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GENERATING VARIETIES FOR AFFINE GRASSMANNIANS

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Abstract. We study Schubert varieties that generate the affine Grassmannian under the loop group product, and in particular generate the homology ring. There is a canonical such Schubert generating variety in each Lie type. The canonical generating varieties are not smooth, and in fact smooth Schubert generating varieties exist only if the group is not of type E_8, F_4 or G_2.

1. Introduction

Let G be a simply connected compact Lie group with simple Lie algebra, and let $G_{\mathbb{C}}$ be its complexification. The affine Grassmannian associated to G is the homogeneous space $\mathcal{L}_G = \tilde{G}_{\mathbb{C}}/P$, where $\tilde{G}_{\mathbb{C}}$ is the affine ind-group of regular maps $\mathbb{C}^\times \longrightarrow G_{\mathbb{C}}$ and P is the subgroup of maps $\mathbb{C} \longrightarrow G_{\mathbb{C}}$. From the algebraic-geometric point of view, $\mathcal{L}_G$ is a projective ind-variety which, despite being infinite-dimensional, behaves much like an ordinary Grassmannian. In particular, $\mathcal{L}_G$ contains a family of finite-dimensional, complex projective varieties, called *affine Schubert varieties*.[1] These varieties are indexed by the elements of $\mathcal{Q}^\vee$, the coroot lattice associated to G. We write X_λ for the variety corresponding to $\lambda \in \mathcal{Q}^\vee$.

There are, however, two striking differences between $\mathcal{L}_G$ and its finite-dimensional cousins: (1) $\mathcal{L}_G$ is a topological group; and (2) the principal bundle defining $\mathcal{L}_G$ as a homogeneous space is topologically trivial. No finite-dimensional flag variety $G_{\mathbb{C}}/Q$ has these properties. These two properties of $\mathcal{L}_G$ give rise to interesting new phenomena that have no analogue in the classical setting. The most obvious is that there is a Schubert calculus not only for the cup product in cohomology, but also for the Pontrjagin product in homology (see [21], [16], [17], and [18]).

In the present paper we consider the Pontrjagin product on the point-set, geometric level. A finite-dimensional projective subvariety $X \subset \mathcal{L}_G$ is a *generating variety* if the image of $H_*X \longrightarrow H_*\mathcal{L}_G$ generates $H_*\mathcal{L}_G$ as a ring. A Schubert variety X_λ with $dim\, X_\lambda > 0$ is a *geometric generating variety* if (1) for all $n \geq 1$, the image of the n-fold multiplication $X_\lambda^n \longrightarrow \mathcal{L}_G$ is a Schubert variety X_{λ_n} with $\dim X_{\lambda_n} = n \cdot \dim X_\lambda$; and (2) $H_*X_\lambda^n \longrightarrow H_*X_{\lambda_n}$ is surjective. Let α_0 denote the highest root of G.

Theorem 1.1. *$X_{-\alpha_0^\vee}$ is a geometric generating variety, with $\lambda_n = -n\alpha_0^\vee$.*

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[1]These varieties, along with several other objects discussed in the introduction—e.g., the coroot lattice $\mathcal{Q}^\vee$ and the highest root α_0, depend upon the choice of a maximal torus in G and a Borel subgroup of $G_{\mathbb{C}}$. See section 2 for details.

A classical theorem of Bott [3] can be reinterpreted to say that every $\mathcal{L}_G$ admits a smooth generating variety (Theorem 6.2). In contrast, the generating varieties of the theorem above are all singular. Their advantage is that they are Schubert varieties and give one canonical generating variety in each Lie type. It turns out that $\mathcal{L}_G$ admits a smooth Schubert generating variety if and only if G is not of type E_8, F_4, or G_2 (Theorem 6.3). In type C_n, it turns out that $X_{-\alpha_0^\vee}$ is essentially the generating complex for $\Omega Sp(n)$ considered by Hopkins [10].

The proof of Theorem 1.1 makes use of the balanced product $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$. We construct an ind-variety structure on the balanced product whose filtration consists of topologically trivial algebraic fiber bundles $E_{\sigma,\lambda}$ with base X_σ and fiber X_λ (λ an antidominant element of the coroot lattice). Then we relate the group multiplication $\mathcal{L}_G \times \mathcal{L}_G \longrightarrow \mathcal{L}_G$ to the action map $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G \longrightarrow \mathcal{L}_G$.

Along the way we consider the surprising topological properties (1) and (2) and how they fail in the category of ind-varieties. In particular, we show that there is no topological group structure on $\mathcal{L}_G$ whose multiplication map is a morphism of ind-varieties (Corollary 3.4). Regarding property (2), although the principal P-bundle $\tilde{G}_{\mathbb{C}} \longrightarrow \tilde{G}_{\mathbb{C}}/P$ is not trivial in the category of ind-varieties (see §4.1), we show that the balanced product bundle is isomorphic to $\mathcal{L}_G \times \mathcal{L}_G$ as an ind-variety (§4.2).

2. Notation and conventions

Throughout this paper, G is a simple, simply connected, compact Lie group with fixed maximal torus T, Weyl group $W = W(G,T)$, and complexification $G_{\mathbb{C}}$. We fix a Borel group B containing the Cartan subgroup $H = T_{\mathbb{C}}$ and let B^- denote the opposite Borel group. Let U be the unipotent radical of B so that $B = HU$. Let $\Phi = \Phi(G,T)$ denote the root system associated to (G,T). We denote by $S \subset W$ the set of simple reflections determined by B and write α_s for the simple root corresponding to $s \in S$. We write α_0 for the highest root in Φ. Let $\mathcal{Q}^\vee$ denote the coroot lattice and let $\mathcal{C}$ denote the dominant Weyl chamber determined by B.

Topologies on ind-varieties. By definition, an ind-variety X over $\mathbb{C}$ is an ascending union $X_0 \subset X_1 \subset \cdots$ of varieties X_n, with each X_n a closed subvariety of X_{n+1}. Each X_n has two natural topologies: The Zariski topology and the usual Hausdorff topology. Following [25], we call the latter the *complex* topology. Passing to the respective direct limit topologies, we obtain Zariski and complex topologies on X. (By direct limit topology we mean that a subset of X is closed if and only if its intersection with each X_n is closed.)

Loop groups. The primary references for this section are [24] and [15]; see also [22].

A *loop in* G is simply a continuous map $\lambda : S^1 \longrightarrow G$. If λ sends the identity element of S^1 to the identity element of G, then we say that λ is *based*. We denote by ΩG the set of all based loops in G and give it the subspace topology inherited from $LG = G^{S^1}$.

An *algebraic loop in* $G_{\mathbb{C}}$ is a regular map $\mathbb{C}^\times \longrightarrow G_{\mathbb{C}}$. The set of all such loops, which we denote by $\tilde{G}_{\mathbb{C}}$, is a topological group with respect to pointwise multiplication. We are interested in several subgroups of $\tilde{G}_{\mathbb{C}}$. Let $L_{alg}G = \{f \in \tilde{G}_{\mathbb{C}} : f|_{S^1} \in LG\}$ and $\Omega_{alg}G = \{f \in \tilde{G}_{\mathbb{C}} : f|_{S^1} \in \Omega G\}$. Elements of this latter group are called *based algebraic loops in* G. Let P be the subgroup of $\tilde{G}_{\mathbb{C}}$ of algebraic loops $\mathbb{C} \longrightarrow G_{\mathbb{C}}$ (i.e., algebraic loops that may be extended over the origin). As an example, we note that in type A_{n-1} we have $G = SU(n), G_{\mathbb{C}} = SL_n\mathbb{C}, \tilde{G}_{\mathbb{C}} = SL_n(\mathbb{C}[z, z^{-1}])$, and $P = SL_n(\mathbb{C}[z])$. In general Lie type, $\tilde{G}_{\mathbb{C}}$ can be shown to be a complex affine ind-group and P a closed sub-ind-group.

The following decomposition theorem is a fundamental part of the structure theory of loop groups (see for example [14], [24]).

Theorem 2.1 (Iwasawa decomposition). *The multiplication map $\Omega_{alg}G \times P \longrightarrow \tilde{G}_{\mathbb{C}}$ is a homeomorphism.*

We call this the Iwasawa decomposition because it is a modified analogue of Iwasawa's KAN decomposition for real Lie groups. The unmodified analogue takes the form $L_{alg}G \times A \times \tilde{U} \xrightarrow{\cong} \tilde{G}_{\mathbb{C}}$, where A comes from the KAN decomposition of $G_{\mathbb{C}}$ (here thought of as the group of constant loops) and $\tilde{U} = \{f \in P : f(0) \in U^-\}$.

Two corollaries of Theorem 2.1 are particularly important for our purposes. First, the homogeneous space $\mathcal{L}_G$ is homeomorphic to $\Omega_{alg}G$ and hence carries the structure of a topological group. As we shall observe later, such a group structure is wholly absent in the classical setting of homogeneous spaces of complex algebraic groups. Second, the principal bundle $P \longrightarrow \tilde{G}_{\mathbb{C}} \longrightarrow \mathcal{L}_G$ is trivial in the complex topology.

The following theorem, due, independently, to Quillen and Garland-Raghunathan, establishes the connection between $\Omega_{alg}G$ and the ordinary loop group ΩG.

Theorem 2.2 (Quillen and Garland-Raghunathan). *The inclusion $\Omega_{alg}G \longrightarrow \Omega G$ is a homotopy equivalence.*

The affine Weyl group. Let $\tilde{B} = \{f \in P : f(0) \in B^-\}$. The group $\tilde{B}$ is called the *Iwahori subgroup* of $\tilde{G}_{\mathbb{C}}$ and may be thought of as an "affine" analogue of a Borel subgroup of $G_{\mathbb{C}}$. Let $\tilde{N}_{\mathbb{C}}$ denote the group of algebraic loops in $G_{\mathbb{C}}$ that take their image in $N_{\mathbb{C}} = N_{G_{\mathbb{C}}}(T_{\mathbb{C}})$. The intersection $\tilde{B} \cap \tilde{N}_{\mathbb{C}}$ is normal in $\tilde{N}_{\mathbb{C}}$, and the quotient group $\tilde{W} = \tilde{N}_{\mathbb{C}}/(\tilde{B} \cap \tilde{N}_{\mathbb{C}})$ is canonically identified with the affine Weyl group associated to G. Thus $\tilde{W}$ has a semi-direct product decomposition $\tilde{W} = \mathcal{Q}^\vee \rtimes W$. Here we note that since G is simply connected, the lattice $\mathcal{Q}^\vee \subset \mathfrak{t}_{\mathbb{C}}$ can be identified with the subgroup $\mathrm{Hom}(S^1, T) = \mathrm{Hom}(\mathbb{C}^\times, T_{\mathbb{C}}) \subset \Omega_{alg}G$. Consequently, we alternate frequently between additive notation and multiplicative notation; the meaning will be clear from the context.

As a Coxeter system, the affine Weyl group has generating set $\tilde{S} = S \cup \{s_0\}$, where s_0 is a reflection across the hyperplane $\alpha_0 = 1$. The length of an element $w \in \tilde{W}$, written $\ell(w)$, is defined to be the smallest r for which there exists an expression $w = s_1 s_2 \cdots s_r$. By convention, the length of the identity element is taken to be zero. An expression $w = s_1 s_2 \cdots s_r$ is called *reduced* if $\ell(w) = r$. More generally, for $w_1, w_2, \ldots, w_k \in \tilde{W}$, we say that the product $w = w_1 w_2 \cdots w_k$ is *reduced* if $\ell(w) = \ell(w_1) + \cdots + \ell(w_k)$.

For a subset $I \subset \tilde{S}$, we let $\tilde{W}_I$ denote the subgroup of $\tilde{W}$ generated by I, and we note that the pair $(\tilde{W}_I, I)$ is a Coxeter system. Observe that $W = \tilde{W}_S$. It turns out that the length of an element $w \in \tilde{W}_I$ is given by $\ell(w)$. In other words, w cannot be represented as a product of fewer than $\ell(w)$ generators, even if one uses elements of $\tilde{S} - I$. Each coset $w\tilde{W}_I \in \tilde{W}/\tilde{W}_I$ contains a unique element of minimal length, which we denote by w^I. We write $\tilde{W}^I$ for the collection of all such coset representatives. We say that an expression $w = s_1 \cdots s_r$ is *I-reduced* if $\ell(w^I) = r$. In general, a product $w = w_1 w_2 \cdots w_k$ is *I-reduced* if $w = w^I$ and $\ell(w) = \ell(w_1) + \cdots + \ell(w_k)$.

The cosets $\tilde{W}/W$ have two canonical sets of representatives: The coroot lattice $\mathcal{Q}^\vee$ and the minimal length elements $\tilde{W}^S$. An element λ of the coroot lattice is said to be *antidominant* if it lies in the closure of $-\mathcal{C}$, the *antidominant* Weyl chamber. We write $\mathcal{Q}^\vee_-$ for the set of antidominant elements in $\mathcal{Q}^\vee$. For an element $\lambda \in \mathcal{Q}^\vee$, the following are equivalent: (1) λ is antidominant; (2) $\lambda \in \tilde{W}^S$; and (3) λW is maximal in $W \cdot \lambda W/W$. We note that if $\sigma \in \tilde{W}^S$ and $\lambda \in \mathcal{Q}^\vee_-$, then $\ell(\sigma\lambda) = \ell(\sigma) + \ell(\lambda)$; that is, the product $\sigma\lambda$ is reduced.

The Bruhat-Chevalley order on $\tilde{W}$ is the partial order defined as follows: For $w, x \in \tilde{W}$, we define $w \leq x$ if, given a reduced expression $x = s_1 s_2 \dots s_r$, one may obtain an expression (not necessarily reduced) for w by simply deleting (zero or more) factors from the expression for x. In other words, $w \leq x$ if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq r$ for which $w = s_{i_1} \cdots s_{i_k}$.

The affine Grassmannian. The affine Grassmannian associated to G is the homogeneous space $\mathcal{L}_G = \tilde{G}_\mathbb{C}/P$. By the Iwasawa decomposition, the natural map $\Omega_{alg}G \longrightarrow \mathcal{L}_G$ is a homeomorphism, and hence $\mathcal{L}_G$ is homotopy equivalent to ΩG by the Quillen/Garland-Raghunathan theorem. As the notation suggests, $\tilde{G}_\mathbb{C}/P$ is closely analogous to a flag variety of a linear algebraic group. The $\tilde{B}$-orbits in $\tilde{G}_\mathbb{C}/P$ are finite-dimensional affine spaces called *Schubert cells*. Each of these cells is of the form $e_w = \tilde{B} \cdot wP/P$, where w is an element of $\tilde{W}^S$. More precisely, let $P^- \subset \tilde{G}_\mathbb{C}$ denote the subgroup of maps regular at infinity (i.e., those $f \in \tilde{G}_\mathbb{C}$ for which $z \mapsto f(\frac{1}{z})$ is in P). Let $P^{(-1)} = \{f \in P^- : f(\infty) = I\}$, and let $\tilde{U}_w = \{u \in \tilde{U} : w^{-1}uw \in P^{(-1)}\}$. Then $\tilde{U}_w$ is a finite-dimensional unipotent group and hence is isomorphic to an affine space. Furthermore, the map $\tilde{U}_w w \longrightarrow e_w$ given by $u \mapsto uwP/P$ is an isomorphism of varieties.

The closure of e_w, which is denoted X_w, is called an *affine Schubert variety* or, simply, a Schubert variety. These varieties are finite-dimensional complex projective varieties with $\dim_\mathbb{C} X_w = \ell(w^I)$. The Bruhat-Chevalley order appears in this context as follows: We have $X_{w_1} \subset X_{w_2}$ if and only if $w_1 \leq w_2$. On the level of Schubert cells, this amounts to the statement that e_{w_1} is contained in the closure of e_{w_2} if and only if $w_1 \leq w_2$.

3. $\mathcal{L}_G$ as a topological group

Since $\Omega_{alg}G$ is a topological group under pointwise multiplication, $\mathcal{L}_G$ inherits a topological group structure from the canonical homeomorphism $\Omega_{alg}G \xrightarrow{\cong} \mathcal{L}_G$. This group structure is a special feature of our present, affine setting. Indeed, a classical homogeneous space of the form $G_\mathbb{C}/P$ cannot even be given the structure of an H-space. A classical theorem of Hopf shows that the rational cohomology ring of a finite H-space is isomorphic to an exterior algebra on odd-dimensional classes ([9], §3.C). The cohomology of $G_\mathbb{C}/P$, however, is concentrated in even dimensions.

On the other hand, there is no algebraic ind-group structure on $\mathcal{L}_G$. Here *algebraic ind-group* means "group object in the category of ind-varieties". It does not mean that the filtrations defining the ind-variety structure are subgroups. We say that an ind-variety X is irreducible if it cannot be written as a union of two proper Zariski closed subsets. It is easy to show that X is irreducible if and only if it admits a filtration by finite-dimensional irreducible subvarieties.

Proposition 3.1. *Let Γ be an irreducible projective algebraic ind-group. Then Γ is abelian.*

Proof. The proof follows the classical argument showing that a complete algebraic group is abelian ([25], §4.3). Let Γ_n be a filtration of Γ by irreducible projective varieties (note that Γ_n need not be a subgroup). Let $\gamma : \Gamma \times \Gamma \longrightarrow \Gamma$ be the map that sends (g,h) to its commutator $ghg^{-1}h^{-1}$ and observe that γ is a morphism of ind-varieties. For $h \in \Gamma$, we let $\gamma_h(g) = \gamma(g,h)$ and note that $\gamma_e(g) = e$ for every $g \in \Gamma$. Since each Γ_n is irreducible and projective (Lemma 1 of [25]), §4.3 applies to the restriction of γ to $(\Gamma \times \Gamma)_n = \Gamma_n \times \Gamma_n$. Thus, γ_h is constant for each $h \in \Gamma_n$. Since $\gamma_h(e) = e$, it follows that each $\gamma_h \equiv e$, whence $\gamma(g,h) = e$ for all g, h in Γ_n. Since n may be chosen arbitrarily, we conclude that Γ is abelian.

Corollary 3.2. *The topological group structure on $\mathcal{L}_G$ is not an algebraic ind-group structure.*

We can say more, however.

Proposition 3.3. *Let Γ be an ind-variety with filtration (Γ_n). Suppose that Γ admits a topological group structure with a regular multiplication map. If each Γ_n is an irreducible normal projective variety, then the inversion map is also regular.*

Proof. Let Γ satisfy the hypotheses stated in the proposition. Let $\mu : \Gamma \times \Gamma \longrightarrow \Gamma$ be the multiplication map and $\chi : \Gamma \longrightarrow \Gamma$ be the inversion map. For a fixed level Γ_p of the filtration, there exist integers q and n for which $\chi(\Gamma_p) \subset \Gamma_q$ and $\mu : \Gamma_p \times \Gamma_q \longrightarrow \Gamma_n$ is a regular map of finite-dimensional varieties. Let $V \subset \Gamma_p \times \Gamma_q$ be the fiber of μ over the identity element e. Since μ is regular, V is a subvariety of $\Gamma_p \times \Gamma_q$. The projection $\pi_1 : V \longrightarrow \Gamma_p$ is bijective and proper, and since Γ_p is normal, it is an isomorphism ([15], p. 514). We conclude that $\chi = \pi_2 \pi_1^{-1} : \Gamma_p \longrightarrow \Gamma_q$ is algebraic.

Corollary 3.4. *There does not exist a topological group structure on $\mathcal{L}_G$ with a regular multiplication map.*

Proof. Suppose that such a structure exists. Since the levels of the standard filtration on $\mathcal{L}_G$ are Schubert varieties—and thus normal ([15], p. 274)—Propositions 3.3 and 3.1 imply that the group structure is abelian. It follows that $\mathcal{L}_G$ is homotopy-equivalent to a product of Eilenberg-MacLane spaces ([9], 4K.7). Since the cohomology of $\mathcal{L}_G$ is torsion-free and concentrated in even degrees, the Eilenberg-MacLane spaces in question must be of type $K(\mathbb{Z}, 2)$. Thus $\mathcal{L}_G$ is homotopy-equivalent to a product of $\mathbb{C}P^\infty$'s. Hence for any prime p, $H^*(\mathcal{L}_G; \mathbb{Z}/p)$ has no nilpotent elements. But it is well known that in fact *every* positive-dimensional element of $H^*(\Omega G; \mathbb{Z}/p)$ is nilpotent (for a proof see Proposition 5.3 below), so this is a contradiction.

We next show that the inversion map on $\mathcal{L}_G$ is not regular. The proof makes use of the following lemma.

Lemma 3.5. *For $w \in W$, let $\varphi_w : G/T \longrightarrow G/T$ denote the map $gT \mapsto gwT$. If $w \neq 1$, then φ_w is not a morphism of varieties $G_{\mathbb{C}}/B \longrightarrow G_{\mathbb{C}}/B$.*

Proof. Suppose φ_w is a morphism of varieties. Then it is $G_{\mathbb{C}}$-equivariant, since it is G-equivariant and G is Zariski dense in $G_{\mathbb{C}}$. But any $G_{\mathbb{C}}$-equivariant map has the form $gB \mapsto gxB$ for some $x \in N_{G_{\mathbb{C}}}B$, and $N_{G_{\mathbb{C}}}B = B$. Hence φ_w is the identity and $w = 1$.

Proposition 3.6. *The inversion map $\chi : \mathcal{L}_G \longrightarrow \mathcal{L}_G$ is not regular.*

Proof. Note that, for $\lambda \in \mathcal{Q}^\vee$, χ takes the P-orbit $\mathcal{O}_\lambda$ to the P-orbit $\mathcal{O}_{\lambda^{-1}}$. Fix $\lambda \in \mathcal{Q}^\vee_-$ with $\alpha_s \lambda < 0$ for all $s \in S$, and let Y_λ denote the $G_{\mathbb{C}}$-orbit $G_{\mathbb{C}} \cdot \lambda$. The corresponding subset of $\Omega_{alg}G$ is Z_λ, the image of the map $\psi_\lambda : G/T \longrightarrow \Omega_{alg}G$ given by

$\psi_\lambda(gT) = g\lambda g^{-1}$. Since λ is antidominant, the antidominant representative of the orbit $\mathcal{O}_{\lambda^{-1}}$ is $w_0 \cdot \lambda^{-1}$. We thus have a commutative diagram of homeomorphisms:

$$\begin{tikzcd}
G/T \arrow[r, "\psi_\lambda"] \arrow[d, "\varphi_{w_0}"'] & Z_\lambda \arrow[d, "\chi"] \\
G/T \arrow[r, "\psi_{\omega_0\cdot\lambda^{-1}}"'] & Z_{\psi_{\omega_0}\cdot\lambda^{-1}}
\end{tikzcd}$$

Expressing this in terms of varieties yields

$$\begin{tikzcd}
G_C/B \arrow[r] \arrow[d, "\varphi_{w_0}"'] & Y_\lambda \arrow[d, "\chi"] \\
G_C/B \arrow[r] & Y_\lambda
\end{tikzcd}$$

where Y_λ is the variety $G_{\mathbb{C}} \cdot \lambda$ (with underlying space Z_λ) and the horizontal maps are just the usual orbit maps $gB \mapsto g\lambda$, $gB \mapsto gw_0\lambda^{-1}$, respectively. If χ is algebraic in this diagram, then it is an isomorphism of varieties. This implies φ_{w_0} is algebraic, contradicting the lemma.

We conclude this section with a remark: The multiplication on the loop space of any topological group G (or even an H-space G) is homotopy-commutative; i.e. $m \circ T \sim m$, where T switches the two factors in $\Omega G \times \Omega G$. Moreover, pointwise multiplication using the group structure on G is homotopic to the loop product defined by concatenation of paths, so "multiplication" can be interpreted in either sense. The proofs of these assertions are straightforward strengthenings of the usual argument showing that $\pi_1 G$ is abelian ([9], 3C, Exercise 5, or see the purely categorical argument in [26], §1.6, Theorem 8). Thus the Pontrjagin product on $\mathcal{L}_G$ is homotopy-commutative and $H_*\mathcal{L}_G$ is a commutative graded ring.

4. The balanced product $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$

In this section we study the balanced product $\mathcal{E}_G = \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$. We begin with a discussion of the orbit map $\tilde{G}_{\mathbb{C}} \longrightarrow \mathcal{L}_G$. Next we define a natural ind-variety structure on $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$. Finally, we discuss the Pontrjagin product and the balanced product in homology.

4.1. **The action map $\tilde{G}_{\mathbb{C}} \times \mathcal{L}_G \longrightarrow \mathcal{L}_G$.** The following proposition is a special case of [15], Lemma 7.4.10 and Proposition 7.4.11.

Proposition 4.1. *The orbit map $\pi : \tilde{G}_{\mathbb{C}} \longrightarrow \mathcal{L}_G$ is a morphism of ind-varieties. Furthermore, π is a Zariski locally trivial principal P-bundle in the category of ind-varieties.*

Note that π is not globally trivial as a map of ind-varieties, since $\tilde{G}_{\mathbb{C}}$ is affine and $\mathcal{L}_G$ is projective. On the other hand, it is a striking fact that π *is* globally trivial in the complex topology; this follows immediately from the Iwasawa decomposition. In contrast, for a finite-dimensional flag variety $G_{\mathbb{C}}/Q_I$, the principal Q_I-bundle $G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}/Q_I$ is never topologically trivial since $\pi_2 G_{\mathbb{C}} \cong \pi_2 G = 0$, whereas $\pi_2(G_{\mathbb{C}}/Q_I) \cong H_2(G_{\mathbb{C}}/Q_I) \cong \mathbb{Z}^{|S|-|I|}$.

Call a morphism of ind-varieties $f : X \longrightarrow Y$ a *quotient morphism* if whenever Z is an ind-variety and $g : Y \longrightarrow Z$ is a set map such that gf is a morphism of ind-varieties, we have g a morphism of ind-varieties. Any regular map with local sections is a quotient morphism. Hence:

Corollary 4.2. *π is a quotient morphism.*

This yields:

Proposition 4.3. *The action map $\theta : \tilde{G}_{\mathbb{C}} \times \mathcal{L}_G \longrightarrow \mathcal{L}_G$ is a morphism of ind-varieties.*

The $\tilde{G}_{\mathbb{C}}$-action on $\mathcal{L}_G$ has two nice properties: First, if $Q \subset \tilde{G}_{\mathbb{C}}$ is any proper, standard parabolic subgroup, then $\mathcal{L}_G$ is exhaustively filtered by Q-invariant Schubert varieties X_λ. Second, the action of Q on such an X_λ factors through a finite-dimensional quotient algebraic group Q/Q_λ. In other words, the action is "regular" in the sense of [15], p. 213, except that our parabolics are not pro-groups, but rather subgroups of certain associated pro-groups.

In the case of interest to us, namely $Q = P$, there is a simple explicit description of the finite-dimensional quotients and the factored action. Thinking of P as $G(\mathbb{C}[z])$, there are natural surjective morphisms of algebraic ind-groups $\epsilon_k : P \longrightarrow G[k]$, where $G[k] = G(\mathbb{C}[z]/z^k)$. Let $P^{(k)}$ denote the kernel of ϵ_k. Then if $\lambda \in \mathcal{Q}_-^\vee$, it is not hard to show directly that $P^{(k)}$ acts trivially on X_λ for all sufficiently large k. Hence the finite-dimensional algebraic group $G[k]$ acts algebraically on X_λ.

We also note that in the category of ind-varieties, $\tilde{G}_{\mathbb{C}}/P^{(k)} \longrightarrow \mathcal{L}_G$ is a Zariski locally trivial principal $G[k]$-bundle. Indeed, for this it is enough to show that ϵ_k is locally trivial, but in fact it is globally trivial, as a principal $P^{(k)}$-bundle of ind-varieties ([7]).

4.2. **Ind-variety structure on $\mathcal{E}_G$.** As topological spaces, we may filter $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$ by the subbundles $\tilde{G}_{\mathbb{C}} \times_P X_\lambda \longrightarrow \mathcal{L}_G$, where λ ranges over $\mathcal{Q}_-^\vee$. Restricting these subbundles to X_σ, $\sigma \in \tilde{W}^S$, we obtain a local product $E_{\sigma,\lambda} \longrightarrow X_\sigma$ with fiber X_λ. In fact $E_{\sigma,\lambda}$ has a unique structure of projective variety such that $E_{\sigma,\lambda} \longrightarrow X_\sigma$ is locally trivial in the Zariski topology as varieties. To prove this we need two lemmas.

Lemma 4.4. *Let $\pi : E \longrightarrow X$ be a morphism of noetherian schemes over a noetherian base scheme A, and suppose that π is locally trivial with fiber F. Let $\mathcal{P}$ be any class of morphisms over A that is closed under composition and base change, and local in the target. Then if $F \longrightarrow A$ and $X \longrightarrow A$ are in $\mathcal{P}$, so is $E \longrightarrow A$.*

In particular, this holds when $\mathcal{P}$ is the class of separated, finite type, or proper morphisms.

Proof. The first part is an easy exercise. The last assertion follows from standard facts ([8], §II.4).

Lemma 4.5. *Let $\pi : E \longrightarrow X$ be a local product with fiber F in the complex topology, and suppose that (i) X, F are varieties; (ii) there are local trivializations on a Zariski open cover U_α of X; and (iii) the transition functions $U_\alpha \cap U_\beta \times F \longrightarrow U_\alpha \cap U_\beta \times F$ associated to the cover in (ii) are regular. Then E has a unique algebraic variety structure compatible with the trivializations in (ii), so that π is a local product as varieties. Moreover, if X and F are complete, then so is E.*

Proof. Identify the category of varieties over $\mathbb{C}$ with the category of reduced separated schemes of finite type over $\operatorname{Spec} \mathbb{C}$. By the gluing lemma ([8], II.2.2) we obtain a reduced scheme E' over $\operatorname{Spec}\mathbb{C}$ and a morphism of schemes $E'\longrightarrow X$ that is locally trivial with fiber F. By the previous lemma, E' is separated of finite type and is complete if X and F are complete. Furthermore, the set of complex points $E'(\mathbb{C})$ is naturally identified with E. This proves the lemma.

We apply the lemma to our situation as follows: The canonical morphism $\tilde{G}_{\mathbb{C}}\longrightarrow\mathcal{L}_G$ is a Zariski locally trivial principal P-bundle, as ind-varieties. Moreover, the action of P on $\mathcal{L}_G$ has the properties (1) $\mathcal{L}_G$ is filtered by finite-dimensional P-invariant projective subvarieties, namely the X_λ's with $\lambda \in \mathcal{Q}_-^\vee$; and (2) the action of P on any fixed X_λ as in (1) factors through a finite-dimensional quotient algebraic group P/N for some Zariski closed normal ind-subgroup N (depending on λ). Property (2) is most transparent using the "lattice" model of $\mathcal{L}_G$; see [15]. Thus $E_{\sigma,\lambda}$ is the algebraic fiber bundle with fiber X_λ associated to a principal P/N-bundle over X_σ. This yields a complete ind-variety structure on $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$, indexed by the poset $\tilde{W}^S \times \mathcal{Q}_-^\vee$.

Note that the maps $\pi, \phi : \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G\longrightarrow\mathcal{L}_G$ given by $\pi([g,x]) = gP$, $\phi([g,x]) = gxP$ are morphisms of ind-varieties. Then $\Psi = (\pi,\phi)$ defines an isomorphism of ind-varieties

$$\Psi : \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G \xrightarrow{\cong} \mathcal{L}_G \times \mathcal{L}_G.$$

The inverse map is given by $(gP, xP) \mapsto [gP, g^{-1}xP]$. Since $E_{\sigma,\lambda}$ is complete, it maps isomorphically onto its image in the projective ind-variety $\mathcal{L}_G \times \mathcal{L}_G$. Hence $E_{\sigma,\lambda}$ is projective and $\tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$ is a projective ind-variety.

Remark. We could have used Ψ to define the ind-variety structure, but the direct construction given above is more natural and has more general applicability. Note that Ψ embeds $E_{\sigma,\lambda}$ as a subvariety of $X_\sigma \times X_{\sigma\lambda}$.

Fix $\sigma \in \tilde{W}^S$ and $\lambda \in \mathcal{Q}_-^\vee$. Then $E_{\sigma,\lambda}$ is partitioned into *bi-Schubert cells*,

$$E_{\sigma,\lambda} = \coprod_{\tau\leq\sigma,\nu\leq\lambda} e_{\tau,\nu},$$

where $e_{\tau,\nu} = U_\tau\tau \times e_\nu$, identifying the latter with a subspace of $E_{\sigma,\lambda}$ in the evident way. Note that each bi-Schubert cell is a locally closed subvariety of $E_{\sigma,\lambda}$.

4.3. **The Pontrjagin product.** Let $F : \Omega_{alg}G \times P\longrightarrow\tilde{G}_{\mathbb{C}}$ denote the Iwasawa homeomorphism and write the components of its inverse as $F^{-1} = (\omega, p)$. Then the Pontrjagin product $m : \mathcal{L}_G \times \mathcal{L}_G\longrightarrow\mathcal{L}_G$ can be written as

$$m(xP, yP) = \omega(x)\omega(y)P = \omega(x)yP,$$

where alternatively we may write $\omega(x) = xp(x)^{-1}$. Note also that if $x \in L_{alg}G$, then $p(x)$ is just $x(1)$. It follows that m is not a cellular map. For example, consider the case $G = SU(2)$ and $x = s_0 = y$. Then $\omega(x) = s_0s_1$ and hence $m(s_0P, s_0P) = s_0s_1s_0P$, so that m fails to preserve 2-skeleta.

Thus the Pontrjagin product m is neither a morphism of varieties nor a cellular map, which makes it rather awkward to analyze. We will get around this by comparing it to the balanced product $\phi : \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G\longrightarrow\mathcal{L}_G$, which enjoys both properties. We first note:

Lemma 4.6. *Suppose $\tau, \nu \in \tilde{W}^S$. Then*

(a) If $\tau\nu$ is S-reduced, then $\phi(E_{\tau,\nu}) = X_{\tau\nu}$ and $E_{\tau,\nu}\longrightarrow X_{\tau\nu}$ is an isomorphism over the top cell.

(b) If $\tau\nu$ is not S-reduced, then $\phi(E_{\tau,\nu}) \subset \bigcup_{\mu<\tau\nu} X_\mu$.

Part (a) follows from the Steinberg lemma ([27], Theorem 15, and [14], Proposition 3.1), while part (b) follows easily from the Tits system axioms. Now define the *balanced product* on $H_*\mathcal{L}_G$ by

$$[X_\tau] \star [X_\nu] = \phi_*[E_{\tau,\nu}].$$

Then the lemma immediately implies:

Proposition 4.7. *The balanced product is associative but not commutative, and it satisfies*

$$[X_\tau] \star [X_\nu] = \begin{cases} [X_{\tau\nu}] & \text{if } \tau\nu \text{ is } S\text{-reduced,} \\ 0 & \text{otherwise.} \end{cases}$$

The product is not commutative, since $\tau\nu$ S-reduced need not imply $\nu\tau$ S-reduced. Next, we have:

Proposition 4.8. *Suppose $\sigma \in \tilde{W}^S$, $\lambda \in \mathcal{Q}^\vee_-$. Then for every $\omega \leq \sigma\lambda$ there exists $\tau \leq \sigma$ and $\nu \leq \lambda$ in $\tilde{W}^S$ such that $[X_\tau] \star [X_\nu] = [X_\omega]$. In particular, $\phi_* : H_*E_{\sigma,\lambda} \longrightarrow H_*X_{\sigma\lambda}$ is onto.*

Proof. Suppose $\omega \leq \sigma\lambda$, and choose an S-reduced product decomposition $\omega = \tau\nu$ with $\tau \leq \sigma$ and $\nu \leq \lambda$. Then $\nu \in \tilde{W}^S$. We claim that if we choose such a decomposition with ν of maximal length, then also $\tau \in \tilde{W}^S$. For if not, then $\tau \downarrow \tau s$ for some $s \in S$, and hence there is a reduced product $\tau = \tau' s$. But then $\omega = \tau' s\nu$ is reduced, and by a general property of Bruhat order ([11], 5.9) either $s\nu \leq s\lambda$ or $s\nu \leq \lambda$. Since $\lambda \in \mathcal{Q}^\vee_-$ we have $s\lambda \leq \lambda$, so in either case $s\nu \leq \lambda$. This contradicts the maximality of ν, proving our claim. Thus $[X_\tau] \star [X_\nu] = [X_\omega]$.

In order to compare the Pontrjagin and balanced products, consider the homeomorphisms

$$\mathcal{L}_G \times \mathcal{L}_G \xleftarrow{\cong} \Omega_{alg}G \times \Omega_{alg}G \xrightarrow{\cong} \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G,$$

where the left arrow is given by $(a, b) \mapsto (aP, bP)$ and the right arrow by $(a, b) \mapsto [a, bP]$. By composition we obtain a homeomorphism

$$\eta : \mathcal{L}_G \times \mathcal{L}_G \xrightarrow{\cong} \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G$$

given by $\eta(xP, yP) = [\omega(x), \omega(y)P]$, with $\psi = \eta^{-1}$ given by $\psi([g, z]) = (gP, p(g)zP)$. This map is a topological trivialization of the bundle $\pi : \tilde{G}_{\mathbb{C}} \times_P \mathcal{L}_G \longrightarrow \tilde{G}_{\mathbb{C}}/P$ and by construction satisfies $\phi \circ \eta = m$. In particular, it is *not* the same as the algebraic trivialization considered earlier. If it were, then m would be a morphism of varieties, contradicting Corollary 3.2.

Proposition 4.9. *Suppose $\eta \in \tilde{W}^S$ and $\lambda \in \mathcal{Q}^\vee_-$. Then $\eta(X_\sigma \times X_\lambda) = E_{\sigma,\lambda}$. Hence η defines a topological trivialization of the bundle $E_{\sigma,\lambda} \longrightarrow X_\sigma$:*

$$\begin{tikzcd} X_\sigma \times X_\lambda \arrow[r, "\eta"] \arrow[d, "m"'] & E_{\sigma,\lambda} \arrow[dl, "\phi"] \\ X_\sigma & \end{tikzcd}$$

Proof. We have

$$\eta(xP, yP) = [xp(x)^{-1}, yP] = [x, p(x)^{-1}yP],$$

proving that $\eta(X_\sigma \times X_\lambda) \subset E_{\sigma,\lambda}$. Equality then follows from the formula $\psi([g, zP]) = (gP, p(g)zP)$.

Remark. These trivializations are not regular maps, since if they were, then the product map m would be algebraic. We conjecture that $E_{\sigma,\lambda} \longrightarrow X_\sigma$ is never a product bundle as varieties, nor indeed a product at all. We note also that η does not map $e_\sigma \times e_\lambda$ into $e_{\sigma,\lambda}$; we only have that $\eta(e_\sigma \times e_\lambda) \subset U_\sigma \sigma \times \mathcal{O}_\lambda$. For example, when $G = SU(2)$ we have $\eta(s_0 P, s_1 s_0 P) = [s_0 s_1, s_1 s_0 P] = [s_0, s_0 P]$.

Proposition 4.10. *Suppose $\sigma \in \tilde{W}^S$, $\lambda \in \mathcal{Q}^\vee_-$. Then*
(a) $m(X_\sigma \times X_\lambda) = X_{\sigma\lambda}$,
(b) $m_*([X_\sigma] \otimes [X_\lambda]) = [X_{\sigma\lambda}]$.

Proof. Part (a) follows from Proposition 4.8 and Proposition 4.9. Since η is a map of X_λ-bundles over X_σ, having degree one on the base and the fiber, η itself has degree one. Part (b) then follows similarly.

5. The canonical generating variety

A *generating complex* for an H-space Y consists of a CW-complex K and a map $f : K \longrightarrow Y$ such that the image of $H_* f$ generates $H_* Y$ as a ring. A *generating variety* for $\mathcal{L}_G$ is a finite-dimensional projective subvariety X such that $X \subset \mathcal{L}_G$ is a generating complex. A Schubert variety X_λ with $\dim X_\lambda > 0$ is a *geometric generating variety* if (1) for all $n \geq 1$ the image of the n-fold Pontrjagin multiplication $m^n : X^n_\lambda \longrightarrow \mathcal{L}_G$ is a Schubert variety X_{λ_n} with $\dim X_{\lambda_n} = n \cdot \dim X_\lambda$; and (2) $m^n_* : H_* X^n_\lambda \longrightarrow H_* X_{\lambda_n}$ is surjective. Since any infinite subset of $\tilde{W}^S$ is cofinal in the Bruhat order (i.e., the lower order ideal it generates is all of $\tilde{W}^S$), a geometric generating variety is in particular a generating variety.

Now let $\lambda_0 = -\alpha_0^\vee \in \mathcal{Q}^\vee_-$. Here $\alpha_0^\vee$ is the coroot of the highest root α_0, or equivalently the highest short coroot. The length of λ_0 in $\tilde{W}^S$, i.e. the dimension of X_{λ_0}, is computed as follows: Write $\alpha_0^\vee = \sum_{s \in S} m_s^\vee \alpha_s^\vee$. Then

$$\ell(\lambda_0) = \sum_{\alpha \in \Phi^+} \alpha(\alpha_0^\vee) = 2 \sum_{s \in S} m_s^\vee,$$

where the first equality comes from a standard length formula [12] and the second from the classical formula $\sum_{\alpha \in \Phi^+} \alpha = 2\rho$, with ρ the sum of the fundamental weights ([4], Ch. VI, §1.10, Proposition 29).

The following theorem provides a canonical geometric generating variety in every Lie type.

Theorem 5.1. *X_{λ_0} is a geometric generating variety for $\mathcal{L}_G$.*

Proof. Take $\mu_n = n\lambda_0$, and proceed by induction on n. At the inductive step, we apply Proposition 4.10 with $\sigma = \lambda_0$ and $\lambda = \mu_n$.

As a corollary we obtain a classical theorem of Bott [3].

Corollary 5.2. *The homology ring $H_* \Omega G$ is finitely-generated.*

Bott's proof uses generating complexes; indeed he introduced the concept partly for this purpose. His generating complexes, however, are smooth manifolds, whereas ours are always singular varieties as we will shortly see. We will also see that smooth

Schubert generating varieties exist if and only if G is not of type E_8, F_4, or G_2. Our canonical generating varieties have a number of interesting properties:

X_{λ_0} *as a Thom space*: Taking P-orbits instead of $\tilde{B}$-orbits in $\mathcal{L}_G$ yields a stratification

$$\mathcal{L}_G = \coprod_{\lambda \in \mathcal{Q}^\vee_-} \mathcal{O}_\lambda,$$

where $\mathcal{O}_\lambda$ is isomorphic as an algebraic variety to a vector bundle ξ_λ over the *Levi orbit* $M_\lambda = G_{\mathbb{C}} \cdot \lambda P$. Here M_λ is a flag variety $G_{\mathbb{C}}/Q_{I(\lambda)}$, where $I(\lambda) = \{s \in S : s\lambda W = \lambda W\}$. For details of this construction, see [23].

Taking $\lambda = \lambda_0$, one finds that $I(\lambda_0)$ corresponds to the nodes on the ordinary Dynkin diagram that are not adjacent to s_0 in the affine diagram. The adjoint action of $Q = Q_{I(\lambda_0)}$ on $\mathfrak{g}_{\mathbb{C}}$ stabilizes the one-dimensional root subalgebra $\mathfrak{u}_{\alpha_0}$ so that (cf. [23])

$$\xi_{\lambda_0} = G_{\mathbb{C}} \times_Q \mathfrak{u}_{\alpha_0}.$$

Furthermore, since λ_0 is the minimal nonzero element of $\mathcal{Q}^\vee_-$, we have $X_{\lambda_0} = \overline{\mathcal{O}}_{\lambda_0} = \mathcal{O}_{\lambda_0} \cup *$, where $* = eP$ is the basepoint. Hence as a topological space, X_{λ_0} is the one-point compactification of $\mathcal{O}_{\lambda_0}$ and so can be identified with the Thom space $T(\xi_{\lambda_0})$.

Minimal nilpotent orbits. Let $Y_0 \subset \mathfrak{g}_{\mathbb{C}}$ denote the minimal nontrivial nilpotent orbit; that is, the orbit of a nonzero element $x \in \mathfrak{u}_{\alpha_0}$. Then

$$Y_0 \cong G_{\mathbb{C}} \times_Q (\mathfrak{u}_{\alpha_0} - \{0\}).$$

In other words, Y_0 is the $\mathbb{C}^\times$ bundle associated to ξ_{λ_0}. Hence $X_{\lambda_0} - (M_{\lambda_0} \cup *) \cong Y_0$. Similarly, $\overline{Y}_0 \cong X_{\lambda_0} - M_{\lambda_0}$, where $\overline{Y}_0$ denotes the Zariski closure in $\mathfrak{g}_{\mathbb{C}}$. In particular, $\overline{Y}_0$ and X_{λ_0} have equivalent singular points. Since the cone point in $\overline{Y}_0$ is known to be singular, this gives one way to see that X_{λ_0} is singular at the basepoint. We will give a different proof below. For an elegant computation of H^*Y_0, see [13].

Birkhoff strata. The Birkhoff stratification of $\mathcal{L}_G$ is "dual" to the Schubert cells and defines a descending filtration by ind-subvarieties Z_λ ($\lambda \in \mathcal{Q}^\vee \cong \tilde{W}^S$) [15], [24], [7]. There is a unique codimension one Birkhoff variety, namely $Z_{s_0} = Z_{\alpha_0^\vee}$. It is P-invariant, and hence $Z_{\alpha_0^\vee} \cap X_{\lambda_0} \subset M_{\lambda_0}$. Since both varieties are irreducible of codimension one in X_{λ_0}, it follows that the inclusion is an equality. Thus if $\mathcal{U}_0$ is the Zariski open complement $\mathcal{L}_G - Z_{\alpha_0^\vee}$, we have $\mathcal{U}_0 \cap X_{\lambda_0} = \overline{Y}_0$. This observation enters into the beautiful classification of minimal degeneration singularities in [19].

Segments. Every $w \in \tilde{W}^S$ has a canonical reduced factorization $w = \sigma_1\sigma_2 \ldots \sigma_n$ into *segments*, defined in [2]. For present purposes we will use a slightly different definition, ignoring the refinement for types BD of [2]. We say that $\sigma \in \tilde{W}^S$ is a *segment* if it is in the left W-orbit of s_0 in $\tilde{W}^S \cong \tilde{W}/W$. In other words, the segments are precisely the elements that index the cells of X_{λ_0}, excluding the basepoint. Each such σ has a unique reduced factorization $\sigma = \nu s_0$ with $\nu \in W^J$, where J is the complement in S of the set of nodes adjacent to s_0 in $\tilde{S}$ and W^J denotes the set of minimal length representatives for the cosets W/W_J. It is then easy to show that each $w \in \tilde{W}^S$ has a unique factorization into segments $w = \sigma_1\sigma_2 \ldots \sigma_n$ such that each partial product $\sigma_1 \ldots \sigma_k$ is also in $\tilde{W}^S$. The canonical generating variety can be viewed as a geometric expression of this factorization. For example, it follows that there is a corresponding unique $\star$-factorization of homology classes

$$[X_w] = [X_{\sigma_1}] \star [X_{\sigma_2}] \star \cdots \star [X_{\sigma_n}].$$

No such factorization holds for the Pontrjagin product, however ([16], [17]).

Examples. (i) Type A_n, $n \geq 2$. This is the only type in which s_0 is adjacent to more than one node in the affine Dynkin diagram, and hence is the only type in which the flag variety M_{λ_0} is of nonmaximal type. Explicitly, M_{λ_0} is the variety of flags of type $V^1 \subset V^n \subset \mathbb{C}^{n+1}$. Now let ξ_1, ξ_{n+1} denote the line bundles corresponding to V^1 and $\mathbb{C}^{n+1}/V^n$, respectively. Then $\xi_{\lambda_0} = \mathrm{Hom}(\xi_{n+1}, \xi_1)$. In this case the canonical generating variety is not very efficient; the smallest generating varieties are the minuscule generating varieties $\mathbb{C}P^n \subset \mathcal{L}_G$.

(ii) Type C_n, $n \geq 1$ (type A_1 is best thought of as type C_1 in this context). Here M_{λ_0} has type C_n/C_{n-1}; thus $M_{\lambda_0} \cong \mathbb{C}P^{2n-1}$. Then $\xi_{\lambda_0} \cong (\eta^*)^2$, where η is the canonical line bundle (in topologists' but not geometers' terminology; i.e., η^* is the hyperplane section bundle). Hence $c_1(\xi_{\lambda_0}) \in H^2\mathbb{C}P^{2n-1}$ is twice a generator. For later reference, we note that it follows that X_{λ_0} satisfies Poincaré duality rationally but not integrally.

Now let $\mu \in \mathcal{Q}^\vee$ be the unique element immediately below $\lambda_0 = -\alpha_0^\vee$; that is, $\mu = s_1\lambda_0$. Since $H_*\Omega Sp(n) \cong \mathbb{Z}[a_1, \ldots, a_n]$ with $|a_i| = 4i - 2$, where $|a|$ denotes the dimension of an element of a graded group, it follows that X_μ is a Schubert generating variety for $\mathcal{L}_{Sp(n)}$. In fact $X_\mu = T(\xi_{\lambda_0} \downarrow \mathbb{C}P^{2n-2})$, where $T(\xi) = T(\xi \downarrow M)$ denotes the Thom space of a vector bundle ξ over a space M. This is the generating complex of [10].

(iii) Type G_2. The flag variety M_{λ_0} is $(G_2)_\mathbb{C}/Q$, where Q is the parabolic omitting the long node s_2. It is a chain with cells indexed by $1, s_2, s_1s_2, s_2s_1s_2, s_1s_2s_1s_2, s_2s_1s_2s_1s_2$. For short, let X_k denote the Schubert variety of dimension k in $G_\mathbb{C}/Q_{s_0}$ and define $y_k \in H^{2k}(G_\mathbb{C}/Q_{s_0})$ by $\langle y_k, [X_k]\rangle = 1$. Then the Chevalley formula ([5], or see e.g. [15]) shows that $y_1 = c_1(\xi_{\lambda_0})$ and the cup products $y_1y_{k-1} = a_ky_k$ with $a_k = 1, 3, 2, 3, 1$ for $1 \leq k \leq 5$. It follows easily that X_{λ_0} satisfies Poincaré duality rationally but not integrally, and in particular is not smooth.

We conclude this section with the following result, a proof of which was promised in §3.

Proposition 5.3. *Let p be a prime and let x be a positive-dimensional element of $H^*(\mathcal{L}_G; \mathbb{Z}/p)$. Then if $p^k > 2\sum_i m_i^\vee$, we have $x^{p^k} = 0$.*

Since $\dim X_{\lambda_0} = 2\sum_i m_i^\vee$, the proposition follows from:

Lemma 5.4. *Let A be a bicommutative graded Hopf algebra of finite type over a field κ of characteristic $p > 0$, and suppose A is finitely-generated as a graded algebra. Then every positive-dimensional element of the dual A^* is nilpotent. In fact if S is a finite set of homogeneous generators and $m = max\,\{|a| : a \in S\}$, then for any positive-dimensional $x \in A^*$ and $p^k > m$ we have $x^{p^k} = 0$.*

Proof. Let $m_d : \otimes^d A \longrightarrow A$ denote the d-fold multiplication map and let $\langle -, - \rangle$ denote the canonical pairing $A^* \otimes A \longrightarrow \kappa$. Then for any d and for any $a_1, \ldots, a_d \in S$, we have

$$\langle x^{p^k}, m_d(a_1 \otimes \cdots \otimes a_d)\rangle = \langle (m_d^* x)^{p^k}, a_1 \otimes \cdots \otimes a_d\rangle.$$

Now $(m_d^* x)^{p^k}$ is a sum of terms of the form $x_1^{p^k} \otimes \cdots \otimes x_d^{p^k}$. For each i we have either $|x_i| = 0$ or $p^k|x_i| > |a_i|$, and hence in all cases $\langle x_i^{p^k}, a_i\rangle = 0$. Hence $\langle x^{p^k}, m_d(a_1 \otimes \cdots \otimes a_d)\rangle = 0$ for all $d > 0$ and $a_i \in S$, and $x^{p^k} = 0$.

6. Smooth vs. singular generating varieties

The varieties X_{λ_0} are always singular. This follows from the results of [6] or [1], but it is also easy to see directly. Indeed, using the fact that X_{λ_0} is a Thom space, one can easily show that (a) X_{λ_0} satisfies Poincaré duality over $\mathbb{Q}$ if and only if G has type A_1, C_n, or G_2; and (b) X_{λ_0} never satisfies Poincaré duality over $\mathbb{Z}$. On the other hand, X_{λ_0} is evidently smooth away from the basepoint (i.e., the point at infinity).

We next recall the classical construction of Bott [3], which yields certain smooth generating varieties for $\mathcal{L}_G$. These generating varieties are not always Schubert varieties, but they are always either Schubert varieties or Levi orbits. Bott works with the adjoint form G^{ad} of G so that $\mathrm{Hom}(S^1, T^{ad})$ can be identified with the coweight lattice $\mathcal{P}^\vee$. If $\lambda \in \mathcal{P}^\vee$, let C_λ^{ad} denote the centralizer of λ in G. The *Bott map*

$$B_\lambda : G^{ad}/C_\lambda^{ad} \longrightarrow \Omega_0 G^{ad}$$

is defined by $B_\lambda(gC_\lambda^{ad}) = \lambda g \lambda^{-1} g^{-1}$. Here Ω_0 denotes the component of the loop space containing the constant loop. Thus by elementary covering space theory there is a natural homeomorphism $\Omega G \xrightarrow{\cong} \Omega_0 G^{ad}$. In fact Bott used the group-theoretic inverse of this map, but $H_* B_\lambda$ and $H_* B_\lambda^{-1}$ generate the same subring of $H_* \Omega_0 G^{ad}$, so the distinction will not be important. Note that (i) the canonical map $\Omega G \longrightarrow \Omega_0 G^{ad}$ is a homeomorphism; and (ii) B_λ is in fact a map into the *algebraic* loops $\Omega_{alg,0} G^{ad}$.

The main result of [3] is that whenever λ is a "short minimal circle", $G^{ad}/C_\lambda^{ad} \longrightarrow \Omega_0 G^{ad}$ is a generating complex. In particular, the fundamental coweight $\omega_s^\vee$ associated to a long simple root α_s is a short minimal circle; we will confine our attention to this special case. If $\lambda = \omega_s^\vee$ happens to lie in the coroot lattice $\mathcal{Q}^\vee$ (examples occur in every type except A, C), then B_λ can be viewed in an evident way as a commutator map $G/C_\lambda \longrightarrow \Omega_{alg} G$. Identifying $\Omega_{alg} G = L_{alg} G/G = \mathcal{L}_G$, we may also view B_λ as the map $gC_\lambda \mapsto \lambda g \lambda^{-1} P$. Since we are assuming $\lambda \in \mathcal{Q}^\vee$ (as opposed to merely $\lambda \in \mathcal{P}^\vee$), multiplication by λ is homotopic to the identity, and hence this last map is in turn homotopic to the map $gC_\lambda \mapsto g\lambda^{-1}P$. In other words, Bott's generating complex corresponds to the $G_{\mathbb{C}}$-Levi orbit $M_{-\lambda}$. This yields the following corollary of Bott's work:

Proposition 6.1. *Suppose α_s is a long simple root and $\omega_s^\vee \in \mathcal{Q}^\vee$. Then the $G_{\mathbb{C}}$-Levi orbit $M_{-\alpha_s^\vee}$ is a smooth generating variety for $\mathcal{L}_G$.*

In particular, if G is not of type A_n or C_n, then $M_{-\alpha_0^\vee}$ is a generating variety.

The second assertion follows because, except in types AC, there is a unique node $t \in S$ adjacent to s_0, linked to s_0 by a single edge. Hence α_t is long and $\omega_t^\vee = \alpha_0^\vee$.

At the opposite extreme, we can take s to be a minuscule node; that is, a node of S that is in the orbit of s_0 under the action of the automorphism group of the affine Dynkin diagram. (Warning: This is a possibly nonstandard use of the term *minuscule*.) The fundamental coweights corresponding to the minuscule nodes are not in $\mathcal{Q}^\vee$and indeed form a complete set of coset representatives for $\mathcal{P}^\vee/\mathcal{Q}^\vee$.

Since these nodes always correspond to long roots, again Bott's construction yields generating varieties for $\Omega_0 G^{ad}$. The second author showed in [21] that these "minuscule generating complexes" correspond to smooth Schubert varieties that are in fact geometric generating varieties. In types E_8, F_4 and G_2, however, there are no minuscule nodes, so this construction does not apply. In view of the preceding proposition, we nevertheless obtain:

Theorem 6.2. *In all Lie types, $\mathcal{L}_G$ admits a smooth generating variety.*

Finally, the following proposition ties up a loose end from [21].

Theorem 6.3. *$\mathcal{L}_G$ admits a smooth Schubert generating variety if and only if G is not of type E_8, F_4 or G_2.*

Proof. What remains to be shown is that in types E_8, F_4 and G_2 no smooth Schubert variety generates the homology. In fact we will show that no smooth Schubert variety can even generate the rational homology. Let $e_1 \leq e_2 \leq \cdots \leq e_r$ denote the exponents of W, where $r = |S|$, and recall that $H_*(\Omega G; \mathbb{Q}) \cong \mathbb{Q}[a_1, \ldots, a_r]$, where $|a_i| = 2e_i$. Hence if X_λ is a rational generating complex, we must have $\dim X_\lambda \geq e_r$. In the three exceptional types $e_r = 29, 11, 5$, respectively [4]. The smooth Schubert varieties were classified in [1]; in particular there are only finitely many in each Lie type. The maximal dimension d of a smooth Schubert variety X_λ is given by:

E_8: $d = 14$ (X_λ is a quadric of type D_8/D_7),

F_4: $d = 7$ (X_λ is a quadric of type B_4/B_3),

G_2: $d = 2$ ($X_\lambda \cong \mathbb{C}P^2$).

Hence none of these three types has a smooth Schubert generating variety.

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