+1}r^nK(r)u^{p+1}(r) \\ &\leq 0 \end{aligned}$$

and hence

$$\left(\frac{ru'(r)}{u(r)}\right)' \leq 0 \text{ for } r \geq 0.$$

We complete the proof of this lemma. □

Lemma 2.2. *Suppose $p > 3$ and $u(r;\alpha)$ is a positive solution of* (1.2). *Then $\phi(r) \equiv \phi(r;\alpha)$ has exactly one zero on $(0,\infty)$ and $\lim_{r\to\infty} r^{n-1}\phi'(r) > 0$.*

Proof. We prove this lemma by the following three steps.

Step 1. $\phi(r)$ changes sign on $(0,\infty)$. Suppose $\phi(r) > 0$ on $(0,\infty)$. Then $\phi(r)$ is bounded on $[0,\infty)$ by (2.1). Since $p > 3$ and due to (2.1) again, we see that $r^{n-1}\phi'(r)$ is also bounded on $(0,\infty)$. Therefore,

$$\begin{aligned} 0 &= \lim_{r\to\infty}\left(r^{n-1}\phi'(r)V_0(r) - r^{n-1}V_0'(r)\phi(r)\right) \\ &= \lim_{r\to\infty}\int_0^r \frac{(n-2)(p-1)}{2}s^{n-1}K(s)u^p(s)\phi(s)\,ds \\ &> 0, \end{aligned}$$

where $V_0(r)$ is defined as in the proof of Lemma 2.1. This leads to a contradiction, and we finish this step.

Step 2. $\phi(r)$ has exactly one zero on $(0,\infty)$.

Let r_1 be the first zero of $\phi(r)$ on $(0,\infty)$ and set $c = -\frac{r_1u'(r_1)}{u(r_1)}$. Then, applying the comparison method to V_c and ϕ, we get $c = \frac{2+\lambda}{p-1}$. Moreover, we also obtain that $\phi(r)$ has exactly one zero on $(0,\infty)$ via the comparison method, Lemma 2.1 and the fact that $f(r;c) \equiv 0$. Step 2 is established.

Step 3. $\lim_{r\to\infty} r^{n-1}\phi'(r) > 0$.

Let $w(r) = V_{n-2}(r) + C$, where V_{n-2} is defined as in the proof of Lemma 2.1, $C = -V_{n-2}(r_1)$, and r_1 is set in Step 1. Then $w(r)$ verifies

$$\begin{cases} w'' + \dfrac{n-1}{r}w' + pK(r)u^{p-1}w = Q(r)K(r)u^p, \\ w(0) = (n-2)\alpha + C, \quad w'(0) = 0, \end{cases} \tag{2.3}$$

where

$$Q(r) = (n-2)p - (n+\lambda) + \frac{C}{u}.$$

Since $\lim_{r\to\infty} Q(r) = -\infty$ and $Q'(r) \le 0$ on $[0,\infty)$, we see that $Q(0) > 0$ and $Q(r_1) < 0$. Hence

$$Q(r) < 0 \ \text{ on } \ [r_1, \infty). \tag{2.4}$$

From (2.4) and using the comparison method, we attain that

$$\begin{aligned} -V_{n-2}(r_1) \lim_{r\to\infty} r^{n-1}\phi'(r) &= \lim_{r\to\infty} (r^{n-1}\phi'(r)w(r) - r^{n-1}w'(r)\phi(r)) \\ &= -\int_{r_1}^{\infty} Q(s)s^{n-1}K(s)u^p(s)\phi(s)\, ds \\ &< 0, \end{aligned}$$

which shows that

$$\lim_{r\to\infty} r^{n-1}\phi'(r) > 0.$$

According to Steps 1-3, we complete the proof. □

Lemma 2.3. *Suppose $u(r;\alpha)$ is a positive solution of* (1.2). *If $p > 3$, then $\mathcal{T}(\alpha) \in C^1(0,\infty)$ and*

$$\begin{aligned} \mathcal{T}'(\alpha) &= p\int_0^{\infty} s^{n-1}K(s)u^{p-1}(s;\alpha)\phi(s;\alpha)\, ds \\ &= -\lim_{r\to\infty} r^{n-1}\phi'(r;\alpha) \end{aligned}$$

for $\alpha > 0$.

Proof. Let $\gamma(\alpha) = \lim_{r\to\infty} r^{n-2}u(r;\alpha)$. Then $\gamma(\alpha) \in C(0,\infty)$ by Lemma 2.1 and the Lebesgue dominated convergence theorem. Moreover, since $(r^{n-2}u(r;\alpha))' \ge 0$ on $(0,\infty)$, we obtain that $\mathcal{T}(\alpha) \in C[0,\infty)$ by the Lebesgue dominated convergence theorem again. □

Claim. $\lim_{r\to\infty} \phi(r;\alpha)$ is continuous for $\alpha \in (0,\infty)$.

Proof of the Claim. Let $u_i(r) = u(r;\alpha_i)$ and $\phi_i(r) = \phi(r;\alpha_i)$ for $\alpha_i > 0$, $i = 1,2$. Then, by (2.1),

$$\begin{aligned} \big|\phi_1(r) - \phi_2(r)\big| &\le \int_0^r ps^{n-1}K(s)u_1^{p-1}(s)\big|\phi_1(s) - \phi_2(s)\big|\, ds \\ &\qquad + \int_0^r ps^{n-1}K(s)\big|u_1^{p-1}(s) - u_2^{p-1}(s)\big|\big|\phi_2(s)\big|\, ds \\ &\le \int_0^r ps^{n-1}K(s)u_1^{p-1}(s)\big|\phi_1(s) - \phi_2(s)\big|\, ds \\ &\qquad + C_2\int_0^{\infty} ps^{n-1}K(s)\big|u_1^{p-1}(s) - u_2^{p-1}(s)\big|\, ds \end{aligned}$$

for some $C_2 > 0$. Thus for $r > 0$,

$$\big|\phi_1(r) - \phi_2(r)\big| \le C_2 g(\alpha_1,\alpha_2)e^{\int_0^{\infty} ps^{n-1}K(s)u_1^{p-1}(s)\, ds}, \tag{2.5}$$

where

$$g(\alpha_1,\alpha_2) = \int_0^{\infty} ps^{n-1}K(s)\big|u_1^{p-1}(s) - u_2^{p-1}(s)\big|\, ds. \tag{2.6}$$

Combining (2.6) and the fact that $\mathcal{T}(\alpha) \in C(0,\infty)$, and by using the Lebesgue dominated convergence theorem again, we have that $g(\alpha_1,\alpha_2) \to 0$ as $\alpha_1 \to \alpha_2$. Therefore, $|\phi_1(r) - \phi_2(r)| \to 0$ uniformly on $[0,\infty)$ as $\alpha_1 \to \alpha_2$ from (2.5). We finish the proof of this claim.

From the above claim and standard arguments, we easily obtain that $\mathcal{T}(\alpha) \in C^1(0,\infty)$, and the proof of this lemma is complete. □

Due to Lemma 2.2, for any $\alpha > 0$ there exists a unique $\widetilde{R}(\alpha) > 0$ such that $\phi(\widetilde{R}(\alpha);\alpha) = 0$. The following fact gives us the monotonicity for $\widetilde{R}(\alpha)$ with respect to $\alpha > 0$.

Lemma 2.4. *$\widetilde{R}(\alpha)$ is strictly decreasing on $(0,\infty)$.*

Proof. Let $\alpha_2 > \alpha_1 > 0$ and there exists $0 < R \le \infty$ such that

$$\begin{cases} u(r;\alpha_2) > u(r;\alpha_1) & \text{for } 0 \le r < R, \\ u(R;\alpha_2) = u(R;\alpha_1). \end{cases}$$

We divide the proof into two steps.

Step 1. $R \le \widetilde{R}(\alpha_1)$.

To establish this step, we need the following assertion.

Claim. If $R \ge \widetilde{R}(\alpha_1)$, then $\widetilde{R}(\alpha_2) < \widetilde{R}(\alpha_1)$.

Proof of the Claim. On the contrary, suppose $\widetilde{R}(\alpha_2) \ge \widetilde{R}(\alpha_1)$. Then applying the comparison method to $\phi(r;\alpha_1)$ and $\phi(r;\alpha_2)$, we obtain that

$$\begin{aligned} 0 &\ge \widetilde{R}^{n-1}(\alpha_1)\phi'(\widetilde{R}(\alpha_1);\alpha_1)\phi(\widetilde{R}(\alpha_1);\alpha_2) \\ &= \int_0^{\widetilde{R}(\alpha_1)} r^{n-1} pK(r)\phi(r;\alpha_1)\phi(r;\alpha_2)\big(u^{p-1}(r;\alpha_2) - u^{p-1}(r;\alpha_1)\big)\,dr \\ &> 0. \end{aligned}$$

This contradiction completes the proof of this claim. □

Now, suppose $R > \widetilde{R}(\alpha_1)$. Then there exist $\widetilde{R}(\alpha_1) < R_0 < R$ and $0 < \varepsilon < \alpha_2$ such that

$$u(r;\alpha) > u(r;\alpha_1) + \varepsilon \quad \text{on } [0,R_0] \text{ for } \alpha_2 - \varepsilon < \alpha \le \alpha_2. \tag{2.7}$$

Combining (2.7) and the claim above with the continuous dependence of solutions on initial values, we see that, for $\alpha_1 < \alpha \le \alpha_2$,

$$\begin{cases} \widetilde{R}(\alpha) < \widetilde{R}(\alpha_1), \\ u(R_0;\alpha) > u(R_0;\alpha_1) + \varepsilon, \\ u(r;\alpha) > u(r;\alpha_1) & \text{on } [0,R_0], \end{cases}$$

which implies

$$u(R_0;\alpha_1) = \lim_{\alpha \searrow \alpha_1} u(R_0;\alpha) \ge u(R_0;\alpha_1) + \varepsilon.$$

This contradiction finishes this step.

Next, let $\delta_0 > 0$ be fixed and define

$$\widehat{R} = \inf\big\{\widetilde{R}(\alpha_1 + \delta) \ : \ \text{for } \ 0 \le \delta \le \delta_0\big\}.$$

Then $\widehat{R} = \widetilde{R}(\alpha_1 + \delta_1) > 0$ for some $\delta_1 \in [0, \delta_0]$ due to the fact that $\phi(0; \alpha) = 1$ for $\alpha > 0$ and the continuity of $\phi(r; \alpha)$ with respect to α.

Step 2. $\delta_1 = \delta_0$.

If $\delta_1 < \delta_0$, then

$$u(r; \alpha_1 + \delta_1 + \delta) > u(r; \alpha_1 + \delta_1) \text{ for } 0 \le r \le \widehat{R}, \ 0 < \delta < \delta_0 - \delta_1,$$

which contradicts the assertion mentioned in Step 1, and thus this step is done.

Since $\alpha_1 > 0$ and $\delta_0 > 0$ are arbitrary, we complete the proof of this lemma owing to Step 2. □

Lemma 2.5. *Suppose* $u(r; \alpha_1)$ *and* $u(r; \alpha_2)$ *are solutions of* (1.2) *with* $\alpha_2 > \alpha_1 > 0$. *If* $u(r; \alpha_1)$ *and* $u(r; \alpha_2)$ *intersect on* $(0, \infty)$, *then* $\widetilde{R}(\alpha_2) < r_0$, *where* $r_0 = r_0(\alpha_2)$ *is the first intersection point of* $u(r; \alpha_1)$ *and* $u(r; \alpha_2)$.

Proof. On the contrary, suppose $r_0 \le \widetilde{R}(\alpha_2)$. Then we set $w(r) = u(r; \alpha_2) - u(r; \alpha_1)$ and $w(r)$ satisfies

$$w''(r) + \frac{n-1}{r} w'(r) + pK(r) u^{p-1}(r; \alpha_2) w(r) > 0 \text{ on } [0, r_0).$$

Applying the comparison theory to $w(r)$ and $\phi(r; \alpha_2)$, we obtain

$$\begin{aligned} 0 &\ge r_0^{n-1} w'(r_0) \phi(r_0; \alpha_2) \\ &= \int_0^{r_0} \left[\left(r^{n-1} w'(r) \right)' \phi(r; \alpha_2) - \left(r^{n-1} \phi'(r; \alpha_2) \right)' w(r) \right] dr \\ &> 0, \end{aligned}$$

which is impossible. The proof of this lemma is complete. □

Now, we are in a position to show Theorem 1.1.

Proof of Theorem 1.1. Suppose $u(r; \alpha_1)$ and $u(r; \alpha_2)$ intersect at least twice on $(0, \infty)$ for some $\alpha_2 > \alpha_1 > 0$. Then we define the set

$$S = \{\alpha_0 \in (\alpha_1, \alpha_2] \ : \ u(r; \alpha) \text{ and } u(r; \alpha_1) \text{ intersect at least twice on } (0, \infty) \text{ for all } \alpha \in [\alpha_0, \alpha_2]\}.$$

Obviously, S is not empty, and thus there exists $\alpha^* \in [\alpha_1, \alpha_2)$ such that $\alpha^* = \inf S$. We want to show that $\alpha^* = \alpha_1$. If not, then $u(r; \alpha^*)$ and $u(r; \alpha_1)$ intersect exactly once on $(0, \infty)$ because of the uniqueness of solutions with respect to initial values. Hence, there exists a sequence $\{\alpha_i^*\} \subset S$ such that $\alpha_i^* \to \alpha^*$ and $\widetilde{R}(\alpha_i^*) \to \infty$ as $i \to \infty$. However, this is impossible due to Lemma 2.4. Therefore, we assert that $\alpha^* = \alpha_1$ and $S = (\alpha_1, \alpha_2]$.

Moreover, from Lemma 2.4 again,

$$r_2(\alpha) < \widetilde{R}(\alpha_1) \text{ for } \alpha \in (\alpha_1, \alpha_2], \tag{2.8}$$

where $r_2(\alpha)$ is the second intersection point of $u(r; \alpha)$ and $u(r; \alpha_1)$. Since $r_2(\alpha)$ is strictly decreasing on $(\alpha_1, \alpha_2]$ by Lemma 2.5, there exists a sequence $\{\bar{\alpha}_i\}$ such that $\bar{\alpha}_i \to \alpha_1$ and $r_2(\bar{\alpha}_i) \to \hat{r}$ as $i \to \infty$ for some $\hat{r} < \widetilde{R}(\alpha_1)$ due to (2.8) and Lemma 2.4. Hence, $\phi(\hat{r}, \alpha_1) = 0$, which yields a contradiction, and we conclude that any two distinct solutions of (1.2) intersect exactly once on $(0, \infty)$. In addition, for any

$\alpha_0 > 0$, we also obtain that $R(\alpha)$ is strictly decreasing on (α_0, ∞) and tends to zero as $\alpha \to \infty$ by Lemmas 2.5 and 2.4. Therefore, we prove (c).

Finally, combining Lemmas 2.2-2.3, we attain that $\phi(r;\alpha)$ has exactly one zero on $[0,\infty)$ and $\mathcal{T}'(\alpha) < 0$ for all $\alpha > 0$. Then, the uniqueness consequence in (b) is proved. Indeed, if $u(r;\alpha_1) > u(r;\alpha_2)$ on $[0,\infty)$, then $\mathcal{T}(\alpha_1) > \mathcal{T}(\alpha_2)$. This is impossible since $\alpha_1 > \alpha_2$. We complete the proof of this theorem except (a) and the existence consequence in (b) which will be assured in Section 4. □

3. Zeros and structure of entire solutions

In this section, we derive some fundamental properties of solutions for (1.10) involving the openness consequences about the regions of initial data corresponding to certain types of solutions in the (θ, η)-plane and the increment of the number of zeros with respect to the η-axis.

To this end, we need to introduce an auxiliary function related to solutions of (1.10), which yields a well-known identity to help us characterize the behaviors of solutions at the origin and infinity. For any solution $u(r)$ of (1.10), define

$$P(r;u) = \frac{r^{n-1}u'(r)}{2}\{ru'(r) + (n-2)u(r)\} + \frac{r^n K(r)}{p+1}|u|^{p+1}(r) \tag{3.1}$$

for $r > 0$. Then, it is easy to verify that

$$\frac{d}{dr}P(r;u) = \frac{r^{n-1}K(r)}{p+1}\left\{\frac{rK'(r)}{K(r)} - \lambda\right\}|u|^{p+1}(r) \tag{3.2}$$

for $r > 0$, which is known as the Pohozaev identity associated with the solution $u(r)$. Occasionally, we denote $P(r; u(r;\theta,\eta))$ by $P(r;\theta,\eta)$ for convenience.

Lemma 3.1. *Suppose $u(r)$ is a solution of* (1.10). *Then the following properties hold.*

(a) *If $u(r) > 0$ (resp., $u(r) < 0$) on $[R,\infty)$ for some $R > 0$, then $ru'(r) + (n-2)u(r) > 0$ (resp., $ru'(r) + (n-2)u(r) < 0$) on $[R_1,\infty)$ for some $R_1 \geq R$.*

(b) *If $p \geq \frac{2+\sigma}{n-2}$ and $u(r) > 0$ (resp., $u(r) < 0$) on $(0,R]$ for some $R > 0$, then $ru'(r)+(n-2)u(r) > 0$ (resp., $ru'(r)+(n-2)u(r) < 0$) on $(0,R_1]$ for some $R_1 \leq R$.*

Proof. In the proofs of (a) and (b), we only show the case of $u(r) > 0$; the proofs for the case of $u(r) < 0$ are similar.

(a) On the contrary, suppose there exists $r_0 \geq R$ such that $r_0u'(r_0) + (n-2)u(r_0) \leq 0$. Then, from (1.10), $u(r)$ verifies

$$[ru'(r) + (n-2)u(r)]' = -rK(r)|u|^{p-1}u < 0 \quad \text{for } r \geq R,$$

which implies

$$ru'(r) < ru'(r) + (n-2)u(r) \leq -C \quad \text{for } r > r_0 \text{ large}$$

with some $C > 0$. Therefore, we deduce $u(r) < 0$ for $r > r_0$ large, which yields a contradiction.

(b) We use the so-called Kelvin transformation. Set $s = r^{-1}$ and $v(s) = r^{n-2}u(r)$. Then $v(s)$ satisfies

$$v''(s) + \frac{n-1}{s}v'(s) + L(s)|v|^{p-1}v(s) = 0 \quad \text{for } s > 0, \tag{3.3}$$

where

$$L(s) = s^{2\lambda} K\left(\frac{1}{s}\right), \tag{3.4}$$

and $v(s) > 0$ on $(\frac{1}{R}, \infty)$. Due to (3.3)-(3.4) and the fact of $K(r) \sim r^\sigma$ at $r = 0$, there exists $0 < R_1 \le R$ such that $s^{n-1}v'(s) < 0$ on $[\frac{1}{R_1}, \infty)$ if $p \ge \frac{2+\sigma}{n-2}$. Therefore,

$$ru'(r) + (n-2)u(r) = -s^{n-1}v'(s) > 0 \text{ for } r \in (0, R_1].$$

□

Remark 3.1. We note that $L(s)$, defined in (3.4), also satisfies (1.3) and (1.4) with respect to s.

The following assertions give us the classification of solutions of (1.10) in terms of $P(r;u)$, which plays a significant role in clarifying the whole structure of solutions.

Lemma 3.2. *Suppose $u(r)$ is a solution of* (1.10). *Then the following properties hold.*

(a) *If $u(r)$ is of Type $*$-$R^\pm$, then there exists a sequence $\{r_j\}$ such that $r_j \to \infty$ and $P(r_j;u) \to 0$ as $j \to \infty$.*
(b) *If $u(r)$ is of Type $*$-$S^\pm$, then there exists a sequence $\{r_j\}$ such that $r_j \to \infty$ as $j \to \infty$ and $P(r_j;u) < 0$ for all j.*
(c) *If $u(r)$ is of Type $*$-O, then there exists a sequence $\{r_j\}$ such that $r_j \to \infty$ as $j \to \infty$ and $P(r_j;u) > 0$ for all j.*
(d) *If $p \ge \frac{2+\sigma}{n-2}$ and $u(r)$ is of Type $S^\pm$-$*$, then there exists a sequence $\{r_j\}$ such that $r_j \to 0$ as $j \to \infty$ and $P(r_j;u) < 0$ for all j.*
(e) *If $u(r)$ is of Type O-$*$, then there exists a sequence $\{r_j\}$ such that $r_j \to \infty$ as $j \to 0$ and $P(r_j;u) > 0$ for all j.*
(f) *If $u(r)$ is of Type $R^\pm$-$*$, then $P(r;u) \to 0$ as $r \to 0$.*

Proof. We refer the proofs of (a)-(c) to Lemma 2.3 in [16] or Lemmas 2.3-2.4 in [5]. Moreover, by using the Kelvin transform as described in the proof of Lemma 3.1(b), we omit the details of the proofs for (d)-(f). □

In order to manage the zeros of solutions and simplify the statements, we say that a solution of (1.10) is of Type X^{k_1}-Y^{k_2} if it is of Type X-Y and has exactly k_1 zeros on $(0,1]$ and k_2 zeros on $[1,\infty)$, where $\mathrm{X}, \mathrm{Y} \in \{\mathrm{R}^\pm, \mathrm{S}^\pm\}$. Obviously, a solution $u(r;\theta,\eta)$ (resp., $u(r;0,\eta)$) which is of Type X^{k_1}-Y^{k_2} is of Type $\mathrm{X}\underline{\ k_1+k_2\ }\mathrm{Y}$ (resp., $\mathrm{X}\underline{\ k_1+k_2-1\ }\mathrm{Y}$) if $\theta \ne 0$.

The following lemma is also crucial for us to conduct the structure of solutions, which depicts the openness of the regions of initial data associated with Types $*$-$\mathrm{S}^\pm$ and $\mathrm{S}^\pm$-$*$ of solutions.

Lemma 3.3. *The following facts concerning solutions of* (1.10) *hold.*

(a) *Suppose $u(r;\theta_0,\eta_0)$ is of Type $*$-$S^{\pm k}$ with either $\theta_0 > 0$ or $\theta_0 < 0$ and*

$$\frac{d}{dr}P(r;u) \le 0 \text{ for } r \ge R \text{ and all solutions } u \text{ of } (1.10) \tag{3.5}$$

for some $R > 0$. Then there exists $\delta > 0$ such that $u(r;\theta,\eta)$ is also of Type $*$-$S^{\pm k}$ for $(\theta,\eta) \in B_\delta((\theta_0,\eta_0))$.

(b) *Suppose* $u(r;\theta_0,\eta_0)$ *is of Type* $S^{\pm k}$-$*$ *with either* $\theta_0>0$ *or* $\theta_0<0$ *and*

$$\frac{d}{dr}P(r;u)\geq 0 \text{ for } 0<r\leq R \text{ and all solutions } u \text{ of } (1.10) \tag{3.6}$$

for some $R>0$. *Then there exists* $\delta>0$ *such that* $u(r;\theta,\eta)$ *is also of Type* $S^{\pm k}$-$*$ *for* $(\theta,\eta)\in B_\delta((\theta_0,\eta_0))$.

Proof. (a) We divide this proof into two steps.

Step 1. There exists $\delta_0>0$ such that $u(r;\theta,\eta)$ is of Type $*$-$S^\pm$ for $(\theta,\eta)\in B_{\delta_0}((\theta_0,\eta_0))$.

From Lemma 3.2(b), there exists a sequence $\{r_j\}$ such that $r_j\to\infty$ as $j\to\infty$ and $P(r_j;\theta_0,\eta_0)<0$ for all j. Choosing j_0 so that $r_{j_0}>R$, then $P(r_{j_0};\theta,\eta)<0$ for $(\theta,\eta)\in B_{\delta_0}((\theta_0,\eta_0))$ with some $\delta_0>0$. Hence, (3.5) and (3.2) show that $P(r;\theta,\eta)<0$ and $P(r;\theta,\eta)$ is nonincreasing for $r\geq r_{j_0}$ and $(\theta,\eta)\in B_{\delta_0}((\theta_0,\eta_0))$. We finish this step by Lemma 3.2 again.

Step 2. There exists $0<\delta<\delta_0$ such that $u(r;\theta,\eta)$ has exactly k zeros on $[1,\infty)$ for $(\theta,\eta)\in B_\delta((\theta_0,\eta_0))$.

Let $r_i>1$ $(i=1,\cdots,k)$ be the i-th zero of $u(r;\theta_0,\eta_0)$ on $[1,\infty)$. Choosing $0<\varepsilon<\min_{1\leq j\leq k-1}\left\{\frac{r_{j+1}-r_j}{2},r_1-1\right\}$ small and $b>r_k+\varepsilon$ such that

$$u(r;\theta_0,\eta_0)\neq 0 \text{ and } u'(r;\theta_0,\eta_0)\neq 0 \quad \text{on } I, \tag{3.7}$$

where

$$I=\bigcup_{j=1}^{k-1}[r_j+\varepsilon,r_{j+1}-\varepsilon]\cup[1,r_1-\varepsilon]\cup[r_k+\varepsilon,b],$$

then there exists $0<\delta_b<\delta_0$, where δ_0 is selected in Step 1, such that $u(r;\theta,\eta)$ has at least k zeros on $[1,\infty)$ for $(\theta,\eta)\in B_{\delta_b}((\theta_0,\eta_0))$.

Without loss of generality, suppose there exists a sequence $\{(\theta_m,\eta_m)\}$ with $(\theta_m,\eta_m)\to(\theta_0,\eta_0)$ as $m\to\infty$ such that $u(r;\theta_m,\eta_m)$ has at least $k+1$ zeros on $[1,\infty)$ and (3.7) holds for $u(r;\theta_m,\eta_m)$ for all m.

Claim. There exists a sequence $\{c_m\}$ with $u(c_m;\theta_m,\eta_m)=0$ for all m such that $c_m\to\infty$ as $m\to\infty$.

Proof of the Claim. On the contrary, if all zeros on $[1,\infty)$ of $u(r;\theta_m,\eta_m)$ for all m stay in a compact set $[a_1,a_2]$ with some $1\leq a_1<b<a_2<\infty$, then, passing to a subsequence of $\{(\theta_m,\eta_m)\}$ if necessary, there exists a positive number $1\leq q\leq k$ such that $u(r;\theta_m,\eta_m)$ has at least two zeros in $(r_q-\varepsilon,r_q+\varepsilon)$ for all m. Hence, we have $u'(\hat{r}_m;\theta_m,\eta_m)=0$ with some $\hat{r}_m\in(r_q-\varepsilon,r_q+\varepsilon)$ for all m.

Now, passing to a subsequence if necessary, let $\hat{r}_m$ converge to $\hat{r}\in[r_q-\varepsilon,r_q+\varepsilon]$ as $m\to\infty$. Then we obtain that

$$\begin{aligned} 0 &= \lim_{m\to\infty}u'(\hat{r}_m;\theta_m,\eta_m)\\ &= u'(\hat{r};\theta_0,\eta_0)\\ &\neq 0, \end{aligned}$$

which yields a contradiction. The proof of this claim is complete.

Finally, for all m we have that

$$P(c_m;\theta_m,\eta_m)=\frac{1}{2}c_m^{n-1}[u'(c_m;\theta_m,\eta_m)]^2>0,$$

which is impossible due to the Claim and the fact in the proof of Step 1. Therefore, Step 2 is established.

According to Steps 1 and 2, we complete the proof of (a).

(b) By using the Kelvin transformation in (3.3)-(3.4) and the assertion of (a), we omit the details of the proof for (b). □

Lemma 3.4. *Suppose* (3.5) (*resp.,* (3.6)) *holds. Let* C *be a compact and connected set in* $\{(\theta,\eta) : \theta > 0$ (*resp.,* $\theta < 0)\}$, *and let* (1.10) *have no solutions of Type* $*$-$R^{\pm}$ (*resp.,* $R^{\pm}$-$*$) *associated with initial data in* C. *If* C *contains an initial data corresponding to a solution of Type* $*$-$S^{\pm k}$ (*resp.,* $S^{\pm k}$-$*$), *then all solutions of* (1.10) *associated with initial data in* C *are also of Type* $*$-$S^{\pm k}$ (*resp.,* $S^{\pm k}$-$*$).

Proof. By means of Lemma 3.3(a), we may consider, without loss of generality, that C exactly contains the initial data corresponding to solutions of Types $*$-O, $*$-S^{+k}, and $*$-S$^{-(k+1)}$. Writing $C = O \cup S_1 \cup S_2$, where O, S_1, and S_2 are the respective regions corresponding to the above three types of solutions, then $O \cap \partial\overline{S}_1 \neq \emptyset$ or $O \cap \partial\overline{S}_2 \neq \emptyset$ due to Lemma 3.3(a) again. Suppose $O \cap \partial\overline{S}_1 \neq \emptyset$. Hence, there exist $\{(\theta_i, \eta_i)\} \subset S_1$ and $(\theta,\eta) \in O$ such that $(\theta_i,\eta_i) \to (\theta,\eta)$ as $i \to \infty$, which is impossible by the continuous dependence of solutions on initial data. Therefore, C only contains the initial data corresponding to solutions of either Type $*$-S^{+k} or Type $*$-S$^{-(k+1)}$, which leads to a contradiction as well.

The respective cases can be done by using the Kelvin transformation and Lemma 3.3(b), and we finish the proof of this lemma. □

The following two lemmas are related to the monotonicity of the number of zeros on $[1,\infty)$ and $(0,1]$, respectively, for solutions of (1.10) with respect to the η-axis.

Lemma 3.5. *Suppose* (3.5) *holds. Let* C *be a compact and connected set in the* (θ,η)-*plane which is contained entirely in the lower* (*resp., upper*) *side of the* θ-*axis, and the intersection of* C *with the* η^--*axis* (*resp.,* η^+-*axis*) *is nonempty and connected. If the solution* $u(r;\theta,\eta)$ *of* (1.10) *is of Type* $*$-S^+ *for all* $(\theta,\eta) \in C$, *then the following are true.*

(a) *The number of zeros on* $[1,\infty)$ *for* $u(r;0,\eta)$ *is the same as that for* $u(r;\theta,\eta)$ *if* $(\theta,\eta),(0,\eta) \in C$ *with* $\theta > 0$ (*resp.,* $\theta < 0$).
(b) $u(r;0,\eta)$ *has exactly one more zero on* $[1,\infty)$ *than* $u(r;\theta,\eta)$ *if* $(\theta,\eta),(0,\eta) \in C$ *with* $\theta < 0$ (*resp.,* $\theta > 0$).

The situation for Type $*$-S^- *is the same as that for Type* $*$-S^+.

Proof. We first establish the following assertion.

Claim. If $u(r;0,\eta_0)$ is of Type $*$-S^{+k} for some $(0,\eta_0) \in C$, then $u(r;\theta,\eta)$ is also of Type $*$-S^{+k} for all $(\theta,\eta) \in C$ with $\theta > 0$.

Proof of the Claim. On the contrary, we consider two cases below:

Case 1. $u(r;\theta_1,\eta_1)$ *is of Type* $*$-$S^{+(k+m)}$ *for some* $(\theta_1,\eta_1) \in C$ *with* $\theta_1 > 0$, $\eta_1 < 0$, *and* m *is an even positive integer.* Then Lemma 3.4 implies that $u(r;\theta,\eta)$ is of Type $*$-S$^{+(k+m)}$ for all $(\theta,\eta) \in C$ with $\theta > 0$. Now, choose a sequence $\{(\theta_i,\eta_i)\} \subset C$ with $\theta_i > 0$ for all i such that $(\theta_i,\eta_i) \to (0,\eta_0)$ as $i \to \infty$. Then, by virtue of the Claim in Step 2 of the proof of Lemma 3.3(a), we see that $r_i \to \infty$ as $i \to \infty$, where r_i is the largest zero of $u(r;\theta_i,\eta_i)$ for all i. Moreover, for all i, we obtain that

$$P(r_i;\theta_i,\eta_i)=\frac{1}{2}r_i^{n-1}[u'(r_i;\theta_i,\eta_i)]^2>0,$$

which leads to a contradiction by Lemma 3.2. Hence, this case cannot occur. The situation that $u(r;\theta_1,\eta_1)$ is of Type $*$-S$^{-(k+m)}$ with m being an odd positive integer is similar.

Case 2. *$u(r;\theta_2,\eta_2)$ is of Type $*$-$S^{+(k-m)}$ for some $(\theta_2,\eta_2)\in C$ with $\theta_2>0$, $\eta_2<0$, and $1\le m\le k$ being an even positive integer.* Then $u(r;\theta,\eta)$ is of Type $*$-S$^{+(k-m)}$ for all $(\theta,\eta)\in C$ with $\theta>0$ by Lemma 3.4 again. Select a sequence $\{(\theta_j,\eta_j)\}\subset C$ with $\theta_j>0$ for all j such that $(\theta_j,\eta_j)\to(0,\eta_0)$ as $j\to\infty$. Then depending on the continuity of solutions with respect to initial data, we conclude that $m=1$ and $r_j\to\hat{r}$ as $j\to\infty$, where r_j and $\hat{r}$ are the largest zeros of $u(r;\theta_j,\eta_j)$ and $u(r;0,\eta_0)$, respectively. Now, without loss of generality, we may assume that $u(r;0,\eta_0)>0$ for $r>\hat{r}$. Then, for j large, we have that $u(r;\theta_j,\eta_j)>0$ for $r>\bar{r}$ with some $\bar{r}>0$. This is impossible since $u(r;\theta_j,\eta_j)<0$ for $r>r_j$ for all j. Therefore, this case cannot happen as well. The situation that $u(r;\theta_2,\eta_2)$ is of Type $*$-S$^{-(k-m)}$ with $1\le m\le k$ being an odd positive integer is similar. We complete the proof of this claim.

We note that the above Claim itself also assures that $u(r;0,\eta)$ is of Type $*$-S^{+k} for all $(0,\eta)\in C$ if $u(r;0,\eta_0)$ is of Type $*$-S^{+k} for some $(0,\eta_0)\in C$.

Using similar arguments based on the proof of the above Claim, the remaining respective situations can also be attained.

We complete the proof of Lemma 3.5. □

Lemma 3.6. *Suppose* (3.6) *holds. Let C be a compact and connected set in the (θ,η)-plane which is contained entirely in the lower* (*resp., upper*) *side of the θ-axis, and the intersection of C with the η^--axis* (*resp., η^+-axis*) *is nonempty and connected. If the solution $u(r;\theta,\eta)$ of* (1.10) *is of Type S^+-$*$ for all $(\theta,\eta)\in C$, then the following are true.*

(a) *The number of zeros on* $(0,1]$ *for $u(r;0,\eta)$ is the same as that for $u(r;\theta,\eta)$ if $(\theta,\eta),(0,\eta)\in C$ with $\theta<0$* (*resp., $\theta>0$*).
(b) *$u(r;0,\eta)$ has exactly one more zero on* $(0,1]$ *than $u(r;\theta,\eta)$ if $(\theta,\eta),(0,\eta)\in C$ with $\theta>0$* (*resp., $\theta<0$*).

The situation for Type S^--$*$ is the same as that for Type S^+-$*$.

Proof. We obtain these results by using the Kelvin transformation and Lemma 3.5. □

4. Proofs of Theorems 1.2-1.5

We give the complete verifications of our main consequences, Theorems 1.2-1.5, in this section.

Proof of Theorem 1.2. From Step 1 in the proof of Lemma 3.3(a), we conclude (a) and (b). Moreover, combining Lemmas 3.4-3.6, the assertions of (d) and (e) are attained. Finally, (c) follows Theorem A, and we complete the proof of this theorem. □

To demonstrate Theorems 1.3 and 1.4, an auxiliary lemma is needed.

Lemma 4.1. *Suppose* (1.6) (*resp.,* (1.7)) *holds and* $u(r;\alpha)$ (*resp.,* $u(r;\beta)$) *is a solution of* (1.2) (*resp.,* (1.11)). *If* $\sup_{\alpha>0}|u(1;\alpha)|$ (*resp.,* $\sup_{\beta>0}|u(1;\beta)|$) *is finite, then so is* $\sup_{\alpha>0}|u'(1;\alpha)|$ (*resp.,* $\sup_{\beta>0}|u'(1;\beta)|$).

Proof. Suppose there exists a sequence $\{\alpha_i\}$ such that $|u'(1;\alpha_i)|\to\infty$ as $i\to\infty$. Since (1.6) holds and by (3.2), we have that $P(1;u(r;\alpha_i))\le 0$; that is,

$$\frac{1}{2}u'(1;\alpha_i)[u'(1;\alpha_i)+(n-2)u(1;\alpha_i)]\le -\frac{K(1)}{p+1}|u(1;\alpha_i)|^{p+1}$$

for all i. Therefore,

$$\begin{aligned}\infty &= \frac{1}{2}\lim_{i\to\infty}u'(1;\alpha_i)[u'(1;\alpha_i)+(n-2)u(1;\alpha_i)]\\ &\le -\frac{K(1)}{p+1}\lim_{i\to\infty}|u(1;\alpha_i)|^{p+1}\\ &< \infty,\end{aligned}$$

which yields a contradiction.

The respective situation can be obtained in a similar way. Indeed, combining (1.7), (3.2), and the fact that $\lim_{r\to\infty}P(r;u(r;\beta))=0$ for all $\beta>0$, we also get that $P(1;u(r;\beta))\le 0$ for all $\beta>0$. We complete the proof of this lemma. □

Now, we are in a position to show the remaining main results described in Section 1.

Proof of Theorem 1.3. Without loss of generality, we consider the curves Γ_1^+ and Γ_2^+. Since $\frac{dP}{dr}\le 0$ by (1.6), $u(r;\theta,\eta)$ is a solution of Type R$^+$-S$^+$ and has no zeros on $(0,\infty)$ for $(\theta,\eta)\in\Gamma_1^+$; that is, $\Gamma_1^+\cap\Gamma_2^+=\emptyset$. Now, according to Lemma 3.5, we see that the origin cannot be the limit point of $\gamma_2^+(\beta)$ as $\beta\to\infty$. Suppose $\lim_{\beta\to\infty}\gamma_2^+(\beta)=\boldsymbol{p}\neq\mathbf{0}$. Then $u(r;\boldsymbol{p})$ must be either of Type $*$-O or Type $*$-S$^\pm$, and $u(r;\gamma_2^+(\beta))$ has at most k zeros on $[1,\infty)$ for all $\beta>0$ for some $k\in\mathbf{N}$. If $u(r;\boldsymbol{p})$ is of Type $*$-O, then there exists a neighborhood U of $\boldsymbol{p}$ such that $u(r;\theta,\eta)$ has at least $k+1$ zeros on $[1,\infty)$ for all $(\theta,\eta)\in U$, which is impossible. Hence, $u(r;\boldsymbol{p})$ is of Type $*$-S$^\pm$. However, this contradicts the fact in Lemma 3.3(a), and we prove Theorem 1.3(a).

Moreover, by means of Theorem 1 and Corollary 4.3 in [10] with Lemma 4.1, (b) is proved. Next, due to Corollary 4.3 in [10], we get that $u(r;\theta,\eta)$ is of Type O-$*$ for all (θ,η) which does not lie on $\Gamma_1^\pm$. To accomplish (c), we need the following fact.

Claim. A sequence of solutions of Type $*$-$S^\mp$ cannot converge to a solution of Type $*$-$S^\pm$.

Proof of the Claim. Let $u_i(r)\equiv u(r;\theta_i,\eta_i)$ be of Type $*$-S$^-$ for all i. Suppose $(\theta_i,\eta_i)\to(\theta,\eta)$ as $i\to\infty$ and $u(r)\equiv u(r;\theta,\eta)$ is of Type $*$-S$^+$. Without loss of generality, we may assume that $u_i(r)$ has zeros on $(1,\infty)$ for all i. Set r_i to be the largest zero of $u_i(r)$ for all i. Then it is easy to see that r_i goes to the infinity as $i\to\infty$. From (1.6), we have that $P(r;u_i)>0$ on $(0,r_i)$ for all i, which implies $P(r;u)\ge 0$ on $(0,\infty)$. This contradicts the fact that $u(r)$ is of Type $*$-S$^+$, and we complete the proof of this claim and hence (c).

Finally, (d) follows (1.6) and Lemma 3.2. We finish the proof of Theorem 1.3. □

Proof of Theorem 1.4. We first note that (1.10) can be transformed into (3.3). Let

$$\begin{cases} m = \dfrac{2\lambda - \ell + 2}{p-1};\ L = [m(n-2-m)]^{\frac{1}{p}-1}; \\ b_0 = n-2-2m;\ c_0 = (p-1)L^{p-1}. \end{cases}$$

To prove (b), since $p < p_-$, we have that $b_0^2 - 4c_0 > 0$. Combining Theorem 1 and Corollary 4.3 in [10] with Lemma 4.1, (b) is attained. The remaining assertions follow from Theorem 1.3, and we complete this proof. □

Proof of Theorem 1.5. Since (1.8) holds, $P(r;\theta,\eta)$ equals identically to some constant depending on (θ,η). Hence, we get that $\Gamma_1^+ = \Gamma_2^+ = \Gamma_3^+ \equiv \Gamma^+$ and $\Gamma_1^- = \Gamma_2^- = \Gamma_3^- \equiv \Gamma^-$, where

$$\Gamma_3^{\pm} = \{(\theta,\eta) : P(1;\theta,\eta) = 0 \text{ and } \theta \in \mathbf{R}^{\pm}\}.$$

This proves (a). Moreover, since $P(1;\theta,\eta) < 0$ (resp., $P(1;\theta,\eta) > 0$) and hence $P(r;\theta,\eta) < 0$ (resp., $P(r;\theta,\eta) > 0$) on $(0,\infty)$ if $(\theta,\eta) \in F^+$ (resp., $(\theta,\eta) \in F^-$), $u(r;\theta,\eta)$ is a solution of Type S^+-S^+ (resp., Type S^--S^-) by Lemma 3.2. On the other hand, if $(\theta,\eta) \notin \overline{F^+} \cup \overline{F^-}$, then $P(r;\theta,\eta) = P(1;\theta,\eta) > 0$ on $(0,\infty)$ and thus $u(r;\theta,\eta)$ is a solution of Type O-O due to Lemma 3.2 again. This proves (b) and (c). The proof of this theorem is complete. □

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Department of Mathematics, National Central University, Chung-Li 32001, Taiwan
E-mail address: chern@math.ncu.edu.tw

Department of Mathematics, National Central University, Chung-Li 32001, Taiwan
E-mail address: zhiyou@math.ncu.edu.tw
Current address: Department of Mathematics, National Tsing Hua University, Hsin-Chu 30013, Taiwan

Department of Mathematics, National Central University, Chung-Li 32001, Taiwan
E-mail address: tangyl@math.ncu.edu.tw