ON SYMMETRIC COMMUTATOR SUBGROUPS, BRAIDS, LINKS AND HOMOTOPY GROUPS

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Abstract. In this paper, we investigate some applications of commutator subgroups to homotopy groups and geometric groups. In particular, we show that the intersection subgroups of some canonical subgroups in certain link groups modulo their symmetric commutator subgroups are isomorphic to the (higher) homotopy groups. This gives a connection between links and homotopy groups. Similar results hold for braid and surface groups.

1. Introduction

The purpose of this article is to investigate some applications of commutator subgroups to homotopy groups and geometric groups.

Recall [16, p. 288-289] that a bracket arrangement of weight \(n\) in a group \(G\) is a map \(\beta^n: G^n \to G\) which is defined inductively as follows:

\[
\beta^1 = \text{id}_G, \quad \beta^2(a_1, a_2) = [a_1, a_2]
\]

for any \(a_1, a_2 \in G\), where \([a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2\). Suppose that the bracket arrangements of weight \(k\) are defined for \(1 \leq k < n\) with \(n \geq 3\). A map \(\beta^n: G^n \to G\) is called a bracket arrangement of weight \(n\) if \(\beta^n\) is the composite

\[
G^n = G^k \times G^{n-k} \xrightarrow{\beta^k \times \beta^{n-k}} G \times G \xrightarrow{\beta^2} G
\]

for some bracket arrangements \(\beta^k\) and \(\beta^{n-k}\) of weight \(k\) and \(n-k\), respectively, with \(1 \leq k < n\). For instance, if \(n = 3\), there are two bracket arrangements given by \([a_1, a_2, a_3]\) and \([a_1, [a_2, a_3]]\).

Let \(R_j\) be a sequence of subgroups of \(G\) for \(1 \leq j \leq n\). The fat commutator subgroup \([[R_1, R_2, \ldots, R_n]]\) is defined to be the subgroup of \(G\) generated by all of the commutators

\[
\beta^t(g_{i_1}, \ldots, g_{i_t}),
\]

Received by the editors August 4, 2009 and, in revised form, January 25, 2010 and February 28, 2010.

2010 Mathematics Subject Classification. Primary 55Q40, 20F12; Secondary 20F36, 57M25.

Key words and phrases. Symmetric commutator subgroup, homotopy group, link group, Brunian braid, free group, surface group.

The first author was partially supported by the National Natural Science Foundation of China 10971050.

The second author was partially supported by the AcRF Tier 1 (WBS No. R-146-000-101-112 and R-146-000-137-112) of MOE of Singapore and a grant (No. 11028104) of NSFC.

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where

1. \(1 \leq i_s \leq n\);
2. \(\{i_1, \ldots, i_t\} = \{1, \ldots, n\}\); that is, each integer in \(\{1, 2, \ldots, n\}\) appears at least one of the integers \(i_s\);
3. \(g_j \in R_j\);
4. \(\beta^t\) runs over all of the bracket arrangements of weight \(t\) (with \(t \geq n\)).

For convenience, let \([[[R_1]]] = R_1\). The fat commutator subgroups have important applications in homotopy theory \([1, 3, 6, 19]\). Let \(G = F_n\) be the free group of rank \(n\) with a basis \(x_1, \ldots, x_n\). Let \(R_1 = \langle \langle x_1 \rangle \rangle\) be the normal closure of \(x_1\) in \(F_n\) for \(1 \leq i \leq n\) and let \(R_{n+1} = \langle \langle x_1 \cdots x_n \rangle \rangle\) be the normal closure of the product element \(x_1x_2 \cdots x_n\) in \(F_n\). According to \([19, \text{Theorem 1.7}]\), the homotopy group \(\pi_{n+1}(S^2)\) is isomorphic to the quotient group

\[
(R_1 \cap R_2 \cap \cdots \cap R_{n+1})/[[[R_1], R_2, \ldots, R_{n+1}]]
\]

for each \(n\).

To understand the fat commutator subgroup, we consider the symmetric commutator subgroup\([1, 3, 6, 19]\) defined by

\[
[[[R_1, R_2, \ldots, R_n]]]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],
\]

where \([[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]]\) is the subgroup generated by the left iterated commutators

\[
[[[g_1, g_2], g_3], \ldots, g_n]
\]

with \(g_i \in R_{\sigma(i)}\). For convenience, let \([R_1]_S = R_1\). From the definition, the symmetric commutator subgroup is a subgroup of the fat commutator subgroup. Our first result states that the fat commutator subgroup is in fact the same as the symmetric commutator subgroup.

**Theorem 1.1.** Let \(R_j\) be any normal subgroup of a group \(G\) with \(1 \leq j \leq n\). Then

\[
[[[R_1, R_2, \ldots, R_n]]] = [[R_1, R_2, \ldots, R_n]]_S.
\]

The symmetric commutator subgroup can be simplified as follows:

**Theorem 1.2.** Let \(R_j\) be any normal subgroup of a group \(G\) with \(1 \leq j \leq n\). Then

\[
[[[R_1, R_2, \ldots, R_n]]]_S = \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],
\]

where \(\Sigma_{n-1}\) acts on \(\{2, 3, \ldots, n\}\).

Our next step is to give a generalization of \([19, \text{Theorem 1.7}]\). This will give more connections between homotopy groups and symmetric commutator subgroups.

Let \((X, A)\) be a pair of spaces. An \(n\)-partition of \(X\) relative to \(A\) means a sequence of subspaces \((A_1, \ldots, A_n)\) of \(X\) such that

1. \(A = A_i \cap A_j\) for each \(1 \leq i < j \leq n\)
2. \(X = \bigcup_{i=1}^{n} A_i\)

\(^1\)The symmetric commutator subgroup was named by Roman Mikhailov during personal communications.
An $n$-partition $(A_1, \ldots, A_n)$ of $X$ relative to $A$ is called cofibrant if the inclusion
\[ \bigcup_{i \in I} A_i \hookrightarrow \bigcup_{j \in J} A_j \]
is a cofibration for any $I \subseteq J \subseteq \{1, 2, \ldots, n\}$. Note that for a cofibrant partition, each union $A_I = \bigcup_{i \in I} A_i$ is the homotopy colimit of the diagram given by the inclusions $A_I \hookrightarrow A_J$ for $\emptyset \subseteq I' \subseteq I$.

**Theorem 1.3.** Let $(X, A)$ be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant $n$-partition of $X$ relative to $A$ with $n \geq 2$. Suppose that

(i) For any proper subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, the union $\bigcup_{i \in I} A_i$ is a path-connected $K(\pi, 1)$-space.

(ii) The inclusion $A \rightarrow A_i$ induces an epimorphism of the corresponding fundamental groups for each $1 \leq i \leq n$.

Let $R_i$ be the kernel of $\pi_1(A) \rightarrow \pi_1(A_i)$ for $1 \leq i \leq n$. Then

(1) For any proper subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$,
\[ R_{i_1} \cap \cdots \cap R_{i_k} = ([R_{i_1}, R_{i_2}], \ldots, R_{i_k})_S. \]

(2) For any $1 < k \leq n$ and any subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, there is an isomorphism of groups
\[ \pi_k(X) \cong \left( \bigcap_{s=1}^k \left( R_{i_s} \cdot \prod_{j \in J} R_j \right) \right) / \left( [R_{i_1}, R_{i_2}], \ldots, R_{i_k} \right)_S \cdot \prod_{j \in J} R_j, \]
where $J = \{1, 2, \ldots, n\} - I$. In particular,
\[ \pi_n(X) \cong (R_1 \cap R_2 \cap \cdots \cap R_n) / ([R_1, R_2], \ldots, R_n)_S. \]

For the special case where $n = 2$, Theorem 1.3 is the classical Brown-Loday Theorem 3.4 with a generalization recently given in [9]. But the connectivity hypothesis on the subgroups in [9] seems difficult to check. Theorem 1.3 emphasizes the point that, for any space $X$ that admits a cofibrant $K(\pi, 1)$-partition, the higher homotopy groups of $X$ measure the difference between the intersection subgroups and the symmetric commutator subgroups for certain subgroups in the fundamental groups of the partition spaces. Moreover, Theorem 1.3 admits many applications. A direct consequence is to give an interesting connection between link groups and higher homotopy groups.

**Corollary 1.4.** Let $M$ be a path-connected 3-manifold and $L$ be a proper $m$-link in $M$ with $m \geq 2$. Suppose that for any non-empty sub-link $L'$ of $L$, the link complement $M \setminus |L'|$ is a $K(\pi, 1)$-space. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be any subsets of $\{1, 2, \ldots, m\}$, with $n \geq 2$, such that

(i) $\Lambda_i \neq \emptyset$ for each $1 \leq i \leq n$.

(ii) $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$.

(iii) $\bigcup_{i=1}^n \Lambda_i = \{1, 2, \ldots, m\}$.

Let $\alpha_j$ be the $j$-th meridian of the link $L$ and let
\[ R_i = \langle \langle \alpha_j \mid j \in \Lambda_i \rangle \rangle \]
be the normal closure of $\alpha_j$ with $j \in \Lambda_i$ in $\pi_1(M \setminus |L|)$. Then

1. For any proper subset $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$,
$$R_{i_1} \cap \cdots \cap R_{i_k} = ([R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$ 
2. There is an isomorphism of groups
$$\langle R_1 \cap R_2 \cap \cdots \cap R_n \rangle / ([R_1, R_2], \ldots, R_n]_S \cong \pi_n(M).$$

There are examples of links whose complements are $K(\pi, 1)$-spaces. For instance, let $M = S^3$ and let $\pi : S^3 \to S^2$ be the Hopf fibration. Let $q_1, \ldots, q_m$ be $m$ distinct points in $S^2$. Let $L = \pi^{-1}(\{q_1, q_2, \ldots, q_m\})$. Then $L$ is an $m$-link in $S^3$ with the property that $S^3 \setminus |L|$ is a $K(\pi, 1)$-space for any non-empty sub-link $L'$ of $L$.

Consider the special case where $n = m$ with $\Lambda_i = \{i\}$ in Corollary 1.4. Then $R_i$ is the normal closure of $\alpha_i$ in $\pi_1(M \setminus |L|)$. Any element in the intersection subgroup $R_1 \cap R_2 \cap \cdots \cap R_n$ can be represented by a 1-link $l$. The union $\bigcup \{l\}$ gives an $(n + 1)$-link in $M$ related to Brunnian links. Thus Corollary 1.4 gives a connection between higher homotopy groups and Brunnian links.

For more applications of Theorem 1.3, we consider certain subgroups of surface groups and braid groups whose intersection subgroups modulo symmetric commutator subgroups are given by the homotopy groups. Our results on the braid groups are described as follows.

Let $M$ be a manifold. Recall that the $m$-th ordered configuration space $F(M, m)$ is defined by

$$F(M, m) = \{(z_1, \ldots, z_m) \in M^m \mid z_i \neq z_j \text{ for } i \neq j\}$$

with subspace topology. Recall that the Artin braid group $B_n$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with defining relations given by $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}$ for each $i$. The pure braid group $P_n$ is defined to be the kernel of the canonical quotient homomorphism from $B_n$ to the symmetric group $\Sigma_n$, with a set of generators given by

$$A_{i,j} = \sigma_{j-1}^{-1} \sigma_j^{-2} \cdots \sigma_{i+1}^{-2} \sigma_i^{-1} \sigma_{i+2}^{-1} \cdots \sigma_{j-1}^{-1}$$

for $1 \leq i < j \leq n$. Let

$$A_{0,j} = (A_{j,j+1} A_{j,j+2} \cdots A_{j,n})^{-1} (A_{1,j} \cdots A_{j-1,j})^{-1}$$
$$= (\sigma_j \sigma_{j+1} \cdots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2} \cdots \sigma_j)^{-1} \cdot (\sigma_{j-1} \cdots \sigma_2 \sigma_1 \cdots \sigma_{j-1})^{-1}.$$
(2) There is an isomorphism of groups
\[(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]] \cong \pi_n(F(S^2, m)).\]

For instance, for \(n = 2\) with \(m \geq 3\), \((R_1 \cap R_2)/[R_1, R_2] = \pi_2(F(S^2, m)) = 0\),
where \(\pi_2(F(S^2, m)) = 0\) for \(m \geq 3\) is given in [3, p. 244]. For \(m \geq n \geq 3\), from [7, Theorem 1], we have \(\pi_n(F(S^2, m)) \cong \pi_n(S^2)\), and so
\[(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]] \cong \pi_n(F(S^2, m)) \cong \pi_n(S^2).\]

The braided descriptions of the homotopy groups \(\pi_\ast(S^2)\) have been studied in [1
2 4 9 15 20]. Theorem 1.3 is a new braided description of the homotopy groups.

The article is organized as follows. The proofs of Theorems 1.1 and 1.2 are given in section 2. In section 3, we provide the proof of Theorem 1.3. In section 4, we discuss some applications to the free groups and surface groups. The applications to the braid groups and the proof of Theorem 1.3 are given in section 5.

2. The proofs of Theorems 1.1 and 1.2

2.1. Some lemmas. In this subsection, we give some useful lemmas on commutator subgroups. The following lemma is elementary.

Lemma 2.1. Let \(G\) be a group and let \(A, B, C\) be normal subgroups of \(G\). Then \([AB, C] = [A, C][B, C]\). □

The following classical theorem can be found in [16, Theorem 5.2, p. 290].

Theorem 2.2 (Hall’s Theorem). Let \(G\) be a group and let \(A, B, C\) be normal subgroups of \(G\). Then any one of the subgroups \([A, [B, C]], [[A, B], C]\) and \([[A, C], B]\) is a subgroup of the product of the other two. □

Now let \(R_1, \ldots, R_n\) be subgroups of \(G\). Recall that the symmetric commutator subgroup \([[R_1, R_2], \ldots, R_n]]\) is defined by
\[[[R_1, R_2], \ldots, R_n]] = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}].\]

If each \(R_i\) is a normal subgroup of \(G\), then \([[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]]\) is normal in \(G\) and so is \([[R_1, R_2], \ldots, R_n]]\).

Lemma 2.3. Let \(R_1, \ldots, R_n\) be normal subgroups of \(G\). Let \(g_j \in R_j\) for \(1 \leq j \leq n\). Then
\[\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \in [[R_1, R_2], \ldots, R_n]]\]
for any \(\sigma \in \Sigma_n\) and any bracket arrangement \(\beta^n\) of weight \(n\).

Proof. The proof is given by double induction. The first induction is on \(n\). Clearly the assertion holds for \(n = 1\). Suppose that the assertion holds for \(m\) with \(m < n\). Given an element \(\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)})\) as in the lemma,
\[\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = [\beta^p(g_{\sigma(1)}, \ldots, g_{\sigma(p)}), \beta^{n-p}(g_{\sigma(p+1)}, \ldots, g_{\sigma(n)})]\]
for some bracket arrangements \(\beta^p\) and \(\beta^{n-p}\) with \(1 \leq p \leq n - 1\). The second induction is on \(q = n - p\). If \(q = 1\), we have
\[\beta^{n-1}(g_{\sigma(1)}, \ldots, g_{\sigma(n-1)}) \in [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n-1)}]]\]
by the first induction, and so
\[\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = [\beta^{n-1}(g_{\sigma(1)}, \ldots, g_{\sigma(n-1)}), g_{\sigma(n)}]\]
\[\in [[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n-1)}]], R_{\sigma(n)}]]\]
with

$$
\begin{align*}
\prod_{\tau \in \Sigma_{n-1}} & \left[ \left[ R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}, \ldots, R_{\sigma(\tau(n-1))}, R_{\sigma(n)} \right] \right] \\
\leq & \left[ \left[ R_1, R_2, \ldots, R_n \right] S \right].
\end{align*}
$$

Now suppose that the assertion holds for \( q' = n - p < q \). By the first induction, we have

$$
\beta^p(g_{\sigma(1)}, \ldots, g_{\sigma(p)}) \in \left[ \left[ R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(p)} \right] S \right]
$$

and

$$
\beta^{n-p}(g_{\sigma(p+1)}, \ldots, g_{\sigma(n)}) \in \left[ \left[ R_{\sigma(p+1)}, R_{\sigma(p+2)}, \ldots, R_{\sigma(n)} \right] S \right].
$$

Thus

$$
\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \in \left[ \left[ R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(p)} \right] S, \left[ R_{\sigma(p+1)}, R_{\sigma(p+2)}, \ldots, R_{\sigma(n)} \right] S \right].
$$

By Lemma 2.1, \( \beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \) lies in the product subgroup

$$
T = \prod_{\tau \in \Sigma_p} \prod_{\rho \in \Sigma_{n-p}} \left[ \left[ R_{\sigma(\rho(\tau(1)))}, R_{\sigma(\rho(\tau(2)))}, \ldots, R_{\sigma(\rho(\tau(p)))} \right] \right],
$$

where \( \Sigma_{n-p} \) acts on \( \{p+1, \ldots, n\} \). By applying Hall’s Theorem, we have

$$
\begin{align*}
\prod_{\tau \in \Sigma_p} & \left[ \left[ R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}, \ldots, R_{\sigma(\tau(p))} \right] \right] \\
\leq & \left[ \left[ R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}, \ldots, R_{\sigma(\tau(p))} \right] \right].
\end{align*}
$$

Note that

$$
A = \left[ \left[ R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}, \ldots, R_{\sigma(\tau(p))} \right] \right],
\left[ R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}, \ldots, R_{\sigma(\rho(n-1))} \right], R_{\sigma(\rho(n))}
$$

is generated by the elements of the form

$$
\left[ \left[ g_{\sigma(\tau(1))}, g_{\sigma(\tau(2))}, \ldots, g_{\sigma(\tau(p))} \right] \right],
\left[ g_{\sigma(\rho(p+1))}, g_{\sigma(\rho(p+2))}, \ldots, g_{\sigma(\rho(n-1))} \right], g_{\sigma(\rho(n))},
$$

with \( g_{\sigma} \in R_j \). By the second induction in the case where \( q = 1 \), the above elements lie in \( \left[ R_1, R_2, \ldots, R_n \right] S \), and so

$$
A \leq \left[ R_1, R_2, \ldots, R_n \right] S.
$$

Similarly, by the second induction hypothesis,

$$
\left[ \left[ R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}, \ldots, R_{\sigma(\tau(p))} \right] \right],
\left[ R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}, \ldots, R_{\sigma(\rho(n-1))} \right]
$$

is a subgroup of \( \left[ R_1, R_2, \ldots, R_n \right] S \). It follows that

$$
T \leq \left[ R_1, R_2, \ldots, R_n \right] S.
$$
and so
\[ \beta^n (g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \in [[R_1, R_2], \ldots, R_n]S. \]
Both the first and second inductions are finished, hence the result. \( \square \)

**Lemma 2.4.** Let \( G \) be a group and let \( R_1, \ldots, R_n \) be normal subgroups of \( G \). Let \((i_1, i_2, \ldots, i_p)\) be a sequence of integers with \( 1 \leq i_s \leq n \). Suppose that
\[
\{i_1, i_2, \ldots, i_p\} = \{1, 2, \ldots, n\}. 
\]
Then
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}] \leq [[R_1, R_2], \ldots, R_n]S.
\]

**Proof.** The proof is given by double induction. The first induction is on \( n \). The assertion clearly holds for \( n = 1 \). Suppose that the assertion holds for \( n - 1 \) with \( n > 1 \). From the hypothesis that \( \{i_1, i_2, \ldots, i_p\} = \{1, 2, \ldots, n\} \), we have \( p \geq n \). When \( p = n \), \( \{i_1, \ldots, i_n\} \) is a permutation of \( \{1, \ldots, n\} \), and so
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_n}] \leq [[R_1, R_2], \ldots, R_n]S.
\]
Suppose that
\[
[[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \leq [[R_1, R_2], \ldots, R_n]S
\]
for any sequence \((j_1, \ldots, j_q)\) with \( q < p \) and \( \{j_1, \ldots, j_q\} = \{1, \ldots, n\} \). Let \((i_1, \ldots, i_p)\) be a sequence with \( \{i_1, \ldots, i_p\} = \{1, \ldots, n\} \).
If \( i_p \in \{i_1, \ldots, i_{p-1}\} \), then \( \{i_1, \ldots, i_{p-1}\} = \{1, \ldots, n\} \), and so
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_{p-1}}] \leq [[R_1, R_2], \ldots, R_n]S
\]
by the second induction hypothesis. It follows that
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}] \leq [[R_1, R_2], \ldots, R_n]S.
\]
If \( i_p \notin \{i_1, \ldots, i_{p-1}\} \), we may assume that \( i_p = n \). Then
\[
\{i_1, \ldots, i_{p-1}\} = \{1, \ldots, n-1\},
\]
and so
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_{p-1}}] \leq [[R_1, R_2], \ldots, R_{n-1}]S
\]
by the first induction hypothesis. From Lemma 2.3 we have
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}] \leq [[R_1, R_2], \ldots, R_n]S.
\]
The inductions are finished, hence the result holds. \( \square \)

**Lemma 2.5.** Let \( G \) be a group and let \( R_1, \ldots, R_n \) be normal subgroups of \( G \) with \( n \geq 2 \). Let \((i_1, \ldots, i_p)\) and \((j_1, \ldots, j_q)\) be sequences of integers such that \( \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2, \ldots, n\} \). Then
\[
[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]] \leq [[R_1, R_2], \ldots, R_n]S.
\]

**Proof.** The proof is given by double induction on \( n \) and \( q \) with \( n \geq 2 \) and \( q \geq 1 \).
First we prove the assertion holds for \( n = 2 \). If \( \{i_1, \ldots, i_p\} = \{1, 2\} \) or \( \{j_1, \ldots, j_q\} = \{1, 2\} \), we have
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}] \leq [[R_1, R_2], S \quad \text{or} \quad [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \leq [[R_1, R_2], S
\]
by Lemma 2.3 and so
\[
[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]] \leq [[R_1, R_2], S.
\]
Otherwise, \( i_1 = \cdots = i_p \) and \( j_1 = \cdots = j_q \). Since \( \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2\} \), we may assume that \( i_1 = \cdots = i_p = 1, j_1 = \cdots = j_q = 2 \). Then

\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}] \leq R_1 \text{ and } [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \leq R_2,
\]

and so

\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \leq [[R_1, R_2]_S].
\]

Suppose the assertion holds for \( n - 1 \); that is,

\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \leq [[R_1, R_2, \ldots, R_{n-1}]_S]
\]

when \( \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2, \ldots, n-1\} \). We will use the second induction on \( q \) to prove the assertion holds for \( n \).

When \( q = 1 \), the assertion follows by Lemma 2.4. Suppose that the assertion holds for \( q = 1 \). By Hall’s Theorem, \( [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \text{ is a subgroup of the product}

\[
 [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}], [R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]]].
\]

By the second induction we have

\[
 [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]] \leq [[[R_1, R_2], \ldots, R_n]_S].
\]

If \( \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} = \{1, 2, \ldots, n\} \), by the second induction

\[
 [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]] \leq [[[R_1, R_2], \ldots, R_n]_S],
\]

and hence

\[
 [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}], R_{j_q}] \leq [[[R_1, R_2], \ldots, R_n]_S].
\]

If \( \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} \neq \{1, 2, \ldots, n\} \), we may assume that

\[
 \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} = \{1, 2, \ldots, n-1\}
\]

and \( j_q = n \). By the first induction,

\[
 [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]] \leq [[[R_1, R_2], \ldots, R_{n-1}]_S].
\]

Then

\[
 [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}], R_{j_q}] \leq [[[R_1, R_2], \ldots, R_n]_S].
\]

It follows that \( [[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]] \leq [[[R_1, R_2], \ldots, R_n]_S]. \)

The double induction is finished, hence the result.

\[ \square \]

2.2. **Proof of Theorem** 1.1 Clearly \([R_1, R_2, \ldots, R_n]_S \leq [[R_1, R_2, \ldots, R_n]] \). We prove by induction on \( n \) that

\[
 [[R_1, R_2, \ldots, R_n]] \leq [[R_1, R_2, \ldots, R_n]_S].
\]

The assertion holds for \( n = 1 \).

**Hypothesis 2.1.** Suppose that

\[
[[R_1, R_2, \ldots, R_n]] \leq [[R_1, R_2, \ldots, R_s]_S]
\]

for any normal subgroups \( R_1, \ldots, R_n \) of \( G \) with \( 1 \leq s < n \).
Let $R_1,\ldots, R_n$ be any normal subgroups of $G$. By definition, $[[R_1,R_2,\ldots, R_n]]$ is generated by all commutators
\[\beta^t(g_{i_1},\ldots, g_{i_t})\]
of weight $t$ such that $\{i_1,i_2,\ldots,i_t\} = \{1,2,\ldots,n\}$ with $g_j \in R_j$. To prove that each generator $\beta^t(g_{i_1},\ldots, g_{i_t}) \in [[R_1,R_2,\ldots, R_n]]$, we start the second induction on the weight $t$ of $\beta^t$ with $t \geq 1$. If $t = n$, then $(i_1,\ldots,i_n)$ is a permutation of $(1,\ldots,n)$, and so the assertion holds by Lemma 2.3. Now assume that the following hypothesis holds:

**Hypothesis 2.2.** Let $n \leq k < t$ and let
\[\beta^k(g'_{i_1},\ldots, g'_{i_k})\]
be any bracket arrangement of weight $k$ such that where
1. $1 \leq i_s \leq n$,
2. $\{i_1,\ldots, i_k\} = \{1,\ldots, n\}$,
3. $g'_{i_s} \in R_{j_s}$,
then $\beta^k(g'_{i_1},\ldots, g'_{i_k}) \in [[R_1, R_2],\ldots, R_n]]$.

Let $\beta^t(g_{i_1},\ldots, g_{i_t})$ be any bracket arrangement of weight $t$ with $\{i_1,\ldots, i_t\} = \{1,\ldots, n\}$ and $g_j \in R_j$ for $1 \leq j \leq n$. From the definition of bracket arrangement, we have
\[\beta^t(g_{i_1},\ldots, g_{i_t}) = [\beta^p(g_{i_1},\ldots, g_{i_p}), \beta^{t-p}(g_{i_{p+1}},\ldots, g_{i_t})]\]
for some bracket arrangements $\beta^p$ and $\beta^{t-p}$ of weight $p$ and $t-p$, respectively, with $1 \leq p \leq n-1$. Let
\[A = \{i_1,\ldots, i_p\} \text{ and } B = \{i_{p+1},\ldots, i_t\}.\]
Then both $A$ and $B$ are subsets of $\{1,\ldots, n\}$ with $A \cup B = \{1,\ldots, n\}$.

Suppose that the cardinality $|A| = n$ or $|B| = n$. We may assume that $|A| = n$. By Hypothesis 2.2
\[\beta^p(g_{i_1},\ldots, g_{i_p}) \in [[R_1, R_2],\ldots, R_n]]\]
Since $[[R_1,R_2,\ldots, R_n]]$ is a normal subgroup of $G$, we have
\[\beta^t(g_{i_1},\ldots, g_{i_t}) = [\beta^p(g_{i_1},\ldots, g_{i_p}), \beta^{t-p}(g_{i_{p+1}},\ldots, g_{i_t})] \in [[R_1, R_2],\ldots, R_n]]\]
This proves the result in this case.

Suppose that $|A| < n$ and $|B| < n$. Let $A = \{l_1,\ldots, l_a\}$ with $1 \leq l_1 < l_2 < \ldots < l_a \leq n$ and $1 \leq a < n$, and let $B = \{k_1,\ldots, k_b\}$ with $1 \leq k_1 < k_2 < \ldots < k_b$ and $1 \leq b < n$. Observe that
\[\beta^p(g_{i_1},\ldots, g_{i_p}) \in [[R_{l_1}, R_{l_2},\ldots, R_{l_a}]].\]
By Hypothesis 2.1
\[[R_{l_1}, R_{l_2},\ldots, R_{l_a}] = [[R_1,R_2],\ldots, R_n]]\]
Thus
\[\beta^p(g_{i_1},\ldots, g_{i_p}) \in [[R_1, R_{l_2},\ldots, R_{l_a}]].\]
Similarly
\[\beta^{t-p}(g_{i_{p+1}},\ldots, g_{i_t}) \in [[R_{k_1}, R_{k_2},\ldots, R_{k_b}}]S.\]
It follows that the element $\beta^t(g_{i_1},\ldots, g_{i_t})$ lies in the commutator subgroup
\[[[R_{l_1}, R_{l_2},\ldots, R_{l_a}], [R_{k_1}, R_{k_2},\ldots, R_{k_b}]]S].\]
From Lemma 2.5 we have
\[[[R_{i_{a(1)}}, R_{i_{a(2)}}, \ldots, R_{i_{a(n)}}], [R_{k_{\tau(1)}}, R_{k_{\tau(2)}}, \ldots, R_{k_{\tau(n)}}]] \leq [[R_1, R_2], \ldots, R_n]_S\]
for all \(\sigma \in \Sigma_a\) and \(\tau \in \Sigma_b\) because \(\{l_1, \ldots, l_a\} \cup \{k_1, \ldots, k_b\} = A \cup B = \{1, 2, \ldots, n\}\).

It follows from Lemma 2.1 that
\[[[R_1, R_{l_1}], \ldots, R_{l_n}]_S, [[R_{k_1}, R_{k_2}], \ldots, R_{k_l}]_S] \leq [[R_1, R_2], \ldots, R_n]_S\]
Thus
\[\beta^t(g_1, \ldots, g_i) \in [[R_1, R_2], \ldots, R_n]_S\]
The inductions are finished, hence the proof of Theorem 1.1.

2.3. Proof of Theorem 1.2. Clearly
\[\prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}] \leq [[R_1, R_2], \ldots, R_n]_S\]
We prove by induction on \(n\) that
\[[[R_1, R_2], \ldots, R_n]_S \leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]\]
The assertion holds for \(n = 1\).

Suppose that
\[[[R_1, R_2], \ldots, R_n]_S \leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(s)}]\]
for any normal subgroups \(R_1, \ldots, R_n\) of \(G\) with \(1 \leq s < n\).

Let \(R_1, \ldots, R_n\) be any normal subgroups of \(G\). By definition,
\[[[R_1, R_2], \ldots, R_n]_S = \prod_{\tau \in \Sigma_n} [[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(n)}]]
It suffices to prove that for any \(\tau \in \Sigma_n\),
\[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(n)}] \leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]\]
The assertion holds for \(\tau(1) = 1\). Suppose that the assertion holds for \(\tau(k-1) = 1\) with \(1 \leq k-1 < n\). When \(\tau(k) = 1\), consider the subgroup
\[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_{\tau(k-1)}], R_1], R_{\tau(k+1)}], \ldots, R_{\tau(n)}]]
Following from Hall's Theorem,
\[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_{\tau(k-1)}], R_1], R_{\tau(k+1)}], \ldots, R_{\tau(n)}]]
\leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]]
By the second induction
\[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_1], R_{\tau(k-1)}], R_{\tau(k+1)}], \ldots, R_{\tau(n)}]]
\leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]]
From Lemma 2.5 and the first induction, we have
\[
[[R_{\tau(k-1)}, R_1], [R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}]] \leq \left[\left[ [R_1, R_{\sigma(1)}], \ldots, R_{\sigma(k-1)} \right] \right]_S
\]
and so
\[
\leq \prod_{\sigma \in \Sigma_{k-1}} [[R_1, R_{\sigma(1)}], \ldots, R_{\sigma(k-1)}],
\]
where \( \{1, \tau(1), \ldots, \tau(k-1)\} = \{1, l_1, l_2, \ldots, l_{k-1}\} \) with \( 1 < l_1 < l_2 < \cdots < l_{k-1} \). By Lemma 2.1
\[
\left[ \left[ \left[ R_{\tau(1)}, R_{\tau(2)}, \ldots, R_{\tau(k-2)} \right] , R_{\tau(k-1)} \right] , \ldots, R_{\tau(n)} \right] \leq 
\prod_{\sigma \in \Sigma_{k-1}} \left[ \left[ R_1, R_{\sigma(1)} \right] , \ldots, R_{\sigma(k-1)} \right],
\]
because for each \( \sigma \in \Sigma_{k-1} \), the sequence \((\sigma(l_1), \ldots, \sigma(l_{k-1}), \tau(k+1), \ldots, \tau(n))\) is a permutation of \((\tau(1), \ldots, \tau(k-1), \tau(k+1), \ldots, \tau(n))\) which is a permutation of \((2, \ldots, n)\). It follows that
\[
\left[ \left[ \left[ R_{\tau(1)}, R_{\tau(2)}, \ldots, R_{\tau(k-2)} \right] , R_{\tau(k-1)} \right] , \ldots, R_{\tau(n)} \right] \leq 
\prod_{\sigma \in \Sigma_{n-1}} \left[ R_1, R_{\sigma(2)}, \ldots, R_{\sigma(n)} \right],
\]
and so
\[
[[R_1, R_2], \ldots, R_n]_S \leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}].
\]
This finishes the proof.

3. Proof of Theorem 1.3

3.1. Ellis-Mikhailov Theorem. In this subsection, we review some terminology and the main result in [6]. Let \( G \) be a group. An \( m \)-tuple of normal subgroups \( (R_1, \ldots, R_m) \) of \( G \) is called connected if either

(1) \( m \leq 2 \) or
(2) \( m \geq 3 \) with the property that: for all subsets \( I, J \subseteq \{1, \cdots, m\} \) with \( |I| \geq 2, |J| \geq 1 \),

(3.1)
\[
\left( \bigcap_{i \in I} R_i \right) \cap \bigcup_{j \in J} R_j = \bigcap_{i \in I} \left( R_i \cap \bigcup_{j \in J} R_j \right).
\]

Let \( G \) be a group with normal subgroups \( R_1, \ldots, R_n \). Let \( X(G; R_1, \ldots, R_n) \) be the homotopy colimit of the cubical diagram obtained by classifying spaces \( B(G/ \prod_{i \in I} R_i) \) with the maps
\[
B(G/ \prod_{i \in I} R_i) \to B(G/ \prod_{i' \in I'} R_{i'})
\]
induced by the canonical quotient homomorphism \( G/ \prod_{i \in I} R_i \to G/ \prod_{i' \in I'} R_{i'} \) for \( I \subseteq I' \), where \( I \) ranges over all proper subsets \( I \subseteq \{1, \ldots, n\} \).
In the following theorem and the rest of the article, the notation $\cdots \ast \cdots$ means that the letter $a$ is removed.

**Theorem 3.1** ([6] Theorem 1). Let $G$ be a group with normal subgroups $R_1, \ldots, R_n$ with $n \geq 2$. Let $X = X(G; R_1, \ldots, R_n)$. Suppose that the $(n-1)$-tuple

$$(R_1, \ldots, \hat{R}_i, \ldots, R_n)$$

is connected for each $1 \leq i \leq n$. Then

$$\pi_n(X) \cong \prod_{I \cup J = \{1, \ldots, n\}, I \cap J = \emptyset} (\cap_{i \in I} R_i, \cap_{j \in J} R_j).$$

\[\square\]

3.2. **Proof of Theorem 1.3** Theorem 1.3 is part of the following statement.

**Theorem 3.2.** Let $(X, A)$ be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant $n$-partition of $X$ relative to $A$ with $n \geq 2$. Suppose that

(i) For any proper subset $I = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, the union $\bigcup_{i \in I} A_i$ is a path-connected $K(\pi, 1)$-space.

(ii) The inclusion $A \to A_i$ induces an epimorphism of the corresponding fundamental groups for each $1 \leq i \leq n$.

Let $R_i$ be the kernel of $\pi_1(A) \to \pi_1(A_i)$ for $1 \leq i \leq n$. Then

1. For any proper subset $I = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,

$$R_{i_1} \cap \cdots \cap R_{i_k} = [R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$  

2. The $(n-1)$-tuple $(R_1, \ldots, \hat{R}_i, \ldots, R_n)$ is connected for each $1 \leq i \leq n$.

3. There is an isomorphism of groups

$$\pi_n(X) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S.$$  

4. For $1 < k < n$, $I = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, n\} \setminus I$, there is an isomorphism

$$\pi_k(X) \cong \left( \bigcap_{s=1}^{k} (R_{i_s} \cdot \prod_{j \in J} R_j) \right) / \left( [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S \cdot \prod_{j \in J} R_j \right).$$

**Proof.** We prove assertions (1)-(3) by induction on $n$. If $n = 2$, assertions (1) and (2) obviously hold and assertion (3) follows from the classical Brown-Loday Theorem 3.

Suppose that assertions (1)-(3) hold for all cofibrant $m$-partitions $(B_1, \ldots, B_m)$ of any space $Y$ relative to $B$ satisfying conditions (i) and (ii) with $m < n$.

Let $(X, A)$ be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant $n$-partition of $X$ relative to $A$ with $n > 2$.

1. Let $I = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ be a proper subset. If $k = 1$, then $R_{i_1} = [R_{i_1}]_S$ by definition. We may assume that $2 \leq k < n$. Let $B_s = A_{i_s}$ for $s = 1, \ldots, k$ and let $Y = B_1 \cup \cdots \cup B_k$. Then $(B_1, \ldots, B_k)$ is a cofibrant $k$-partition of $Y$ relative to $A$ satisfying conditions (i) and (ii). Notice that

$$\text{Ker}(\pi_1(A) \to \pi_1(B_s)) = \text{Ker}(\pi_1(A) \to \pi_1(A_{i_s})) = R_{i_s}.$$  

By the induction hypothesis, there is an isomorphism

$$\pi_k(Y) \cong (R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k})/[[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$
From condition (i), \( Y = A_1 \cup \cdots \cup A_{i_k} \) is a \( K(\pi, 1) \)-space (as \( k < n \)), and so \( \pi_k(Y) = 0 \) (as \( k \geq 2 \)). Thus
\[
R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k} = ([R_{i_1}, R_{i_2}, \ldots, R_{i_k}]_S,
\]
which is assertion (1).

(2) We show that the \((n-1)\)-tuple \((R_1, \ldots, R_{n-1})\) is connected. Let \( m = n - 1 \).
If \( m = 2 \), then \((R_1, R_2)\) is connected by definition. Assume that \( m \geq 3 \). Let
\( I, J \subseteq \{1, \ldots, m\} \) with \(|I| \geq 2\) and \(|J| \geq 2\). We have to show the connectivity condition that
\[
\bigcap_{i \in I} R_i \cdot \prod_{j \in J} R_j = \bigcap_{i \in I} \left( R_i \cdot \prod_{j \in J} R_j \right).
\]
Clearly
\[
\bigcap_{i \in I} R_i \cdot \prod_{j \in J} R_j \leq \bigcap_{i \in I} \left( R_i \cdot \prod_{j \in J} R_j \right).
\]
Now we show the other direction. Let \( I = \{i_1, \ldots, i_k\} \) with \( 2 \leq k < n \). Let
\( B = \bigcup_{j \in J} A_j \) and let \( B_s = B \cup A_s \) for \( s = 1, 2, \ldots, k \). Let
\[
Y = \bigcup_{s=1}^k B_s = \bigcup_{i \in I, j \in J} A_i \cup A_j.
\]
Then \((B_1, \ldots, B_k)\) is a cofibrant \(k\)-partition of \( Y\) relative to \( B\). Clearly condition (i) holds. Let \( G = \pi_1(A)\). Then \( \pi_1(B) = G/\prod_{j \in J} R_j \) and
\[
\pi_1(B_s) = G/(R_{i_s} \cdot \prod_{j \in J} R_j).
\]
Thus condition (ii) holds. Let
\[
N_s = \text{Ker}(\pi_1(B) \rightarrow \pi_1(B_s)) = \text{Ker}(G/(\prod_{j \in J} R_j) \rightarrow G/(R_{i_s} \cdot \prod_{j \in J} R_j)) = (R_{i_s} \cdot \prod_{j \in J} R_j)/\prod_{j \in J} R_j = R_{i_s}/(R_{i_s} \cap \prod_{j \in J} R_j).
\]
From the induction hypothesis, we have
\[
\pi_k(Y) = (N_1 \cap N_2 \cap \cdots \cap N_k)/[[N_1, N_2], \ldots, N_k]_S.
\]
Since \( Y = \bigcup_{i \in I, j \in J} A_i \cup A_j \) with \( I, J \subseteq \{1, 2, \ldots, n-1\} \), the space \( Y \) is a \( K(\pi, 1) \)-space from condition (i). Thus \( \pi_k(Y) = 0 \), and so
\[
N_s = R_{i_s}/(R_{i_s} \cap \prod_{j \in J} R_j) \text{ in } G/\prod_{j \in J} R_j.
\]
Consider the quotient homomorphism
\[
\phi: G \rightarrow G/\prod_{j \in J} R_j.
\]
Then
\[
N_s = R_{i_s}/(R_{i_s} \cap \prod_{j \in J} R_j) = \phi(R_{i_s}),
\]
and so
\[ \phi \left( [R_{i_1}, R_{i_2}, \ldots, R_{i_k}]_S \right) = [\phi(R_{i_1}), \phi(R_{i_2}), \ldots, \phi(R_{i_k})]_S \]
\[ = [N_1, N_2, \ldots, N_k]_S \]
\[ = N_1 \cap N_2 \cap \cdots \cap N_k \text{ by equation (3.2).} \]

From the fact that
\[ \phi \left( \bigcap_{i \in I} \left( R_i \cdot \prod_{j \in J} R_j \right) \right) \leq \bigcap_{s=1}^k N_s, \]

we have
\[ \bigcap_{i \in I} \left( R_i \cdot \prod_{j \in J} R_j \right) \leq \phi^{-1} \left( [\phi([R_{i_1}, R_{i_2}, \ldots, R_{i_k}])_S \right) \]
\[ = [R_{i_1}, R_{i_2}, \ldots, R_{i_k}]_S \cdot \prod_{j \in J} R_j \]
\[ \leq \left( \bigcap_{s=1}^k R_i \right) \cdot \prod_{j \in J} R_j. \]

This proves that \((R_1, \ldots, R_{n-1})\) is connected. Similarly, each \((R_1, \ldots, \hat{R}_i, \ldots, R_n)\) is connected for \(1 \leq i < n\), hence assertion (2).

(3) From assertion (2), each \((R_1, \ldots, \hat{R}_i, \ldots, R_n)\) is connected for \(1 \leq i \leq n\). Since \((A_1, \ldots, A_n)\) is a cofibrant partition of \(X\) relative to \(A\), \(X\) is the homotopy colimit of the diagram given by the inclusions
\[ A \subseteq A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_q} \subseteq A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_{k-1}} \]
with \(\{j_1, \ldots, j_q\} \subseteq \{i_1, \ldots, i_{k-1}\} \subseteq \{1, 2, \ldots, n\}\). From Van Kampen’s theorem
\[ \pi_1(A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_{k-1}}) = \pi_1(A) / \prod_{s=1}^k R_i. \]

Thus \(X = X(\pi_1(A); R_1, R_2, \ldots, R_n)\). From Theorem 3.1 we have
\[ \pi_n(X) \cong \prod_{I \cup J = \{1, \ldots, n\}, I \cap J = \emptyset} [R_i \cap \cdots \cap R_n]. \]

It suffices to show that
\[ \prod_{I \cup J = \{1, \ldots, n\}, I \cap J = \emptyset} \left[ \bigcap_{i \in I} R_i \cap \bigcap_{j \in J} R_j \right] = [R_1, R_2, \ldots, R_n]. \]

Recall that
\[ [R_1, R_2, \ldots, R_n]_S = \prod_{\sigma \in \Sigma_n} [R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(n)}]. \]

For each \(\sigma \in \Sigma_n\), let \(I = \{\sigma(1), \ldots, \sigma(n-1)\}\) and \(J = \{\sigma(n)\}\). Then
\[ [R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(n)}] = \prod_{\sigma \in \Sigma_n} [R_{\sigma(1)}, R_{\sigma(2)}, \ldots, R_{\sigma(n-1)}] \]
\[ \leq \prod_{i \in I} R_i \cap \bigcap_{j \in J} R_j, \]

and so
\[ [R_1, R_2, \ldots, R_n]_S \leq \prod_{I \cup J = \{1, \ldots, n\}, I \cap J = \emptyset} \left[ \bigcap_{i \in I} R_i \cap \bigcap_{j \in J} R_j \right]. \]
Conversely let \( I = \{i_1, \ldots, i_p\} \) and \( J = \{j_1, \ldots, j_q\} \) with \( 1 \leq p, q \leq n - 1 \), \( I \cup J = \{1, \ldots, n\} \) and \( I \cap J = \emptyset \). By assertion (1),
\[
\bigcap_{i \in I} R_i = [[R_{i_1}, R_{i_2}], \ldots, R_{i_p}]_S \quad \text{and} \quad \bigcap_{j \in J} R_j = [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]_S.
\]
Thus
\[
\left[ \bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j \right] = \left[ [[R_{i_1}, R_{i_2}], \ldots, R_{i_p}]_S, [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]_S \right] \leq \left[ [R_1, R_2], \ldots, R_n \right]_S
\]
by Theorem \ref{thm:main}, because
\[
[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}]_S, [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]_S \leq [[R_1, R_2], \ldots, R_n].
\]
This finishes the proof of assertion (3).

Assertion (4) follows from assertion (3) by constructing a new partition as follows. Let \( B = \bigcup_{j \in J} A_j \) and let \( B_s = B \cup A_i \) for \( s = 1, 2, \ldots, k \). Then
\[
X = \bigcup_{s=1}^k B_s = \bigcup_{i \in I, j \in J} A_i \cup A_j
\]
and \( (B_1, \ldots, B_k) \) is a cofibrant \( k \)-partition of \( X \) relative to \( B \). Conditions (i) and (ii) hold, similar to the proof of assertion (2). Let
\[
N_s = \operatorname{Ker}(\pi_1(B) \to \pi_1(B_s)) = R_i / (R_i \cap \prod_{j \in J} R_j).
\]
From assertion (3), we have
\[
\pi_k(X) \cong \left( \bigcap_{s=1}^k N_s \right) / [[N_1, N_2], \ldots, N_k].
\]
To finish the proof, it suffices to show that
\[
\left( \bigcap_{s=1}^k N_s \right) / [[N_1, N_2], \ldots, N_k] \cong \left( \prod_{i=1}^k (R_i \cdot \prod_{j \in J} R_j) / \left( [[R_{i_1}, R_{i_2}], \ldots, R_{i_p}]_S \cdot \prod_{j \in J} R_j \right) \right).
\]
Let \( \phi : G \to G / \prod_{j \in J} R_j \) be the quotient homomorphism. Then it is straightforward to check that
\[
\phi^{-1}(N_s) = R_i \cdot \prod_{j \in J} R_j,
\]
\[
\phi^{-1} \left( \bigcap_{s=1}^k N_s \right) = \bigcap_{s=1}^k (R_i \cdot \prod_{j \in J} R_j),
\]
\[
\phi^{-1}([[[N_1, N_2], \ldots, N_k]]_S) = [[R_{i_1}, R_{i_2}], \ldots, R_{i_p}]_S \cdot \prod_{j \in J} R_j.
\]
Thus equation (3.3) holds, hence assertion (4).

The proof is finished. \( \square \)

4. Applications to the free groups and surface groups

4.1. Subgroups of the surface groups. Let \( X = S \) be a path-connected compact 2-manifold with or without boundary. Let \( Q_i \) be a set of finite points in \( S \setminus \partial S \), \( 1 \leq i \leq n \), such that

1. \( Q_i \neq \emptyset \) for each \( 1 \leq i \leq n \)
2. \( Q_i \cap Q_j = \emptyset \) for \( i \neq j \).

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Let $A = S \setminus (\bigcup_{i=1}^n Q_i)$ be the punctured surface and let $A_i = A \cup Q_i$. Then $(A_1, \ldots, A_n)$ is a cofibrant $n$-partition of $S$ relative to $A$. For any subset
\[ \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}, \]
the space
\[ \bigcup_{s=1}^k A_{i_s} \]
is a $K(\pi, 1)$-space because it is a surface punctured by at least one point. Observe that each homomorphism
\[ \pi_1(A) \longrightarrow \pi_1(A_i) \]
is an epimorphism. Let $R_i$ be the kernel of the epimorphism $\pi_1(A) \rightarrow \pi_1(A_i)$. By Theorem 1.3, we have
\[ \pi_n(S) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S. \]

We give a group theoretic interpretation of this isomorphism.

Case 1. $X = S^2$. Let $Q = \bigcup_{i=1}^n Q_i$ with
\[ Q = \{q_1, q_2, \ldots, q_m\}, \]
with a choice of order that $q_i < q_j$ for $i < j$. For each $q \in Q$, let $c_q$ be a generator in $\pi_1(A)$ represented by a small circle around the point $q$ with a choice of orientation such that $\pi_1(A)$ admits the presentation
\[ \pi_1(A) = \left\langle c_q \right\mid q \in Q \prod_{j=1}^m c_{q_j} = 1 \right\rangle. \]

Then $R_i = \langle \langle c_q \mid q \in Q_i \rangle \rangle$. From equation (4.1), we have the following result.

**Theorem 4.1.** Let $G = \langle x_1, x_2, \ldots, x_m \mid x_1x_2\cdots x_m = 1 \rangle$ be the free group of rank $m - 1$ with $m \geq 2$. Let $n \geq 2$ and let $P_i$ be any subset of $\{x_1, x_2, \ldots, x_m\}$, $1 \leq i \leq n$, such that
\[ (i) \text{ } P_i \neq \emptyset \text{ for each } 1 \leq i \leq n, \]
\[ (ii) \text{ } P_i \cap P_j = \emptyset \text{ for } i \neq j \text{ and } \]
\[ (iii) \text{ } \biguplus_{i=1}^n P_i = \{x_1, x_2, \ldots, x_m\}. \]
Let $R_i = \langle \langle P_i \rangle \rangle$ be the normal closure of $P_i$ in $G$. Then there is an isomorphism of groups
\[ (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(S^2). \]

If $n = m$ with $P_i = \{x_i\}$, then there is an isomorphism of groups
\[ (\langle \langle x_1 \rangle \rangle \cap \langle \langle x_2 \rangle \rangle \cap \cdots \cap \langle \langle x_n \rangle \rangle)/[[\langle \langle x_1 \rangle \rangle, \langle \langle x_2 \rangle \rangle], \ldots, \langle \langle x_n \rangle \rangle]]_S \cong \pi_n(S^2), \]
which is [19 Theorem 1.7]. One interesting point of Theorem 1.3 is that the factor group
\[ (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \]
only depends on the length of the partition $(P_i, \ldots, P_n)$ of $\{x_1, \ldots, x_m\}$, which does not seem obvious from the group theoretic point of view.

---

2One needs to do some modifications such that the cofibrant hypothesis holds: By replacing each punctured point by a small open disk in $S$, the resulting punctured surfaces become compact 2-manifolds, and so the cofibrant hypothesis for the partition $(A_1, \ldots, A_n)$ holds.
Case 2. \( X = \mathbb{R}P^2 \). Let \( Q = \bigcup_{i=1}^n Q_i \) with
\[
Q = \{ q_1, q_2, \ldots, q_m \}.
\]
Then \( \pi_1(A) \) admits a presentation
\[
\pi_1(A) = \left< a_1, c_q \mid q \in Q, \ a_1^2 = \prod_{i=1}^m c_{q_i} \right>
\]
with \( R_i = \langle \langle c_q \mid q \in Q_i \rangle \rangle \). Note that \( \pi_n(\mathbb{R}P^2) \cong \pi_n(S^2) \) for \( n \geq 2 \). From equation (4.1), we have the following result.

**Theorem 4.2.** Let \( G = \langle a_1, x_1, x_2, \ldots, x_m \mid a_1^2 = x_1 x_2 \cdots x_m \rangle \) with \( m \geq 2 \). Let \( n \geq 2 \) and let \( P_i \) be any subset of \( \{ x_1, x_2, \ldots, x_m \} \), \( 1 \leq i \leq n \), such that

(i) \( P_i \neq \emptyset \) for each \( 1 \leq i \leq n \),
(ii) \( P_i \cap P_j = \emptyset \) for \( i \neq j \) and
(iii) \( \bigcup_{i=1}^n P_i = \{ x_1, x_2, \ldots, x_m \} \).

Let \( R_i = \langle \langle P_i \rangle \rangle \) be the normal closure of \( P_i \) in \( G \). Then there is an isomorphism of groups
\[
(R_1 \cap R_2 \cap \cdots \cap R_n)/[R_1, R_2, \ldots, R_n] \cong \pi_n(S^2).
\]

**Remark.** In the above theorem, \( G \cong \pi_1(\mathbb{R}P^2 \setminus Q_m) \) is a free group of rank \( m \) with the presentation given in the same form as in the statement. Thus the subgroups \( R_i \) are different from those given in Theorem 4.1.

Case 3. \( X \neq S^2 \) or \( \mathbb{R}P^2 \). Let \( Q = \bigcup_{i=1}^n Q_i \) with
\[
Q = \{ q_1, q_2, \ldots, q_m \}.
\]
If \( X \) is an oriented surface of genus \( g \) with \( t \) boundary components, then \( \pi_1(A) \) admits a presentation
\[
\pi_1(A) = \left< a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_t, c_q \mid q \in Q, \ a_1 \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^t d_j \prod_{i=1}^m c_{q_i} \right>
\]
with \( R_i = \langle \langle c_q \mid q \in Q_i \rangle \rangle \). If \( X \) is a non-oriented surface of genus \( h \) with \( t \) boundary components, then \( \pi_1(A) \) admits a presentation
\[
\pi_1(A) = \left< a_1, \ldots, a_h, d_1, \ldots, d_t, c_q \mid q \in Q, \ a_1^{\prod_{i=1}^h a_i^2} = \prod_{j=1}^t d_j \prod_{i=1}^m c_{q_i} \right>
\]
with \( R_i = \langle \langle c_q \mid q \in Q_i \rangle \rangle \). Note that \( \pi_n(X) = 0 \) for \( n \geq 2 \). From equation (4.1), we have the following result.

**Theorem 4.3.** Let \( G \) be one of the following surface groups:

(a) \( \langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_t, x_1, \ldots, x_m \mid \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^t y_j \cdot \prod_{i=1}^m x_i \rangle \)
with \( g > 0 \) or \( t > 0 \).

(b) \( \langle a_1, \ldots, a_h, y_1, \ldots, y_t, x_1, \ldots, x_m \mid \prod_{i=1}^h a_i^2 = \prod_{j=1}^t y_j \cdot \prod_{i=1}^m x_i \rangle \) with \( h > 1 \) or \( t > 0 \).
Let $P_i$ be any subset of $\{x_1, x_2, \ldots, x_m\}$, $1 \leq i \leq n$, such that

1. $P_i \neq \emptyset$ for each $1 \leq i \leq n$,
2. $P_i \cap P_j = \emptyset$ for $i \neq j$ and
3. $\bigcup_{i=1}^{n} P_i = \{x_1, x_2, \ldots, x_m\}$.

Let $R_i = \langle \langle P_i \rangle \rangle$ be the normal closure of $P_i$ in $G$. Then

$$R_1 \cap R_2 \cap \cdots \cap R_n = [\cdots[R_1, R_2], \ldots, R_n].$$

\[\square\]

4.2. Homotopy groups of higher dimensional spheres and free products of surface groups. It is a natural question as to whether one can get similar group theoretical descriptions of the (general) homotopy groups of higher-dimensional spheres. We give some remarks that the homotopy groups of certain 2-dimensional complexes contain the homotopy groups of all of higher-dimensional spheres as summands. From this, we can answer the above question in some sense.

Let $X$ and $Y$ be path-connected spaces. According to [10], there is a homotopy decomposition

\[\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega \Sigma(\Omega X \wedge \Omega Y).\]

Now let $X = S^2$ and let $Y$ be any surface. Equation (4.2) implies the following homotopy decomposition:

\[\Omega(S^2 \vee Y) \simeq \Omega S^2 \times \Omega Y \times \Omega(\Sigma \Omega S^2 \wedge \Omega Y).\]

From the classical James Theorem [11], there is homotopy decomposition

\[\Sigma \Omega S^2 = \Sigma \Omega S^1 \simeq \bigvee_{k=1}^{\infty} \Sigma S^k = \bigvee_{k=2}^{\infty} S^k.\]

By substituting this decomposition into formula (4.3), there is a homotopy decomposition

\[\Omega(S^2 \vee Y) \simeq \Omega S^2 \times \Omega Y \times \Omega \left(\bigvee_{k=2}^{\infty} S^k \wedge \Omega Y\right) \simeq \Omega S^2 \times \Omega Y \times \left(\prod_{k=2}^{\infty} \Omega(S^k \wedge \Omega Y)\right) \times \text{other factors}.

If $Y$ is a surface with $Y \neq S^2$, $\mathbb{R}P^2$ and $D^2$, then $\Omega Y$ is homotopy equivalent to a discrete space of a countably infinite set. Then $S^k \wedge Y$ is a wedge of countably infinite copies of $S^k$. Thus given any $k \geq 2$, $\pi_q(S^k)$ is a summand of $\pi_q(S^2 \vee Y)$ for each $q$ with infinite multiplicity. If $Y = S^2$ or $\mathbb{R}P^2$, then we can repeat the above decomposition formula and obtain the fact that, given any $k \geq 2$, $\pi_q(S^k)$ is a summand of $\pi_q(S^2 \vee Y)$ for each $q$ (with infinite multiplicity).

Hence it suffices to consider $\pi_q(S^2 \vee Y)$ for any surface $Y \neq D^2$. By the same arguments as in the previous subsection, we can get partitions of the space $S^2 \vee Y$ by removing points from $S^2$ and $Y$. This gives the following result.
Theorem 4.4. Let \( G_1 = \langle x_{11}, x_{12}, \ldots, x_{1m} \mid x_{11}x_{12} \cdots x_{1m} = 1 \rangle \) and let \( G_2 \) be one of the following surface groups:

(a) \( \langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_t, x_{21}, \ldots, x_{2m} \mid \prod_{i=1}^{g} [a_i, b_i] = \prod_{j=1}^{t} y_j \cdot \prod_{i=1}^{m} x_{2i} \rangle \),

(b) \( \langle a_1, \ldots, a_h, y_1, \ldots, y_t, x_{21}, \ldots, x_{2m} \mid \prod_{i=1}^{h} a_i^2 = \prod_{j=1}^{t} y_j \cdot \prod_{i=1}^{m} x_{2i} \rangle \).

Let \( P_{ji} \) be any subset of \( \{x_{j1}, x_{j2}, \ldots, x_{jm}\} \), \( j = 1, 2 \) and \( 1 \leq i \leq n \), such that

(i) \( P_{ji} \neq \emptyset \) for each \( 1 \leq i \leq n \),

(ii) \( P_{ji} \cap P_{jk} = \emptyset \) for \( i \neq k \) and

(iii) \( \bigcup_{j=1}^{2} P_{ji} = \{x_{j1}, x_{j2}, \ldots, x_{jm}\} \).

Let \( R_i = \langle \langle P_{1i}, P_{2i} \rangle \rangle \) be the normal closure of \( P_{1i} \cup P_{2i} \) in the free product \( G = G_1 * G_2 \). Then there is an isomorphism of groups

\[
\pi_n(S^2 \vee Y) \cong (R_1 \cap R_2 \cap \cdots \cap R_n) /[\langle R_1, R_2, \ldots, R_n \rangle S],
\]

where \( Y \) is a surface such that the fundamental group of \( Y \) punctured by \( m \) points is the group \( G_2 \). \( \square \)

5. Applications to braid groups and the proof of Theorem 4.4

5.1. Applications to the braid groups. Recall that a braid of \( n \) strands is called Brunnian if deleting any one of the strands produces a trivial braid of \((n-1)\)-strands. Let Brun\(_n\) denote the Brunnian subgroup of the \( n \)-th pure Artin braid group \( P_n \). \( \text{(Note. An \( n \)-strand Brunnian braid is always a pure braid for \( n \geq 3 \) according to \[1\] Proposition 4.2.2.)} \) The group Brun\(_n\) has been characterized by Levinson \[13, 14\] for \( n \leq 4 \). A classical question proposed by Makanin \[17\] in 1980 is to determine a set of generators for Brun\(_n\). This question has been answered by Johnson \[12\] and Stanford \[18\]. An answer using fat commutators was given in \[14\] Theorem 8.6.1. As an application, we gave a generalization of Levinson's result \[14\] Theorem 2.

Recall that the braid group \( B_n \) is generated by \( \sigma_1, \ldots, \sigma_{n-1} \) with defining relations given by

\[
\begin{align*}
& (1) \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i - j| \geq 2, \\
& (2) \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.
\end{align*}
\]

Following Levinson’s notation in \[14\], let

\[
t_i = \sigma_{i} \sigma_{i+1} \cdots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2}^{-1} \cdots \sigma_{i}^{-1}
\]

for \( 1 \leq i \leq n - 1 \). Intuitively \( t_i \) is the braid that links strand \( i \) and strand \( n \) in front of all other strands. Let \( R_i = \langle \langle t_i \rangle \rangle \) be the normal closure of \( t_i \) in \( P_n \). \( \text{(Note. Levinson uses the notation } \theta_i \).)

Theorem 5.1. For each \( n \geq 2 \), Brun\(_n\) = \([R_1, R_2, \ldots, R_{n-1}]S\).

When \( n = 4 \), we have Brun\(_4\) = \([R_1, R_2, R_3]S\) = \([R_1, R_2]S\) \cdot \([R_1, R_3]S\) \cdot \([R_1, R_4]S\), which is exactly Levinson’s theorem \[14\] Theorem 2. Let \( M \) be any manifold. Recall that the coordinate projection

\[
F(M, n) \to F(M, k) \quad (z_1, \ldots, z_n) \mapsto (z_i, \ldots, z_{i_k}) \text{ for given } i_1 < \cdots < i_k
\]

is a fibration by Fadell-Neuwirth’s Theorem \[9\]. The coordinate projection

\[
d_j : F(\mathbb{R}^2, n) \to F(\mathbb{R}^2, n - 1), \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_j, \ldots, z_n)
\]
induces a group homomorphism
\[ d_{j*} : \pi_1(F(\mathbb{R}^2, n)) \to \pi_1(F(\mathbb{R}^2, n-1)) \]
which is given by removing the \( j \)-th strand.

**Proof of Theorem 5.1.** The basepoint \((q_1, q_2, \ldots, q_n)\) of \(F(\mathbb{R}^2, n)\) is chosen so that the points \(q_i \in \mathbb{R}^1 \subseteq \mathbb{R}^2\) with the order \(q_1 < q_2 < \cdots < q_n\). Let \(Q_n = \{q_1, \ldots, q_n\}\) and let \(Q_{n,i} = \{q_1, \ldots, q_i, \ldots, q_n\}\). For each \(1 \leq i \leq n-1\), there is a commutative diagram of fibrations

\[
\begin{array}{ccc}
\mathbb{R}^2 \setminus Q_{n-1,i} & \xrightarrow{f} & F(\mathbb{R}^2, n) \\
\downarrow g_i & & \downarrow d_i \\
\mathbb{R}^2 \setminus Q_{n-1} & \xrightarrow{h_i} & F(\mathbb{R}^2, n-1) \\
\end{array}
\]

By taking fundamental groups and using the fact that \(F(\mathbb{R}^2, m)\) is a \(K(\pi, 1)\)-space, there is a commutative diagram of short exact sequences of groups

\[
\begin{array}{ccc}
\pi_1(\mathbb{R}^2 \setminus Q_{n-1}) & \xrightarrow{f*} & \pi_1(F(\mathbb{R}^2, n)) \\
\downarrow g_* & & \downarrow d_* \\
\pi_1(\mathbb{R}^2 \setminus Q_{n-1,i}) & \xrightarrow{h_*} & \pi_1(F(\mathbb{R}^2, n-1)) \\
\end{array}
\]

Let
\[ R'_i = \text{Ker}(g_{i*} : \pi_1(\mathbb{R}^2 \setminus Q_{n-1}) \to \pi_1(\mathbb{R}^2 \setminus Q_{n-1,i})) \]

Since \(h_{i*}\) is a monomorphism, the restriction
\[ f_{i*} = f_*|_{R'_i} : R'_i \to \text{Ker}(d_{n*} : P_n \to P_{n-1}) \cap \text{Ker}(d_{i*} : P_n \to P_{n-1}) \]
is an isomorphism. Observe that \(R'_i\) is the normal closure of the elements in \(\pi_1(\mathbb{R}^2 \setminus Q_{n-1})\) represented by the small circles around the point \(q_i\). By using the terminology of braids, the small circle around the point \(q_i\) is represented by \(t_i^\pm\). Thus
\[ \text{Ker}(d_{n*} : P_n \to P_{n-1}) \cap \text{Ker}(d_{i*} : P_n \to P_{n-1}) \leq R_i \]
on the other hand, since \(d_{n*}(t_i) = d_{i*}(t_i) = 1\), we have
\[ R_i \leq \text{Ker}(d_{n*} : P_n \to P_{n-1}) \cap \text{Ker}(d_{i*} : P_n \to P_{n-1}), \]
and so
\[ (5.1) \quad \text{Ker}(d_{n*} : P_n \to P_{n-1}) \cap \text{Ker}(d_{i*} : P_n \to P_{n-1}) = R_i \]
with the property that
\[ f_{i*} : R'_i \to R_i \]
is an isomorphism. If \(w \in \bigcap_{i=1}^{n-1} R_i\), then \(w \in \text{Im}(f_*)\) and \(f^{-1}_i(w) \in R'_i\) for each \(1 \leq i \leq n-1\). Thus the monomorphism \(f_*\) induces an isomorphism
\[ f_* : \bigcap_{i=1}^{n-1} R'_i \to \bigcap_{i=1}^{n-1} R_i. \]

By applying Theorem 4.3 to the punctured disks, we have
\[ R'_1 \cap R'_2 \cap \cdots \cap R'_{n-1} = \left[ [R'_1, R'_2], \ldots, R'_{n-1} \right]. \]
It follows that
\[ R_1 \cap R_2 \cap \cdots \cap R_{n-1} = [[R_1, R_2, \ldots, R_{n-1}]]. \]

From equality (5.1), we have
\[ \bigcap_{i=1}^{n-1} R_i = \bigcap_{i=1}^{n-1} \text{Ker}(d_{n,i}) \cap \text{Ker}(d_{i,*}) = \bigcap_{j=1}^{n} \text{Ker}(d_{j,*}: P_n \to P_{n-1}) = \text{Brun}_n, \]

hence the result. \( \square \)

5.2. **Proof of Theorem 1.5.** The key point to proving Theorem 1.5 is to construct a \( K(\pi, 1) \)-partition for the configuration space \( F(S^2, n) \). Before giving the proof, we need some lemmas. Let \( q_1, q_2, \ldots \) be strictly increasing real numbers that lie in the open interval \((-1, 1) \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq S^2\). The base point of the configuration spaces \( F(\mathbb{R}^2, m) \subseteq F(S^2, m) \) is chosen to be \( (q_1, \ldots, q_m) \) for each \( m \geq 1 \).

**Lemma 5.2.** Let \( 1 \leq i \leq m \) and let
\[ B_i = \{(z_1, \ldots, z_m) \in F(S^2, m) \mid z_p \neq \infty \text{ for } p \neq i\}. \]

Let \( f: F(\mathbb{R}^2, m) \to B_i \) be the inclusion. Then the homomorphism
\[ f_*: P_m = \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_i) \]
is an epimorphism with \( \text{Ker}(f_*) = \langle (A_{0,i})^{P_m}, \text{the normal closure of } A_{0,i} \rangle \text{ in } P_m \).

**Proof.** The assertion holds trivially for \( m = 1 \). We may assume that \( m \geq 2 \).

Consider the Fadell-Neuwirth fibration
\[ d_i: F(S^2, m) \to F(S^2, m-1) \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_i, \ldots, z_m). \]

Then \( B_i = d_i^{-1}(F(\mathbb{R}^2, m-1)) \) by the construction of \( B_i \). Consider the commutative diagram of the fibrations
\[
\begin{array}{ccc}
\mathbb{R}^2 \setminus Q_{m,i} & \longrightarrow & F(\mathbb{R}^2, m) \\
| f |_| & & | f |
\end{array}
\]
\[
\begin{array}{ccc}
S^2 \setminus Q_{m,i} & \longrightarrow & B_i \\
| f |_| & & | f |
\end{array}
\]
where \( Q_{m,i} = \{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m\} \). Thus there is a commutative diagram of short exact sequence of groups
\[
\begin{array}{ccc}
\pi_1(\mathbb{R}^2 \setminus Q_{m,i}) & \longrightarrow & \pi_1(F(\mathbb{R}^2, m)) \\
| f |_| & & | f |
\end{array}
\]
\[
\begin{array}{ccc}
\pi_1(S^2 \setminus Q_{m,i}) & \longrightarrow & \pi_1(B_i) \\
| f |_| & & | f |
\end{array}
\]
Since \( f_*: \pi_1(\mathbb{R}^2 \setminus Q_{m,i}) \to \pi_1(S^2 \setminus Q_{m,i}) \) is an epimorphism, the middle map
\[ f_*: \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_i) \]
is an epimorphism with \( \text{Ker}(f_*) = \text{Ker}(f_*|_{\pi_1(\mathbb{R}^2 \setminus Q_{m,i})}) \). Note that \( \text{Ker}(f_*|_{\pi_1(\mathbb{R}^2 \setminus Q_{m,i})}) \) is the normal closure of the element \([\omega_i]\) in \( \pi_1(\mathbb{R}^2 \setminus Q_{m,i}) \), where the loop \( \omega_i \) is given in Figure 1. In that figure, the braid \( A_{0,i} \) is obtained from the loop \( \omega_i \) by drawing it as a braid. Thus \( [\omega_i] = A_{0,i} \), and so
\[ \text{Ker}(f_*) = \text{Ker}(f_*|_{\pi_1(\mathbb{R}^2 \setminus Q_{m,i})}) = \langle (A_{0,i})^{P_m}, \text{the normal closure of } A_{0,i} \rangle \subseteq \langle (A_{0,i})^{P_m} \rangle. \]
On the other hand, since $\ker(f^*)$ is normal in $P_m$ with $A_{0,i} \in \ker(f_*)$, we have $\langle\langle A_{0,i} \rangle\rangle_{P_m} \leq \ker(f_*)$. It follows that
\[ \ker(f_*) = \langle\langle A_{0,i} \rangle\rangle_{P_m}, \]
hence the result holds.

\textbf{Lemma 5.3.} Let $I = \{i_1, i_2, \ldots, i_k\}$ be a subset of $\{1, 2, \ldots, m\}$. Let
\[ B_I = \{(z_1, z_2, \ldots, z_m) \in F(S^2, m) \mid z_p \neq \infty \text{ for } p \notin I\} \]
and let $f : F(\mathbb{R}^2, m) \to B_I$ be the inclusion. Then the homomorphism
\[ f_* : \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_I) \]
is an epimorphism whose kernel $\ker(f_*) = \langle\langle A_{0,i} \mid i \in I \rangle\rangle_{P_m}$.

\textbf{Proof.} Note that
\[ B_I = \bigcup_{s=1}^{k} B_{i_s}. \]
By Lemma 5.2, $P_m = \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_{i_s})$ is an epimorphism whose kernel is given by $\langle\langle A_{0,i_s} \rangle\rangle_{P_m}$ for each $1 \leq s \leq k$. From Van Kampen’s Theorem,
\[ f_* : P_m = \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_I) \]
is an epimorphism with
\[ \ker(f_*) = \prod_{s=1}^{k} \langle\langle A_{0,i_s} \rangle\rangle_{P_m} = \langle\langle A_{0,i_s} \mid 1 \leq s \leq k \rangle\rangle_{P_m}, \]
hence the result.

\textbf{Proof of Theorem 1.5.} Let $A = F(\mathbb{R}^2, m)$ be the subspace of $F(S^2, m)$. Let
\[ A_i = \{(z_1, z_2, \ldots, z_m) \in F(S^2, m) \mid z_p \neq \infty \text{ for } p \notin \Lambda_i\}, \]
where the $\Lambda_i$ are as defined in the statement of the theorem. Since $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$, we have
\[ A_i \cap A_j = F(\mathbb{R}^2, m) = A \]
for $i \neq j$. From the assumption that $\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, m\}$, we have

$$\bigcup_{i=1}^{n} A_i = F(S^2, m).$$

Thus $(A_1, A_2, \ldots, A_n)$ is a cofibrant $n$-partition of the space $F(S^2, m)$ relative to $F(\mathbb{R}^2, m)$. For a subset $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, let

$$A_I = \bigcup_{s=1}^{k} A_{i_s}$$

with $A_0 = A$. We now check that $A_I$ is a $K(\pi, 1)$-space for $I \subseteq \{1, 2, \ldots, n\}$. Let

$$J = \{j_1, j_2, \ldots, j_t\} = \{1, 2, \ldots, m\} \setminus \left( \bigcup_{s=1}^{k} A_{i_s} \right)$$

with $j_1 < j_2 < \cdots < j_t$. Since $I \neq \{1, 2, \ldots, n\}$, $J$ is not empty. Let $(z_1, \ldots, z_m) \in F(S^2, m)$. Observe that $(z_1, \ldots, z_m) \in A_I$ if and only if $z_j \neq \infty$ for $1 \leq s \leq t$.

Consider the Fadell-Neuwirth fibration $F(S^2, m) \to F(S^2, t)$.

There is a pull-back diagram

$$\begin{array}{ccc}
A_I & \xrightarrow{f} & F(S^2, m) \\
\downarrow{d_I} & & \downarrow{d_I} \\
F(\mathbb{R}^2, t) & \xrightarrow{\text{pull}} & F(S^2, t)
\end{array}$$

with a fibration

$$F(S^2 \setminus Q_t, m - t) \to A_I \to F(\mathbb{R}^2, t),$$

where $Q_t$ is a set of $t$ distinct points in $S^2$. Since $t \geq 1$,

$$F(S^2 \setminus Q_t, m - t) \cong F(\mathbb{R}^2 \setminus Q_{t-1}, m - t)$$

is a $K(\pi, 1)$-space. Together with the fact that the base space $F(\mathbb{R}^2, t)$ is a $K(\pi, 1)$-space, the total space $A_I$ is a $K(\pi, 1)$-space for any $I \subseteq \{1, 2, \ldots, n\}$.

By Lemma 5.3, the inclusion $f: A = F(\mathbb{R}^2, m) \to A_I$ induces an epimorphism

$$f_*: P_m = \pi_1(A) \to \pi_1(A_I)$$

whose kernel

$$R_\ell = \text{Ker}(f_*) = \langle \langle A_{0,j} \mid j \in \Lambda_i \rangle \rangle P_m.$$}

The assertion follows from Theorem 1.3.

**Acknowledgements**

The authors would like to thank the referee for pointing out some errors in the original manuscript. The authors would also like to thank Fred Cohen for his help in the writing of this article.

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3Similar to footnote 2, one needs to do some modifications such that the cofibrant hypothesis holds: Take a triangulation on $F(S^2, m)$ and deform linearly $A$ and each $A_i$ into certain subcomplexes with necessary subdivisions for having a cofibrant $n$-partition of $F(S^2, m)$.
References


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