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BOUNDARY ORBIT STRATA AND FACES OF INVARIANT CONES AND COMPLEX OL'SHANSKIĬ SEMIGROUPS

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Abstract. Let $D = G/K$ be an irreducible Hermitian symmetric domain. Then G is contained in a complexification $G_{\mathbb{C}}$, and there exists a closed complex subsemigroup $G \subset \Gamma \subset G_{\mathbb{C}}$, the so-called *minimal Ol'shanskiĭ* semigroup, characterised by the fact that all holomorphic discrete series representations of G extend holomorphically to Γ°.

Parallel to the classical theory of boundary strata for the symmetric domain D, due to Wolf and Korányi, we give a detailed and complete description of the K-orbit type strata of Γ as K-equivariant fibre bundles. They are given by the conjugacy classes of faces of the minimal invariant cone in the Lie algebra.

1. Introduction

The boundary structure of Hermitian symmetric domains $D = G/K$ is well understood through the work of Pjateckiĭ-Shapiro (for the classical domains), and of Wolf and Korányi, in their seminal papers from 1965 [31, 64]: Each of the strata is a K-equivariant fibre bundle whose fibres are Hermitian symmetric domains of lower rank; moreover, the strata are in bijection with maximal parabolic subalgebras. This detailed understanding of the geometry of $\overline{D}$ has been fruitful and is the basis of a variety of developments in representation theory, harmonic analysis, complex and differential geometry, Lie theory, and operator algebras. Let us mention a few.

The original motivation of Wolf–Korányi was to provide Siegel domain realisations for Hermitian symmetric domains, without recourse to their classification. The existence of such realisations alone has led to an extensive literature well beyond the scope of this introduction.

The study of compactifications of (locally and globally) symmetric spaces is of current and continued interest. (We mention the recent monograph [4].) As a

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prominent example, the Baily–Borel compactification of Hermitian symmetric domains has been studied intensely, with applications to moduli spaces of $K3$ surfaces, variation of Hodge structure, and modular forms, among others. Its understanding relies essentially on the Wolf–Korányi result.

The Wolf–Korányi theory has been generalised to complex flag manifolds [65, 68] and has thus played an important role in the realisation theory of tempered representations of semi-simple Lie groups (see the references in [66]); it has found applications to cycle spaces [22, 69, 67] and orbit duality in flag manifolds [8, 48].

Further applications of the original Wolf–Korányi theory include unitary highest weight representations [23, 11, 2]; Poisson integrals [30, 24, 33, 6]; Hardy spaces on various domains [9, 50, 3]; parahermitian or Cayley type symmetric spaces [25, 26]; Toeplitz operators [58, 60].

In 1977, Gel'fand and Gindikin [13] proposed an approach to the study of harmonic analysis on Lie groups of Hermitian type G by considering them as extreme boundaries of complex domains in $G_{\mathbb{C}}$, to which certain series of representations should extend holomorphically. This programme has been widely investigated; notably, it has led to the definition of the so-called Ol'shanskiĭ semigroups and to Hardy type spaces of holomorphic functions on their interiors [51, 52, 56]. More recent progress has been made through the study of so-called complex crowns [38, 39, 37].

Although a great deal is known about Ol'shanskiĭ domains [32, 46, 34, 35, 36], their boundary structure has as yet not been completely investigated. As in the case of Hermitian symmetric domains, we would expect that detailed and complete information on the K-orbit type strata ($K \subset G$ maximal compact) could lead to a better understanding of the geometry and analysis on these domains.

To be more specific, fix an irreducible Hermitian symmetric domain $D = G/K$. The Lie algebra $\mathfrak{g}$ of G contains a minimal G-invariant closed convex cone Ω^- and $\Gamma = G \cdot \exp i\Omega^- \subset G_{\mathbb{C}}$ is the minimal Ol'shanskiĭ semigroup. We classify the faces of Ω^- (Theorem 4.26); each of them can be described explicitly and gives rise to an Ol'shanskiĭ semigroup in the complexification of a certain subgroup of G. These subgroups are semidirect products $S \ltimes H$ where S is the connected automorphism group of a (convex) face of D and H is a certain generalised Heisenberg group related to the intersection of two maximal parabolic subalgebras of $\mathfrak{g}$.

The relative interiors of the faces fall into G- (equivalently, K-) conjugacy classes, each of which forms exactly one of the K-orbit type strata of Ω^- (Theorem 4.28). One immediately deduces the K-orbit type stratification of Γ (Theorem 5.4). Each stratum is a K-equivariant fibre bundle whose fibres are the G-orbits of the 'little Ol'shanskiĭ' semigroups alluded to above. In particular, the fibres are K-equivariantly homotopy equivalent to K itself.

While this result is in beautiful analogy to that of Wolf–Korányi, we stress that the structure of the Ol'shanskiĭ semigroups occurring as fibres is more complicated than that of Γ—their unit groups are not all Hermitian simple. This is already exemplied in the case of the unit disc: here, $G = \mathrm{PSL}(2, \mathbb{R})$, and the only non-trivial faces of the unique invariant cone are the rays spanning nilpotent Iwasawa subalgebras. In general, the unit groups have the semi-direct product structure of a 'generalised Jacobi group' (*i.e.* of a simple Hermitian group acting *via* a vector-valued symplectic group on a generalised Heisenberg group). Another difference to the Korányi–Wolf Theorem is that the strata in the Ol'shanskiĭ correspond to

intersections of *two* maximal parabolics. We remark that all of the above statements can and will be made entirely explicit in the main text of this paper, by the use of the Jordan algebraic structure of the Harish-Chandra embedding of D.

Let us give a more detailed overview of our paper. In Section 2, we collect several basic facts about symmetric domains, symmetric cones, and the associated Lie and Jordan algebraic objects. While most of the information we recall here can be easily extracted from the literature, some items are more specific. So, although this accounts for a rather lengthy glossary of results, we feel that it may serve as a useful reference, in particular with regard to some of the more technical arguments.

Section 3 contains an account of the classification of nilpotent faces. In fact, in the course of its proof, we reprove the classification of conal nilpotent *orbits*. Assuming the latter would not simplify our argument; indeed, our proof of the more precise result (Theorem 3.27) is shorter than the existing proof of the classification of conal nilpotent orbits. The theorem gives the description of all faces of the minimal (or maximal) invariant cone which contain a nilpotent element in their relative interior, and the decomposition of the nilpotent variety in the minimal cone into K-orbit type strata (which are the same as the conal nilpotent G-orbits).

The main body of our work is the content of Section 4. It culminates in the classification of the faces of the minimal invariant cone (Theorem 4.26), the characterisation of their conjugacy, and the description of the K-orbit type strata (Theorem 4.28). The basic observation is that each face generates a subalgebra (the *face algebra*), and its structure is well understood due to the work of Hofmann, Hilgert, Neeb *et al.* on invariant cones in Lie algebras. We show that the Levi complements of the face algebras are exactly the Lie algebras of complete holomorphic vector fields on the faces of the domain D. On the other hand, the centres and nilradicals of the face algebras can be understood through the classification of nilpotent faces. These considerations suffice to complete the classification of all those faces—both of the minimal and the maximal invariant cone—whose face algebra is *non*-reductive. The classification in the case of *reductive* face algebras only works well in the case of the minimal cone; it relies on the observation (Lemma 4.23) that all extreme rays of the minimal cone are nilpotent (a fact which follows directly from the Jordan–Chevalley decomposition).

Finally, in Section 5, we globalise the results of Section 4 to the minimal Ol'shanskiĭ semigroup (Theorem 5.4). Although the global results on the level of the semigroup may ultimately be of greater interest than the infinitesimal results on the level of the minimal cone, the globalisation follows essentially by standard procedures. At this point, all the hard work has been done.

2. Bounded symmetric domains and Jordan triples

We begin with a revision of basic facts about bounded symmetric domains and related matters. We apologise for the tedium, but we will need the details.

2.1. **Bounded symmetric domains and their automorphism groups.** Let Z, $\dim Z = n$, be a complex vector space, $D \subset Z$ a circular bounded symmetric domain. Let G be the connected component of $\mathrm{Aut}(D)$. Then G is a Lie group whose Lie algebra $\mathfrak{g}$ is the set of *complete* holomorphic vector fields on D. The bracket is $\left[h(z)\frac{\partial}{\partial z}, k(z)\frac{\partial}{\partial z}\right] = \big(h'(z)k(z) - k'(z)h(z)\big)\frac{\partial}{\partial z}$, where $\big(h(z)\frac{\partial}{\partial z}\big)f(z) = f'(z)h(z)$ for $h, f : D \to Z$. For $\xi \in \mathfrak{g}$, $g \in \mathrm{Aut}(D)$, $z \in D$, the adjoint action of $\mathrm{Aut}(D)$ is $\mathrm{Ad}(g^{-1})(\xi)(z) = g'(z)^{-1}\xi(g(z))$.

Let $\mathfrak{k}$ be the set of *linear* vector fields. The vector field $h_0 = iz\frac{\partial}{\partial z} \in \mathfrak{z}(\mathfrak{k}) \setminus 0$ generates the U(1)-action. The isotropy K of G at 0 has the Lie algebra $\mathfrak{k}$. Moreover, $D = G/K$; we have $Z(G) = 1$; elements of K are linear; K is maximal compact and equals the fixed group of the Cartan involution $\vartheta = \mathrm{Ad}(-\mathrm{id}_D)$ [57, Lemma 1.7], [41, Sect. 1.2], [16, Chapter III, §7, Proposition 7.4].

Jordan triples. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$; $\mathfrak{p} \to Z : \xi \mapsto \xi(0)$ is a real linear isomorphism. Define $\xi_z^- \in \mathfrak{p}$ by $\xi_z^-(0) = z$, further $Q_z(w) = z - \xi_z^-(w)$, $2Q_{u,w} = Q_{u+w} - Q_u - Q_w$; then $\{uv^*w\} = Q_{u,w}(v)$ is linear in u and w, conjugate linear in v, and $(u \,\square\, v^*)(w) = \{uv^*w\}$ satisfies [41, Lemma 2.6]

$$\{uv^*w\} = \{wv^*u\}\ ,\ \left[u \,\square\, v^*, z \,\square\, w^*\right] = \{uv^*z\} \,\square\, w^* - z \,\square\, \{wu^*v\}^*\ , \tag{2.1}$$

so Z is a *Jordan triple*. We have $\xi_u^- = (u - \{zu^*z\})\frac{\partial}{\partial z}$ and the identities [41, Lemma 2.6]

$$\left[\xi_u^-, \xi_v^-\right] = 2(u \,\square\, v^* - v \,\square\, u^*) \quad \text{and} \quad \left[\left[\xi_u^-, \xi_v^-\right], \xi_w^-\right] = 2\xi^-_{\{uv^*w\} - \{vu^*w\}}\ . \tag{2.2}$$

Here, we identify $\mathfrak{k}$ with a subset of $\mathrm{End}(Z)$.

The trace form $\mathrm{tr}_Z(u \,\square\, v^*)$ defines a positive Hermitian inner product on Z such that $v \,\square\, u^* = (u \,\square\, v^*)^*$, so the Jordan triple Z is *Hermitian* [41, Lemma 2.6], [59]. W.r.t. a certain norm, D is the unit ball, and this sets up a bijection between isomorphism classes of Hermitian Jordan triples and of bounded symmetric domains [41, Theorem 4.1]. Note that D is convex and K is connected.

Triple automorphisms. Any $k \in \mathrm{GL}(Z)$ such that $k(u \,\square\, v^*)k^{-1} = ku \,\square\, (kv)^*$ is called a *triple automorphism*. K is the connected component of the set $\mathrm{Aut}(Z)$ of triple automorphisms [41, Corollary 4.9]. The Lie algebra $\mathfrak{k}$ of $\mathrm{Aut}(Z)$ coincides with the *triple derivations* $\mathrm{aut}(Z) = \left\{\delta \in \mathrm{End}(Z) \;\middle|\; \left[\delta, u \,\square\, v^*\right] = (\delta u) \,\square\, v^* + u \,\square\, (\delta v)^*\right\}$. All triple derivations are inner, i.e. $\mathfrak{k} = \mathrm{aut}(Z) = \langle u \,\square\, v^* - v \,\square\, u^* \mid u, v \in Z\rangle_{\mathbb{R}}$.

2.2. **Cartan decomposition and Killing form.** Denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_{\mathbb{C}}$, *etc.* The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ gives $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$; ϑ extends to the conjugation of $\mathfrak{g}_{\mathbb{C}}$ w.r.t. $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$.

Lemma 2.1. *We have the vector space decomposition* $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, *where*

$$\mathfrak{p}^+ = \left\{u\tfrac{\partial}{\partial z} \;\middle|\; u \in Z\right\} \quad \textit{and} \quad \mathfrak{p}^- = \left\{\{zu^*z\}\tfrac{\partial}{\partial z} \;\middle|\; u \in Z\right\}\ .$$

Moreover, $\vartheta(\mathfrak{p}^+) = \mathfrak{p}^-$, *and* $\mathfrak{p}^\pm$ *are* $\mathfrak{k}_{\mathbb{C}}$*-invariant and Abelian.*

Proof. We have

$$\vartheta\big(u\tfrac{\partial}{\partial z}\big) = \tfrac{1}{2}\vartheta(\xi_u^- - i\xi_{iu}^-) = -\tfrac{1}{2}(\xi_u^- + i\xi_{iu}^-) = \{zu^*z\}\tfrac{\partial}{\partial z}\ . \tag{2.3}$$

The vector fields in $\mathfrak{p}^+$ are constant, so $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$. Applying ϑ gives $\left[\mathfrak{p}^-, \mathfrak{p}^-\right] = 0$.

Any $\delta \in \mathfrak{k}$ is linear, so $\left[\delta, u\frac{\partial}{\partial z}\right] = (\delta u)\frac{\partial}{\partial z}$ for all $u \in Z$. Since $\mathfrak{k}$ leaves u invariant and hence commutes with ϑ, the assertion follows. □

Lemma 2.2. *The centraliser of* $h_0 = iz\frac{\partial}{\partial z}$ *in* $\mathfrak{g}$ *is* $\mathfrak{k}$. *More precisely,* $\mathrm{ad}\, h_0 = \pm i$ *on* $\mathfrak{p}^\pm$.

Proof. Clearly, $\mathrm{ad}\, h_0 = i$ on $\mathfrak{p}^+$. The assertion follows from (2.3). □

Killing form. The Killing form B of $\mathfrak{g}$ is given by $B(\xi, \eta) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\,\xi\,\mathrm{ad}\,\eta)$ for all $\xi, \eta \in \mathfrak{g}$. Its complex bilinear extension to $\mathfrak{g}_{\mathbb{C}}$ will also be denoted by B.

Lemma 2.3 ([29, Lemma 4.2], [57, Lemma 6.1])**.** *The splitting* $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$ *is B-orthogonal, and* $\mathfrak{p}^\pm$ *are B-isotropic. We have, for all* $\delta \in \operatorname{aut}(Z)$, $u, v \in Z$,

$$B(\delta, u \,\square\, v^*) = 2 \operatorname{tr}_Z\big((\delta u) \,\square\, v^*\big) \ , \ B\big(u\tfrac{\partial}{\partial z}, \{zv^*z\}\tfrac{\partial}{\partial z}\big) = -4 \operatorname{tr}_Z(u \,\square\, v^*) \ ,$$
$$B\big(\xi_u^-, \xi_v^-\big) = 4 \operatorname{tr}_Z(u \,\square\, v^* + v \,\square\, u^*) \ .$$

Proof. The decomposition is orthogonal since $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{p}^\pm$ are distinct $\operatorname{ad} h_0$-eigenspaces. We have $\big[u\frac{\partial}{\partial z}, \{zv^*z\}\frac{\partial}{\partial z}\big] = -2 \cdot u \,\square\, v^*$ from (2.2). Thus,

$$B(\delta, u \,\square\, v^*) = -\tfrac{1}{2} B\big(\big[\delta, u\tfrac{\partial}{\partial z}\big], \{zv^*z\}\tfrac{\partial}{\partial z}\big) = -\tfrac{1}{2} B\big((\delta u)\tfrac{\partial}{\partial z}, \{zv^*z\}\tfrac{\partial}{\partial z}\big) \ .$$

The 2nd equation implies the 1st; the 1st with $\delta = h_0$ implies the 2nd. But

$$B(h_0, u \,\square\, v^*) = i \operatorname{tr}_{\mathfrak{p}^+} \operatorname{ad}(u \,\square\, v^*) - i \operatorname{tr}_{\mathfrak{p}^-} \operatorname{ad}(u \,\square\, v^*)$$

by Lemma 2.2. Moreover, by (2.3) and $\vartheta(u \,\square\, v^*) = -v \,\square\, u^*$,

$$\big[u \,\square\, v^*, w\tfrac{\partial}{\partial z}\big] = \{uv^*w\}\tfrac{\partial}{\partial z} \ , \ \big[u \,\square\, v^*, \{zw^*z\}\tfrac{\partial}{\partial z}\big] = -\big\{z\{vu^*w\}^*z\big\}\tfrac{\partial}{\partial z} \ .$$

Because $\operatorname{tr}_Z u \,\square\, v^* = \overline{\operatorname{tr}_Z v \,\square\, u^*}$, we have $B(h_0, u \,\square\, v^*) = 2i \operatorname{tr}_Z(u \,\square\, v^*)$.

The subspaces $\mathfrak{p}^\pm$ are isotropic, since $B(\mathfrak{p}^\pm, \mathfrak{p}^\pm) = \mp i B(h_0, [\mathfrak{p}^\pm, \mathfrak{p}^\pm]) = 0$. Now,

$$B(\xi_u^-, \xi_v^-) = -B\big(u\tfrac{\partial}{\partial z}, \{zv^*z\}\tfrac{\partial}{\partial z}\big) - B\big(\{zu^*z\}\tfrac{\partial}{\partial z}, v\tfrac{\partial}{\partial z}\big) = 4 \operatorname{tr}_Z(u \,\square\, v^* + v \,\square\, u^*) \ . \ \square$$

We remark that if Z is a simple Jordan triple, then $\mathfrak{g}$ is simple [29, Theorem 4.4].

2.3. **Tripotents and faces of D.** An $e \in Z$ is a *tripotent* if $\{ee^*e\} = e$. The $Z_\lambda(e) = \ker(e \,\square\, e^* - \lambda)$ are called *Peirce λ-spaces.* Then $Z = Z_0(e) \oplus Z_{1/2}(e) \oplus Z_1(e)$, the orthogonal sum w.r.t. the trace form [41, Theorem 3.13]. We have the *Peirce rules* [59, Proposition 21.9]

$$\{Z_\alpha(e) Z_\beta(e)^* Z_\gamma(e)\} \subset Z_{\alpha-\beta+\gamma}(e) \ , \ \{Z_0(e) Z_1(e)^* Z\} = \{Z_1(e) Z_0(e)^* Z\} = 0 \ .$$

In particular, $Z_1(e)$ and $Z_0(e)$ are then subtriples. For tripotents e, c, $e \,\square\, c^* = 0$ if and only if $\{ee^*c\} = 0$ [41, Lemma 3.9]; we call e, c *orthogonal* ($e \perp c$). Define an order,

$$c \leqslant e \ :\Leftrightarrow \ \{(e-c)(e-c)^*(e-c)\} = e - c \text{ and } c \perp e - c \ .$$

We call non-zero minimal (maximal) tripotents *primitive* (*maximal*); e is primitive (maximal) if and only if $Z_1(e) = \mathbb{C}e$ ($Z_0(e) = 0$). *Unitary* tripotents ($Z = Z_1(e)$) are maximal; the converse holds for Z a Jordan algebra (D of tube type).

Frames, joint Peirce spaces. A maximal orthogonal set $e_1, \dots, e_r$ of primitive tripotents is a *frame.* In this case, $r = \operatorname{rk} D$. Define $\operatorname{rk} Z = r$, $\operatorname{rk} e = \operatorname{rk} Z_1(e)$. Any tripotent equals $e_1 + \cdots + e_k$ for orthogonal primitive e_j [41, 5.1, Theorem 3.11]. Given a frame, the *joint Peirce spaces* are

$$Z_{ij} = \{z \in Z | \{e_k e_k^* z\} = \tfrac{1}{2}(\delta_{ik} + \delta_{jk}) \cdot z \ \forall k\} \ , \ 0 \leqslant i \leqslant j \leqslant r \ .$$

Then $Z = \bigoplus_{0 \leqslant i \leqslant j \leqslant r} Z_{ij}$, $Z_{00} = 0$, and $Z_{ii} = \mathbb{C}e_i$ ($i > 0$). If Z is simple, then $a = \dim Z_{ij}$, $b = \dim Z_{0j}$ are independent of i, j and the frame, and $b = 0$ exactly if Z is a Jordan algebra. The *canonical inner product* $(\cdot|\cdot)$ is the unique K-invariant inner product on Z for which $v \,\square\, u^* = (u \,\square\, v^*)^*$ and $(e|e) = 1$ for every primitive tripotent e. Its restriction to any subtriple is canonical. For simple Z, $(u|v) = \frac{2r}{2n - rb} \cdot \operatorname{tr}_Z(u \,\square\, v^*)$.

Faces of D. Given a convex set $C \subset \mathbb{R}^d$, a subset $F \subset \overline{C}$ is a *face* if any open line segment in C intersecting F lies in F. We let F° (the *relative interior*) denote the

union of these open line segments. A hyperplane is *supporting* if C lies on one side of it. A proper face is *exposed* if $F = \overline{C} \cap H$ for a supporting hyperplane H.

For any tripotent $e \in Z$, define $D_0(e) = D \cap Z_0(e)$, the symmetric domain associated with $Z_0(e)$. Then $e + D_0(e)$ is a face of D, and this defines a bijection between tripotents of Z and faces of D [41, Theorem 6.3], and all faces are exposed.

Definition 2.4. For any tripotent e, let $G_0(e) = \mathrm{Aut}_0(D_0(e))$. Then $G_0(e) = K_0(e) \cdot \exp \mathfrak{p}_0(e)$ where $K_0(e) = \mathrm{Aut}_0(Z_0(e))$, $\mathfrak{p}_0(e) = \{\xi_u^- | u \in Z_0(e)\}$. The Lie algebra of $K_0(e)$ is $\mathfrak{k}_0(e) = \mathrm{aut}(Z_0(e)) \subset \mathfrak{k}$. In particular, $K_0(e) \subset K$, $G_0(e)$ is a closed subgroup of G, and $e + D_0(e) = G_0(e).e \cong G_0(e)/K_0(e)$. If $c \leqslant e$, then $G_0(e) \subset G_0(c)$.

2.4. **Symmetric cones and formally real Jordan algebras.** We conclude our preliminaries with a short section on symmetric cones. This may seem to be somewhat of a digression, but will be important in what follows.

Unitary tripotents and Jordan algebras. A tripotent $e \in Z$ is *unitary* if $Z = Z_1(e)$. It defines a composition $z \circ w = \{ze^*w\}$ and an involution $z^* = \{ez^*e\}$. Then $\circ$ is commutative, e is a unit, and $z^2 \circ (z \circ w) = z \circ (z^2 \circ w)$, so Z is a *complex Jordan algebra* [59, Proposition 13]. The triple product is recovered by

$$\{uv^*w\} = u \circ (v^* \circ w) - v^* \circ (w \circ u) + w \circ (u \circ v^*) \quad \text{for all} \quad u, v, w \in Z \ . \tag{2.4}$$

The $\circ$-closed real form $X = \{x \in Z \mid x^* = x\}$ is a real *Jordan algebra*. Furthermore, $x^2 + y^2 = 0$ implies $x = y = 0$, *i.e.* X is *formally real* [41, Theorem 3.13]. Conversely, for formally real X, the underlying triple of the complex Jordan algebra $X_{\mathbb{C}}$ is Hermitian. We write $X_1(e)$ for the canonical real form of $Z_1(e)$.

Symmetric cones. Let X, $\dim X = n$, be a real vector space with an inner product $(\cdot : \cdot)$. A convex cone $\Omega \subset X$ is *pointed* if it contains no affine line, *solid* if its interior in X is non-void, and *regular* if it is both pointed and solid. The *dual cone* $\Omega^* = \left\{x \in X \;\middle|\; (x : \Omega) \geqslant 0\right\}$ is pointed (solid) if and only if Ω is solid (pointed). Let $\Omega \subset X$ be a closed solid cone. Then $\mathrm{GL}(\Omega) = \{g \in \mathrm{GL}(X) | g\Omega = \Omega\}$ is a closed subgroup of $\mathrm{GL}(X)$; denote its Lie algebra by $\mathfrak{gl}(\Omega)$. Ω is *symmetric* if $\Omega^* = \Omega$ and $\mathrm{GL}(\Omega)$ acts transitively on Ω°. Any symmetric cone is pointed.

Assume Ω is symmetric. Then $\vartheta(g) = (g^{-1})^t$ is a Cartan involution of the reductive group $\mathrm{GL}(\Omega)$, with compact fixed group $\mathrm{O}(\Omega) = \mathrm{O}(X) \cap \mathrm{GL}(\Omega)$, and we may fix $e \in \Omega^\circ$ such that the stabiliser $\mathrm{GL}(\Omega)_e = \mathrm{O}(\Omega)$ [12, Proposition I.1.8]. Denote the Cartan decomposition by $\mathfrak{gl}(\Omega) = \mathfrak{o}(\Omega) \oplus \mathfrak{p}(\Omega)$. The linear map $\xi \mapsto \xi(e) : \mathfrak{p}(\Omega) \to X$ is an isomorphism. Define $M_x \in \mathfrak{p}(\Omega)$ by $M_x(e) = x$. Then $x \circ y = M_x y$ makes X a formally real Jordan algebra with identity e [12, Theorem III.3.1]. On the other hand, $\Omega = \{x^2 \mid x \in X\}$. This sets up a bijection between isomorphism classes of symmetric cones and of formally real Jordan algebras [12, Theorem III.2.1].

The connected component $\mathrm{GL}_+(\Omega)$ of $\mathrm{GL}(\Omega)$ is transitive on Ω°, and $\mathrm{O}(\Omega)$ is the set $\mathrm{Aut}(X)$ of *Jordan algebra automorphisms* ($k \in \mathrm{GL}(X)$, $k(x \circ y) = (kx) \circ (ky)$) [12, Theorem III.5.1]. Its Lie algebra $\mathfrak{o}(\Omega)$ is the set $\mathrm{aut}(X)$ of all *Jordan algebra derivations* ($\delta \in \mathrm{End}(X)$ such that $\delta(x \circ y) = (\delta x) \circ y + x \circ (\delta y)$). It can be seen that $\mathrm{Aut}(X)$ is the set of those triple automorphisms k of $X \otimes \mathbb{C}$ such that $ke = e$, and that $\mathrm{aut}(X)$ consists of all triple derivations δ such that $\delta(e) = 0$.

Idempotents and Peirce decomposition. Any $c \in X$ such that $c^2 = c$ is an *idempotent*. Let $X_\lambda(c) = \ker(M_c - \lambda)$ be the *Peirce λ-space*; the *Peirce decomposition* $X = X_0(c) \oplus X_{1/2}(c) \oplus X_1(c)$ is orthogonal w.r.t. the trace form $\mathrm{tr}_X(M_{x \circ y})$. The latter is an $\mathrm{O}(\Omega)$-invariant inner product and thus is proportional to $(\cdot : \cdot)$.

As above, we define orthogonality and the ordering of idempotents. We call non-zero minimal idempotents *primitive*, and maximal orthogonal sets of primitive idempotents *frames*. Their common cardinality is $r = \mathrm{rk}\, X$. Then $\mathrm{rk}\, c = \mathrm{rk}\, X_1(c) = k$ if and only if $c = c_1 + \cdots + c_k$ for orthogonal primitive c_j. The *canonical inner product* $(\cdot|\cdot)$ is the unique $\mathrm{O}(\Omega)$-invariant inner product for which $(u \circ v|w) = (v|u \circ w)$ and $(c|c) = 1$ for any primitive c. If X is simple, then $(x|y) = \frac{r}{n} \cdot \mathrm{tr}_X(M_{x \circ y})$.

Orbits and faces of symmetric cones. Let X be a simple formally real Jordan algebra, and fix a frame $c_1, \ldots, c_r$. Let $c^k = c_1 + \cdots + c_k$. The cone Ω of squares decomposes into the $r+1$ orbits $\mathrm{GL}_+(\Omega).c^k$, $k = 0, \ldots, r$ [12, Proposition IV.3.1]. With any idempotent c, we associate the cone of squares $\Omega_0(c) \subset X_0(c) = X_1(e-c)$.

Proposition 2.5 ([7]). *The set of faces of Ω consists of*

$$\Omega_0(c) = X_0(c) \cap \Omega = c^\perp \cap \Omega = \left\{ x^2 \;\middle|\; x \in X_0(c) \right\} \ , \ c = c^2 \in X \ .$$

In particular, all the faces of Ω are exposed. The dual face of $\Omega_0(c)$ is $\Omega_0(e-c)$. Two faces $\Omega_0(c)$ and $\Omega_0(c')$ are $\mathrm{GL}_+(\Omega)$-conjugate if and only if $\mathrm{rk}\, c = \mathrm{rk}\, c'$.

Proof. The set of elements of rank k in Ω is $\mathrm{GL}_+(\Omega).c^k$; hence the conjugacy [12, Proposition IV.3.1] holds. For $x \in X$, $x \in \Omega$ if and only if M_x is positive semi-definite [12, Proposition III.2.2], so $\Omega \cap X_0(c) = \Omega_0(c)$. Since $c \in \Omega = \Omega^*$, $c^\perp$ is a supporting hyperplane, and $c^\perp \cap \Omega$ is an exposed face. We have $c^\perp \supset \Omega_0(c)$. On the other hand, $\Omega \cap c^\perp \subset X_0(c)$ if $c^2 = c$ [12, Exercise III.3], so $\Omega_0(c) = c^\perp \cap \Omega$.

More generally, $\Omega_0(e-c) = \Omega \cap \Omega_0(c)^\perp$, and $\Omega_0(e-c)$, $\Omega_0(c)$ are dual faces. The extreme rays of Ω are the $\Omega_0(c)$, where $\mathrm{rk}\, c = r-1$ [12, Proposition IV.3.2]. Since Ω is self-dual, any proper face $F \subsetneq \Omega$ has a non-trivial dual face. Hence, $\Omega_0(e-c)$ is a maximal proper face. The faces of Ω contained in $\Omega_0(e-c)$ are exactly the faces of $\Omega_0(e-c)$. The claim follows by induction. $\square$

3. Nilpotent orbits and faces, maximal parabolics, and principal faces

We now return to our setting of a Hermitian Jordan triple Z of dimension n and the associated circular bounded symmetric domain $D \subset Z$. In this section, we introduce the minimal and maximal invariant cones in $\mathfrak{g}$ and classify their nilpotent faces. On the way, we reprove the classification of conal nilpotent *orbits*. We also introduce a class of faces (called *principal*) which are associated to maximal parabolic subalgebras.

3.1. Weyl group-invariant cones.

Consider the positive symmetric invariant form defined by

$$(\xi : \eta) = -B(\xi, \vartheta\eta) \quad \text{for all} \quad \xi, \eta \in \mathfrak{g} \ . \tag{3.1}$$

Toral Cartan subalgebra. Fix a frame $e_1, \ldots, e_r$ of Z. By [61, Lemma 1.1-2], there exists a Cartan subalgebra $\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{t}^- \subset \mathfrak{k}$, where

$$\mathfrak{t}^- = \langle ie_j \,\square\, e_j^* \mid j = 1, \ldots, r \rangle_{\mathbb{R}} \ , \ \mathfrak{t}^+ = \{ \delta \in \mathfrak{t} | \delta e_j = 0 \ \text{ for all } \ j = 1, \ldots, r \} \ . \tag{3.2}$$

By Lemma 2.2, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$, and $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}} : \mathfrak{t}_{\mathbb{C}})$ be the associated root system. Let $h_0 = iz\frac{\partial}{\partial z}$. By Lemma 2.2, for $\alpha \in \Delta$,

$$\mathfrak{g}^{\alpha}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}} \;\Leftrightarrow\; \alpha(h_0) = 0 \quad \text{and} \quad \mathfrak{g}^{\alpha}_{\mathbb{C}} \subset \mathfrak{p}_{\mathbb{C}} \;\Leftrightarrow\; \alpha(h_0) \neq 0 \;.$$

This gives a partition of Δ into subsets Δ_c and Δ_n of *compact* and *non-compact* roots, respectively. We consider the Weyl groups $W = W(\Delta)$ and $W_c = W(\Delta_c)$. For $\alpha \in \Delta$, let $H_\alpha \in i\mathfrak{t}$ be determined by the fact that $B(H_\alpha, \cdot) \in \mathbb{R}\alpha$ and $\alpha(H_\alpha) = 2$ [27, ch. IV].

Definition 3.1. Let Φ be a positive system of Δ. Let $\Phi_c = \Delta_c \cap \Phi$ and $\Phi_n = \Delta_n \cap \Phi$. The positive system Φ is *adapted* [47] if for all $\alpha, \beta \in \Phi_n$, $\alpha + \beta \notin \Delta$. Equivalently, any Φ_c-simple root is Φ-simple; Φ_n is W_c-invariant; for some (any) total order on $\langle \Delta \rangle_{\mathbb{R}}$ defining Φ, $\Phi_c < \Phi_n$; the set $\Delta_c \cup \Phi_n$ is parabolic [47, Proposition VII.2.12].

Lemma 3.2. *Let $\Delta_c^{++} \subset \Delta_c$ be a positive system, $\Delta_n^{++} = \{\alpha \in \Delta | -i\alpha(h_0) > 0\}$ and $\Delta^{++} = \Delta_c^{++} \cup \Delta_n^{++}$. Then Δ^{++} is adapted, and $\mathfrak{p}^{\pm} = \bigoplus_{\alpha \in \Delta_n^{++}} \mathfrak{g}^{\pm\alpha}_{\mathbb{C}}$.*

Remark 3.3. For simple Z, adapted positive systems are $\{\pm 1\} \times W_c$-conjugate [47, Lemma VII.2.16]. Moreover, a non-compact simple Lie algebra has an adapted positive system if and only if it is Hermitian if and only if it is associated with an irreducible bounded symmetric domain [47, Proposition VII.2.14].

Minimal W_c-invariant cone. Consider the following polyhedral cones in $\mathfrak{t}$,

$$\omega^- = \operatorname{cone} \langle iH_\alpha \mid \alpha \in \Delta_n^{++} \rangle \;,\; \omega^+ = \left\{ H \in \mathfrak{t} \;\middle|\; -i\alpha(H) \geqslant 0 \;\text{ for all }\; \alpha \in \Delta_n^{++} \right\} \;.$$

Then $\omega^+ = (-i\Delta_n^{++})^*$ is the dual cone of ω^-, and both cones are regular. By Lemma 3.2, $\omega^{\pm}$ are W_c-invariant. We have $\omega^- \subset \omega^+$ [15, Lemma 10]. For $k = 1, \ldots, r$, define $\gamma_k \in i\mathfrak{t}^*$ by $\gamma_k(e_\ell \,\square\, e_\ell^*) = \delta_{k\ell}$ and $\gamma_k(\mathfrak{t}^+) = 0$. Then (γ_k) is a strongly orthogonal set [61, Lemma 1.3], *i.e.* $\gamma_k \pm \gamma_\ell \notin \Delta$, $k \neq \ell$.

Since $iz\frac{\partial}{\partial z} \equiv \sum_{k=1}^r ie_k \,\square\, e_k^* \; (\mathfrak{t}^+)$, we find $\gamma_k \in \Delta_n^{++}$. There is a total vector space order on $i\mathfrak{t}^*$ defining Δ^{++}, such that $0 < \gamma_1 < \cdots < \gamma_r$. Consequently, $\gamma_1, \ldots, \gamma_r$ is the *Harish-Chandra fundamental sequence* [15, II.6]. In particular, (γ_k) is a strongly orthogonal set of maximal cardinality [15, Lemma 8 and Corollary].

Definition 3.4. A root $\alpha \in \Delta$ is *long* if $|\alpha| \geqslant |\beta|$ for all $\beta \in \Delta$ contained in the same irreducible subsystem of Δ as α.[1]

The γ_k are long [49, Theorem 2], [53, Lemma 1]. All positive, long non-compact roots lying in the same irreducible subsystem of Δ are W_c-conjugate [53, Lemma 2].

Lemma 3.5. *The extreme rays of ω^- are generated exactly by iH_α, $\alpha \in \Delta_n^{++}$, α long. In particular, $\omega^- = \operatorname{cone} \langle i\sigma(e_j \,\square\, e_j^*) \mid \sigma \in W_c \,,\, j = 1, \ldots, r \rangle$.*

Proof. By definition, the generators of the extreme rays of ω^- are among the H_α, $\alpha \in \Delta_n^{++}$. Since Δ, and hence ω^-, decompose according to $\mathfrak{g}$ into simple factors, we may assume w.l.o.g. that $\mathfrak{g}$ is simple. For any short $\gamma \in \Delta_n^{++}$, $\gamma = \frac{1}{2}(\gamma_k + \gamma_\ell)$, some $k \neq \ell$ [53, Lemma 1]. Hence, $4|\gamma|^2 = |\gamma_k|^2 + |\gamma_\ell|^2 = 2|\gamma_k|^2$, and

$$(H_\gamma : \xi) = 2|\gamma|^{-2}\gamma(\xi) = 2|\gamma_k|^{-2}\gamma_k(\xi) + 2|\gamma_\ell|^{-2}\gamma_\ell(\xi) = (H_{\gamma_k} + H_{\gamma_\ell} : \xi)$$

for all $\xi \in \mathfrak{t}$. Hence, $iH_\gamma = iH_{\gamma_k} + iH_{\gamma_\ell}$ lies in the interior of a face of dimension at least 2. On the other hand, ω^- being polyhedral, there is $\alpha \in \Delta_n^{++}$, necessarily long, such that $i\mathbb{R}_{\geqslant 0} \cdot H_\alpha$ is extreme, but all such iH_α are W_c-conjugate.

[1] An irreducible root system has at most two root lengths [5, Chapter VI, §1.4, Proposition 12].

Returning to the semi-simple case, by maximality, any irreducible factor of Δ contains some γ_k. Moreover, any long $\alpha \in \Delta_n^{++}$ is W_c-conjugate to any γ_k contained in the same irreducible factor. By Lemma 2.3 and (3.2), $e_j \,\square\, e_j^*$ is proportional to H_{γ_j}. Finally, observe that any polyhedral cone is generated by its extreme rays. $\square$

Lemma 3.6. *Let $\gamma \in \Delta_n^{++}$ be long. There is a frame $c_1, \dots, c_r$ such that $\mathfrak{t}$ is given by (3.2) (for $e_j = c_j$), and an integer ℓ such that $\gamma(c_k \square c_k^*) = \delta_{k\ell}$ and $\gamma(\mathfrak{t}^+) = 0$.*

Proof. For some ℓ, γ_ℓ and γ lie in the same irreducible factor of Δ; there is some $\sigma \in W_c$ such that $\sigma\gamma_\ell = \gamma$. Then $\sigma = \mathrm{Ad}(k)$ for some $k \in N_K(\mathfrak{t})$ [28, Theorem 4.54]. Since $k \in \mathrm{Aut}(Z)$, the ke_j, $j = 1, \dots, r$, are orthogonal primitive tripotents, and

$$\mathrm{Ad}(k)\big(e_j \square e_j^* \tfrac{\partial}{\partial z}\big) = k^{-1\prime}(z)^{-1}\{e_j e_j^* k^{-1}(z)\}\tfrac{\partial}{\partial z} = k(e_j \square e_j^*)k^{-1} = (ke_j) \square (ke_j)^*,$$

where we identify linear maps and linear vector fields. Since $\mathrm{Ad}(k)$ normalises $\mathfrak{t}$, we have a decomposition as stated. By the definition of γ_ℓ, the lemma follows. $\square$

Corollary 3.7. *The extreme rays of ω^- are generated by the $i \cdot e \,\square\, e^*$, where e is a primitive tripotent W_c-conjugate to an element of the frame $e_1, \dots, e_r$.*

Relation to the Weyl chamber. The Weyl chamber associated to Δ^{++} is

$$c_+ = \big\{H \in \mathfrak{t} \;\big|\; -i\alpha(H) > 0 \;\text{ for all }\; \alpha \in \Delta^{++}\big\}\ .$$

By definition, it is obvious that $c_+ \subset \omega^{+\circ}$. In fact, $\overline{c_+}$ is a fundamental domain for the action of W_c on ω^+ [45, Lemma I.5]. From this, one immediately deduces the following statement.

Lemma 3.8. *Let $\Pi = (\alpha_k)$ be the simple system defining Δ^{++}, and define $\omega_k \in \mathfrak{t}$ by $\alpha_k(\omega_\ell) = i\delta_{k\ell}$. Then the generators of extreme rays of ω^+ belong to $\bigcup_k W_c.\omega_k$.*

3.2. **Minimal and maximal invariant cones.** From now on, we assume that Z is simple. Then $\mathfrak{z}(\mathfrak{k}) = \mathbb{R} \cdot h_0$, where $h_0 = iz\frac{\partial}{\partial z}$.

Maximal and minimal cone. Consider the map $\Omega \mapsto \omega = \Omega \cap \mathfrak{t}$ from the set of closed regular G-invariant convex cones $\Omega \subset \mathfrak{g}$ to the set of closed W_c-invariant convex cones ω such that $\omega^- \subset \omega \subset \omega^+$. It is a lattice isomorphism [53, Theorem 2], and $\Omega = \{\xi \in \mathfrak{g} | p_{\mathfrak{t}}(\mathcal{O}_\xi) \subset \omega\}$, where $\mathcal{O}_\xi = \mathrm{Ad}(G)(\xi)$ and $p_{\mathfrak{t}}$ is the orthogonal projection onto $\mathfrak{t}$. Moreover, $\Omega^* \cap \mathfrak{t} = (\Omega \cap \mathfrak{t})^*$ [53, Theorem 3] and any orbit in Ω° intersects the relative interior of ω non-trivially.

Let Ω^- be the closed G-invariant convex cone generated by $iz\frac{\partial}{\partial z}$. Then we have $iz\,\frac{\partial}{\partial z} \in \omega^- \cap \Omega^{-\circ}$ [53, Lemma 3]. All solid invariant cones have a K-fixed vector [62, § 2], so Ω^- is minimal among solid invariant cones, and its dual Ω^+ is maximal among pointed invariant cones.[2] From this, it follows that $\Omega^\pm \cap \mathfrak{t} = \omega^\pm$. The following result relates $\Omega^\pm$ to the set of semi-simple elements.

Proposition 3.9. *Let $\xi \in \Omega^\pm$ be semi-simple. Then ξ is conjugate to an element of $\omega^\pm$. If, in addition, ξ is regular, then ξ is conjugate to an element of $\omega^{+\circ}$ and hence is contained in $\Omega^{+\circ}$.*

Proof. The orbit $\mathcal{O}_\xi$ is closed [63, Proposition 1.3.5.5]. Hence, it intersects $\mathfrak{t}$ [21, Theorem 5.11]. This proves the first statement. The second statement now follows immediately from the fact that the set of regular semi-simple elements is open [63, Proposition 1.3.4.1], and that the centraliser of ξ is a compact Cartan subalgebra. $\square$

[2]For the invariance observe that $(\mathrm{Ad}(g)(x) : y) = (x : \mathrm{Ad}(\vartheta(g))(y))$ for all $x, y \in \mathfrak{g}$, $g \in G$.

3.3. **Tripotents, nilpotent faces, and nilpotent orbits of convex type.** Although Cayley triples have already been studied extensively, we have to redo some of their theory to derive our result. In particular, we are interested in the following subclass of Cayley triples.

(H_1)-Cayley triples. A Lie algebra $\mathfrak{a}$ is *quasihermitian* if $\mathfrak{b} = \mathfrak{z}_{\mathfrak{a}}(\mathfrak{z}(\mathfrak{b}))$ for some maximal compact subalgebra $\mathfrak{b} \subset \mathfrak{a}$ containing a Cartan subalgebra of $\mathfrak{a}$. If $\mathfrak{a}$ is simple and non-compact, it is called *Hermitian* if some maximal compact subalgebra has non-trivial centre. A reductive Lie algebra $\mathfrak{a}$ is quasihermitian if and only if it is the direct sum of a maximal compact ideal and Hermitian simple ideals [44].

Consider the basis of $\mathfrak{sl}(2,\mathbb{R})$ given by $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $X^+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $X^- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Let $\mathfrak{h}$ be a quasihermitian reductive Lie algebra. Recall that $(h, x^+, x^-) \in \mathfrak{h}^3$ is called an $\mathfrak{sl}_2$*-triple* if the associated linear map, defined by $H \mapsto h$ and $X^\pm \mapsto x^\pm$, is a Lie algebra monomorphism. x^+ is called the *nilpositive* element of the triple. Given a Cartan involution θ, an $\mathfrak{sl}_2$-triple (h, x^+, x^-) is called a *Cayley triple* if $\theta(x^+) = -x^-$.

An element $h_0 \in \mathfrak{h}$ is called an H*-element* if $\mathfrak{z}_{\mathfrak{h}}(h_0)$ is maximally compactly embedded and $\mathrm{Sp}(\operatorname{ad} h_0) = \{0, \pm i\}$. Any H-element is semi-simple. With any H-element h_0, there is associated a unique Cartan involution $\theta = 2\operatorname{ad}(h_0)^2 + 1$. For example, $iz\frac{\partial}{\partial z}$ is an H-element of $\mathfrak{g}$, and $Z = \frac{1}{2}(X^+ - X^-)$ is an H-element of $\mathfrak{sl}_2$. A homomorphism $\phi : \mathfrak{h} \to \mathfrak{h}'$ of quasihermitian reductive Lie algebras with fixed H-elements $h_0 \in \mathfrak{h}$ and $h_0' \in \mathfrak{h}'$ is called an (H_1)-homomorphism if $\operatorname{ad} h_0' \circ \phi = \phi \circ \operatorname{ad} h_0$.

Given an H-element $h_0 \in \mathfrak{h}$ with associated Cartan involution θ, any Cayley triple (h, x^+, x^-) will be called an (H_1)*-Cayley triple* if the associated homomorphism $\mathfrak{sl}_2 \to \mathfrak{h}$ is an (H_1)-homomorphism (relative to the H-elements Z and h_0).

Lemma 3.10. *Let h_0 be an H-element in the quasihermitian reductive Lie algebra $\mathfrak{h}$ with associated Cartan involution θ, and let $x \in \mathfrak{h}$. Then $x = x^+$ for some Cayley triple (h, x^+, x^-) if and only if the following equation holds:*

$$[[\theta(x), x], x] = 2x \ . \tag{3.3}$$

This Cayley triple is unique. In this case, $\langle h, x^\pm\rangle_{\mathbb{R}}$ is $\operatorname{ad} h_0$-invariant if and only if $[h_0, x] = \pm\frac{1}{2}[\theta(x), x] = \pm\frac{1}{2}h$, and the triple is (H_1) if and only if the sign is $+$.

Proof. If $x = x^+$ for some Cayley triple (h, x^+, x^-), then $h = [x^+, x^-] = -[x, \theta(x)]$ and of course $x^- = -\theta(x)$. In particular, (h, x^+, x^-) is unique, and (3.3) holds.

If equation (3.3) holds, we *define* $x^+ = x$, $x^- = -\theta(x)$, $h = -[x, \theta(x)]$. Then we have $\theta(h) = -\theta([x, \theta(x)]) = [x, \theta(x)] = -h$,

$$[h, x] = 2x \quad \text{and} \quad [h, y] = \theta([h, x]) = 2\theta(x) = -2y \ .$$

Thus, in this case, (h, x^+, x^-) is a Cayley triple.

Next, observe that $\mathfrak{z}_{\mathfrak{h}}(h_0) = \ker(1 - \theta)$. Setting $z = \frac{1}{2}(x^+ - x^-)$, this implies that $[h_0, z] = 0$, so that $\operatorname{ad}(h_0)$ leaves the eigenspaces of $\operatorname{ad} z$ invariant. We have

$$[z, h \pm i(x^+ + x^-)] = \pm i(h \pm i(x^+ + x^-)) \ .$$

If $\operatorname{ad}(h_0)$ leaves $\langle h, x^\pm\rangle_{\mathbb{R}}$ invariant, this implies that $h \pm i(x^+ + x^-)$ is an eigenvector of $\operatorname{ad} h_0$, for the eigenvalue i or $-i$. The triple is (H_1) if and only if the sign of the eigenvalue is the same as for $\operatorname{ad} z$. Moreover, because $\ker(1 - \theta)$ centralises h_0, $[h_0, x] = \frac{1}{2}[h_0, x - \theta(x)]$, so $[h_0, x] = -[h_0, \theta(x)$, and $2[h_0, x] = [h_0, x^+ + x^-]$. Taking imaginary parts in the eigenvalue equation, $[h_0, x] = \pm\frac{1}{2}[\theta(x), x]$, and the (H_1) condition amounts to the requirement that the sign be $+$. $\square$

Proposition 3.11. *Fix the H-element $h_0 = iz\frac{\partial}{\partial z}$, and let $x \in \mathfrak{g}\backslash 0$ be the nilpositive element of some Cayley triple. This triple is (H_1) if and only if $x \in \Omega^-$ and if and only if $x \in \Omega^+$. In particular, the nilpotent elements of Ω^+ belong to Ω^-.*

Proof. Let (h, x^+, x^-) be the Cayley triple with $x = x^+$. If $\langle h, x^\pm\rangle_\mathbb{R}$ is $\operatorname{ad} h_0$-invariant, then $[h_0, x] = \pm\frac{1}{2}h$ by Lemma 3.10. Thus $e^{t\operatorname{ad}(x)}(h_0) = h_0 \mp \frac{t}{2}h \pm \frac{t^2}{2}x$, and $\pm x = \lim_{t\to\infty} 2t^{-2}e^{t\operatorname{ad}(x)}(h_0) \in \Omega^-$.

By Lemma 3.10, if the triple is (H_1), then $x \in \Omega^-$. If $\langle h, x^\pm\rangle_\mathbb{R}$ is $\operatorname{ad} h_0$-invariant, let $x \in \Omega^+$, and assume that (H_1) fails. Then $[h_0, x] = -\frac{1}{2}[\vartheta(x), x]$ and $-x \in \Omega^-$. But $\Omega^- \subset \Omega^+$ and Ω^+ is pointed, a contradiction! Hence, the triple must be (H_1).

We need to check that $x \in \Omega^+$ implies that $\langle h, x^\pm\rangle_\mathbb{R}$ is $\operatorname{ad} h_0$-invariant. It is sufficient to prove that $u^+ = h + i(x^+ + x^-) \in \mathfrak{p}^+ \cup \mathfrak{p}^-$. Up to K-conjugacy, we may assume that $x^+ - x^- \in \mathfrak{t}$. Since $-x^- = \vartheta(x) \in \Omega^+$, we have $z = \frac{1}{2}(x^+ - x^-) \in \omega^+$, so $-i\alpha(z) \geqslant 0$ for all $\alpha \in \Delta_n^{++}$. Since $[z, u^+] = iu^+$, we see that $z \in \mathfrak{p}^+ = \bigoplus_{\alpha\in\Delta_n^{++}} \mathfrak{g}_\mathbb{C}^\alpha$. Hence, $x \in \Omega^+$ implies that $\langle h, x^\pm\rangle_\mathbb{R}$ is $\operatorname{ad} h_0$-invariant.

Finally, any nilpotent element is G-conjugate to a nilpotent element belonging to a Cayley triple [10, Theorems 9.2.1, 9.4.1], so the claim follows. □

For $u \in Z$, define the *Cayley vector field* $\xi_u^+ = -i\xi_{iu}^- = (u + \{zu^*z\})\frac{\partial}{\partial z}$, and

$$X_u^\pm = \tfrac{1}{2}\big(\xi_{-iu}^- \pm \tfrac{1}{2}[\xi_u^-, \xi_{-iu}^-]\big) = \tfrac{1}{2}(\xi_{-iu}^- \pm 2iu\,\square\,u^*)\ .$$

For later use, we record the following simple formula:

$$\operatorname{Ad}(k)(X_u^\pm) = \tfrac{1}{2}\xi_{-iku}^- \pm \operatorname{Ad}(k)(iu\,\square\,u^*) = \tfrac{1}{2}\xi_{-iku}^- \pm i(ku)\,\square\,(ku)^* = X_{ku}^\pm\ . \tag{3.4}$$

Proposition 3.12. *Let $e, c \neq 0$ be tripotents, $\mathfrak{s}^e = \langle \xi_e^-, X_e^\pm\rangle_\mathbb{R}$. Then (ξ_e^-, X_e^+, X_e^-) is an (H_1)-Cayley triple and $\pm X_e^\pm \in \Omega^-$. Moreover, $[\mathfrak{s}^e, \mathfrak{s}^c] = 0$ if only if $e \perp c$.*

Proof. First, note $[[\xi_{ae}^-, \xi_{be}^-], \xi_{ce}^-] = 4\operatorname{Im}(a\bar{b})\xi_{ic\cdot e}^-$ for all $a, b, c \in \mathbb{C}$, whence

$$[\xi_e^-, X_e^\pm] = \tfrac{1}{2}[\xi_e^-, \xi_{-ie}^-] \mp \xi_{ie}^- = \pm 2X_e^\pm\ ,\ [X_e^+, X_e^-] = \tfrac{1}{4}[[\xi_e^-, \xi_{-ie}^-], \xi_{-ie}^-] = \xi_e^-\ .$$

Clearly, $X_e^- = \vartheta(X_e^+)$ and $\big[iz\frac{\partial}{\partial z}, X_e^-\big] = \frac{1}{2}\big[iz\frac{\partial}{\partial z}, \xi_{-ie}^-\big] = \frac{1}{2}\xi_e^-$. Hence, the triple (ξ_e^-, X_e^+, X_e^-) is an (H_1)-Cayley triple, and $\pm X_e^\pm \in \Omega^-$ by Lemma 3.10 and Proposition 3.11. Next, observe that $\xi_a^\pm \in \mathfrak{s}^a$ for $a = e, c$. Since

$$[\xi_e^-, \xi_c^-] - [\xi_e^-, \xi_c^+] = [\xi_e^-, \xi_c^- + i\xi_{ic}^-] = 4e\,\square\,c^*\ ,\ [\xi_e^-, \xi_c^-] = [e\,\square\,e^*, c\,\square\,c^*]\ ,$$

e and c are orthogonal if and only if $[\mathfrak{s}^e, \mathfrak{s}^c] = 0$. □

Remark 3.13. Paneitz [53, Lemma 4] proves that $X_e^+ \in \Omega^-$ for e primitive.

Proposition 3.14. *Let (h, x^+, x^-) be an (H_1)-Cayley triple. Then there exists a unique non-zero tripotent $e \in Z$ such that $h = \xi_e^-$ and $x^\pm = X_e^\pm$.*

Proof. We have $h \in \mathfrak{p}$, so $h = \xi_e^-$ (some $e \in Z\backslash 0$). Set $z = \frac{1}{2}(x^+ - x^-) \in \mathfrak{k} = \operatorname{aut}(Z)$. The value $z(e) \in Z$ makes sense, and $\xi_{z(e)}^- = [z, h] = -(x^+ + x^-)$.

By assumption, $\operatorname{ad} z$ and $\operatorname{ad} h_0$ (where $h_0 = iz\frac{\partial}{\partial z}$) coincide on $\langle h, x^+, x^-\rangle_\mathbb{C}$. Thus, $\xi_{z(e)}^- = [h_0, \xi_e^-] = \xi_{ie}^-$. This shows that $z(e) = ie$. Next,

$$\xi_{i\{ee^*e\}}^- = \tfrac{1}{4}[[\xi_e^-, \xi_{-ie}^*], \xi_e^-] = \tfrac{1}{4}[[h, x^+ + x^-], h] = [z, h] = \xi_{ie}^-\ ,$$

so $e = \{ee^*e\}$. We have $x^+ + x^- = \xi_{ie}^-$, $x^+ - x^- = \frac{1}{2}[h, x^+ + x^-] = \frac{1}{2}[\xi_e^-, \xi_{-ie}^-]$. □

Remark 3.15. The result [55, Proposition 4.1] seems to be somewhat similar.

We now introduce certain Heisenberg algebras associated to tripotents of Z. They will be important in the determination and description of the faces of $\Omega^{\pm}$.

Conal Heisenberg algebras. In what follows, e, c shall denote tripotents.

Definition 3.16. Given e, and any set $A \subset \mathbb{C}$, let $\mathfrak{g}^e[A] = \bigoplus_{\lambda \in A} \ker(\operatorname{ad} \xi_e^- - \lambda)$. Then $\mathfrak{g} = \mathfrak{g}^e[-2,-1,0,1,2]$ by [41, Lemma 9.14]. Moreover (*loc.cit.*),

$$\mathfrak{g}^e[0] = \mathfrak{k}^e \oplus \{\xi_u^- | u \in Z_0(e) \oplus X_1(e)\} \quad \text{where} \quad \mathfrak{k}^e = \{\delta \in \mathfrak{k} | \delta e = 0\} \ .$$

Let

$$\eta_u^e = \xi_u^- + [\xi_e^-, \xi_u^-] = \xi_u^- + 2(e \,\square\, u^* - u \,\square\, e^*)$$
$$\zeta_u^e = \xi_u^- + \tfrac{1}{2}[\xi_e^-, \xi_u^-] = \xi_u^- + (e \,\square\, u^* - u \,\square\, e^*) \quad \text{for all} \quad u \in Z \ .$$

Then $X_e^{\pm} = \frac{1}{2}\zeta_{\mp ie}^{\pm e}$, and (*loc.cit.*)

$$\mathfrak{g}^e[\pm 1] = \left\{\eta_u^{\pm e} \;\middle|\; u \in Z_{1/2}(e)\right\} \ , \ \mathfrak{g}^e[\pm 2] = \left\{\zeta_u^{\pm e} \;\middle|\; u \in iX_1(e)\right\} \ .$$

Furthermore, $\mathfrak{q}^e = \mathfrak{g}^e[0,1,2]$ is a maximal parabolic subalgebra [41, Proposition 9.21], and $\mathfrak{h}^e = \mathfrak{g}^e[1,2]$ is its nilradical. We call $\mathfrak{h}^e$ a *conal Heisenberg algebra.*

Recall that $\mathfrak{k}_0(e) = \operatorname{aut}(Z_0(e))$ and $\mathfrak{k}_1(e) = \operatorname{aut}(X_1(e))$ are, respectively, the set of triple derivations of $Z_0(e)$, and the set of algebra derivations of $X_1(e)$. Similarly, we consider $\mathfrak{p}_0(e) = \{\xi_u^- | u \in Z_0(e)\}$ and $\mathfrak{p}_1(e) = \{\xi_u^- | u \in X_1(e)\}$.

We already know that $\mathfrak{g}_0(e) = \mathfrak{k}_0(e) \oplus \mathfrak{p}_0(e)$ is the set of complete holomorphic vector fields on $D_0(e)$. Let $\mathfrak{g}_1(e) = \mathfrak{k}_1(e) \oplus \mathfrak{p}_1(e)$. Then by [59, Lemma 21.16],

$$\operatorname{Ad}(\gamma_e)(\xi_u^-) = 2M_u \quad \text{and} \quad \operatorname{Ad}(\gamma_e)(\delta) = \delta \quad \text{for all} \quad u \in X_1(e)\,,\ \delta \in \operatorname{aut}(X_1(e)),$$

where $M_u(v) = u \circ v$, so that $\operatorname{Ad}(\gamma_e)(\mathfrak{g}_1(e)) = \mathfrak{gl}(\Omega_1(e))$. Here, $\Omega_1(e)$ denotes the cone of squares of the Jordan algebra $X_1(e)$.

We have $[\mathfrak{g}_0(e), \mathfrak{g}_1(e)] = 0$ by the Peirce rules. Let $\mathfrak{m}^e = \mathfrak{k}^e \cap (\mathfrak{k}_0(e) \oplus \mathfrak{k}_1(e))^{\perp}$. If we let $\mathfrak{a} = \langle \xi_{e_1}^-, \ldots, \xi_{e_r}^- \rangle_{\mathbb{R}}$ for some frame such that $e_j \leqslant e$ or $e_j \perp e$ for all j, then $\mathfrak{m}^e \subset \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$. Using this fact, it is easy to see that $\mathfrak{m}^e$ leaves $\mathfrak{g}_i(e)$ $(i = 0, 1)$ invariant, so that, as Lie algebras,

$$\mathfrak{g}^e[0] = \mathfrak{g}_0(e) \oplus \mathfrak{g}_1(e) \oplus \mathfrak{m}^e = \operatorname{aut}(D_0(e)) \oplus \operatorname{Ad}(\gamma_e^{-1})(\mathfrak{gl}(\Omega_1(e)) \oplus \mathfrak{m}^e \ . \tag{3.5}$$

Define a linear isomorphism $\phi^e : Z_{1/2}(e) \oplus X_1(e) \to \mathfrak{h}^e$ by

$$\phi^e(u, v) = \eta_u^e + \zeta_{-iv/2}^e \ . \tag{3.6}$$

Definition 3.17. Let U, V be complex vector spaces, V be endowed with an antilinear involution *, and K be a closed convex cone such that $x^* = x$ for all $x \in K$. A sesquilinear map $\phi : U \times U \to V$ such that $\phi(u, v)^* = \phi(v, u)$ and $\phi(u, u) \in K \setminus 0$ for all $u \neq 0$ is called *K-positive Hermitian.*

Proposition 3.18. *Define $h_e : Z_{1/2}(e) \times Z_{1/2}(e) \to Z_1(e)$ by $h_e(u, v) = 8 \cdot \{uv^*e\}$, and $q_e(u, v) = \operatorname{Im} h_e(u, v) = 4i \cdot (\{vu^*e\} - \{uv^*e\}) \in X_1(e)$. Then h_e is $\Omega_1(e)$-positive Hermitian, and if we let*

$$[(u, v), (u', v')] = (0, q_e(u, u')) \quad \textit{for all} \quad u, u' \in Z_{1/2}(e)\,,\ v, v' \in X_1(e) \ , \tag{3.7}$$

then $Z_{1/2}(e) \oplus X_1(e)$ is a Lie algebra isomorphic to $\mathfrak{h}^e$ by the map ϕ^e from (3.6).

The subspaces $\mathfrak{g}^e[\lambda]$, $\lambda = 1, 2$, are $\mathfrak{g}^e[0]$-invariant. By transport of structure, $Z_{1/2}(e)$ and $X_1(e)$ may be turned into $\mathfrak{g}^e[0]$-modules. Here, $\mathfrak{l}^e = \mathfrak{g}_0(e) \oplus \mathfrak{m}^e$ centralises $\mathfrak{g}^e[2]$, and $\mathfrak{g}_1(e)$ acts on $X_1(e)$ via

$$\delta.v = \delta(v) \ , \ \xi_u^-.v = 2(u \circ v) \quad \textit{for all} \quad u, v \in X_1(e)\,,\ \delta \in \mathfrak{k}_1(e) \ .$$

In particular, the action of $\mathfrak{g}_1(e)$ *on* $\mathfrak{g}^e[2]$ *is equivalent to the action of* $\mathfrak{gl}(\Omega_1(e))$ *on* $X_1(e)$ *and is therefore faithful.*

Futhermore, $\mathfrak{g}^e[0]$ *acts on* $Z_{1/2}(e)$ *via*

$$\delta.v = \delta(v)\ ,\ \xi_u^-.v = -2\{uv^*e\} \quad \forall \delta \in \mathfrak{k}^e\ ,\ u \in Z_0(e) \oplus X_1(e)\ ,\ v \in Z_{1/2}(e)\ . \tag{3.8}$$

In particular, $\mathfrak{z}(\mathfrak{l}^e \ltimes \mathfrak{h}^e) = \mathfrak{z}(\mathfrak{g}_0(e) \ltimes \mathfrak{h}^e) = \mathfrak{z}(\mathfrak{h}^e) = \mathfrak{g}^e[2]$.

Proof. The map h_e is positive Hermitian [41, 10.4]. Clearly, $[\mathfrak{h}^e, \mathfrak{h}^e] \subset \mathfrak{g}^e[2] \subset \mathfrak{z}(\mathfrak{h}^e)$, and $\mathfrak{h}^e$ is a generalised Heisenberg algebra. For $u, v \in Z_{1/2}(e)$, $[\eta_u^e, \eta_v^e] \in \mathfrak{g}^e[2]$ and hence equals $\zeta^e_{-iw/2}$ for some $w \in X_1(e)$. Since $\zeta^e_{-iw/2}(0) = \xi^-_{-iw/2}(0) = -\frac{i}{2}w$,

$$\begin{aligned} -\tfrac{i}{2}w &= [\eta_u^e, \eta_v^e](0) = [\xi_u^-, [\xi_e^-, \xi_v^-]](0) + [[\xi_e^-, \xi_u^-], \xi_v^-](0) \\ &= 2 \cdot (\{ve^*u\} - \{ev^*u\} + \{eu^*v\} - \{ue^*v\}) = 2 \cdot (\{vu^*e\} - \{uv^*e\})\ . \end{aligned}$$

This proves that $Z_{1/2}(e) \oplus X_1(e)$ is a Lie algebra isomorphic to $\mathfrak{h}^e$.

For $x \in \Omega_1(e)^\circ$, let $b_x(u, v) = (q_e(iu, v)|x)$ for all $u, v \in Z_{1/2}(e)$. Then b_e is a symmetric bilinear form, positive definite since $\Omega_1(e)$ is regular and self-dual. Since $[(iu, 0), (u, 0)] = (0, q_e(iu, u))$ for all $u \in Z_{1/2}(e)$, we find that $\mathfrak{z}(\mathfrak{h}^e) = \mathfrak{g}^e[2] = X_1(e)$.

Next, we consider the $\mathfrak{g}^e[0]$-action on $X_1(e)$. If $\xi_u^- \in \mathfrak{p}_0(e) \oplus \mathfrak{p}_1(e)$ and $v \in X_1(e)$, then $[\xi_u^-, \zeta^e_{-iv/2}] = \zeta^e_{-iw/2}$ for some $w \in X_1(e)$. We have

$$-\tfrac{i}{2}w = \tfrac{1}{2}[\xi_u^-, [\xi_e^-, \xi^-_{-iv/2}]] = \tfrac{1}{2}\xi_{-i\{ev^*u\}-i\{ve^*u\}}(0) = -\tfrac{i}{2}(\{ev^*u\} + \{ve^*u\})\ .$$

For $u \in Z_0(e)$, this is zero, and for $u \in X_1(e)$, it equals $-i(u \circ v)$ by (2.4). Similarly, for $\delta \in \mathfrak{k}^e$, $[\delta, \zeta^e_{-iv/2}] = \zeta^e_{-iw/2}$ gives $-\frac{i}{2}w = [\delta, \xi^-_{-iv/2}](0) = -\frac{i}{2}\delta(v)$, so $w = \delta(v)$. For $\delta \in \mathfrak{k}_0(e)$, this is zero, and so it is if $\delta \in \mathfrak{k}^e$ is arbitrary and $v = e$. We have shown that $\mathfrak{g}_0(e)$ centralises $\mathfrak{g}^e[2]$, and that the $\mathfrak{g}_1(e)$-action on $\mathfrak{g}^e[2]$ is equivalent to the $\mathfrak{gl}(\Omega_1(e))$-action on $X_1(e)$. In particular, X_e^+ is a cyclic vector of the $\mathfrak{g}^e[0]$-module $\mathfrak{g}^e[2]$. Since it is annihilated by $\mathfrak{m}^e \subset \mathfrak{k}^e$ and $\mathfrak{m}^e$ is an ideal of $\mathfrak{g}^e[0]$, we see that $\mathfrak{m}^e$ centralises $\mathfrak{g}^e[2]$. Evaluating $[\delta, \eta_v^e]$ and $[\xi_u^-, \eta_v^e]$ at zero for all $\delta \in \mathfrak{k}^e$, $u \in Z_0(e) \oplus X_1(e)$, and $v \in Z_{1/2}(e)$ gives the remaining relations. □

Lemma 3.19. *The centre of* $\mathfrak{k}_1(e)$ *is trivial. In particular, if* $\operatorname{rk} e \geqslant 2$*, then the derived algebra* $\mathfrak{g}_1(e)' = [\mathfrak{g}_1(e), \mathfrak{g}_1(e)]$ *is a non-compact, non-Hermitian simple Lie algebra. If* $\operatorname{rk} e \leqslant 1$*, then* $\mathfrak{g}_1(e) = \mathbb{R}\xi_e^-$ *is Abelian.*

Proof. By (3.5), $\operatorname{Ad}(\gamma_e)(\mathfrak{g}_1(e)) = \mathfrak{gl}(\Omega_1(e))$. The Lie algebra $\mathfrak{gl}(\Omega_1(e))$ is reductive with centre $\mathbb{R}M_e$. Because $X_1(e)$ is a simple Jordan algebra for $e \neq 0$, $\mathfrak{gl}(\Omega_1(e))'$ is a simple Lie algebra or zero. If $\operatorname{rk} e \geqslant 2$, then there exists an idempotent $c \in X_1(e)$, $0 < c < e$, and $M_c \subset \mathfrak{gl}(\Omega_1(e))'$ generates an unbounded one-parameter group, so $\mathfrak{gl}(\Omega_1(e))'$ is non-compact. Finally, let $\delta \in \mathfrak{z}(\mathfrak{k}_1(e))$ and $u \in X_1(e)$. We have $0 = [\delta, u \,\square\, e^*] = (\delta u) \,\square\, e^* = M_{\delta u}$, since $\delta e = 0$, so $\delta u = 0$. Thus, $\delta = 0$. □

Principal faces. Using the identification $\phi^e : Z_{1/2}(e) \oplus X_1(e) \to \mathfrak{h}^e$ from Proposition 3.18, we consider the cone $\Omega_1(e) \subset X_1(e)$ as a subset of $\mathfrak{g}^e[2] = \mathfrak{z}(\mathfrak{h}^e)$. We point out that this notation is only meaningful if we keep the embedding ϕ^e attached to the tripotent e in mind. (Indeed, $\phi^e(-\Omega_1(e))$ and $\phi^{-e}(-\Omega_1(e))$ are distinct!) In what follows, the chosen embedding will always be clear from the context.

Proposition 3.20. *We have* $\Omega^\pm \cap \mathfrak{h}^e = \Omega_1(e)$.

Proof. Let Ω be one of $\Omega^\pm \cap \mathfrak{h}^e$. Then Ω is a closed pointed cone invariant under $N_G(\mathfrak{h}^e)$, in particular, under inner automorphisms of $\mathfrak{h}^e$. Hence $\Omega \subset \mathfrak{z}(\mathfrak{h}^e) = \mathfrak{g}^e[2]$ [20, Lemma I.13], and by Proposition 3.18, Ω is invariant under $\mathrm{GL}(\Omega_1(e))$. On the other hand, $X_e^+ = \phi^e(e) \in \Omega$. Identifying Ω with its image in $X_1(e)$, this implies $\Omega_1(e) \subset \Omega$ and $\Omega^* \subset \Omega_1(e)^* = \Omega_1(e)$. Since Ω is pointed, the interior of Ω^* in $\mathfrak{g}^e[2]$ is non-void. Hence, there is some $x \in \Omega^* \cap \Omega_1(e)^\circ$, and $\Omega_1(e)^\circ \subset \Omega^*$ since $\Omega_1(e)^\circ$ is homogeneous. It follows that $\Omega^* = \Omega_1(e)$, and by duality, $\Omega = \Omega_1(e)$. □

Definition 3.21. Define $F_e^\pm = \Omega^\pm \cap (X_e^-)^\perp$. Since $-X_e^- \in \Omega^- \subset \Omega^+$, this is an exposed face of $\Omega^\pm$. We call $F_e^\pm$ a *principal face*.

Proposition 3.22. *We have $F_e^\pm = \Omega^\pm \cap \mathfrak{q}^e$, and this is an exposed face of $\Omega^\pm$.*

The *proof* is preceded by two lemmata.

Lemma 3.23. *Let $e \geqslant c$ be non-zero tripotents, $n = \dim X_1(e)$, $k = \operatorname{rk} e$. Denote the canonical inner product of $X_1(e)$ by $(\cdot|\cdot)$. Then, for all $u \in X_1(e)$, $v \in X_1(c)$,*

$$(\phi^e(u) : \phi^c(v)) = \tfrac{4n}{k} \cdot (u|v) \quad \textit{and} \quad (\phi^e(u) : \phi^{-c}(v)) = 0 \ .$$

Proof. Let $c \leqslant e$, $u \in X_1(e)$, $v \in X_1(c)$. Then $\vartheta(\zeta_{iu}^e) = -\xi_{iu}^- + \frac{1}{2}[\xi_e^-, \xi_{iu}^-]$, so

$$\begin{aligned}(\phi^e(u) : \phi^{\pm c}(v)) &= (\zeta_{-iu/2}^e : \zeta_{-iv/2}^{\pm c}) \\ &= -B(\xi_{iu/2}^- - \tfrac{1}{2}[\xi_e^-, \xi_{iu/2}^-], -\xi_{iv/2}^- \mp \tfrac{1}{2}[\xi_c^-, \xi_{iv/2}^-]) \\ &= \tfrac{1}{4}B(\xi_{iu}^-, \xi_{iv}^-) \mp \tfrac{1}{16}B(\xi_{iu}^-, [[\xi_c^-, \xi_{iv}^-], \xi_e^-]) \ .\end{aligned}$$

Since $\frac{1}{4}[[\xi_c^-, \xi_{iv}^-], \xi_e^-] = -\xi_{iv}^-$ by (2.2), this is 0 for $\phi^{-c}(v)$. For $\phi^c(v)$, by Lemma 2.3,

$$= \tfrac{1}{2}B(\xi_{iu}^-, \xi_{iv}^-) = 2\operatorname{tr}_Z(u \,\square\, v^* + v \,\square\, u^*) = \tfrac{4n}{k} \cdot (u|v) \ . \qquad \square$$

Lemma 3.24. *Let $\Omega \subset \mathfrak{g}$ be a closed set invariant under $\mathbb{R}_{\geqslant 0}$ and $\operatorname{Ad}(\exp t\xi_e^-)$ for all $t \in \mathbb{R}$. If $\xi = \sum_{j=k}^\ell \xi_j \in \Omega$ where $\xi_j \in \mathfrak{g}^e[j]$, then $\xi_k, \xi_\ell \in \Omega$.*

Proof. We have $\operatorname{Ad}(\exp t\xi_e^-)(\xi) = \sum_{j=k}^\ell e^{jt} \cdot \xi_j \in \Omega$ for all $t \in \mathbb{R}$. Hence, we have $\xi_k = \lim_{t\to\infty} e^{kt} \operatorname{Ad}(\exp -t\xi_e^-)(\xi) \in \Omega$ and $\xi_\ell = \lim_{t\to\infty} e^{-\ell t} \operatorname{Ad}(\exp t\xi_e^-)(\xi) \in \Omega$, proving the lemma. □

Proof of Proposition 3.22. If $e = 0$, then $X_e^- = 0$, $F_e^\pm = \Omega^\pm$, and $\mathfrak{q}^e = \mathfrak{g}$. W.l.o.g., we may assume $k = \operatorname{rk} e > 0$. Since $\vartheta(\xi_e^-) = -\xi_e^-$, $\operatorname{ad}\xi_e^-$ is symmetric, and its eigenspaces are orthogonal. In particular, $\mathfrak{q}^e \perp \mathfrak{g}^e[-2] \ni X_e^-$, and $\Omega^\pm \cap \mathfrak{q}^e \subset F_e^\pm$.

For the converse, let $\xi \in F_e^\pm$, and write $\xi = \sum_{j=-2}^2 \xi_j$ where $\xi_j \in \mathfrak{g}^e[j]$. Since X_e^- is an eigenvector of $\operatorname{ad}\xi_e^-$, $F_e^\pm$ is invariant under $\operatorname{Ad}(\exp t\xi_e^-)$ for all $t \in \mathbb{R}$, so we can employ Lemma 3.24. In particular, $\xi_{-2} \in F_e^\pm \cap \mathfrak{h}^{-e}$. By Proposition 3.20, $\xi_{-2} = \phi^{-e}(u)$ for a unique $u \in -\Omega_1(e)$. By Lemma 3.23,

$$(\xi_{-2} : X_e^-) = -(\phi^{-e}(u) : \phi^{-e}(e)) = -\tfrac{2n}{k}(u|e),$$

where $n = \dim X_1(e)$. This is positive if $u \neq 0$, so $u = 0$ and $\xi_{-2} = 0$. Therefore, $\xi_{-1} \in \Omega^\pm \cap \mathfrak{h}^{-e}$ by Lemma 3.24. By Proposition 3.20, $\xi_{-1} = 0$, and $\xi \in \mathfrak{q}^e$. □

Corollary 3.25. *For any tripotent e, $\Omega_1(e) \subset \Omega^-$ is a face of Ω^+ and Ω^-.*

Proof. It is sufficient to show that $\Omega_1(e)$ is a face of $F_e^\pm$. Hence, let $\xi, \eta \in F_e^\pm$ such that $\xi + \eta \in \Omega_1(e)$ and decompose $\xi = \sum_{j=0}^2 \xi_j$, $\eta = \sum_{j=0}^2 \eta_j$, according to the grading of $\mathfrak{q}^e$. Then $\xi_0 + \eta_0 = 0$ by assumption, and $\xi_0, \eta_0 \in \Omega^\pm$ by Lemma 3.24. This implies $\xi_0 = \eta_0 = 0$, and $\xi_1, \eta_1 \in \Omega^\pm$ by the same lemma. But then Proposition 3.20 implies that $\xi_1 = \eta_1 = 0$. Hence the claim. □

Nilpotent faces and nilpotent orbits. We will now give a precise description of the conal nilpotent orbits. They are closely related to the *nilpotent faces* of $\Omega^\pm$.

Definition 3.26. Let $F \subset \Omega^\pm$ be a face. If F° contains a nilpotent (semi-simple) element of $\mathfrak{g}$, we will call F a *nilpotent face* (*semi-simple face*).

For any tripotent e, let $\mathcal{O}_e = \mathrm{Ad}(G)(X_e^+)$. Let M_k be the set of rank k tripotents.

Theorem 3.27. *Let e be a tripotent of* $\operatorname{rk} e = k$*, and let $K^e = Z_K(e)$. Then*

$$\mathcal{O}_e = \bigcup_{c \in M_k} \Omega_1(c)^\circ = K \times_{K^e} \Omega_1(e)^\circ \ . \tag{3.9}$$

In particular, $\mathcal{O}_e$ depends only on the rank of e; moreover,

$$\operatorname{rk} e \neq \operatorname{rk} c \ \Rightarrow\ \mathcal{O}_e \cap \mathcal{O}_c = \varnothing \quad \textit{and} \quad \operatorname{rk} e \leqslant \operatorname{rk} c \ \Rightarrow\ \mathcal{O}_e \subset \overline{\mathcal{O}_c} \ .$$

The $\mathcal{O}_c$ exhaust the nilpotent orbits of Ω^+; the $\Omega_1(c)$ exhaust the nilpotent faces.

Remark 3.28. The classification of conal nilpotent orbits is contained in [62, Theorem 2], [19, Theorem III.9], [53, Lemma 4]. The description in terms of tripotents and the connection to nilpotent faces is, however, new. While the parametrisation of the conal nilpotent orbits by tripotents might be deduced from [19, Theorem III.9] by applying Proposition 3.14, our *proof* of the more precise result is independent of existing results, and at the same time, is shorter and more elementary.

Proof of Theorem 3.27. By Proposition 3.12, $\mathcal{O}_e \subset \Omega^-$. If $\operatorname{rk} e = \operatorname{rk} c$, then $\ell(e) = c$ for some $\ell \in K$ [41, Corollary 5.12]. Then $\mathrm{Ad}(\ell)(X_e^+) = X_c^+$ by (3.4), so $\mathcal{O}_e = \mathcal{O}_c$.

Let $x \in \Omega^+$ be nilpotent, $x \neq 0$. Then $x \in \Omega^-$ and there exists $g \in G$ such that $\mathrm{Ad}(g)(x) = x^+$ for some Cayley triple (h, x^+, x^-) [10, Theorems 9.2.1, 9.4.1]. By Proposition 3.11, the triple is (H_1), so $\mathrm{Ad}(g)(x) = X_e^+$ for some tripotent e, by Proposition 3.14. Let F be the face of $\Omega = \Omega^\pm$ generated by x. Since $\Omega \cap \mathfrak{g}^e[2] = \Omega_1(e)$ is a face of Ω by Corollary 3.25, $\mathrm{Ad}(g)(F)$ equals the face of $\Omega_1(e)$ generated by $x = X_e^+$. But this face is $\Omega_1(e)$ itself.

By the Iwasawa decomposition, G is generated by K and the analytic subgroup Q^e associated with $\mathfrak{q}^e$. Since Q^e normalises $\mathfrak{g}^e[2] = X_1(e)$, $\mathrm{Ad}(\ell)(F) = \Omega_1(e)$ for some $\ell \in K$. From (3.4), $F = \Omega_1(c)$ for some tripotent $c = \ell^{-1}(e)$ with $\operatorname{rk} c = k$. Let $G_1(c)$ be the analytic subgroup of G associated with $\mathfrak{g}_1(c)$. By Proposition 3.18, the action of $G_1(c)$ on $\mathfrak{g}^c[2]$ corresponds to the action of $\mathrm{GL}(\Omega_1(c))$ on $X_1(c)$, and is thus transitive on F°. This proves the equation (3.9), the exhaustion of nilpotent orbits in Ω^+, and the exhaustion of nilpotent faces. Since c is the only tripotent contained in $\Omega_1(c)^\circ$, $\mathrm{Ad}(G)(x) = \mathcal{O}_c$ does not contain any rank $k-1$ tripotent. Similarly, any tripotent $c' \leqslant c$ is contained in $\Omega_1(c)$, and therefore in $\overline{\mathcal{O}_c}$. □

Corollary 3.29. *Let e be primitive. Then $\Omega^- = \overline{\mathrm{co}(\mathcal{O}_e)} = 0 \cup \mathrm{co}(\mathcal{O}_e)$.*

Proof. Let $C = \overline{\mathrm{co}(\mathcal{O}_e)} \subset \Omega^-$. Then C is a G-invariant closed convex cone. We have $\mathcal{O}_e = K \times_{K^e} (\mathbb{R}_{>0} \cdot X_e^+)$, so $C = 0 \cup \mathrm{co}(\mathcal{O}_e) = \mathbb{R}_{\geqslant 0} \cdot \mathrm{co}(\mathrm{Ad}(K)(X_e^+))$.

To see that $C = \Omega^-$, it remains to be shown that $iz\,\frac{\partial}{\partial z} \in C$. We have $\pm X_e^\pm \in \mathcal{O}_e$, so $ie\,\square\, e^* = \frac{1}{2}(X_e^+ - X_e^-) \in C$. By Lemma 3.5 and the K-invariance of C, $\omega^- \subset C$. But $iz\,\frac{\partial}{\partial z} \in \omega^-$, and therefore, $C = \Omega^-$. $\square$

Corollary 3.30. *Every conal nilpotent orbit $\mathcal{O}_e$ is a K-equivariant fibre bundle over $K/K^e = M_k$ (for $k = \operatorname{rk} e$), with contractible fibres. In particular, $\mathcal{O}_e$ is K-equivariantly homotopy equivalent to M_k.*

Remark 3.31. The projection of the fibre bundle $\Omega_1(e)^\circ \to \mathcal{O}_e \to M_k$ ($k = \operatorname{rk} e$) associates to a nilpotent x the *unique* y which generates the same face of Ω^- as x and is the nilpositive element of a Cayley triple. In particular, with any nilpotent element of Ω^-, we may associate a *canonical* Cayley triple.

4. Classification of the faces of the minimal invariant cone

In this section, we classify all faces of Ω^-. First, we study $F_e^\pm$ in detail.

4.1. **Fine structure of the principal faces.** We have seen that the exposed face $F_e^\pm$ is contained in the maximal parabolic subalgebra $\mathfrak{q}^e$, and in particular, is invariant under inner automorphisms of $\mathfrak{q}^e$. However, this is not the definitive statement on $F_e^\pm$: the linear span of $F_e^\pm$ is a proper ideal of $\mathfrak{q}^e$.

Proposition 4.1. *We have $F_e^\pm = \Omega^\pm \cap (\mathfrak{g}_0(e) \ltimes \mathfrak{h}^e)$. If $\operatorname{rk} e < r$, then both of the faces $F_e^\pm$ span $\mathfrak{g}_0(e) \ltimes \mathfrak{h}^e$. If $\operatorname{rk} e = r$, then $F_e^\pm = \Omega^\pm \cap \mathfrak{g}^e[2] = \Omega_1(e) \subset \Omega^-$.*

The *proof* requires a preparatory lemma. Fix a frame $e_1, \dots, e_r$ and recall the Cartan subalgebra $\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{t}^-$ from (3.2). Let $\mathfrak{a} = \langle \xi_{e_1}^-, \dots, \xi_{e_r}^- \rangle_\mathbb{R}$, $\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$.

Lemma 4.2. *Let $e = \mathbf{e}_k = e_1 + \dots + e_k$. We have $\mathfrak{t} \cap \mathfrak{m} = \mathfrak{t}^+$ and*

$$\mathfrak{t}^e[0] = \mathfrak{t} \cap \mathfrak{g}^e[0] = \langle i \cdot e_j \,\square\, e_j^* \mid j = k+1, \dots, r \rangle \oplus \mathfrak{t}^+ \subset \mathfrak{l}^e \ .$$

The subalgebras $\mathfrak{g}^e[0]$, $\mathfrak{g}_0(e)$, and $\mathfrak{m}^e$ of $\mathfrak{g}$ are $\mathfrak{t}$-invariant. Moreover, $\mathfrak{t}_0(e) = \mathfrak{t} \cap \mathfrak{g}_0(e)$ and $\mathfrak{t}^+ \cap \mathfrak{m}^e$ are Cartan subalgebras of $\mathfrak{g}_0(e)$ and $\mathfrak{m}^e$, respectively.

Proof. Since $[\delta, \xi_{e_j}^-] = \xi_{\delta e_j}^-$ for all $\delta \in \mathfrak{k}$, $\mathfrak{t}^+ \subset \mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$. For the converse, we have $\{cc^*c\} = c \neq 0$ for $c = e_j$, so $\mathfrak{m} \cap \mathfrak{t}^- = 0$ and $\mathfrak{t} \cap \mathfrak{m} = \mathfrak{t}^+$. Since $\mathfrak{m}^e \subset \mathfrak{m}$, $\mathfrak{t} \cap \mathfrak{m}^e \subset \mathfrak{t}^+$.

Moreover, $ie_j \,\square\, e_j^* \in \mathfrak{k}_0(e)$ if $j > k$, and if $j \leqslant k$, then

$$[\delta, ie_j \,\square\, e_j^*] = i \cdot (\delta e_j) \,\square\, e_j^* + i \cdot e_j \,\square\, (\delta e_j)^* = 0 \quad \text{for all} \quad \delta \in \mathfrak{k}^e \ ,$$

and $[\xi_u^-, i \cdot e_j \,\square\, e_j^*] = -\xi_{i\{e_j e_j^* u\}}^- = 0$ for all $u \in Z_0(e)$.

We conclude that $\mathfrak{l}^e$ is $\mathfrak{t}$-invariant, and $\mathfrak{t}^e[0] = \langle ie_j \,\square\, e_j^* \mid j = k+1, \dots, r \rangle \oplus \mathfrak{t}^+ \subset \mathfrak{l}^e$. In addition, $\mathfrak{t}_0(e)$ and $\mathfrak{t}^+ \cap \mathfrak{m}^e$ are Cartan subalgebras of $\mathfrak{g}_0(e)$ and $\mathfrak{m}^e$, respectively, [5, Chapter VIII, §3.1, Proposition 3]. $\square$

Let $\Omega_0^\pm(e)$ denote the minimal and maximal cone of the Lie algebra $\mathfrak{g}_0(e)$, respectively, *cf.* Definition 2.4. Likewise, set $\omega_0^\pm(e) = \Omega_0^\pm(e) \cap \mathfrak{t}_0(e)$. Then

$$\omega_0^+(e) = \omega_0^-(e)^* \quad \text{and} \quad \omega_0^-(e) = \langle iH_\alpha \mid \alpha \in \Delta_n^{++} \,,\, \mathfrak{g}_\mathbb{C}^\alpha \subset \mathfrak{g}_{0,\mathbb{C}}(e) \rangle \ .$$

Here, $\mathfrak{g}_{0,\mathbb{C}}(e) = \mathfrak{g}_0(e) \otimes \mathbb{C}$. The set $\{\alpha \in \Delta_n^{++} \mid \mathfrak{g}_\mathbb{C}^\alpha \subset \mathfrak{g}_0(e)\}$ coincides with the set of positive non-compact roots for $\mathfrak{g}_0(e)$, since this algebra is $\mathfrak{t}$- and ϑ-invariant [5, Chapter VIII, §3.1, Proposition 3].

Proof of Proposition 4.1. We have $\operatorname{rk} e < r$ if and only if $\mathfrak{g}_0(e) \neq 0$. In this case, $\mathfrak{h} = \mathfrak{t}_0(e) \oplus X_1(e)$ is a compact Cartan subalgebra of $\mathfrak{g}_0(e) \ltimes \mathfrak{h}^e$. The intersection of a solid cone with such a Cartan subalgebra completely determines the cone [17, Proposition III.5.14 (ii)]. Thus, we claim that $F_e^\pm = \Omega^\pm \cap (\mathfrak{g}_0(e) \ltimes \mathfrak{h}^e)$, independent of the rank of e. This will imply the assertion for $\operatorname{rk} e < r$; for $\operatorname{rk} e = r$, it follows from Proposition 3.20.

Assume that we have shown $F_e^\pm \subset \mathfrak{l} = \mathfrak{g}_0(e) \ltimes \mathfrak{h}^e$ and that $F_e^\pm \cap \mathfrak{h}$ is solid in $\mathfrak{h}$. Since $F_e^\pm = \Omega^\pm \cap \mathfrak{q}^e$, $F_e^\pm$ is invariant under inner automorphisms of $\mathfrak{q}^e$, and in particular, of $\mathfrak{l}$. It follows that $F_e^\pm$ is the unique pointed invariant cone in $\mathfrak{l}$ whose intersection with $\mathfrak{h}$ is $F_e^\pm \cap \mathfrak{h}$, and this intersection is regular in $\mathfrak{l}$ [17, Theorem III.5.15, Proposition III.5.14 (iii)]. Thus, once we have shown our assumption, it is clear that $\mathfrak{l}$ is spanned by $F_e^\pm$.

In view of Lemma 3.24, it is sufficient to prove that $\Omega^\pm \cap \mathfrak{g}^e[0] = \Omega_0^\pm(e)$ and that $\omega^\pm \cap \mathfrak{g}^e[0] = \omega_0^\pm(e)$. Moreover, we may assume $e = \mathbf{e}_k = e_1 + \cdots + e_k$ and $k \geqslant 1$. From (3.5), we have $\mathfrak{g}^e[0] = \mathfrak{g}_0(e) \oplus \mathfrak{m}^e \oplus \mathfrak{g}_1(e)$ for some compact reductive ideal $\mathfrak{m}^e \subset \mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{g}_0(e)$. Moreover, $\mathfrak{g}_0(e) \oplus \mathfrak{m}^e$ is invariant under $\mathfrak{t}$ by Lemma 4.2. Let $p_{\mathfrak{t}}$ be the orthogonal projection onto $\mathfrak{t}$. Since $\mathfrak{k} \perp \mathfrak{p}$ and

$$(i \cdot e_j \,\square\, e_j^* : \delta) = -2i \operatorname{tr}_Z((\delta e_j) \,\square\, e_j^*) = 0 \quad \text{for all} \quad \delta \in \mathfrak{k} \cap \mathfrak{g}^e[0]\,,\ j \leqslant k\ ,$$

by Lemma 2.3, $p_{\mathfrak{t}}$ leaves $\mathfrak{g}^e[0]$ invariant. Thus, $p_{\mathfrak{t}}(\Omega^- \cap \mathfrak{g}^e[0]) = \omega^- \cap \mathfrak{g}^e[0] \subset \mathfrak{t}^e[0]$.

Lemmata 3.5 and 4.2 give $\omega^- \cap \mathfrak{g}_0(e) = \omega_0^-(e)$. Hence, $\Omega^- \cap \mathfrak{g}_0(e) = \Omega_0^-(e)$ [53, Theorem 2]. Let $\widetilde{\Omega}^\pm = \Omega^\pm \cap \mathfrak{g}^e[0]$. Then $\widetilde{\Omega}^\pm$ is closed, pointed, and invariant under inner automorphisms. Hence, $\mathfrak{a}^\pm = \widetilde{\Omega}^\pm - \widetilde{\Omega}^\pm$ is an ideal of $\mathfrak{g}^e[0]$. Since $\mathfrak{g}^e[0]$ is reductive, $\mathfrak{a}^\pm$ is reductive and quasihermitian [44, Proposition II.2 and Lemma II.4]. Lemma 3.19 gives $\mathfrak{a}^\pm \cap \mathfrak{g}_1(e) = 0$, since $\mathfrak{a}^\pm$ has neither proper non-compact Abelian nor non-Hermitian simple ideals. We conclude $\widetilde{\Omega}^\pm \subset \mathfrak{a}^\pm \subset \mathfrak{g}_0(e) \oplus \mathfrak{m}^e$.

Let $\xi \in \omega^+ \cap \mathfrak{m}^e$. Seeking a contradiction, assume $\xi \neq 0$. Then there is $\alpha \in \Delta_n^{++}$ such that $\alpha(\xi) > 0$. Since $[\xi_e^-, \xi] = 0$, $\mathfrak{g}_{\mathbb{C}}^\alpha \subset [\xi, \mathfrak{p}^+] \subset \mathfrak{p}^+$ is $\operatorname{ad} \xi_e^-$-invariant, and hence contained in $\mathfrak{g}^e[\ell]_{\mathbb{C}}$ for some ℓ. But $[\xi_e^-, \mathfrak{g}_{\mathbb{C}}^\alpha] \subset [\mathfrak{p}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}] \subset \mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^\alpha = 0$, so necessarily $\ell = 0$. Since $\mathfrak{m}^e$ is an ideal of $\mathfrak{g}^e[0]$, we infer $\mathfrak{g}_{\mathbb{C}}^\alpha = [\xi, \mathfrak{g}_{\mathbb{C}}^\alpha] \subset \mathfrak{m}^e \cap \mathfrak{p}^+ = 0$, a contradiction. Therefore, $\omega^+ \cap \mathfrak{m}^e = 0$.

Since $\mathfrak{t} \cap (\mathfrak{g}_0(e) \oplus \mathfrak{m}^e) = \mathfrak{t}_0(e) \oplus \mathfrak{m}^e \cap \mathfrak{t}^+$, the projections $p_{\mathfrak{t}}$ and $p_{\mathfrak{m}^e}$ commute, and $p_{\mathfrak{m}^e}(\omega^\pm \cap \mathfrak{g}^e[0]) = \omega^\pm \cap \mathfrak{m}^e = 0$. Consequently, $\omega^- \cap \mathfrak{g}^e[0] = \omega_0^-(e)$, and this entails $\widetilde{\Omega}^- = \Omega_0^-(e)$. As for the dual cone, clearly $\Omega^+ \cap \mathfrak{g}^e[0] \subset \Omega_0^-(e)^* = \Omega_0^+(e)$. In particular, we have the inclusion $\omega^+ \cap \mathfrak{g}^e[0] \subset \omega_0^+(e)$.

Conversely, for $\alpha \in \Delta_n^{++}$, non-vanishing on $\omega_0^+(e)$, we have $\mathfrak{g}_{\mathbb{C}}^\alpha \subset \mathfrak{p}^+ \cap \mathfrak{g}^e[0]_{\mathbb{C}}$. If $\mathfrak{g}_{\mathbb{C}}^\alpha \not\subset \mathfrak{g}_0(e)_{\mathbb{C}}$, then, since $\mathfrak{g}_{\mathbb{C}}^\alpha \cap \mathfrak{m}^e = 0$, $\mathfrak{g}_{\mathbb{C}}^\alpha \subset \mathfrak{g}_1(e)_{\mathbb{C}}$. Because $\alpha(\mathfrak{t}_0(e)) \neq 0$, we find that $\mathfrak{g}_{\mathbb{C}}^\alpha \subset [\mathfrak{t}_0(e), \mathfrak{g}_1(e)_{\mathbb{C}}] \subset [\mathfrak{g}_0(e)_{\mathbb{C}}, \mathfrak{g}_1(e)_{\mathbb{C}}] = 0$, which is a contradiction. Therefore, $\mathfrak{g}_{\mathbb{C}}^\alpha \subset \mathfrak{g}_0(e)_{\mathbb{C}}$. This means that α is a root for $\mathfrak{g}_0(e) : \mathfrak{t}_0(e)$, and hence $-i\alpha \geqslant 0$ on $\omega_0^+(e)$ by definition. We have established that $-i\alpha \geqslant 0$ on $\omega_0^+(e)$, for all $\alpha \in \Delta_n^{++}$. Hence, we have that $\omega_0^+(e) \subset \omega^+ \cap \mathfrak{g}^e[0]$, and equality follows. In particular, we have $\Omega^+ \cap \mathfrak{g}^e[0] = \Omega_0^+(e)$ [53, Theorem 2]. $\square$

Remark 4.3. We use techniques due to Neeb [47, Proposition VIII.3.30].

4.2. **Semi-simple and general faces.** We will now construct the semi-simple faces of $\Omega^\pm$; by the use of the general theory of Lie algebras with invariant cones, we will also determine the structure of arbitrary faces of these cones. In particular,

we will prove that all of the faces span subalgebras of $\mathfrak{g}$ whose Levi complements are among the subalgebras $\mathfrak{g}_0(e)$.

Construction of semi-simple faces.

Proposition 4.4. *We have* $\Omega_0^{\pm}(e) = F_e^{\pm} \cap \Omega_1(e)^{\perp} = \Omega^{\pm} \cap (X_e^-)^{\perp} \cap (X_e^+)^{\perp}$, *and this set is an exposed semi-simple face of* $\Omega^{\pm}$.

Proof. We have $\Omega_1(e) \subset \Omega^- \subset \Omega^+$, so that $\Omega^{\pm} \cap \Omega_1(e)^{\perp}$ is an exposed face of $\Omega^{\pm}$. We have $\Omega^{\pm} \cap \Omega_1(e)^{\perp} = \Omega^{\pm} \cap (X_e^+)^{\perp}$, because $X_e^+ = \phi^e(e) \in \Omega_1(e)^{\circ}$.

As the intersection of exposed faces, $F = F_e^{\pm} \cap \Omega_1(e)^{\perp}$ is exposed. Since $\Omega_1(e)$ spans $\mathfrak{g}^e[2]$, Lemma 3.24 and Propositions 3.20 and 4.1 show that $F = \Omega_0^{\pm}(e)$. □

Corollary 4.5. *The nilpotent faces of* $\Omega^{\pm}$ *are exposed.*

Proof. Note $\Omega_1(e) = F_e^{\pm} \cap \Omega_0^{\mp}(e)^{\perp}$ and that exposed faces form a complete lattice. □

We will show that the $\Omega_0^{\pm}(e)$ exhaust the set of semi-simple faces. In view of the following lemma, it will suffice to show that they exhaust them up to conjugacy.

Lemma 4.6. *Let* $\mathfrak{h}$ *be a subalgebra of* $\mathfrak{g}$ *conjugate to* $\mathfrak{g}_0(e)$ *for some tripotent* e. *Then there exists a tripotent* c, $\operatorname{rk} c = \operatorname{rk} e$, *such that* $\mathfrak{h} = \mathfrak{g}_0(c)$.

Proof. We may assume that $\mathfrak{h} \neq 0$. Recall that $\mathfrak{g}$ is the set of all complete holomorphic vector fields on the bounded symmetric domain $D \subset Z$. The group G acts on the set of faces of D, and each of the faces is of the form $F = e + D_0(e)$ where e is the unique tripotent contained in F. The normaliser of the face F is the parabolic subalgebra $\mathfrak{q}^e$, and the latter is invariant under $\operatorname{ad} \xi_e^-$.

The unique $\operatorname{ad} \xi_e^-$-invariant complement of the nilradical of $\mathfrak{q}^e$ is $\mathfrak{g}^e[0]$, and $\mathfrak{g}_0(e)$ is the unique Hermitian simple ideal therein by (3.5) and Lemma 3.19. By assumption, $\mathfrak{h}$ is the Hermitian simple ideal in the canonical complement of the nilradical of the normaliser of a face of D. □

The structure of general faces.

Lemma 4.7. *Let* H *be a Lie group, and let* $\Omega \subset \mathfrak{h}$ *be a closed convex* $\operatorname{Ad} H$-*invariant cone. Any face* F *of* Ω *spans a subalgebra of* $\mathfrak{h}$. *In fact, if* $\xi \in F^{\circ}$ *and* $\eta \in \mathfrak{n}_{\mathfrak{h}}(\mathbb{R}\xi)$, *then* $\operatorname{ad} \eta$ *leaves* $\langle F \rangle_{\mathbb{R}}$ *invariant.*

Proof. Let $F \subset \Omega$ be a face, and $\xi \in F^{\circ}$. Let $\eta \in \mathfrak{n}_{\mathfrak{h}}(\mathbb{R}\xi)$. Then for all t, $\operatorname{Ad}(\exp t\eta)$ normalises $\mathbb{R}\xi$. Furthermore, $F' = \operatorname{Ad}(t \exp \eta)(F)$ is a face of Ω, since $\operatorname{Ad}(t \exp \eta)$ is a linear automorphism of $\mathfrak{h}$ leaving Ω invariant. Moreover, $\operatorname{Ad}(t \exp \eta)$ is an open map, so $\xi \in F'$. Hence, $F' = F$ [54, Theorem 13.1], and differentiating with respect to t, we obtain $[\eta, F] \subset \overline{\mathbb{R}_{>0} \cdot F - F} = \langle F \rangle_{\mathbb{R}}$. In particular, we may choose $\eta = \xi$. Since F° is dense in F, the claim follows. □

Definition 4.8. Let F be a face of $\Omega^{\pm}$. We let $\mathfrak{g}_F$ be the subalgebra spanned by F and call this the *face algebra*. Furthermore, let $\mathfrak{r}_F$ be the radical of $\mathfrak{g}_F$, $\mathfrak{n}_F$ the nilradical, $\mathfrak{z}_F = \mathfrak{z}(\mathfrak{g}_F)$ the centre, and let G_F be the analytic subgroup of G associated with $\mathfrak{g}_F$.

Proposition 4.9. *Let* $\Omega = \Omega^{\pm}$ *and let* F *be a face of* Ω. *There exists a compact Cartan subalgebra* $\mathfrak{t}_F \subset \mathfrak{g}_F$ *and a unique* $\mathfrak{t}_F$-*invariant Levi complement* $\mathfrak{s}_F$. *Then* $\mathfrak{s}_F$ *is quasihermitian semi-simple and* $\mathfrak{g}_F = \mathfrak{s}_F \ltimes \mathfrak{n}_F$, *where* $[\mathfrak{n}_F, \mathfrak{n}_F] \subset \mathfrak{z}_F$. *Moreover,*

$\mathfrak{n}_F = [\mathfrak{t}_F \cap \mathfrak{s}_F, \mathfrak{n}_F] \oplus \mathfrak{z}_F$ *as vector spaces. If* $\mathfrak{z}_F = 0$*, then* $\mathfrak{g}_F = \mathfrak{s}_F$*; if* $\mathfrak{g}_F$ *is solvable, then* $\mathfrak{g}_F = \mathfrak{z}_F$ *is Abelian.*

If the *proof*, we will need the following definition.

Definition 4.10. Let $\mathfrak{a}$ be a real Lie algebra with compactly embedded Cartan subalgebra $\mathfrak{b}$. Let * be the complex conjugation of $\mathfrak{a}_\mathbb{C}$ with respect to $i\mathfrak{a}$. A root α of $\mathfrak{a} : \mathfrak{b}$ is called *compact* if $\alpha([x, x^*]) > 0$ for some $x \in \mathfrak{a}_\mathbb{C}^\alpha$, and *non-compact* otherwise. Moreover, $\mathfrak{a}$ said to have *cone potential* if $[x, x^*] \neq 0$ for each non-zero non-compact root vector x [47, Definition VII.2.22].

Proof of Proposition 4.9. The face F is an $\operatorname{Ad} G_F$-invariant closed regular convex cone in $\mathfrak{g}_F$. It follows that $\mathfrak{g}_F$ is quasihermitian with a compactly embedded Cartan subalgebra $\mathfrak{t}_F$ and a maximal compactly embedded subalgebra $\mathfrak{k}_F$.

Let $\mathfrak{r}_F$ be the radical of $\mathfrak{g}_F$. There exists a unique $\mathfrak{t}_F$-invariant Levi complement $\mathfrak{s}_F$ of $\mathfrak{r}_F$ [47, Propositions VII.1.9, VII.2.5], and it is also $\mathfrak{k}_F$-invariant. Furthermore, we have $\mathfrak{k}_F = \mathfrak{r}_F \cap \mathfrak{k}_F \oplus \mathfrak{s}_F \cap \mathfrak{k}_F$, and if $\mathfrak{l}_F = \mathfrak{s}_F \oplus \mathfrak{t}_F \cap \mathfrak{r}_F \cap \mathfrak{z}(\mathfrak{g}_F)^\perp$, then $\mathfrak{l}_F$ is a reductive subalgebra which is complementary to $\mathfrak{n}_F$ in $\mathfrak{g}_F$ (*loc. cit.*). This subalgebra is quasihermitian [47, Lemma VIII.3.5, Theorem VIII.3.6], and hence, is the sum of a compact and of Hermitian simple ideals.

Since F is an invariant regular cone in $\mathfrak{g}_F$, this Lie algebra has cone potential [17, Theorem III.6.18]. Then $[\mathfrak{n}_F, \mathfrak{n}_F] \subset \mathfrak{z}_F = \mathfrak{z}(\mathfrak{g}_F)$ [17, Theorem III.6.23]. The subset $\mathfrak{l}_F \cap \mathfrak{t}_F \oplus \mathfrak{z}_F$ is a Cartan subalgebra of $\mathfrak{g}_F$ [47, Theorem VII.2.26], and since it is contained in $\mathfrak{t}_F$, we find $\mathfrak{t}_F = \mathfrak{l}_F \cap \mathfrak{t}_F \oplus \mathfrak{z}_F$.

Let $\Delta_F = \Delta(\mathfrak{g}_{F,\mathbb{C}} : \mathfrak{t}_{F,\mathbb{C}})$, and denote by $\Delta_{F,n}$ and $\Delta_{F,c}$ the subsets of non-compact and compact roots, respectively. There exists a unique adapted positive system Δ_F^{++} such that $\omega_F^- \subset F \cap \mathfrak{t}_F \subset \omega_F^+$, where ω_F^- is the cone spanned by $i[x^*, x]$ for $x \in \mathfrak{g}_{F,\mathbb{C}}^\alpha$, $\alpha \in \Delta_{F,n}^{++}$, and ω_F^+ is the set of all $H \in \mathfrak{t}$ such that $-i\alpha(H) \geqslant 0$, for all $\alpha \in \Delta_{F,n}^{++}$ [47, Theorem VII.3.8]. Let $p_{\mathfrak{t},F} : \mathfrak{g}_F \to \mathfrak{t}_F$ be the projection along $[\mathfrak{t}_F, \mathfrak{g}_F]$. Then we have $F = \left\{\xi \in \mathfrak{g}_F \;\middle|\; p_{\mathfrak{t},F}(\operatorname{Ad}(G_F)(\xi)) \subset \mathfrak{t}_F \cap F\right\}$ (*loc. cit.*).

Thus, $\Omega_F^- = \left\{\xi \in \mathfrak{g}_F \;\middle|\; p_{\mathfrak{t},F}(\operatorname{Ad}(G_F)(\xi)) \subset \omega_F^-\right\}$ is contained in F, and hence, pointed. It follows that Ω_F^- is the closed convex hull of $\operatorname{Ad}(G_F)(\omega_F^-)$ [47, Corollary VIII.3.31], so it is solid in $\mathfrak{g}_F$ [17, Proposition III.5.14]. Consequently, $\mathfrak{l}_F$ has no non-zero compact ideal [47, Proposition III.3.30], and we have $\mathfrak{l}_F = \mathfrak{s}_F$. Hence, $\mathfrak{g}_F = \mathfrak{s}_F \oplus [\mathfrak{t}_F \cap \mathfrak{s}_F, \mathfrak{n}_F] \oplus \mathfrak{z}_F$ as vector spaces, because $\mathfrak{t}_F = \mathfrak{s}_F \cap \mathfrak{t}_F \oplus \mathfrak{z}_F$. Since any Abelian ideal of $\mathfrak{g}_F$ is central [47, Proposition VII.3.23], we deduce that $\mathfrak{g}_F = \mathfrak{s}_F$ if $\mathfrak{z}_F = 0$. By the same token, $\mathfrak{g}_F$ is Abelian if it is solvable. $\square$

Next, we determine the structure of the Levi complement $\mathfrak{s}_F$.

Proposition 4.11. *Let* $\Omega = \Omega^\pm$ *and* $F \subset \Omega$ *be a face. Let* $\mathfrak{t}_F$ *be a compact Cartan subalgebra of* $\mathfrak{g}_F$ *and let* $\mathfrak{s}_F$ *denote the* $\mathfrak{t}_F$*-invariant Levi complement of* $\mathfrak{g}_F$*. There exists a tripotent* e *such that* $\mathfrak{s}_F = \mathfrak{g}_0(e)$*.*

The *proof* requires a little spadework. We begin with three lemmata which reduce the question to the study of the extremal rays of the cone $\mathfrak{s}_F \cap \omega^-$.

Lemma 4.12. *Let* $\mathfrak{s}$ *be an ideal of* $\mathfrak{s}_F$*, and* $i \cdot e \,\square\, e^* \in \mathfrak{s}$*. Then* $\mathfrak{s}^e \subset \mathfrak{s}$ *(where we recall* $\mathfrak{s}^e = \langle \xi_e^-, X_e^\pm \rangle_\mathbb{R}$ *from Proposition* 3.12*).*

Proof. Since $i \cdot e \,\square\, e^* = \frac{1}{2}(X_e^+ - X_e^-)$ and $\pm X_e^\pm \in \Omega^-$, we have $i \cdot e \,\square\, e^* \in \Omega^- \cap \mathfrak{s} \subset F$. Because F is a face, it follows that $\pm X_e^\pm \in F$. Now, $i \cdot e \,\square\, e^*$ remains unchanged if

we replace e by te, where $t\bar{t} = 1$. Theorem 3.27 shows that the minimal nilpotent orbit of $\Omega^- \cap \mathfrak{s}^e$ is the union of the rays spanned by the X^-_{te}, $t\bar{t} = 1$. By Corollary 3.29, the minimal cone $\Omega^- \cap \mathfrak{s}^e$ is generated by this orbit, and hence $\mathfrak{s}^e \subset \mathfrak{g}_F$. Now, $\mathfrak{s}^e$ is not completely contained in $\mathfrak{n}_F$ and is simple, so it is contained in $\mathfrak{s}_F$. Since it intersects $\mathfrak{s}$ non-trivially, we conclude $\mathfrak{s}^e \subset \mathfrak{s}_F$. □

Lemma 4.13. *Let $\mathfrak{s} \subset \mathfrak{s}_F$ be an ideal. If $i \cdot e \,\square\, e^* \in \mathfrak{s}$, then $\mathfrak{s}^c \subset \mathfrak{s}$ for all $0 < c \leqslant e$.*

Proof. From the previous lemma, we have $\pm X^\pm_e \in F \cap \mathfrak{s}_F$. Then F contains the faces $\Omega_1(\pm e)$ of Ω generated by these vectors. In particular, $\pm X^\pm_c \in F$ for all $0 < c \leqslant e$, and $i \cdot c \,\square\, c^* = \frac{1}{2}(X^+_c - X^-_c) \in F$. The simple algebra $\mathfrak{s}^c$ cannot be completely contained in $\mathfrak{n}_F$. Arguing as in the previous lemma, we find $\mathfrak{s}^c \subset \mathfrak{s}_F$. □

Lemma 4.14. *Assume that the span of those $i \cdot e \,\square\, e^*$ which belong to $\mathfrak{s}_F$ contains a Cartan subalgebra of $\mathfrak{s}_F$. Then the algebra $\mathfrak{s}_F$ is simple.*

Proof. Assume that $\mathfrak{s}^F$ splits as the direct sum of ideals $\mathfrak{s}_1 \oplus \mathfrak{s}_2$. By assumption, there exist orthogonal tripotents e_j such that $i \cdot e_j \,\square\, {e_j}^* \in \mathfrak{s}_j$. But then $e = e_1 + e_2$ satisfies $e \,\square\, e^* = e_1 \,\square\, {e_1}^* + e_2 \,\square\, {e_2}^*$. By Lemma 4.12, $\mathfrak{s}^e \subset \mathfrak{s}_F$. Since $\mathfrak{s}^e$ is simple, it must be contained in one of the ideals, $\mathfrak{s}^e \subset \mathfrak{s}_1$ (say). But then $i \cdot e_2 \,\square\, {e_2}^* \in \mathfrak{s}_1$, by Lemma 4.13, a contradiction! □

Proof of Proposition 4.11. The semi-simple subalgebra $\mathfrak{s} = \mathfrak{s}_F$ is reductive in $\mathfrak{g}$ and, possibly replacing F by a G-conjugate, we may assume that it is ϑ-invariant [63, Lemma 1.1.5.5]. Then we have $\mathfrak{k}_F \cap \mathfrak{s} \subset \mathfrak{k}$ [47, Proposition VII.2.5]. Replacing $\mathfrak{t}_F$ by a conjugate under inner automorphisms of $\mathfrak{k}_F \cap \mathfrak{s}$ (which are elements of K), we may assume $\mathfrak{t} \cap \mathfrak{s} \subset \mathfrak{t}_F$. Then $\mathfrak{t}_F \cap \mathfrak{s}$ is contained in a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. Replacing F by a K-conjugate, we may assume that $\mathfrak{t}_F \cap \mathfrak{s} \subset \mathfrak{t}$, so $\mathfrak{t} \cap \mathfrak{s} = \mathfrak{t}_F \cap \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$ contained in $\mathfrak{k}_F \cap \mathfrak{s} \subset \mathfrak{k}$.

It follows that $\Delta_{\mathfrak{s}} = \Delta(\mathfrak{s}_{\mathbb{C}} : \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{t}_{\mathbb{C}}) \subset \Delta$, and that the subsets $\Delta_{\mathfrak{s},c}$ and $\Delta_{\mathfrak{s},n}$ of compact and non-compact roots are, respectively, contained in Δ_c and Δ_n. We may choose an adapted positive system $\Delta^{++}_{\mathfrak{s}}$ contained in Δ^{++}. Let $\omega^-_{\mathfrak{s}}$ be the cone spanned by iH_α where $\alpha \in \Delta^{++}_{\mathfrak{s},n}$, and let $\omega^+_{\mathfrak{s}}$ be the set of all $H \in \mathfrak{t} \cap \mathfrak{s}$ such that $-i\alpha(H) \geqslant 0$ for all $\alpha \in \Delta^{++}_{\mathfrak{s},n}$. It is immediate that $\Delta^{++}_{\mathfrak{s},n}$ is the set of all $\alpha \in \Delta^{++}_n$ for which $\alpha(\mathfrak{t} \cap \mathfrak{s}) \neq 0$, and hence $\mathfrak{s} \cap \omega^\pm = \omega^\pm_{\mathfrak{s}}$. Since ω^- is pointed, $\omega^-_{\mathfrak{s}}$ is pointed, and its dual cone $\omega^+_{\mathfrak{s}}$ is solid in $\mathfrak{t} \cap \mathfrak{s}$. Since ω^+ is pointed, $\omega^+_{\mathfrak{s}}$ is pointed, and its dual cone $\omega^-_{\mathfrak{s}}$ is solid in $\mathfrak{t} \cap \mathfrak{s}$. Therefore, both of $\omega^\pm_{\mathfrak{s}}$ are regular cones in $\mathfrak{t} \cap \mathfrak{s}$.

Since Ω is an invariant regular cone, $\mathfrak{g}$ has cone potential [17, Theorem III.6.18]. We have $\Delta_{\mathfrak{s},n} \subset \Delta_n$, so $\mathfrak{s}$ has cone potential, too. Since $\mathfrak{s}$ is semi-simple, there exist unique invariant convex cones $\Omega^\pm_{\mathfrak{s}} \subset \mathfrak{s}$ such that $\Omega^\pm_{\mathfrak{s}} \cap \mathfrak{t} = \omega^\pm_{\mathfrak{s}}$ [17, Theorem III.5.15, Proposition III.5.14], and they are regular. Since $\Omega \cap \mathfrak{t} \cap \mathfrak{s} = \omega^\pm_{\mathfrak{s}}$, it follows that $\Omega \cap \mathfrak{s} = \Omega^\pm_{\mathfrak{s}}$ is an invariant regular convex cone in $\mathfrak{s}$. Because $\Omega^-_{\mathfrak{s}}$ is pointed, $\mathfrak{s}$ has no compact ideals and is therefore a Hermitan non-compact Lie algebra [47, Proposition VIII.3.30].

Observe now that $\mathfrak{s}$ is ϑ-invariant, and that $\mathfrak{s}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{p}^+ \oplus \mathfrak{s}_{\mathbb{C}} \cap \mathfrak{p}^-$ because $\Delta_{\mathfrak{s},n} \subset \Delta_n$. This decomposition allows the reconstruction of the triple product. Hence, if we let $Z_F = \mathfrak{s}_F \cap \mathfrak{p}$, which is a positive Hermitian Jordan triple in its own right, $\mathfrak{s}_F$ being Hermitian non-compact, then Z_F is a subtriple of $\mathfrak{p} = Z$. Because $\omega^+_{\mathfrak{s}} = \mathfrak{s} \cap \omega^+$, it follows from Corollary 3.7 that $\mathfrak{t} \cap \mathfrak{s}$ is spanned by those $i \cdot e \,\square\, e^*$ which lie in $\mathfrak{t} \cap \mathfrak{s}$. By Lemma 4.14, $\mathfrak{s}$ is simple, so Z_F is simple.

After renumbering, $ie_j \,\square\, e_j^* \in \mathfrak{t} \cap \mathfrak{s}$ for $j = 1, \dots, r_F$, where $e_1, \dots, e_{r_F}$ forms a frame of Z_F. If $e = e_{j+1} + \cdots + e_r$, then the simplicity of Z_F implies $Z_F = Z_0(e)$. Thus, $\mathfrak{s} = \mathfrak{g}_0(e)$. Finally, invoking Lemma 4.6, this conclusion also holds without replacing F by a G-conjugate. $\square$

4.3. **Determination of the faces with non-reductive face algebra.** In order to determine all faces with non-reductive face algebra, the main step is to understand their centres. This is the content of the following proposition, which will also help us determine the faces with reductive face algebra.

Proposition 4.15. *Let $F \subset \Omega = \Omega^{\pm}$ be a face and $\mathfrak{g}_F = \mathfrak{g}_0(e) \ltimes \mathfrak{n}_F$ where $\operatorname{rk} e < r$. Assume that $\mathfrak{z}_F = \mathfrak{z}_{\mathfrak{g}_F}(\mathfrak{g}_0(e))$. Possibly replacing e by $-e$, we have $\mathfrak{g}_F \subset \mathfrak{q}^e$, $\mathfrak{n}_F \subset \mathfrak{h}^e$, and there exists a unique $c \leqslant e$ such that $\mathfrak{z}_F = \mathfrak{g}^c[2]$ and $F \cap \mathfrak{z}_F = \Omega_1(c)$.*

The *proof* requires some preparatory lemmata.

Lemma 4.16. *Let $u, v \in X_1(e)$. Then $[\phi^e(u), \phi^{-e}(v)](0) = u \circ v$ where $\circ$ is the Jordan algebra product of $X_1(e)$.*

Proof. Recall from (3.6) that $\phi^{\pm e}(u) = \zeta^{\pm e}_{-iu/2} = \xi^-_{-iu/2} \pm \frac{1}{2}\big[\xi^-_e, \xi^-_{-iu/2}\big]$. Then

$$\begin{aligned}
[\zeta^e_u, \zeta^{-e}_v](0) &= \tfrac{1}{8}[[\xi^-_e, \xi^-_{iv}], \xi^-_{iu}](0) + \tfrac{1}{8}[[\xi^-_e, \xi^-_{iu}], \xi^-_{iv}](0) \\
&= \tfrac{1}{4}\big(\{e(iv)^*(iu)\} - \{(iv)e^*(iu)\} + \{e(iu)^*(iv)\} - \{(iu)e^*(iv)\}\big) \\
&= \tfrac{1}{4}(2 \cdot u \circ v + \{ev^*u\} + \{eu^*v\}) = u \circ v \ ,
\end{aligned}$$

because $u^* = u$ and (2.4) give

$$\{eu^*v\} = \{ev^*u\} = e \circ (u^* \circ v) - u^* \circ (v \circ e) + v \circ (e \circ u^*) = u \circ v \ . \qquad \square$$

Lemma 4.17. *Under the assumptions of Proposition 4.15, both of $\mathfrak{g}^e[\pm 2] \cap F$ are faces of F, and $\mathfrak{g}^e[-2, 2] \cap F = \mathfrak{g}^e[-2] \cap F \oplus \mathfrak{g}^e[2] \cap F$.*

Proof. Let $p_2^{\pm}$ and $p_2 = p_2^+ + p_2^-$ be the orthogonal projections onto $\mathfrak{g}^e[\pm 2]$ and $\mathfrak{g}^e[-2, 2]$, respectively. By Lemma 3.24, $p_2^{\pm}(\mathfrak{z}_F \cap F) \subset \mathfrak{g}^e[\pm 2] \cap F$, and the converse inclusion is obvious. By the same lemma, $\mathfrak{g}^e[-2, 2] \cap F = \mathfrak{g}^e[-2] \cap F \oplus \mathfrak{g}^e[2] \cap F$.

Now let $x, y \in F$ such that $x + y \in \mathfrak{g}^e[2] \cap F$. Write $x = \sum_{j=-2}^{2} x_j$, $y = \sum_{j=-2}^{2} y_j$ where $x_j, y_j \in \mathfrak{g}^e[j]$. Then $x_{-2} + y_{-2} = 0$, and with $x_{-2}, y_{-2} \in F$, this implies $x_{-2} = y_{-2} = 0$. By Lemma 3.24 and Proposition 3.20, $x_{-1} = y_{-1} = 0$. Then $x_0 + y_0 = 0$, where $x_0, y_0 \in F$; hence $x_0 = y_0 = 0$. Finally, $x_1 = y_1 = 0$ (*loc. cit.*) and $x = x_2, y = y_2 \in \mathfrak{g}^e[2] \cap F$, so this is a face. Analogously, so is $\mathfrak{g}^e[-2] \cap F$. $\square$

Lemma 4.18. *If $c_1 \perp c_2$ are tripotents, then $Z_j(c_1 + c_2) = Z_j(c_1 - c_2)$ for all j.*

Proof. It suffices to observe that $(c_1 \pm c_2) \,\square\, (c_1 \pm c_2)^* = c_1 \,\square\, {c_1}^* + c_2 \,\square\, {c_2}^*$. $\square$

Proof of Proposition 4.15. The spaces $\mathfrak{g}^e[\pm 1]$ are zero or faithful $\mathfrak{g}_0(e)$-modules [55, Chapter III, §4, Proposition 4.4, Corollary 4.5]. Hence,

$$\mathfrak{z}_F = \mathfrak{g}_F \cap (\mathfrak{m}^e \oplus \mathfrak{g}_1(e) \oplus \mathfrak{g}^e[-2, 2]), \tag{4.1}$$

where we recall $\mathfrak{g}^e[0] = \mathfrak{g}_0(e) \oplus \mathfrak{g}_1(e) \oplus \mathfrak{m}^e$ from (3.5).

Let $u^{\pm} \in X_1(e)$ such that $\phi^{\pm e}(u^{\pm}) \in F$. Then $\phi^{\pm e}(u^{\pm}) \in \mathfrak{z}_F$, and by Lemma 4.16, $u^+ \circ u^- = [\phi^e(u^+), \phi^{-e}(u^-)](0) = 0$. On the other hand, $u^{\pm} \in \pm\Omega_1(e)$ by Proposition 3.20, and $0 = (e|u^+ \circ u^-) = (e \circ u^+|u^-) = (u^+|u^-)$. If $c^{\pm} \leqslant e$ are such that $\pm\Omega_1(c^{\pm})$ is the face of $\Omega_1(e)$ generated by $u^{\pm}$, then $\Omega_1(c^+) \perp -\Omega_1(c^-)$. In particular, $c^+ \perp c^-$.

Let $F^\pm = \mathfrak{g}^e[\pm 2] \cap F$. By Lemma 4.17, $F^\pm$ are nilpotent faces of Ω, so by Theorem 3.27, there exist tripotents $c^\pm \leqslant e$ such that $F^\pm = \phi^{\pm e}(\Omega_1(\pm c^\pm))$. Necessarily, $c^+ \perp c^-$. By [41, Corollary 5.12], there exists some $\ell \in K$ such that $\ell(\pm c^\pm) = c^\pm$ and $\ell(e - (c^+ + c^-)) = e - (c^+ + c^-)$.

By the above considerations, whenever $X_1(c) \subset X_F^\pm$, then $X_F^\mp \subset X_1(e-c)$. There exist $c^\pm \leqslant e$ such that $X_1(c^\pm) \subset X_F^\pm$ (e.g. $0 = X_1(0) \subset X_F^+$). Then $c^+ \perp c^-$, and $X_F^+ \times X_F^- \subset X_1(e-c^-) \times X_1(e-c^+)$. By Lemma 4.18, the Peirce decompositions for the tripotents e and $\ell(e)$ are identical. On the other hand, it is clear by (3.4) that $\mathrm{Ad}(\ell)(\mathfrak{g}^{c^-}[-2]) = \mathfrak{g}^{c^-}[2] \subset \mathfrak{g}^e[2]$.

Thus, if we set $F' = \mathrm{Ad}(\ell)(F)$, then we obtain $\mathfrak{g}_{F'} = \mathfrak{g}_0(\ell(e)) \oplus \mathfrak{n}_{F'}$ where the nilradical $\mathfrak{n}_{F'} = [\mathfrak{n}_{F'}, \mathfrak{t}_0(\ell(e))] \oplus \mathfrak{z}_{F'}$, and

$$\mathfrak{z}_{F'} \cap \mathfrak{g}^{\ell(e)}[-2,2] = \mathrm{Ad}(\ell)(F \cap \mathfrak{g}^e[-2,2]) = \mathfrak{g}^{c^+}[2] \oplus \mathfrak{g}^{c^-}[2] \subset \mathfrak{g}^e[2] \ .$$

Furthermore, $\mathfrak{g}^e[2] \cap F'$ is a nilpotent face by Lemma 4.18, and therefore equals $\Omega_1(c)$ for some $c \leqslant e$, by Theorem 3.27. In particular, $\mathfrak{g}^e[2] \cap F' = \mathrm{Ad}(\ell)(\mathfrak{g}^e[-2,2] \cap F)$ is an irreducible cone. Hence, one of the faces $\mathfrak{g}^e[\pm 2] \cap F$ must be trivial.

Possibly replacing e by $-e$ (which does not change F or $\mathfrak{g}_0(e)$), we may assume that $\mathfrak{g}^e[-2] \cap F = 0$. We set $c = c^+$. Arguing as usual with Lemma 3.24 and Proposition 3.20, we find that $\mathfrak{g}_F \subset \mathfrak{q}^e = \mathfrak{g}^e[0,1,2]$, so that $\mathfrak{g}_0(e) \oplus \mathfrak{g}^c[2] \subset \mathfrak{g}_F \subset \mathfrak{g}_0(e) \ltimes \mathfrak{h}^e$, by Proposition 4.1. Moreover, $\mathfrak{z}_F \subset \mathfrak{g}_1(e) \oplus \mathfrak{m}^e \oplus \mathfrak{g}^c[2]$ by (4.1). It follows that $\mathfrak{z}_F = \mathfrak{g}^c[2]$, $F \cap \mathfrak{z}_F = \Omega_1(c)$, and $\mathfrak{n}_F \subset \mathfrak{h}^e$. □

Proposition 4.19. *Let $\Omega = \Omega^\pm$ and $F \subset \Omega$ be a face. Assume that $\mathfrak{g}_F = \mathfrak{g}_0(e) \ltimes \mathfrak{n}_F$ is not reductive (so, in particular, $\operatorname{rk} e < r$). Possibly replacing e by $-e$, we have $\mathfrak{z}_F = \mathfrak{g}^c[2] = X_1(c)$ for a unique $c \leqslant e$, and $\mathfrak{g}_F = \mathfrak{g}_0(e) \ltimes \mathfrak{h}^{e,c}$, where*

$$\mathfrak{h}^{e,c} = \left\{ \eta_u^e \;\middle|\; u \in Z_{1/2}(e) \cap Z_{1/2}(c) \right\} \oplus \mathfrak{g}^c[2] \ .$$

In particular, we have $F = F_{e,c}^\pm = F_e^\pm \cap F_c^\pm$, and this is an exposed face of Ω.

The *proof* requires only the following simple lemma.

Lemma 4.20. *Let $c \leqslant e$, and let $\mathfrak{h} \subset \mathfrak{h}^e$ be a subalgebra such that $\mathfrak{h} \cap \mathfrak{g}^e[2] = \mathfrak{g}^c[2]$. Let I be the complex structure on $\mathfrak{g}^e[1]$ induced by that of $Z_{1/2}(e)$. If $\mathfrak{h} \cap \mathfrak{g}^e[1]$ is I-invariant, then $\eta_u^e \in \mathfrak{h}$ implies $u \in Z_{1/2}(e) \cap Z_{1/2}(c)$.*

Proof. Let $\eta_u^e \in \mathfrak{h}$. Then $\mathfrak{g}^c[2] \ni [I\eta_u^e, \eta_u^e] = \zeta_{-iv/2}^e$, where $v = q_e(iu,u) = 8\{uu^*e\}$ by (3.7). By Proposition 3.18, if $u \neq 0$, then $v \neq 0$. In particular, $v \in X_1(c) \setminus 0$.

Now, $Z_{1/2}(e) = Z_{1/2}(e) \cap Z_{1/2}(c) \oplus Z_{1/2}(e) \cap Z_0(c)$. If a lies in the second summand, then $\{aa^*e\} = \{aa^*(e-c)\} \in Z_0(c)$. Similarly, $\{ab^*e\} \in Z_{1/2}(c)$ if a lies in the first summand, and b lies in the second. Because $h_e(a,b) = 8\{ab^*e\}$ is $\Omega_1(e)$-positive Hermitian by Proposition 3.18, $\{ab^*e\}^* = \{ba^*e\}$, and we conclude that $v \in X_1(c)$ if and only if $u \in Z_{1/2}(e) \cap Z_{1/2}(c)$. □

Proof of Proposition 4.19. If we had $\mathfrak{g}_0(e) = 0$, then $\mathfrak{g}_F$ would be nilpotent and hence Abelian [20, Lemma I.13]. By the assumption, this is excluded, so $\mathfrak{g}_0(e) \neq 0$. Let $\mathfrak{t}$ be chosen according to (3.2) for some frame adapted to e, and $\mathfrak{t}_F = \mathfrak{t}_0(e) \oplus \mathfrak{z}_F$ be the associated compact Cartan subalgebra of $\mathfrak{g}_F$. Since $\mathfrak{g}_F$ is not reductive, we have $\mathfrak{n}_F = [\mathfrak{n}_F, \mathfrak{t}_0(e)] \oplus \mathfrak{z}_F$ and the first summand contains no $\mathfrak{g}_0(e)$-fixed vector [43, Theorem V.1]. Hence, $\mathfrak{z}_F = \mathfrak{z}_{\mathfrak{g}_F}(\mathfrak{g}_0(e))$.

By Proposition 4.15, possibly replacing e by $-e$, we have $\mathfrak{g}_F \subset \mathfrak{q}^e$, $\mathfrak{n}_F \subset \mathfrak{h}^e$, and there exists a unique tripotent $c \leqslant e$ such that $\mathfrak{z}_F = \mathfrak{g}^c[2]$ and $F \cap \mathfrak{z}_F = \Omega_1(c)$.

Hence, $\mathfrak{n}_F \cap \mathfrak{g}^e[2] = \mathfrak{g}^c[2]$. On the other hand, $h_0' = iz\frac{\partial}{\partial z} - ie\,\square\, e^* \in \mathfrak{k}_0(e)$ and $h_0'(z) = \frac{i}{2}z$ for all $z \in Z_{1/2}(e)$ (considering h_0' as an endomorphism of Z). By (3.8), it follows that the $\mathfrak{g}_0(e)$-module $\mathfrak{n}_F \cap \mathfrak{g}^e[1]$ is invariant under the complex structure I of $\mathfrak{g}^e[1]$. Invoking Lemma 4.20, it follows that $\eta_u^e \in \mathfrak{n}_F$ implies $u \in Z_{1/2}(e) \cap Z_{1/2}(c)$. Since $\mathfrak{n}_F = [\mathfrak{t}_0(e), \mathfrak{n}_F] \oplus \mathfrak{z}_F$, we deduce that

$$\mathfrak{n}_F = \mathfrak{n}_F \cap \mathfrak{g}^e[1] \oplus \mathfrak{g}^c[2] \subset Z_{1/2}(e) \cap Z_{1/2}(c) \oplus \mathfrak{g}^c[2]\ .$$

Let $\mathfrak{l} = \mathfrak{g}_0(e) \ltimes (Z_{1/2}(e) \cap Z_{1/2}(c) \oplus \mathfrak{g}^c[2])$. Then $\mathfrak{h} = \mathfrak{t}_0(e) \oplus \mathfrak{g}^c[2]$ is a compact Cartan subalgebra of $\mathfrak{l}$. We have $\mathfrak{g}_F \subset \mathfrak{l}$ and $F_{e,c}^{\pm} = F_e^{\pm} \cap F_c^{\pm} \subset \mathfrak{l}$ since $\mathfrak{g}_0(e) \subset \mathfrak{g}_0(c)$, $\mathfrak{g}^e[2] \supset \mathfrak{g}^c[2]$, $\mathfrak{g}_0(c) \cap \mathfrak{g}^c[2] = 0$, and by the argument in the previous paragraph. The face $F_{e,c}^{\pm}$ is $\mathfrak{l}$-invariant since it is $(\mathfrak{g}_0(e) \ltimes \mathfrak{h}^e) \cap (\mathfrak{g}_0(c) \ltimes \mathfrak{h}^c)$-invariant, and we have $\mathfrak{l} \cap \mathfrak{h} = \omega_0^{\pm}(e) \oplus \Omega_1(c) = F \cap \mathfrak{h}$. It follows that $F \subset F_{e,c}^{\pm}$ and $F_{e,c}^{\pm}$ is regular in $\mathfrak{l}$ (by the same argument as in the proof of Proposition 4.1).

But since $F_{e,c}^{\pm} \cap \mathfrak{h}$ contains an element of the relative interior $F_{e,c}^{\pm\circ}$ [17, Proposition III.5.14 and proof], it follows that the faces F and $F_{e,c}^{\pm}$ are identical. In particular, $\mathfrak{g}_F = \mathfrak{l}$, and since the lattice of exposed faces is complete, F is an exposed face. $\square$

Corollary 4.21. *Let $F \subset \Omega = \Omega^{\pm}$ be a face with a reductive face algebra $\mathfrak{g}_F$. Then F is a semi-simple face of the form $F = \Omega_0^{\pm}(e)$, or $\mathfrak{g}_F$ is Abelian.*

Proof. By the assumption and Proposition 4.11, $\mathfrak{g}_F = \mathfrak{g}_0(e) \oplus \mathfrak{z}_F$. We may assume that $\operatorname{rk} e < r$ since otherwise $\mathfrak{g}_F = \mathfrak{z}_F$ is Abelian. Then Proposition 4.15 implies that (after possibly replacing e by $-e$) there exists a tripotent $c \leqslant e$ such that $\mathfrak{z}_F = \mathfrak{g}^c[2]$. We may assume $c > 0$ since otherwise $F = \Omega_0^{\pm}(e)$. But then $F \supset \Omega_0^{\pm}(e) \oplus \Omega_1(c)$ and the latter cone contains points in the relative interior of $F_{e,c}^{\pm}$ [17, Proposition III.5.14 and proof]. Since $F \subset (\mathfrak{g}_0(e) \oplus \mathfrak{g}^c[2]) \cap \Omega \subset F_{e,c}^{\pm}$ and is a face, we conclude $F = F_{e,c}^{\pm}$. But this is a contradiction, since the face algebra of $F_{e,c}^{\pm}$ is non-reductive. $\square$

4.4. **Exhaustion of the faces of Ω^-.** We are finally ready to describe all the convex faces of Ω^-.

Lemma 4.22. *Let $F \subsetneq \Omega = \Omega^{\pm}$ be a proper face. For any $\xi \in \mathfrak{g}$, denote its Jordan decomposition by $\xi = \xi_s + \xi_n$. For all $\xi \in F^\circ$, we have $\xi_s, \xi_n \in F$.*

Proof. We have $\xi_s \in \Omega$ and $\xi_n \in \Omega^- \subset \Omega$ for all $\xi \in \Omega$ [45, Lemma IV.4]. Let $\xi \in F^\circ$. Elements of Ω° are elliptic and hence semi-simple, and $\partial\Omega$ is closed. Thus, $\xi_s + t\xi_n, t\xi_s + \xi_n \in \partial\Omega$ for all $t \geqslant 0$. This means that the line segments $[\xi|\xi_s]$ and $[\xi|\xi_n]$ lie within a proper face of Ω, and therefore the open segments intersect F°. But this implies $[\xi|\xi_s], [\xi|\xi_n] \subset F$. Hence, $\xi_s, \xi_n \in F$. $\square$

Lemma 4.23. *Let F be an extreme ray of $\Omega^{\pm}$. Then $F \subset \Omega^-$ if and only if F is nilpotent. In this case, $F = \mathbb{R}_{\geqslant 0} \cdot X_c^+$ for some primitive tripotent c. If this is not the case, then F is conjugate to an extreme ray of ω^+ not contained in ω^-.*

Proof. Let $\xi \in F^\circ$, $\xi = \xi_s + \xi_n$. Then $\xi_s, \xi_n \in F = \mathbb{R}_{\geqslant 0} \cdot \xi$ by Lemma 4.22, and ξ is semi-simple or nilpotent. The case of ξ nilpotent is covered by Theorem 3.27. Since $\pm X_e^{\pm} \in \Omega^-$ and $i \cdot e\,\square\, e^* = \frac{1}{2}(X_e^+ - X_e^-)$ for any tripotent e, no extreme ray of ω^- is an extreme ray of $\Omega^{\pm}$, by Corollary 3.7.

Hence, ξ is semi-simple if and only if F is an extreme ray of Ω^+ which is not contained in Ω^-. In this case, ξ is conjugate to an element of ω^+ by Proposition 3.9, so we may assume $F \subset \omega^+$. Since $\omega^+ \subset \Omega^+$, F is then an extreme ray of ω^+. $\square$

Corollary 4.24. *Any face of Ω^- with a solvable face algebra is a nilpotent face.*

Proof. Let $F \subset \Omega^-$ be a face with a solvable face algebra. By Proposition 4.9, $\mathfrak{g}_F$ is Abelian. By Strasziewicz's spanning theorem, the cone spanned by the extreme rays of F is dense in F. Hence, there exists $x \in F^\circ$ which is the positive linear combination of extreme generators. By Lemma 4.23, all of the latter are nilpotent elements of $\mathfrak{g}$. Since they commute, x is nilpotent, and F is a nilpotent face. □

Proposition 4.25. *Let F be a semisimple face of $\Omega = \Omega^\pm$. Then $F = \Omega_0^\pm(e)$ for some tripotent e, or $\mathfrak{g}_F$ is Abelian and contained in a compact Cartan subalgebra of $\mathfrak{g}$. The latter alternative only occurs for $F = 0$ or $\Omega = \Omega^+$, and in this case, F is conjugate to a face of ω^+. In particular, the semi-simple faces of Ω form a lattice.*

Proof. By the semisimplicity of F, $F^\circ \subset \mathfrak{g} \setminus 0$ contains semi-simple elements. By Theorem 3.27, the nilpotent faces consist of nilpotent elements of $\mathfrak{g}$. Hence, F is not contained in a nilpotent face. But then F° cannot intersect any nilpotent face. Since any nilpotent element of Ω^+ is contained in a nilpotent face, F° contains no nilpotent elements.

If $\mathfrak{g}_F$ were non-reductive, then by Proposition 4.19, it would contain a compact Cartan subalgebra of the form $\mathfrak{h} = \mathfrak{t}_0(e) \oplus \mathfrak{z}_F$, where both summands are non-zero, all elements of the first summand are semi-simple, and all elements of the second summand are nilpotent. Since any element of F° is G-conjugate to an element of $F^\circ \cap \mathfrak{h}$, F° could then not contain a semi-simple element, a contradiction.

Hence, $\mathfrak{g}_F$ must be reductive. By Corollary 4.21, $F = \Omega_0^\pm(e)$ for some tripotent e, or $\mathfrak{g}_F$ is Abelian. Let us consider the latter case. As noted above, F° contains no nilpotent elements, so unless $F = 0$, we must have $\Omega = \Omega^+$, by Corollary 4.24.

Assume that $\xi = \xi_s + \xi_n \in F^\circ$ is such that $\xi_n \neq 0$. Then $\xi_s \neq 0$, and by Lemma 4.22, the line $[\xi|\xi_s]$ is contained in F. If ξ_s were contained in F°, then so would $\xi_s - t\xi_n$, for some $t > 0$. But then Lemma 4.22 would imply $-\xi_n \in F$, a contradiction, since F is pointed. Thus, $\xi_s \in \partial F$. Hence, no semi-simple element of F° can commute with a non-zero nilpotent element of F (otherwise we could construct $\xi = \xi_s + \xi_n \in F^\circ$ with $\xi_n \neq 0$ and $\xi_s \in F^\circ$). But since $\mathfrak{g}_F$ is Abelian, F cannot contain any non-zero nilpotent element. This is a contradiction.

Hence, F° and $\mathfrak{g}_F = \langle F^\circ \rangle_\mathbb{R}$ consist entirely of semi-simple elements. By Proposition 3.9, any $\xi \in F^\circ$ is contained in a compact Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{z}_\mathfrak{g}(\mathfrak{g}_F) = \mathfrak{z}_\mathfrak{g}(\xi)$ for any $\xi \in F^\circ$, $\mathfrak{z}_\mathfrak{g}(\mathfrak{g}_F)$ contains a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This is of necessity also a Cartan subalgebra of $\mathfrak{z}_\mathfrak{g}(\mathfrak{g}_F)$. Since $\mathfrak{g}_F$ is contained in the centre of $\mathfrak{z}_\mathfrak{g}(\mathfrak{g}_F)$, $\mathfrak{h}$ must contain $\mathfrak{g}_F$, by [5, Chapter VII, § 2.1, Proposition 5]. The assertion follows from the conjugacy of compact Cartan subalgebras. □

We summarise our considerations for Ω^- in the following theorem and corollary.

Theorem 4.26. *Each face of Ω^- is one of $\Omega_0^-(e)$, $\Omega_1(e)$, e a tripotent; or of $F_{e,c}^-$, $e \geqslant c > 0$ tripotents with $\operatorname{rk} e < r$. In particular, Ω^- is facially exposed.*

Proof. Let F be a face of Ω^-. The face algebra $\mathfrak{g}_F$ is either reductive or not. If $\mathfrak{g}_F$ is non-reductive, then by Proposition 4.19, $F = F_{e,c}^-$ for some tripotents $c \leqslant e$ where $\operatorname{rk} e < r$, and F is exposed. We also observe that $F_{e,0}^- = \Omega_0^-(e)$. Since the latter face has reductive face algebra, c must be non-zero.

Hence, we are reduced to the consideration of the case where $\mathfrak{g}_F$ is reductive. By Corollary 4.21, either $F = \Omega_0^-(e)$ for some tripotent e, or $\mathfrak{g}_F$ is Abelian. In the

former case, F is, in particular, exposed by Proposition 4.4. It remains to show that in the latter case, $F = \Omega_1(e)$ for some tripotent e.

If $\mathfrak{g}_F$ is Abelian, then by Corollary 4.24, F is a nilpotent face. But then Theorem 3.27 shows that $F = \Omega_1(e)$ for some tripotent e. By Corollary 4.5, F is also exposed. Since we have considered all cases and listed all the alternatives in the statement of the theorem, we have proved our claim. □

Since (by Proposition 4.1) $F^-_{e,c} = \Omega_1(c)$ for $\operatorname{rk} e = r$, $e \geqslant c$, and $F^-_{e,0} = \Omega^-_0(e)$ for $\operatorname{rk} e < r$, we may rephrase Theorem 4.26 as follows.

Corollary 4.27. *Every face of Ω^- is one of the faces $F^-_{e,c}$, for tripotents $0 \leqslant c \leqslant e$.*

Conjugacy classes of faces and K-orbit type decomposition of Ω^-.

Theorem 4.28. *Any two faces $F^\pm_{e,c}$ and $F^\pm_{e',c'}$ of $\Omega^\pm$ are G-conjugate if and only if one has $(\operatorname{rk} e, \operatorname{rk} c) = (\operatorname{rk} e', \operatorname{rk} c')$ and if and only if they are K-conjugate. In particular,*

$$\Omega_{k,\ell} = \bigcup_{c \leqslant e\,,\,(\operatorname{rk} e, \operatorname{rk} c) = (k,\ell)} F^{-\circ}_{e,c}, \qquad 0 \leqslant \ell \leqslant k \leqslant r$$

are exactly the orbit types of the K-action on Ω^-. If $M_{k,\ell}$ is the set of pairs (e,c) of tripotents $e \geqslant c$ such that $(\operatorname{rk} e, \operatorname{rk} c) = (k,\ell)$, then $\Omega_{k,\ell}$ is a K-equivariant fibre bundle over the K-homogeneous space $M_{k,\ell}$ with typical fibre $F^{-\circ}_{e,c}$.

Proof. Given the equality of ranks, the faces are K-conjugate, in view of [41, Theorem 5.9]. Moreover, they are certainly G-conjugate if they are K-conjugate. If they are G-conjugate, then the algebras $\mathfrak{g}^c[2]$ and $\mathfrak{g}^{c'}[2]$ are G-conjugate, and so are $\mathfrak{g}_0(e)$ and $\mathfrak{g}_0(e')$, as the centres of the respective face algebras and their Levi complements invariant under compact Cartan subalgebras, respectively. By Theorem 3.27, we have $\operatorname{rk} c = \operatorname{rk} c'$, and by Lemma 4.6, we have $\operatorname{rk} e = \operatorname{rk} e'$.

Any element in the relative interior of the face $F = F^-_{e,c}$ is G-conjugate to an element of the relative interior of $f = F \cap (\mathfrak{t}_0(e) \oplus \mathfrak{g}^c[2]) = \omega^-_0(e) \oplus \Omega_1(c)$. Moreover, $k \in K$ fixes $\xi \in f^\circ$ if and only if k fixes $\xi_s \in \omega^-_0(e)^\circ$ and $\xi_n \in \Omega_1(c)^\circ$, where we denote by $\xi = \xi_s + \xi_n$ the Jordan decomposition. The stabiliser of ξ_s is $N_K(\mathfrak{t}_0(e))$, and the stabiliser of ξ_n is K^c, independent of ξ. This shows that $\Omega_{k,\ell}$, $(k,\ell) = (\operatorname{rk} e, \operatorname{rk} c)$, is exactly a single K-orbit type. By Corollary 4.27, the assertion follows. □

Corollary 4.29. *For any $r \geqslant k \geqslant \ell \geqslant 0$, $\Omega_{k,\ell}$ is K-equivariantly homotopy equivalent to the K-homogeneous space $M_{k,\ell} = K/(K^e \cap K^c)$ (where $(e,c) \in M_{k,\ell}$).*

5. The stratification of the minimal Ol'shanskiĭ semigroup

In this section, we apply our previous considerations to achieve our ultimate goal: namely, the decomposition of the minimal Ol'shanskiĭ semigroup into K-orbit type strata, and their description in terms of K-equivariant fibre bundles.

5.1. **The minimal Ol'shanskiĭ semigroup.** There exists a connected complex Lie group $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{C}$ such that $G \subset G_\mathbb{C}$. (*E.g.*, consider the projective completion D^* of D [41, §§ 8–9] and let $G_\mathbb{C}$ be the connected component of $\operatorname{Aut}(D^*)$ [41, Proposition 9.4]. Alternatively, we may invoke [14, Proposition 25.9].)

By the following proposition, there exists a closed complex semigroup $\Gamma \subset G_\mathbb{C}$ such that $\Gamma = G \cdot \exp i\Omega^-$ and $G \times \Omega^- \to \Gamma : (g,\xi) \mapsto g \exp i\xi$ is a homeomorphism

which restricts to a diffeomorphism $G \times \Omega^{-\circ} \to \Gamma^\circ$. This semigroup is called the *minimal Ol'shanskiĭ semigroup*.

The following proposition is a compilation of known results. We give it for the reader's convenience, since we lack a succinct reference. The existence of Ol'shanskiĭ semigroups is treated in full generality in [18, Chapters 3, 7], [47, Chapter XI].

Proposition 5.1. *Let $H \subset H_\mathbb{C}$ be connected Lie groups where H is closed, $H_\mathbb{C}$ is complex, and the Lie algebra of $H_\mathbb{C}$ is $\mathfrak{h}_\mathbb{C}$. Let $\Omega \subset \mathfrak{h}$ be an invariant regular cone. Then $\psi : H \times \Omega \to H_\mathbb{C} : (h, \xi) \mapsto h \cdot \exp i\xi$ is a homeomorphism onto a closed semigroup Γ and induces a diffeomorphism $H \times \Omega^\circ \to \Gamma^\circ$, where $\Gamma^\circ \subset \Gamma$ is dense.*

Proof. Let $\phi : \widetilde{H}_\mathbb{C} \to H_\mathbb{C}$ be the universal covering. Then $\ker \phi$ is a discrete central subgroup. The $\mathrm{Gal}(\mathbb{C} : \mathbb{R})$-action on $H_\mathbb{C}$ lifts to $\widetilde{H}_\mathbb{C}$. Let $\widetilde{H}$ be the fixed group of this action; then $\widetilde{H}$ is closed and connected [40, Chapter IV, Theorem 3.4]. The adjoint action of $\mathfrak{h}$ has imaginary spectrum [44, Proposition II.2].

Therefore, $\tilde{\psi} : \widetilde{H} \times \Omega \to \widetilde{H}^\mathbb{C} : (h, \xi) \mapsto h \exp i\xi$ (where we take the exponential map of $\widetilde{H}_\mathbb{C}$) is a homeomorphism onto a closed subsemigroup $\widetilde{\Gamma} \subset \widetilde{H}_\mathbb{C}$ which restricts to a diffeomorphism $\widetilde{H} \times \Omega^\circ \to \widetilde{\Gamma}^\circ$ [47, Theorems XI.1.7, XI.1.10]. In particular, $\widetilde{H}$ is a retraction of $\widetilde{H}_\mathbb{C}$ and is therefore simply connected. It follows that H and $H_\mathbb{C}$ are homotopy equivalent [14, Proposition 25.9]. We have canonical isomorphisms

$$\ker \phi \cap \widetilde{H} \to \pi_1(H, 1) \to \pi_1(H_\mathbb{C}, 1) \to \ker \phi \ .$$

This map associates to $h \in \ker \phi$ the homotopy class of $\phi \circ \gamma_h$, γ_h a path in $\widetilde{H}$ from 1 to h; to this is associated the homotopy class in $H_\mathbb{C}$ of $\phi \circ \gamma_h$ and hereto, the end point of a lifting of $\phi \circ \gamma_h$ in $H_\mathbb{C}$. Since γ_h is such a lifting and $\gamma_h(1) = h$, the composite map is the identity, and $\ker \phi \subset \widetilde{H}$. Thus, we conclude that $\tilde{\psi}$ drops to a map ψ with the required properties. (For the statement on Γ°, see [47, Lemma XI.I.9].) □

5.2. **The stratification of the minimal Ol'shanskiĭ semigroup.**

Definition 5.2. A *Lie semigroup* is a pair (S, H), where H is a connected Lie group and $S \subset H$ is a closed subsemigroup which is generated (as a closed semigroup) by the one-parameter semigroups it contains [42, Definition IV.3]. The *tangent wedge* of (S, H) is the convex cone $L(S) = \left\{ \xi \in \mathfrak{h} \;\middle|\; \exp(\mathbb{R}\xi) \subset S \right\}$ [42, Definition IV.2].

Let (S, H) be a Lie semigroup. A *face* of (S, H) is a subsemigroup $F \subset S$ such that $S \setminus F$ is a semigroup ideal.

Proposition 5.3. *The pair $(\Gamma, G_\mathbb{C})$ is a Lie semigroup whose faces are G and Γ.*

Proof. Consider the cone $W = \mathfrak{g} \oplus i\Omega^-$. It is G-invariant and therefore a *Lie wedge* [42, Definition IV.1]. It equals the tangent wedge of Γ and is therefore *global* [42, Definition IV.23, Lemma IV.24]; in particular, $(\Gamma, G_\mathbb{C})$ is a Lie semigroup. By [18, Lemma 7.30], [17, Lemma II.2.11], W is *Lie semialgebra* [42, Definition IV.29]. By [42, Proposition IV.32], the faces of $(\Gamma, G_\mathbb{C})$ are among the closed subsemigroups whose tangent wedges are faces of W, and therefore of the form $\mathfrak{g} \oplus iF$, where $F \subset \Omega^-$ is a face.

Let $S \subset \Gamma$ be a non-trivial face. Then $L(S) = \mathfrak{g} \oplus iF$, where $F \subset \Omega^-$ is a non-trivial face. Hence, F contains an extreme ray: by Lemma 4.23, F intersects the minimal nilpotent orbit of Ω^- non-trivially. Since $G \subset S$, $L(S)$ is G-invariant, and therefore contains the minimal nilpotent orbit in $i\Omega^-$. Since $L(S)$ is a closed

convex cone, we have $i\Omega^- \subset L(S)$ by Corollary 3.29, and thus $L(S) = W$. This implies $\Gamma = S$. □

The stratification of Γ into K-orbit types is more interesting. To describe it, let $F \subset \Omega^-$ be a face. Then $\mathfrak{g}_F = \langle F \rangle_{\mathbb{R}}$ is a subalgebra, and we may consider the analytic subgroups $G_F \subset G$ and $G_{F\mathbb{C}} \subset G_{\mathbb{C}}$ associated with $\mathfrak{g}_F$ and $\mathfrak{g}_{F\mathbb{C}}$, respectively. We have an Ol'shanskiĭ semigroup $\Gamma_F = G \cdot \exp iF \subset G_{F\mathbb{C}}$ whose interior Γ_F° in $G_{F\mathbb{C}}$ is $G \cdot \exp iF^\circ$ (F° denoting the relative interior).

Theorem 5.4. *The subsemigroups $\Gamma_F, \Gamma_{F'} \subset \Gamma$, $F = F^-_{e,c}$ and $F' = F^-_{e',c'}$, are G-conjugate if and only if they are K-conjugate and if and only if $(\operatorname{rk} e, \operatorname{rk} c) = (\operatorname{rk} e', \operatorname{rk} c')$. The orbit type strata for the action of K on Γ by conjugation are*

$$\Gamma_{k,\ell} = G \cdot \exp i\Omega_{k,\ell} = \bigcup_{(e,c)\in M_{k,\ell}} G \cdot \exp iF^{-\circ}_{e,c}, \qquad 0 \leqslant \ell \leqslant k \leqslant r\ .$$

Proof. This follows immediately from Theorem 4.28. □

Corollary 5.5. *The orbit type stratum $\Gamma_{k,\ell}$ is K-equivariantly homotopy equivalent to $K \times M_{k,\ell} = K \times (K/(K^e \cap K^c))$ (where $(e,c) \in M_{k,\ell}$).*

Proof. Clearly, $\Gamma_{k,\ell}$ fibres over $M_{k,\ell}$ with fibre $G \cdot \exp iF^{-\circ}_{e,c}$. Moreover, there exists a K-equivariant homotopy equivalence $G \cdot \exp iF^{-\circ}_{e,c} \simeq G \simeq K$. □

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