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ON SYMMETRIC COMMUTATOR SUBGROUPS, BRAIDS, LINKS AND HOMOTOPY GROUPS

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Abstract. In this paper, we investigate some applications of commutator subgroups to homotopy groups and geometric groups. In particular, we show that the intersection subgroups of some canonical subgroups in certain link groups modulo their symmetric commutator subgroups are isomorphic to the (higher) homotopy groups. This gives a connection between links and homotopy groups. Similar results hold for braid and surface groups.

1. Introduction

The purpose of this article is to investigate some applications of commutator subgroups to homotopy groups and geometric groups.

Recall [16, p. 288-289] that *a bracket arrangement of weight* n in a group G is a map $\beta^n\colon G^n \to G$ which is defined inductively as follows:

$$\beta^1 = \mathrm{id}_G,\ \beta^2(a_1, a_2) = [a_1, a_2]$$

for any $a_1, a_2 \in G$, where $[a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2$. Suppose that the bracket arrangements of weight k are defined for $1 \le k < n$ with $n \ge 3$. A map $\beta^n\colon G^n \to G$ is called a bracket arrangement of weight n if β^n is the composite

$$G^n = G^k \times G^{n-k} \xrightarrow{\beta^k \times \beta^{n-k}} G \times G \xrightarrow{\beta^2} G$$

for some bracket arrangements β^k and β^{n-k} of weight k and $n-k$, respectively, with $1 \le k < n$. For instance, if $n = 3$, there are two bracket arrangements given by $[[a_1, a_2], a_3]$ and $[a_1, [a_2, a_3]]$.

Let R_j be a sequence of subgroups of G for $1 \le j \le n$. The *fat commutator subgroup* $[[R_1, R_2, \ldots, R_n]]$ is defined to be the subgroup of G generated by all of the commutators

$$\beta^t(g_{i_1}, \ldots, g_{i_t}), \tag{1.1}$$

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where

(1) $1 \leq i_s \leq n$;
(2) $\{i_1, \ldots, i_t\} = \{1, \ldots, n\}$; that is, each integer in $\{1, 2, \ldots, n\}$ appears as at least one of the integers i_s;
(3) $g_j \in R_j$;
(4) β^t runs over all of the bracket arrangements of weight t (with $t \geq n$).

For convenience, let $[[R_1]] = R_1$. The fat commutator subgroups have important applications in homotopy theory [1, 3, 6, 19]. Let $G = F_n$ be the free group of rank n with a basis $x_1, \ldots, x_n$. Let $R_i = \langle\langle x_i \rangle\rangle$ be the normal closure of x_i in F_n for $1 \leq i \leq n$ and let $R_{n+1} = \langle\langle x_1 \cdots x_n \rangle\rangle$ be the normal closure of the product element $x_1 x_2 \cdots x_n$ in F_n. According to [19, Theorem 1.7], the homotopy group $\pi_{n+1}(S^2)$ is isomorphic to the quotient group

$$(R_1 \cap R_2 \cap \cdots \cap R_{n+1})/[[R_1, R_2, \ldots, R_{n+1}]]$$

for each n.

To understand the fat commutator subgroup, we consider the *symmetric commutator subgroup*[1] $[[R_1, R_2], \ldots, R_n]_S$ defined by

$$[[R_1, R_2], \ldots, R_n]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],$$

where $[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]$ is the subgroup generated by the left iterated commutators

$$[[[g_1, g_2], g_3], \ldots, g_n]$$

with $g_i \in R_{\sigma(i)}$. For convenience, let $[R_1]_S = R_1$. From the definition, the symmetric commutator subgroup is a subgroup of the fat commutator subgroup. Our first result states that the fat commutator subgroup is in fact the same as the symmetric commutator subgroup.

Theorem 1.1. *Let R_j be any normal subgroup of a group G with $1 \leq j \leq n$. Then*

$$[[R_1, R_2, \ldots, R_n]] = [[R_1, R_2], \ldots, R_n]_S.$$

The symmetric commutator subgroup can be simplified as follows:

Theorem 1.2. *Let R_j be any normal subgroup of a group G with $1 \leq j \leq n$. Then*

$$[[R_1, R_2], \ldots, R_n]_S = \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],$$

where Σ_{n-1} acts on $\{2, 3, \ldots, n\}$.

Our next step is to give a generalization of [19, Theorem 1.7]. This will give more connections between homotopy groups and symmetric commutator subgroups.

Let (X, A) be a pair of spaces. An *n-partition* of X relative to A means a sequence of subspaces $(A_1, \ldots, A_n)$ of X such that

(1) $A = A_i \cap A_j$ for each $1 \leq i < j \leq n$ and
(2) $X = \bigcup_{i=1}^{n} A_i$.

[1]The symmetric commutator subgroup was named by Roman Mikhailov during personal communications.

An n-partition $(A_1, \ldots, A_n)$ of X relative to A is called *cofibrant* if the inclusion

$$\bigcup_{i\in I} A_i \hookrightarrow \bigcup_{j\in J} A_j$$

is a cofibration for any $I \subseteq J \subseteq \{1, 2, \ldots, n\}$. Note that for a cofibrant partition, each union $A_I = \bigcup_{i\in I} A_i$ is the homotopy colimit of the diagram given by the inclusions $A_{I'} \hookrightarrow A_{I''}$ for $\emptyset \subseteq I' \subseteq I'' \subsetneqq I$.

Theorem 1.3. *Let (X, A) be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant n-partition of X relative to A with $n \geq 2$. Suppose that*

(i) *For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$, the union $\bigcup_{i\in I} A_i$ is a path-connected $K(\pi, 1)$-space.*
(ii) *The inclusion $A \to A_i$ induces an epimorphism of the corresponding fundamental groups for each $1 \leq i \leq n$.*

Let R_i be the kernel of $\pi_1(A) \to \pi_1(A_i)$ for $1 \leq i \leq n$. Then

(1) *For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$,*

$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$

(2) *For any $1 < k \leq n$ and any subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, there is an isomorphism of groups*

$$\pi_k(X) \cong \left(\bigcap_{s=1}^{k} \left(R_{i_s} \cdot \prod_{j\in J} R_j\right)\right) \Big/ \left([[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S \cdot \prod_{j\in J} R_j\right),$$

where $J = \{1, 2, \ldots, n\} - I$. In particular,

$$\pi_n(X) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S.$$

For the special case where $n = 2$, Theorem 1.3 is the classical Brown-Loday Theorem [3, Corollary 3.4] with a generalization recently given in [6]. But the connectivity hypothesis on the subgroups in [6] seems difficult to check. Theorem 1.3 emphasizes the point that, for any space X that admits a cofibrant $K(\pi, 1)$-partition, the higher homotopy groups of X measure the difference between the intersection subgroups and the symmetric commutator subgroups for certain subgroups in the fundamental groups of the partition spaces. Moreover, Theorem 1.3 admits many applications. A direct consequence is to give an interesting connection between link groups and higher homotopy groups.

Corollary 1.4. *Let M be a path-connected 3-manifold and L be a proper m-link in M with $m \geq 2$. Suppose that for any non-empty sub-link L' of L, the link complement $M \smallsetminus |L'|$ is a $K(\pi, 1)$-space. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be any subsets of $\{1, 2, \ldots, m\}$, with $n \geq 2$, such that*

(i) *$\Lambda_i \neq \emptyset$ for each $1 \leq i \leq n$.*
(ii) *$\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$.*
(iii) *$\bigcup_{i=1}^{n} \Lambda_i = \{1, 2, \ldots, m\}$.*

Let α_j be the j-th meridian of the link L and let

$$R_i = \langle\langle \alpha_j \mid j \in \Lambda_i \rangle\rangle$$

be the normal closure of α_j *with* $j \in \Lambda_i$ *in* $\pi_1(M \setminus |L|)$. *Then*

(1) *For any proper subset* $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$,

$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$

(2) *There is an isomorphism of groups*

$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(M).$$

□

There are examples of links whose complements are $K(\pi, 1)$-spaces. For instance, let $M = S^3$ and let $\pi\colon S^3 \to S^2$ be the Hopf fibration. Let $q_1, \ldots, q_m$ be m distinct points in S^2. Let $L = \pi^{-1}(\{q_1, q_2, \ldots, q_m\})$. Then L is an m-link in S^3 with the property that $S^3 \setminus |L'|$ is a $K(\pi, 1)$-space for any non-empty sub-link L' of L.

Consider the special case where $n = m$ with $\Lambda_i = \{i\}$ in Corollary 1.4. Then R_i is the normal closure generated by the i-th meridian in $\pi_1(M \setminus |L|)$. Any element in the intersection subgroup $R_1 \cap R_2 \cap \cdots \cap R_n$ can be represented by a 1-link l. The union $L \cup \{l\}$ gives an $(n+1)$-link in M related to Brunnian links. Thus Corollary 1.4 gives a connection between higher homotopy groups and Brunnian links.

For more applications of Theorem 1.3, we consider certain subgroups of surface groups and braid groups whose intersection subgroups modulo symmetric commutator subgroups are given by the homotopy groups. Our results on the braid groups are described as follows.

Let M be a manifold. Recall that the m-th ordered configuration space $F(M, m)$ is defined by

$$F(M, m) = \{(z_1, \ldots, z_m) \in M^m \mid z_i \neq z_j \text{ for } i \neq j\}$$

with subspace topology. Recall that the Artin braid group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with defining relations given by $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for each i. The pure braid group P_n is defined to be the kernel of the canonical quotient homomorphism from B_n to the symmetric group Σ_n, with a set of generators given by

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

for $1 \leq i < j \leq n$. Let

$$\begin{aligned} A_{0,j} &= (A_{j,j+1}A_{j,j+2}\cdots A_{j,n})^{-1}(A_{1,j}\cdots A_{j-1,j})^{-1} \\ &= (\sigma_j\sigma_{j+1}\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_j)^{-1}\cdot(\sigma_{j-1}\cdots\sigma_2\sigma_1^2\sigma_1\cdots\sigma_{j-1})^{-1}. \end{aligned}$$

Then we have the following:

Theorem 1.5. *Let* $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ *be any subsets of* $\{1, 2, \ldots, m\}$, *with* $n \geq 2$, *such that*

(i) $\Lambda_i \neq \emptyset$ *for each* $1 \leq i \leq n$.
(ii) $\Lambda_i \cap \Lambda_j = \emptyset$ *for* $i \neq j$.
(iii) $\bigcup_{i=1}^n \Lambda_i = \{1, 2, \ldots, m\}$.

Let $R_i = \langle\langle A_{0,j} \mid j \in \Lambda_i\rangle\rangle$ *be the normal closure in the pure braid group* P_m. *Then*

(1) *For any proper subset* $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$,

$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$

(2) *There is an isomorphism of groups*

$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(F(S^2, m)).$$

For instance, for $n = 2$ with $m \geq 3$, $(R_1 \cap R_2)/[R_1, R_2] = \pi_2(F(S^2, m)) = 0$, where $\pi_2(F(S^2, m)) = 0$ for $m \geq 3$ is given in [8, p. 244]. For $m \geq n \geq 3$, from [7, Theorem 1], we have $\pi_n(F(S^2, m)) \cong \pi_n(S^2)$, and so

$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(F(S^2, m)) \cong \pi_n(S^2).$$

The braided descriptions of the homotopy groups $\pi_*(S^2)$ have been studied in [1, 2, 4, 5, 15, 20]. Theorem 1.5 is a new braided description of the homotopy groups.

The article is organized as follows. The proofs of Theorems 1.1 and 1.2 are given in section 2. In section 3, we provide the proof of Theorem 1.3. In section 4, we discuss some applications to the free groups and surface groups. The applications to the braid groups and the proof of Theorem 1.5 are given in section 5.

2. The proofs of Theorems 1.1 and 1.2

2.1. **Some lemmas.** In this subsection, we give some useful lemmas on commutator subgroups. The following lemma is elementary.

Lemma 2.1. *Let G be a group and let A, B, C be normal subgroups of G. Then $[AB, C] = [A, C][B, C]$.* □

The following classical theorem can be found in [16, Theorem 5.2, p. 290].

Theorem 2.2 (Hall's Theorem)**.** *Let G be a group and let A, B, C be normal subgroups of G. Then any one of the subgroups $[A, [B, C]]$, $[[A, B], C]$ and $[[A, C], B]$ is a subgroup of the product of the other two.* □

Now let $R_1, \ldots, R_n$ be subgroups of G. Recall that the symmetric commutator subgroup $[[R_1, R_2], \ldots, R_n]_S$ is defined by

$$[[R_1, R_2], \ldots, R_n]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}].$$

If each R_i is a normal subgroup of G, then $[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]$ is normal in G and so is $[[R_1, R_2], \ldots, R_n]_S$.

Lemma 2.3. *Let $R_1, \ldots, R_n$ be normal subgroups of G. Let $g_j \in R_j$ for $1 \leq j \leq n$. Then*

$$\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \in [[R_1, R_2], \ldots, R_n]_S$$

for any $\sigma \in \Sigma_n$ and any bracket arrangement β^n of weight n.

Proof. The proof is given by double induction. The first induction is on n. Clearly the assertion holds for $n = 1$. Suppose that the assertion holds for m with $m < n$. Given an element $\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ as in the lemma,

$$\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = [\beta^p(g_{\sigma(1)}, \ldots, g_{\sigma(p)}), \beta^{n-p}(g_{\sigma(p+1)}, \ldots, g_{\sigma(n)})]$$

for some bracket arrangements β^p and β^{n-p} with $1 \leq p \leq n-1$. The second induction is on $q = n - p$. If $q = 1$, we have

$$\beta^{n-1}(g_{\sigma(1)}, \ldots, g_{\sigma(n-1)}) \in [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n-1)}]_S$$

by the first induction, and so

$$\begin{aligned} \beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) &= [\beta^{n-1}(g_{\sigma(1)}, \ldots, g_{\sigma(n-1)}), g_{\sigma(n)}] \\ &\in [[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n-1)}]_S, R_{\sigma(n)}] \end{aligned}$$

with

$$\begin{aligned}
& [[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n-1)}]_S, R_{\sigma(n)}] \\
= & \left[\prod_{\tau\in\Sigma_{n-1}} [[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(n-1))}], R_{\sigma(n)}\right] \\
\overset{\text{by Lemma 2.1}}{=\!=\!=} & \prod_{\tau\in\Sigma_{n-1}} [[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(n-1))}], R_{\sigma(n)}] \\
\leq & [[R_1, R_2], \ldots, R_n]_S.
\end{aligned}$$

Now suppose that the assertion holds for $q' = n - p < q$. By the first induction, we have

$$\beta^p(g_{\sigma(1)}, \ldots, g_{\sigma(p)}) \in [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(p)}]_S$$

and

$$\beta^{n-p}(g_{\sigma(p+1)}, \ldots, g_{\sigma(n)}) \in [[R_{\sigma(p+1)}, R_{\sigma(p+2)}], \ldots, R_{\sigma(n)}]_S.$$

Thus

$$\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \in \left[[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(p)}]_S, [[R_{\sigma(p+1)}, R_{\sigma(p+2)}], \ldots, R_{\sigma(n)}]_S\right].$$

By Lemma 2.1, $\beta^n(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ lies in the product subgroup

$$\begin{aligned}
T = \prod_{\substack{\tau\in\Sigma_p \\ \rho\in\Sigma_{n-p}}} & \big[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], \\
& [[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n))}]\big],
\end{aligned}$$

where Σ_{n-p} acts on $\{p+1, \ldots, n\}$. By applying Hall's Theorem, we have

$$\begin{aligned}
& \big[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], [[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n))}]\big] \\
= & \big[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], \\
& \quad [[[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n-1))}], R_{\sigma(\rho(n))}]\big] \\
\leq & \big[[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], \\
& \quad [[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n-1))}]], R_{\sigma(\rho(n))}\big] \\
& \cdot \big[[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], R_{\sigma(\rho(n))}], \\
& \quad [[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n-1))}]\big].
\end{aligned}$$

Note that

$$\begin{aligned}
A = \big[[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], \\
[[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n-1))}]], R_{\sigma(\rho(n))}\big]
\end{aligned}$$

is generated by the elements of the form

$$\left[\left[[[g'_{\sigma(\tau(1))}, g'_{\sigma(\tau(2))}], \ldots, g'_{\sigma(\tau(p))}], [[g'_{\sigma(\rho(p+1))}, g'_{\sigma(\rho(p+2))}], \ldots, g'_{\sigma(\rho(n-1))}]\right], g'_{\sigma(\rho(n))}\right],$$

with $g'_j \in R_j$. By the second induction in the case where $q = 1$, the above elements lie in $[[R_1, R_2], \ldots, R_n]_S$, and so

$$A \leq [[R_1, R_2], \ldots, R_n]_S.$$

Similarly, by the second induction hypothesis,

$$\begin{aligned}
\big[[[[R_{\sigma(\tau(1))}, R_{\sigma(\tau(2))}], \ldots, R_{\sigma(\tau(p))}], R_{\sigma(\rho(n))}\big], \\
[[R_{\sigma(\rho(p+1))}, R_{\sigma(\rho(p+2))}], \ldots, R_{\sigma(\rho(n-1))}]\big]
\end{aligned}$$

is a subgroup of $[[R_1, R_2], \ldots, R_n]_S$. It follows that

$$T \leq [[R_1, R_2], \ldots, R_n]_S,$$

and so

$$\beta^n(g_{\sigma(1)}, \dots, g_{\sigma(n)}) \in [[R_1, R_2], \dots, R_n]_S.$$

Both the first and second inductions are finished, hence the result. □

Lemma 2.4. *Let G be a group and let $R_1, \dots, R_n$ be normal subgroups of G. Let $(i_1, i_2, \dots, i_p)$ be a sequence of integers with $1 \le i_s \le n$. Suppose that*

$$\{i_1, i_2, \dots, i_p\} = \{1, 2, \dots, n\}.$$

Then

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_p}] \le [[R_1, R_2], \dots, R_n]_S.$$

Proof. The proof is given by double induction. The first induction is on n. The assertion clearly holds for $n = 1$. Suppose that the assertion holds for $n - 1$ with $n > 1$. From the hypothesis that $\{i_1, i_2, \dots, i_p\} = \{1, 2, \dots, n\}$, we have $p \ge n$. When $p = n$, $(i_1, \dots, i_n)$ is a permutation of $(1, \dots, n)$, and so

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_n}] \le [[R_1, R_2], \dots, R_n]_S.$$

Suppose that

$$[[R_{j_1}, R_{j_2}], \dots, R_{j_q}] \le [[R_1, R_2], \dots, R_n]_S$$

for any sequence $(j_1, \dots, j_q)$ with $q < p$ and $\{j_1, \dots, j_q\} = \{1, \dots, n\}$. Let $(i_1, \dots, i_p)$ be a sequence with $\{i_1, \dots, i_p\} = \{1, \dots, n\}$.

If $i_p \in \{i_1, \dots, i_{p-1}\}$, then $\{i_1, \dots, i_{p-1}\} = \{1, \dots, n\}$, and so

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_{p-1}}] \le [[R_1, R_2], \dots, R_n]_S$$

by the second induction hypothesis. It follows that

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_p}] \le [[R_1, R_2], \dots, R_n]_S.$$

If $i_p \notin \{i_1, \dots, i_{p-1}\}$, we may assume that $i_p = n$. Then

$$\{i_1, \dots, i_{p-1}\} = \{1, \dots, n-1\},$$

and so

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_{p-1}}] \le [[R_1, R_2], \dots, R_{n-1}]_S$$

by the first induction hypothesis. From Lemma 2.3, we have

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_p}] \le [[R_1, R_2], \dots, R_n]_S.$$

The inductions are finished, hence the result holds. □

Lemma 2.5. *Let G be a group and let $R_1, \dots, R_n$ be normal subgroups of G with $n \ge 2$. Let $(i_1, \dots, i_p)$ and $(j_1, \dots, j_q)$ be sequences of integers such that $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\} = \{1, 2, \dots, n\}$. Then*

$$[[[R_{i_1}, R_{i_2}], \dots, R_{i_p}], [[R_{j_1}, R_{j_2}], \dots, R_{j_q}]] \le [[R_1, R_2], \dots, R_n]_S.$$

Proof. The proof is given by double induction on n and q with $n \ge 2$ and $q \ge 1$. First we prove the assertion holds for $n = 2$. If $\{i_1, \dots, i_p\} = \{1, 2\}$ or $\{j_1, \dots, j_q\} = \{1, 2\}$, we have

$$[[R_{i_1}, R_{i_2}], \dots, R_{i_p}] \le [[R_1, R_2]_S \text{ or } [[R_{j_1}, R_{j_2}], \dots, R_{j_q}] \le [[R_1, R_2]_S$$

by Lemma 2.4, and so

$$\big[[[R_{i_1}, R_{i_2}], \dots, R_{i_p}], [[R_{j_1}, R_{j_2}], \dots, R_{j_q}]\big] \le [[R_1, R_2]_S.$$

Otherwise, $i_1 = \cdots = i_p$ and $j_1 = \cdots = j_q$. Since $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2\}$, we may assume that $i_1 = \cdots = i_p = 1, j_1 = \cdots = j_q = 2$. Then

$$[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}] \leq R_1 \text{ and } [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}] \leq R_2,$$

and so

$$\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]\big] \leq [[R_1, R_2]_S.$$

Suppose the assertion holds for $n-1$; that is,

$$\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]\big] \leq [[R_1, R_2], \ldots, R_{n-1}]_S$$

when $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} = \{1, 2, \ldots, n-1\}$. We will use the second induction on q to prove the assertion holds for n.

When $q = 1$, the assertion follows by Lemma 2.4. Suppose that the assertion holds for $q-1$. By Hall's Theorem, $[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]]$ is a subgroup of the product

$$\begin{aligned}&\big[\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]\big], R_{j_q}\big]\\ &\cdot \big[\big[[[R_{i_1}, R_{i_2}] \ldots, R_{i_p}], R_{j_q}\big], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]\big].\end{aligned}$$

By the second induction we have

$$\big[\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], R_{j_q}\big], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]\big] \leq [[R_1, R_2], \ldots, R_n]_S.$$

If $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} = \{1, 2, \ldots, n\}$, by the second induction

$$\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}] \ldots, R_{j_{q-1}}]\big] \leq [[R_1, R_2], \ldots, R_n]_S,$$

and hence

$$\big[\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]\big], R_{j_q}\big] \leq [[R_1, R_2], \ldots, R_n]_S.$$

If $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} \neq \{1, 2, \ldots, n\}$, we may assume that

$$\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_{q-1}\} = \{1, 2, \ldots, n-1\}$$

and $j_q = n$. By the first induction,

$$\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]\big] \leq [[R_1, R_2], \ldots, R_{n-1}]_S.$$

Then

$$\big[\big[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_{q-1}}]\big], R_{j_q}\big] \leq [[R_1, R_2], \ldots, R_n]_S.$$

It follows that $[[[R_{i_1}, R_{i_2}], \ldots, R_{i_p}], [[R_{j_1}, R_{j_2}], \ldots, R_{j_q}]] \leq [[R_1, R_2], \ldots, R_n]_S$. The double induction is finished, hence the result. □

2.2. **Proof of Theorem 1.1.** Clearly $[[R_1, R_2], \ldots, R_n]_S \leq [[R_1, R_2, \ldots, R_n]]$. We prove by induction on n that

$$[[R_1, R_2, \ldots, R_n]] \leq [[R_1, R_2], \ldots, R_n]_S.$$

The assertion holds for $n = 1$.

Hypothesis 2.1. *Suppose that*

$$[[R_1, R_2, \ldots, R_s]] \leq [[R_1, R_2], \ldots, R_s]_S$$

for any normal subgroups $R_1, \ldots, R_s$ of G with $1 \leq s < n$.

Let $R_1, \ldots, R_n$ be any normal subgroups of G. By definition, $[[R_1, R_2, \ldots, R_n]]$ is generated by all commutators

$$\beta^t(g_{i_1}, \ldots, g_{i_t})$$

of weight t such that $\{i_1, i_2, \ldots, i_t\} = \{1, 2, \ldots, n\}$ with $g_j \in R_j$. To prove that each generator $\beta^t(g_{i_1}, \ldots, g_{i_t}) \in [[R_1, R_2], \ldots, R_n]_S$, we start the second induction on the weight t of β^t with $t \geq n$. If $t = n$, then $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$, and so the assertion holds by Lemma 2.3. Now assume that the following hypothesis holds:

Hypothesis 2.2. *Let $n \leq k < t$ and let*

$$\beta^k(g'_{i_1}, \ldots, g'_{i_k})$$

be any bracket arrangement of weight k such that where

(1) $1 \leq i_s \leq n$,
(2) $\{i_1, \ldots, i_k\} = \{1, \ldots, n\}$,
(3) $g'_j \in R_j$,

then $\beta^k(g'_{i_1}, \ldots, g'_{i_k}) \in [[R_1, R_2], \ldots, R_n]_S$.

Let $\beta^t(g_{i_1}, \ldots, g_{i_t})$ be any bracket arrangement of weight t with $\{i_1, \ldots, i_t\} = \{1, \ldots, n\}$ and $g_j \in R_j$ for $1 \leq j \leq n$. From the definition of bracket arrangement, we have

$$\beta^t(g_{i_1}, \ldots, g_{i_t}) = [\beta^p(g_{i_1}, \ldots, g_{i_p}), \beta^{t-p}(g_{i_{p+1}}, \ldots, g_{i_t})]$$

for some bracket arrangements β^p and β^{t-p} of weight p and $t-p$, respectively, with $1 \leq p \leq n-1$. Let

$$A = \{i_1, \ldots, i_p\} \text{ and } B = \{i_{p+1}, \ldots, i_t\}.$$

Then both A and B are subsets of $\{1, \ldots, n\}$ with $A \cup B = \{1, \ldots, n\}$.

Suppose that the cardinality $|A| = n$ or $|B| = n$. We may assume that $|A| = n$. By Hypothesis 2.2,

$$\beta^p(g_{i_1}, \ldots, g_{i_p}) \in [[R_1, R_2], \ldots, R_n]_S.$$

Since $[[R_1, R_2], \ldots, R_n]_S$ is a normal subgroup of G, we have

$$\beta^t(g_{i_1}, \ldots, g_{i_t}) = [\beta^p(g_{i_1}, \ldots, g_{i_p}), \beta^{t-p}(g_{i_{p+1}}, \ldots, g_{i_t})] \in [[R_1, R_2], \ldots, R_n]_S.$$

This proves the result in this case.

Suppose that $|A| < n$ and $|B| < n$. Let $A = \{l_1, \ldots, l_a\}$ with $1 \leq l_1 < l_2 < \cdots < l_a \leq n$ and $1 \leq a < n$, and let $B = \{k_1, \ldots, k_b\}$ with $1 \leq k_1 < k_2 < \cdots < k_b$ and $1 \leq b < n$. Observe that

$$\beta^p(g_{i_1}, \ldots, g_{i_p}) \in [[R_{l_1}, R_{l_2}, \ldots, R_{l_a}]].$$

By Hypothesis 2.1,

$$[[R_{l_1}, R_{l_2}, \ldots, R_{l_a}]] = [[R_{l_1}, R_{l_2}], \ldots, R_{l_a}]_S.$$

Thus

$$\beta^p(g_{i_1}, \ldots, g_{i_p}) \in [[R_{l_1}, R_{l_2}], \ldots, R_{l_a}]_S.$$

Similarly

$$\beta^{t-p}(g_{i_{p+1}}, \ldots, g_{i_t}) \in [[R_{k_1}, R_{k_2}], \ldots, R_{k_b}]_S.$$

It follows that the element $\beta^t(g_{i_1}, \ldots, g_{i_t})$ lies in the commutator subgroup

$$[[[R_{l_1}, R_{l_2}], \ldots, R_{l_a}]_S, [[R_{k_1}, R_{k_2}], \ldots, R_{k_b}]_S].$$

From Lemma 2.5, we have

$$\left[[R_{l_{\sigma(1)}}, R_{l_{\sigma(2)}}], \ldots, R_{l_{\sigma(a)}}], [R_{k_{\tau(1)}}, R_{k_{\tau(2)}}], \ldots, R_{k_{\tau(b)}}]\right] \leq [[R_1, R_2], \ldots, R_n]_S$$

for all $\sigma \in \Sigma_a$ and $\tau \in \Sigma_b$ because $\{l_1, \ldots, l_a\} \cup \{k_1, \ldots, k_b\} = A \cup B = \{1, 2, \ldots, n\}$. It follows from Lemma 2.1 that

$$[[[R_{l_1}, R_{l_2}], \ldots, R_{l_a}]_S, [[R_{k_1}, R_{k_2}], \ldots, R_{k_b}]_S] \leq [[R_1, R_2], \ldots, R_n]_S.$$

Thus

$$\beta^t(g_{i_1}, \ldots, g_{i_t}) \in [[R_1, R_2], \ldots, R_n]_S.$$

The inductions are finished, hence the proof of Theorem 1.1.

2.3. **Proof of Theorem 1.2.** Clearly

$$\prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}] \leq [[R_1, R_2], \ldots, R_n]_S.$$

We prove by induction on n that

$$[[R_1, R_2], \ldots, R_n]_S \leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}].$$

The assertion holds for $n = 1$.

Suppose that

$$[[R_1, R_2], \ldots, R_s]_S \leq \prod_{\sigma \in \Sigma_{s-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(s)}]$$

for any normal subgroups $R_1, \ldots, R_s$ of G with $1 \leq s < n$.

Let $R_1, \ldots, R_n$ be any normal subgroups of G. By definition,

$$[[R_1, R_2], \ldots, R_n]_S = \prod_{\tau \in \Sigma_n} [[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(n)}].$$

It suffices to prove that for any $\tau \in \Sigma_n$,

$$[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(n)}] \leq \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}].$$

The assertion holds for $\tau(1) = 1$. Suppose that the assertion holds for $\tau(k-1) = 1$ with $1 \leq k-1 < n$. When $\tau(k) = 1$, consider the subgroup

$$[[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_{\tau(k-1)}], R_1], R_{\tau(k+1)}], \ldots, R_{\tau(n)}].$$

Following from Hall's Theorem,

$$\begin{aligned} & [[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_{\tau(k-1)}], R_1], R_{\tau(k+1)}], \ldots, R_{\tau(n)}] \\ \leq \; & [[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_1], R_{\tau(k-1)}], R_{\tau(k+1)}], \ldots, R_{\tau(n)}] \\ & \cdot \left[\left[\left[[R_{\tau(k-1)}, R_1], [[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}]\right], R_{\tau(k+1)}\right], \ldots, R_{\tau(n)}\right]. \end{aligned}$$

By the second induction

$$\begin{aligned} & [[[[[[R_{\tau(1)}, R_{\tau(2)}], \ldots, R_{\tau(k-2)}], R_1], R_{\tau(k-1)}], R_{\tau(k+1)}], \ldots, R_{\tau(n)}] \\ \leq \; & \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]. \end{aligned}$$

From Lemma 2.5 and the first induction, we have

$$\begin{aligned}\big[[R_{\tau(k-1)}, R_1], [[R_{\tau(1)}, R_{\tau(2)}], \dots, R_{\tau(k-2)}]\big] &\le \big[[[R_1, R_{l_1}], R_{l_2}], \dots, R_{l_{k-1}}\big]_S \\ &\le \prod_{\sigma\in\Sigma_{k-1}} [[R_1, R_{\sigma(l_1)}], \dots, R_{\sigma(l_{k-1})}],\end{aligned}$$

where $\{1, \tau(1), \dots, \tau(k-1)\} = \{1, l_1, l_2, \dots, l_{k-1}\}$ with $1 < l_1 < l_2 < \cdots < l_{k-1}$. By Lemma 2.1

$$\begin{aligned}&\big[\big[\big[[R_{\tau(k-1)}, R_1], [[R_{\tau(1)}, R_{\tau(2)}] \dots, R_{\tau(k-2)}]\big], R_{\tau(k+1)}\big], \dots, R_{\tau(n)}\big] \\ \le\ & \prod_{\sigma\in\Sigma_{k-1}} [[[[R_1, R_{\sigma(l_1)}], \dots, R_{\sigma(l_{k-1})}], R_{\tau(k+1)}], \dots, R_{\tau(n)}] \\ \le\ & \prod_{\sigma\in\Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \dots, R_{\sigma(n)}],\end{aligned}$$

because for each $\sigma \in \Sigma_{k-1}$, the sequence $(\sigma(l_1), \dots, \sigma(l_{k-1}), \tau(k+1), \dots, \tau(n))$ is a permutation of $(\tau(1), \dots, \tau(k-1), \tau(k+1), \dots, \tau(n))$ which is a permutation of $(2, \dots, n)$. It follows that

$$\begin{aligned}&[[[[[[R_{\tau(1)}, R_{\tau(2)}], \dots, R_{\tau(k-2)}], R_{\tau(k-1)}], R_1], R_{\tau(k+1)}], \dots, R_{\tau(n)}] \\ \le\ & \prod_{\sigma\in\Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \dots, R_{\sigma(n)}],\end{aligned}$$

and so

$$[[R_1, R_2], \dots, R_n]_S \le \prod_{\sigma\in\Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \dots, R_{\sigma(n)}].$$

This finishes the proof.

3. Proof of Theorem 1.3

3.1. **Ellis-Mikhailov Theorem.** In this subsection, we review some terminology and the main result in [6]. Let G be a group. An m-tuple of normal subgroups $(R_1, \dots, R_m)$ of G is called *connected* if either

(1) $m \le 2$ or
(2) $m \ge 3$ with the property that: for all subsets $I, J \subseteq \{1, \cdots, m\}$ with $|I| \ge 2, |J| \ge 1$,

$$\left(\bigcap_{i\in I} R_i\right) \cdot \prod_{j\in J} R_j = \bigcap_{i\in I}\left(R_i \cdot \prod_{j\in J} R_j\right). \tag{3.1}$$

Let G be a group with normal subgroups $R_1, \dots, R_n$. Let $X(G; R_1, \dots, R_n)$ be the homotopy colimit of the cubical diagram obtained by classifying spaces $B(G/\prod_{i\in I} R_i)$ with the maps

$$B(G/\prod_{i\in I} R_i) \to B(G/\prod_{i'\in I'} R_{i'})$$

induced by the canonical quotient homomorphism $G/\prod_{i\in I} R_i \twoheadrightarrow G/\prod_{i'\in I'} R_{i'}$ for $I \subseteq I'$, where I ranges over all proper subsets $I \subsetneq \{1, \dots, n\}$.

In the following theorem and the rest of the article, the notation $(\cdots \hat{a} \cdots)$ means that the letter a is removed.

Theorem 3.1 ([6, Theorem 1])**.** *Let G be a group with normal subgroups $R_1, \ldots, R_n$ with $n \geq 2$. Let $X = X(G; R_1, \ldots, R_n)$. Suppose that the $(n-1)$-tuple*

$$(R_1, \ldots, \hat{R}_i, \ldots, R_n)$$

is connected for each $1 \leq i \leq n$. Then

$$\pi_n(X) \cong \frac{R_1 \cap \cdots \cap R_n}{\prod_{I \cup J = \{1,\ldots,n\}, I \cap J = \emptyset} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]}.$$

□

3.2. **Proof of Theorem 1.3.** Theorem 1.3 is part of the following statement.

Theorem 3.2. *Let (X, A) be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant n-partition of X relative to A with $n \geq 2$. Suppose that*

(i) *For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$, the union $\bigcup_{i \in I} A_i$ is a path-connected $K(\pi, 1)$-space.*
(ii) *The inclusion $A \to A_i$ induces an epimorphism of the corresponding fundamental groups for each $1 \leq i \leq n$.*

Let R_i be the kernel of $\pi_1(A) \to \pi_1(A_i)$ for $1 \leq i \leq n$. Then

(1) *For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$,*
$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$
(2) *The $(n-1)$-tuple $(R_1, \ldots, \hat{R}_i, \ldots, R_n)$ is connected for each $1 \leq i \leq n$.*
(3) *There is an isomorphism of groups*
$$\pi_n(X) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S.$$
(4) *For $1 < k < n$, $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, n\} \smallsetminus I$, there is an isomorphism*
$$\pi_k(X) \cong \left(\bigcap_{s=1}^{k} (R_{i_s} \cdot \prod_{j \in J} R_j) \right) \Big/ \left(([[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S) \cdot \prod_{j \in J} R_j \right).$$

Proof. We prove assertions (1)-(3) by induction on n. If $n = 2$, assertions (1) and (2) obviously hold and assertion (3) follows from the classical Brown-Loday Theorem [3].

Suppose that assertions (1)-(3) hold for all cofibrant m-partitions $(B_1, \ldots, B_m)$ of any space Y relative to B satisfying conditions (i) and (ii) with $m < n$.

Let (X, A) be a pair of spaces and let $(A_1, \ldots, A_n)$ be a cofibrant n-partition of X relative to A with $n > 2$.

(1) Let $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$ be a proper subset. If $k = 1$, then $R_{i_1} = [R_{i_1}]_S$ by definition. We may assume that $2 \leq k < n$. Let $B_s = A_{i_s}$ for $s = 1, \ldots, k$ and let $Y = B_1 \cup \cdots \cup B_k$. Then $(B_1, \ldots, B_k)$ is a cofibrant k-partition of Y relative to A satisfying conditions (i) and (ii). Notice that

$$\mathrm{Ker}(\pi_1(A) \to \pi_1(B_s)) = \mathrm{Ker}(\pi_1(A) \to \pi_1(A_{i_s})) = R_{i_s}.$$

By the induction hypothesis, there is an isomorphism

$$\pi_k(Y) \cong (R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k})/[[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$

From condition (i), $Y = A_{i_1} \cup \cdots \cup A_{i_k}$ is a $K(\pi,1)$-space (as $k < n$), and so $\pi_k(Y) = 0$ (as $k \geq 2$). Thus

$$R_{i_1} \cap R_{i_2} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S,$$

which is assertion (1).

(2) We show that the $(n-1)$-tuple $(R_1, \ldots, R_{n-1})$ is connected. Let $m = n-1$. If $m = 2$, then (R_1, R_2) is connected by definition. Assume that $m \geq 3$. Let $I, J \subseteq \{1, \ldots, m\}$ with $|I| \geq 2$ and $|J| \geq 1$. We have to show the connectivity condition that

$$\left(\bigcap_{i \in I} R_i\right) \cdot \prod_{j \in J} R_j = \bigcap_{i \in I} \left(R_i \cdot \prod_{j \in J} R_j\right).$$

Clearly

$$\left(\bigcap_{i \in I} R_i\right) \cdot \prod_{j \in J} R_j \leq \bigcap_{i \in I} \left(R_i \cdot \prod_{j \in J} R_j\right).$$

Now we show the other direction. Let $I = \{i_1, \ldots, i_k\}$ with $2 \leq k < n$. Let $B = \bigcup_{j \in J} A_j$ and let $B_s = B \cup A_{i_s}$ for $s = 1, 2, \ldots, k$. Let

$$Y = \bigcup_{s=1}^{k} B_s = \bigcup_{i \in I, j \in J} A_i \cup A_j.$$

Then $(B_1, \ldots, B_k)$ is a cofibrant k-partition of Y relative to B. Clearly condition (i) holds. Let $G = \pi_1(A)$. Then $\pi_1(B) = G/\prod_{j \in J} R_j$ and

$$\pi_1(B_s) = G/(R_{i_s} \cdot \prod_{j \in J} R_j).$$

Thus condition (ii) holds. Let

$$\begin{aligned} N_s &= \mathrm{Ker}(\pi_1(B) \to \pi_1(B_s)) \\ &= \mathrm{Ker}(G/(\textstyle\prod_{j \in J} R_j) \to G/(R_{i_s} \cdot \prod_{j \in J} R_j)) \\ &= (R_{i_s} \cdot \textstyle\prod_{j \in J} R_j)/\prod_{j \in J} R_j \\ &= R_{i_s}/(R_{i_s} \cap \textstyle\prod_{j \in J} R_j). \end{aligned}$$

From the induction hypothesis, we have

$$\pi_k(Y) = (N_1 \cap N_2 \cap \cdots \cap N_k)/[[N_1, N_2], \ldots, N_k]_S.$$

Since $Y = \bigcup_{i \in I, j \in J} A_i \cup A_j$ with $I, J \subseteq \{1, 2, \ldots, n-1\}$, the space Y is a $K(\pi,1)$-space from condition (i). Thus $\pi_k(Y) = 0$, and so

$$N_1 \cap N_2 \cap \cdots \cap N_k = [[N_1, N_2], \ldots, N_k]_S \tag{3.2}$$

with $N_s = R_{i_s}/(R_{i_s} \cap \prod_{j \in J} R_j)$ in $G/\prod_{j \in J} R_j$. Consider the quotient homomorphism

$$\phi \colon G \longrightarrow G/\prod_{j \in J} R_j.$$

Then

$$N_s = R_{i_s}/(R_{i_s} \cap \prod_{j \in J} R_j) = \phi(R_{i_s}),$$

and so

$$\begin{aligned}\phi\left([[R_{i_1},R_{i_2}],\ldots,R_{i_k}]_S\right) &= [[\phi(R_{i_1}),\phi(R_{i_2})],\ldots,\phi(R_{i_k})]_S\\ &= [[N_1,N_2],\ldots,N_k]_S\\ &= N_1\cap N_2\cap\cdots\cap N_k \text{ by equation (3.2).}\end{aligned}$$

From the fact that

$$\phi\left(\bigcap_{i\in I}\left(R_i\cdot\prod_{j\in J}R_j\right)\right)\leq\bigcap_{s=1}^{k}N_s,$$

together with equation (3.2), we have

$$\begin{aligned}\textstyle\bigcap_{i\in I}\left(R_i\cdot\prod_{j\in J}R_j\right) &\leq \phi^{-1}\left(\phi\left([[R_{i_1},R_{i_2}],\ldots,R_{i_k}]_S\right)\right)\\ &= [[R_{i_1},R_{i_2}],\ldots,R_{i_k}]_S\cdot\textstyle\prod_{j\in J}R_j\\ &\leq \left(\textstyle\bigcap_{s=1}^{k}R_{i_s}\right)\cdot\textstyle\prod_{j\in J}R_j\\ &= \left(\textstyle\bigcap_{i\in I}R_i\right)\cdot\textstyle\prod_{j\in J}R_j.\end{aligned}$$

This proves that $(R_1,\ldots,R_{n-1})$ is connected. Similarly, each $(R_1,\ldots,\hat{R}_i,\ldots,R_n)$ is connected for $1\leq i<n$, hence assertion (2).

(3) From assertion (2), each $(R_1,\ldots,\hat{R}_i,\ldots,R_n)$ is connected for $1\leq i\leq n$. Since $(A_1,\ldots,A_n)$ is a cofibrant partition of X relative to A, X is the homotopy colimit of the diagram given by the inclusions

$$A\subseteq A_{j_1}\cup A_{j_2}\cup\cdots\cup A_{j_q}\subseteq A_{i_1}\cup A_{i_2}\cup\cdots\cup A_{i_p}$$

with $\{j_1,\ldots,j_q\}\subseteq\{i_1,\ldots,i_p\}\subsetneq\{1,2,\ldots,n\}$. From Van Kampen's theorem

$$\pi_1(A_{i_1}\cup\cdots\cup A_{i_k})=\pi_1(A)/\prod_{s=1}^{k}R_{i_s}.$$

Thus $X=X(\pi_1(A);R_1,R_2,\ldots,R_n)$. From Theorem 3.1, we have

$$\pi_n(X)\cong\frac{R_1\cap\cdots\cap R_n}{\prod_{I\cup J=\{1,\ldots,n\},I\cap J=\emptyset}[\bigcap_{i\in I}R_i,\bigcap_{j\in J}R_j]}.$$

It suffices to show that

$$\prod_{I\cup J=\{1,\ldots,n\},I\cap J=\emptyset}\left[\bigcap_{i\in I}R_i,\bigcap_{j\in J}R_j\right]=[[R_1,R_2],\ldots,R_n]_S.$$

Recall that

$$[[R_1,R_2],\ldots,R_n]_S=\prod_{\sigma\in\Sigma_n}[[R_{\sigma(1)},R_{\sigma(2)}],\ldots,R_{\sigma(n)}].$$

For each $\sigma\in\Sigma_n$, let $I=\{\sigma(1),\ldots,\sigma(n-1)\}$ and $J=\{\sigma(n)\}$. Then

$$\begin{aligned}[[R_{\sigma(1)},R_{\sigma(2)}],\ldots,R_{\sigma(n)}] &= [[[R_{\sigma(1)},R_{\sigma(2)}],\ldots,R_{\sigma(n-1)}],R_{\sigma(n)}]\\ &\leq \left[\textstyle\bigcap_{i\in I}R_i,\bigcap_{j\in J}R_j\right],\end{aligned}$$

and so

$$[[R_1,R_2],\ldots,R_n]_S\leq\prod_{I\cup J=\{1,\ldots,n\},I\cap J=\emptyset}\left[\bigcap_{i\in I}R_i,\bigcap_{j\in J}R_j\right].$$

Conversely let $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$ with $1 \le p, q \le n-1$, $I \cup J = \{1, \dots, n\}$ and $I \cap J = \emptyset$. By assertion (1),

$$\bigcap_{i \in I} R_i = [[R_{i_1}, R_{i_2}], \dots, R_{i_p}]_S \text{ and } \bigcap_{j \in J} R_j = [[R_{j_1}, R_{j_2}], \dots, R_{j_q}]_S.$$

Thus

$$\begin{aligned} \left[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j\right] &= \left[[[R_{i_1}, R_{i_2}], \dots, R_{i_p}]_S, [[R_{j_1}, R_{j_2}], \dots, R_{j_q}]_S\right] \\ &\le [[R_1, R_2], \dots, R_n]_S \end{aligned}$$

by Theorem 1.1 because

$$\left[[[R_{i_1}, R_{i_2}], \dots, R_{i_p}]_S, [[R_{j_1}, R_{j_2}], \dots, R_{j_q}]_S\right] \le [[R_1, R_2, \dots, R_n]].$$

This finishes the proof of assertion (3).

Assertion (4) follows from assertion (3) by constructing a new partition as follows. Let $B = \bigcup_{j \in J} A_j$ and let $B_s = B \cup A_{i_s}$ for $s = 1, 2, \dots, k$. Then

$$X = \bigcup_{s=1}^{k} B_s = \bigcup_{i \in I, j \in J} A_i \cup A_j$$

and $(B_1, \dots, B_k)$ is a cofibrant k-partition of X relative to B. Conditions (i) and (ii) hold, similar to the proof of assertion (2). Let

$$\begin{aligned} N_s &= \operatorname{Ker}(\pi_1(B) \to \pi_1(B_s)) \\ &= R_{i_s}/(R_{i_s} \cap \textstyle\prod_{j \in J} R_j). \end{aligned}$$

From assertion (3), we have

$$\pi_k(X) \cong \left(\bigcap_{s=1}^{k} N_s\right) / [[N_1, N_2], \dots, N_k]_S.$$

To finish the proof, it suffices to show that

$$\begin{aligned} &\left(\textstyle\bigcap_{s=1}^{k} N_s\right) / [[N_1, N_2], \dots, N_k]_S \\ \cong\ &(\textstyle\bigcap_{s=1}^{k} (R_{i_s} \cdot \prod_{j \in J} R_j)) / ([[R_{i_1}, R_{i_2}], \dots, R_{i_k}]_S \cdot \prod_{j \in J} R_j). \end{aligned} \tag{3.3}$$

Let $\phi\colon G \longrightarrow G/\prod_{j \in J} R_j$ be the quotient homomorphism. Then it is straightforward to check that

$$\begin{aligned} \phi^{-1}(N_s) &= R_{i_s} \cdot \textstyle\prod_{j \in J} R_j, \\ \phi^{-1}\left(\textstyle\bigcap_{s=1}^{k} N_s\right) &= \textstyle\bigcap_{s=1}^{k} (R_{i_s} \cdot \prod_{j \in J} R_j), \\ \phi^{-1}([[N_1, N_2], \dots, N_k]_S) &= [[R_{i_1}, R_{i_2}], \dots, R_{i_k}]_S \cdot \textstyle\prod_{j \in J} R_j. \end{aligned}$$

Thus equation (3.3) holds, hence assertion (4).

The proof is finished. □

4. Applications to the free groups and surface groups

4.1. **Subgroups of the surface groups.** Let $X = S$ be a path-connected compact 2-manifold with or without boundary. Let Q_i be a set of finite points in $S \smallsetminus \partial S$, $1 \le i \le n$, such that

(1) $Q_i \ne \emptyset$ for each $1 \le i \le n$ and
(2) $Q_i \cap Q_j = \emptyset$ for $i \ne j$.

Let $A = S \smallsetminus (\bigcup_{i=1}^n Q_i)$ be the punctured surface and let $A_i = A \cup Q_i$. Then $(A_1, \ldots, A_n)$ is a cofibrant n-partition[2] of S relative to A. For any subset

$$\{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\},$$

the space

$$\bigcup_{s=1}^{k} A_{i_s}$$

is a $K(\pi, 1)$-space because it is a surface punctured by at least one point. Observe that each homomorphism

$$\pi_1(A) \longrightarrow \pi_1(A_i)$$

is an epimorphism. Let R_i be the kernel of the epimorphism $\pi_1(A) \to \pi_1(A_i)$. By Theorem 1.3, we have

$$\pi_n(S) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S. \tag{4.1}$$

We give a group theoretic interpretation of this isomorphism.

Case 1. $X = S^2$. Let $Q = \bigcup_{i=1}^n Q_i$ with

$$Q = \{q_1, q_2, \ldots, q_m\},$$

with a choice of order that $q_i < q_j$ for $i < j$. For each $q \in Q$, let c_q be a generator in $\pi_1(A)$ represented by a small circle around the point q with a choice of orientation such that $\pi_1(A)$ admits the presentation

$$\pi_1(A) = \left\langle c_q \;\middle|\; q \in Q \;\; \prod_{j=1}^{m} c_{q_j} = 1 \right\rangle.$$

Then $R_i = \langle\langle c_q \mid q \in Q_i \rangle\rangle$. From equation (4.1), we have the following result.

Theorem 4.1. *Let $G = \langle x_1, x_2, \ldots, x_m \mid x_1 x_2 \cdots x_m = 1 \rangle$ be the free group of rank $m - 1$ with $m \geq 2$. Let $n \geq 2$ and let P_i be any subset of $\{x_1, x_2, \ldots, x_m\}$, $1 \leq i \leq n$, such that*

(i) *$P_i \neq \emptyset$ for each $1 \leq i \leq n$,*
(ii) *$P_i \cap P_j = \emptyset$ for $i \neq j$ and*
(iii) *$\bigcup_{i=1}^n P_i = \{x_1, x_2, \ldots, x_m\}$.*

Let $R_i = \langle\langle P_i \rangle\rangle$ be the normal closure of P_i in G. Then there is an isomorphism of groups

$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(S^2).$$

□

If $n = m$ with $P_i = \{x_i\}$, then there is an isomorphism of groups

$$(\langle\langle x_1 \rangle\rangle \cap \langle\langle x_2 \rangle\rangle \cap \cdots \cap \langle\langle x_n \rangle\rangle)/[[\langle\langle x_1 \rangle\rangle, \langle\langle x_2 \rangle\rangle], \ldots, \langle\langle x_n \rangle\rangle]_S \cong \pi_n(S^2),$$

which is [19, Theorem 1.7]. One interesting point of Theorem 4.1 is that the factor group

$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S$$

only depends on the length of the partition $(P_1, \ldots, P_n)$ of $\{x_1, \ldots, x_m\}$, which does not seem obvious from the group theoretic point of view.

[2]One needs to do some modifications such that the cofibrant hypothesis holds: By replacing each punctured point by a small open disk in S, the resulting punctured surfaces become compact 2-manifolds, and so the cofibrant hypothesis for the partition $(A_1, \ldots, A_n)$ holds.

Case 2. $X = \mathbb{R}\mathrm{P}^2$. Let $Q = \bigcup_{i=1}^n Q_i$ with

$$Q = \{q_1, q_2, \ldots, q_m\}.$$

Then $\pi_1(A)$ admits a presentation

$$\pi_1(A) = \left\langle a_1, c_q \;\middle|\; q \in Q,\ a_1^2 = \prod_{i=1}^m c_{q_i} \right\rangle$$

with $R_i = \langle\langle c_q \mid q \in Q_i \rangle\rangle$. Note that $\pi_n(\mathbb{R}\mathrm{P}^2) \cong \pi_n(S^2)$ for $n \geq 2$. From equation (4.1), we have the following result.

Theorem 4.2. *Let $G = \langle a_1, x_1, x_2, \ldots, x_m \mid a_1^2 = x_1 x_2 \cdots x_m \rangle$ with $m \geq 2$. Let $n \geq 2$ and let P_i be any subset of $\{x_1, x_2, \ldots, x_m\}$, $1 \leq i \leq n$, such that*

(i) *$P_i \neq \emptyset$ for each $1 \leq i \leq n$,*
(ii) *$P_i \cap P_j = \emptyset$ for $i \neq j$ and*
(iii) *$\bigcup_{i=1}^n P_i = \{x_1, x_2, \ldots, x_m\}$.*

Let $R_i = \langle\langle P_i \rangle\rangle$ be the normal closure of P_i in G. Then there is an isomorphism of groups

$$(R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \cong \pi_n(S^2).$$

□

Remark. In the above theorem, $G \cong \pi_1(\mathbb{R}\mathrm{P}^2 \smallsetminus Q_m)$ is a free group of rank m with the presentation given in the same form as in the statement. Thus the subgroups R_i are different from those given in Theorem 4.1.

Case 3. $X \neq S^2$ or $\mathbb{R}\mathrm{P}^2$. Let $Q = \bigcup_{i=1}^n Q_i$ with

$$Q = \{q_1, q_2, \ldots, q_m\}.$$

If X is an oriented surface of genus g with t boundary components, then $\pi_1(A)$ admits a presentation

$$\pi_1(A) = \left\langle a_1, b_1, \ldots, a_g, b_g, d_1, \ldots, d_t, c_q \;\middle|\; q \in Q\ \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^t d_j \cdot \prod_{i=1}^m c_{q_i} \right\rangle$$

with $R_i = \langle\langle c_q \mid q \in Q_i \rangle\rangle$. If X is a non-oriented surface of genus h with t boundary components, then $\pi_1(A)$ admits a presentation

$$\pi_1(A) = \left\langle a_1, \ldots, a_h, d_1, \ldots, d_t, c_q \;\middle|\; q \in Q\ \prod_{i=1}^h a_i^2 = \prod_{j=1}^t d_j \cdot \prod_{i=1}^m c_{q_i} \right\rangle$$

with $R_i = \langle\langle c_q \mid q \in Q_i \rangle\rangle$. Note that $\pi_n(X) = 0$ for $n \geq 2$. From equation (4.1), we have the following result.

Theorem 4.3. *Let G be one of the following surface groups:*

(a) *$\langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_t, x_1, \ldots, x_m \mid \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^t y_j \cdot \prod_{i=1}^m x_i \rangle$ with $g > 0$ or $t > 0$.*
(b) *$\langle a_1, \ldots, a_h, y_1, \ldots, y_t, x_1, \ldots, x_m \mid \prod_{i=1}^h a_i^2 = \prod_{j=1}^t y_j \cdot \prod_{i=1}^m x_i \rangle$ with $h > 1$ or $t > 0$.*

Let P_i be any subset of $\{x_1, x_2, \ldots, x_m\}$, $1 \le i \le n$, such that

(i) $P_i \neq \emptyset$ *for each* $1 \le i \le n$,
(ii) $P_i \cap P_j = \emptyset$ *for* $i \neq j$ *and*
(iii) $\bigcup_{i=1}^n P_i = \{x_1, x_2, \ldots, x_m\}$.

Let $R_i = \langle\langle P_i \rangle\rangle$ be the normal closure of P_i in G. Then

$$R_1 \cap R_2 \cap \cdots \cap R_n = [[R_1, R_2], \ldots, R_n]_S.$$

□

4.2. **Homotopy groups of higher dimensional spheres and free products of surface groups.** It is a natural question as to whether one can get similar group theoretical descriptions of the (general) homotopy groups of higher-dimensional spheres. We give some remarks that the homotopy groups of certain 2-dimensional complexes contain the homotopy groups of all of higher-dimensional spheres as summands. From this, we can answer the above question in some sense.

Let X and Y be path-connected spaces. According to [10], there is a homotopy decomposition

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega\Sigma(\Omega X \wedge \Omega Y). \tag{4.2}$$

Now let $X = S^2$ and let Y be any surface. Equation (4.2) implies the following homotopy decomposition:

$$\Omega(S^2 \vee Y) \simeq \Omega S^2 \times \Omega Y \times \Omega(\Sigma\Omega S^2 \wedge \Omega Y). \tag{4.3}$$

From the classical James Theorem [11], there is homotopy decomposition

$$\Sigma\Omega\Sigma Z \simeq \bigvee_{k=1}^{\infty} \Sigma Z^{\wedge k},$$

where $Z^{\wedge k}$ is the k-fold self smash product of Z. Note that the k-fold self smash product of S^1 is S^k. There is a homotopy decomposition

$$\Sigma\Omega S^2 = \Sigma\Omega\Sigma S^1 \simeq \bigvee_{k=1}^{\infty} \Sigma S^k = \bigvee_{k=2}^{\infty} S^k.$$

By substituting this decomposition into formula (4.3), there is a homotopy decomposition

$$\begin{aligned} \Omega(S^2 \vee Y) &\simeq \Omega S^2 \times \Omega Y \times \Omega\left(\textstyle\bigvee_{k=2}^{\infty} S^k \wedge \Omega Y\right) \\ &\simeq \Omega S^2 \times \Omega Y \times \left(\textstyle\prod_{k=2}^{\infty} \Omega(S^k \wedge \Omega Y)\right) \times \text{ other factors.} \end{aligned}$$

If Y is a surface with $Y \neq S^2$, $\mathbb{R}\mathrm{P}^2$ and D^2, then ΩY is homotopy equivalent to a discrete space of a countably infinite set. Then $S^k \wedge Y$ is a wedge of countably infinite copies of S^k. Thus given any $k \ge 2$, $\pi_q(S^k)$ is a summand of $\pi_q(S^2 \vee Y)$ for each q with infinite multiplicity. If $Y = S^2$ or $\mathbb{R}\mathrm{P}^2$, then we can repeat the above decomposition formula and obtain the fact that, given any $k \ge 2$, $\pi_q(S^k)$ is a summand of $\pi_q(S^2 \vee Y)$ for each q (with infinite multiplicity).

Hence it suffices to consider $\pi_*(S^2 \vee Y)$ for any surface $Y \neq D^2$. By the same arguments as in the previous subsection, we can get partitions of the space $S^2 \vee Y$ by removing points from S^2 and Y. This gives the following result.

Theorem 4.4. *Let $G_1 = \langle x_{11}, x_{12}, \ldots, x_{1m} \mid x_{11}x_{12}\cdots x_{1m} = 1\rangle$ and let G_2 be one of the following surface groups:*

(a) $\langle a_1, b_1, \ldots, a_g, b_g, y_1, \ldots, y_t, x_{21}, \ldots, x_{2m} \mid \prod_{i=1}^{g}[a_i, b_i] = \prod_{j=1}^{t} y_j \cdot \prod_{i=1}^{m} x_{2i}\rangle$,

(b) $\langle a_1, \ldots, a_h, y_1, \ldots, y_t, x_{21}, \ldots, x_{2m} \mid \prod_{i=1}^{h} a_i^2 = \prod_{j=1}^{t} y_j \cdot \prod_{i=1}^{m} x_{2i}\rangle$.

Let P_{ji} be any subset of $\{x_{j1}, x_{j2}, \ldots, x_{jm}\}$, $j = 1, 2$ and $1 \le i \le n$, such that

(i) *$P_{ji} \neq \emptyset$ for each $1 \le i \le n$,*
(ii) *$P_{ji} \cap P_{jk} = \emptyset$ for $i \neq k$ and*
(iii) *$\bigcup_{i=1}^{n} P_{ji} = \{x_{j1}, x_{j2}, \ldots, x_{jm}\}$.*

Let $R_i = \langle\langle P_{1i}, P_{2i}\rangle\rangle$ be the normal closure of $P_{1i} \cup P_{2i}$ in the free product $G = G_1 * G_2$. Then there is an isomorphism of groups

$$\pi_n(S^2 \vee Y) \cong (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S,$$

where Y is a surface such that the fundamental group of Y punctured by m points is the group G_2. □

5. Applications to braid groups and the proof of Theorem 1.5

5.1. **Applications to the braid groups.** Recall that a braid of n strands is called *Brunnian* if deleting any one of the strands produces a trivial braid of $(n-1)$-strands. Let Brun_n denote the Brunnian subgroup of the n-th pure Artin braid group P_n. (**Note.** An n-strand Brunnian braid is always a pure braid for $n \ge 3$ according to [1, Proposition 4.2.2].) The group Brun_n has been characterized by Levinson [13, 14] for $n \le 4$. A classical question proposed by Makanin [17] in 1980 is to determine a set of generators for Brun_n. This question has been answered by Johnson [12] and Stanford [18]. An answer using fat commutators was given in [1, Theorem 8.6.1]. As an application, we gave a generalization of Levinson's result [14, Theorem 2].

Recall that the braid group B_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with defining relations given by

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \ge 2$,
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

Following Levinson's notation in [14], let

$$t_i = \sigma_i\sigma_{i+1}\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}^{-1}\cdots\sigma_i^{-1}$$

for $1 \le i \le n-1$. Intuitively t_i is the braid that links strand i and strand n in front of all other strands. Let $R_i = \langle\langle t_i\rangle\rangle$ be the normal closure of t_i in P_n. (**Note.** Levinson uses the notation θ_i.)

Theorem 5.1. *For each $n \ge 2$, $\mathrm{Brun}_n = [[R_1, R_2], \ldots, R_{n-1}]_S$.*

When $n = 4$, we have $\mathrm{Brun}_4 = [[R_1, R_2], R_3]_S = [[R_1, R_2], R_3] \cdot [[R_1, R_3], R_2]$, which is exactly Levinson's theorem [14, Theorem 2]. Let M be any manifold. Recall that the coordinate projection

$$F(M, n) \longrightarrow F(M, k) \quad (z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_k}) \text{ for given } i_1 < \cdots < i_k$$

is a fibration by Fadell-Neuwirth's Theorem [9]. The coordinate projection

$$d_j \colon F(\mathbb{R}^2, n) \longrightarrow F(\mathbb{R}^2, n-1), \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_j, \ldots, z_n)$$

induces a group homomorphism

$$d_{j*}\colon P_n = \pi_1(F(\mathbb{R}^2, n)) \longrightarrow P_{n-1} = \pi_1(F(\mathbb{R}^2, n-1))$$

which is given by removing the j-th strand.

Proof of Theorem 5.1. The basepoint $(q_1, q_2, \ldots, q_n)$ of $F(\mathbb{R}^2, n)$ is chosen so that the points $q_i \in \mathbb{R}^1 \subseteq \mathbb{R}^2$ with the order $q_1 < q_2 < \cdots < q_n$. Let $Q_n = \{q_1, \ldots, q_n\}$ and let $Q_{n,i} = \{q_1, \ldots, \hat{q}_i, \ldots, q_n\}$. For each $1 \le i \le n-1$, there is a commutative diagram of fibrations

$$\begin{tikzcd}
\mathbb{R}^2 \backslash Q_{n-1} \arrow[r, hook, "f"] \arrow[d, hook, "g_i"] & F(\mathbb{R}^2, n) \arrow[r, two heads, "d_n"] \arrow[d, "d_i"] & F(\mathbb{R}^2, n-1) \arrow[d, "d_i"] \\
\mathbb{R}^2 \backslash Q_{n-1,i} \arrow[r, hook, "h_i"] & F(\mathbb{R}^2, n-1) \arrow[r, two heads, "d_{n-1}"] & F(\mathbb{R}^2, n-2).
\end{tikzcd}$$

By taking fundamental groups and using the fact that $F(\mathbb{R}^2, m)$ is a $K(\pi, 1)$-space, there is a commutative diagram of short exact sequences of groups

$$\begin{tikzcd}
\pi_1(\mathbb{R}^2 \backslash Q_{n-1}) \arrow[r, hook, "f_*"] \arrow[d, "g_{i*}"] & P_n = \pi_1(F(\mathbb{R}^2, n)) \arrow[r, two heads, "d_{n*}"] \arrow[d, "d_{i*}"] & P_{n-1} = \pi_1(F(\mathbb{R}^2, n-1)) \arrow[d, "d_{i*}"] \\
\pi_1(\mathbb{R}^2 \backslash Q_{n-1,i}) \arrow[r, hook, "h_{i*}"] & P_{n-1} = \pi_1(F(\mathbb{R}^2, n-1)) \arrow[r, two heads, "(d_{n-1})_*"] & P_{n-2} = \pi_1(F(\mathbb{R}^2, n-2)).
\end{tikzcd}$$

Let

$$R'_i = \operatorname{Ker}(g_{i*}\colon \pi_1(\mathbb{R}^2 \smallsetminus Q_{n-1}) \to \pi_1(\mathbb{R}^2 \smallsetminus Q_{n-1,i})).$$

Since h_{i*} is a monomorphism, the restriction

$$f_{i*} = f_*|\colon R'_i \longrightarrow \operatorname{Ker}(d_{n*}\colon P_n \to P_{n-1}) \cap \operatorname{Ker}(d_{i*}\colon P_n \to P_{n-1})$$

is an isomorphism. Observe that R'_i is the normal closure of the elements in $\pi_1(\mathbb{R}^2 \smallsetminus Q_{n-1})$ represented by the small circles around the point q_i. By using the terminology of braids, the small circle around the point q_i is represented by $t_i^{\pm}$. Thus

$$\operatorname{Ker}(d_{n*}\colon P_n \to P_{n-1}) \cap \operatorname{Ker}(d_{i*}\colon P_n \to P_{n-1}) \le R_i.$$

On the other hand, since $d_{n*}(t_i) = d_{i*}(t_i) = 1$, we have

$$R_i \le \operatorname{Ker}(d_{n*}\colon P_n \to P_{n-1}) \cap \operatorname{Ker}(d_{i*}\colon P_n \to P_{n-1}),$$

and so

$$\operatorname{Ker}(d_{n*}\colon P_n \to P_{n-1}) \cap \operatorname{Ker}(d_{i*}\colon P_n \to P_{n-1}) = R_i \tag{5.1}$$

with the property that

$$f_{i*}\colon R'_i \longrightarrow R_i$$

is an isomorphism. If $w \in \bigcap_{i=1}^{n-1} R_i$, then $w \in \operatorname{Im}(f_*)$ and $f_{i*}^{-1}(w) \in R'_i$ for each $1 \le i \le n-1$. Thus the monomorphism f_* induces an isomorphism

$$f_*|\colon \bigcap_{i=1}^{n-1} R'_i \longrightarrow \bigcap_{i=1}^{n-1} R_i.$$

By applying Theorem 4.3 to the punctured disks, we have

$$R'_1 \cap R'_2 \cap \cdots \cap R'_{n-1} = [[R'_1, R'_2], \ldots, R'_{n-1}]_S.$$

It follows that

$$R_1 \cap R_2 \cap \cdots \cap R_{n-1} = [[R_1, R_2], \ldots, R_{n-1}]_S.$$

From equality (5.1), we have

$$\bigcap_{i=1}^{n-1} R_i = \bigcap_{i=1}^{n-1} \mathrm{Ker}(d_{n*}) \cap \mathrm{Ker}(d_{i*}) = \bigcap_{j=1}^{n} \mathrm{Ker}(d_{j*}\colon P_n \to P_{n-1}) = \mathrm{Brun}_n,$$

hence the result. □

5.2. **Proof of Theorem 1.5.** The key point to proving Theorem 1.5 is to construct a $K(\pi, 1)$-partition for the configuration space $F(S^2, n)$. Before giving the proof, we need some lemmas. Let $q_1, q_2, \ldots$ be strictly increasing real numbers that lie in the open interval $(-1, 1) \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq S^2$. The base point of the configuration spaces $F(\mathbb{R}^2, m) \subseteq F(S^2, m)$ is chosen to be $(q_1, \ldots, q_m)$ for each $m \geq 1$.

Lemma 5.2. *Let $1 \leq i \leq m$ and let*

$$B_i = \{(z_1, \ldots, z_m) \in F(S^2, m) \mid z_p \neq \infty \text{ for } p \neq i\}.$$

Let $f\colon F(\mathbb{R}^2, m) \to B_i$ be the inclusion. Then the homomorphism

$$f_*\colon P_m = \pi_1(F(\mathbb{R}^2, m)) \longrightarrow \pi_1(B_i)$$

is an epimorphism with $\mathrm{Ker}(f_*) = \langle\langle A_{0,i}\rangle\rangle^{P_m}$, the normal closure of $A_{0,i}$ in P_m.

Proof. The assertion holds trivially for $m = 1$. We may assume that $m \geq 2$. Consider the Fadell-Neuwirth fibration

$$d_i\colon F(S^2, m) \longrightarrow F(S^2, m-1) \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, \hat{z}_i, \ldots, z_m).$$

Then $B_i = d_i^{-1}(F(\mathbb{R}^2, m-1))$ by the construction of B_i. Consider the commutative diagram of the fibrations

$$\begin{tikzcd}
\mathbb{R}^2 \backslash Q_{m,i} \arrow[r, hook] \arrow[d, hook, "f|"] & F(\mathbb{R}^2, m) \arrow[r] \arrow[d, hook, "f"] & F(\mathbb{R}^2, m-1) \arrow[d, equal] \\
S^2 \backslash Q_{m,i} \arrow[r, hook] & B_i \arrow[r] & F(\mathbb{R}^2, m-1),
\end{tikzcd}$$

where $Q_{m,i} = \{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m\}$. Thus there is a commutative diagram of short exact sequence of groups

$$\begin{tikzcd}
\pi_1(\mathbb{R}^2 \backslash Q_{m,i}) \arrow[r, hook] \arrow[d, "f|_*"] & \pi_1(F(\mathbb{R}^2, m)) \arrow[r, two heads] \arrow[d, "f_*"] & \pi_1(F(\mathbb{R}^2, m-1)) \arrow[d, equal] \\
\pi_1(S^2 \backslash Q_{m,i}) \arrow[r, hook] & \pi_1(B_i) \arrow[r, two heads] & \pi_1(F(\mathbb{R}^2, m-1)).
\end{tikzcd}$$

Since $f|_*\colon \pi_1(\mathbb{R}^2 \smallsetminus Q_{m,i}) \to \pi_1(S^2 \smallsetminus Q_{m,i})$ is an epimorphism, the middle map

$$f_*\colon \pi_1(F(\mathbb{R}^2, m)) \longrightarrow \pi_1(B_i)$$

is an epimorphism with $\mathrm{Ker}(f_*) = \mathrm{Ker}(f|_*)$. Note that $\mathrm{Ker}(f|_*)$ is the normal closure of the element $[\omega_i]$ in $\pi_1(\mathbb{R}^2 \smallsetminus Q_{m,i})$, where the loop ω_i is given in Figure 1. In that figure, the braid $A_{0,i}$ is obtained from the loop ω_i by drawing it as a braid. Thus $[\omega_i] = A_{0,i}$, and so

$$\mathrm{Ker}(f_*) = \mathrm{Ker}(f|_*) = \langle\langle A_{0,i}\rangle\rangle^{P_m} \cap \pi_1(\mathbb{R}^2 \smallsetminus Q_{m,i}) \leq \langle\langle A_{0,i}\rangle\rangle^{P_m}.$$

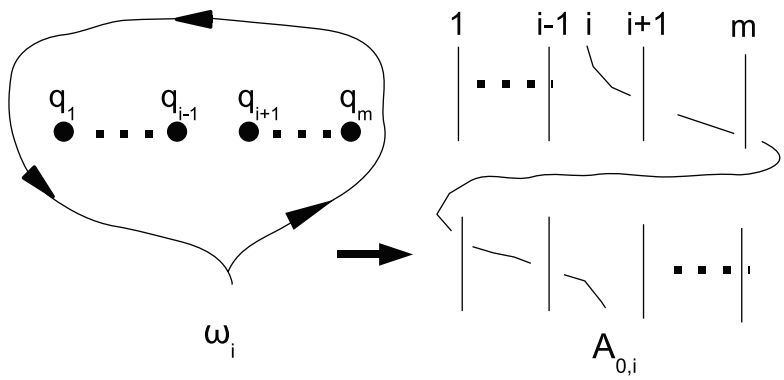

FIGURE 1

On the other hand, since $\mathrm{Ker}(f_*)$ is normal in P_m with $A_{0,i} \in \mathrm{Ker}(f_*)$, we have $\langle\langle A_{0,i}\rangle\rangle^{P_m} \leq \mathrm{Ker}(f_*)$. It follows that

$$\mathrm{Ker}(f_*) = \langle\langle A_{0,i}\rangle\rangle^{P_m},$$

hence the result holds. □

Lemma 5.3. *Let $I = \{i_1, i_2, \ldots, i_k\}$ be a subset of $\{1, 2, \ldots, m\}$. Let*

$$B_I = \{(z_1, \ldots, z_m) \in F(S^2, m) \mid z_p \neq \infty \text{ for } p \notin I\}$$

and let $f\colon F(\mathbb{R}^2, m) \to B_I$ be the inclusion. Then the homomorphism

$$f_*\colon \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_I)$$

is an epimorphism whose kernel $\mathrm{Ker}(f_*) = \langle\langle A_{0,i} \mid i \in I\rangle\rangle^{P_m}$.

Proof. Note that

$$B_I = \bigcup_{s=1}^{k} B_{i_s}.$$

By Lemma 5.2, $P_m = \pi_1(F(\mathbb{R}^2, m)) \to \pi_1(B_{i_s})$ is an epimorphism whose kernel is given by $\langle\langle A_{0,i_s}\rangle\rangle^{P_m}$ for each $1 \leq s \leq k$. From Van Kampen's Theorem,

$$f_*\colon P_m = \pi_1(F(\mathbb{R}^2, m)) \longrightarrow \pi_1(B_I)$$

is an epimorphism with

$$\mathrm{Ker}(f_*) = \prod_{s=1}^{k} \langle\langle A_{0,i_s}\rangle\rangle = \langle\langle A_{0,i_s} \mid 1 \leq s \leq k\rangle\rangle^{P_m},$$

hence the result. □

Proof of Theorem 1.5. Let $A = F(\mathbb{R}^2, m)$ be the subspace of $F(S^2, m)$. Let

$$A_i = \{(z_1, z_2, \ldots, z_m) \in F(S^2, m) \mid z_p \neq \infty \text{ for } p \notin \Lambda_i\},$$

where the Λ_i are as defined in the statement of the theorem. Since $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$, we have

$$A_i \cap A_j = F(\mathbb{R}^2, m) = A$$

for $i \neq j$. From the assumption that $\bigcup_{i=1}^{n} \Lambda_i = \{1, 2, \ldots, m\}$, we have

$$\bigcup_{i=1}^{n} A_i = F(S^2, m).$$

Thus $(A_1, A_2, \ldots, A_n)$ is a cofibrant n-partition[3] of the space $F(S^2, m)$ relative to $F(\mathbb{R}^2, m)$. For a subset $I = \{i_1, i_2, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$, let

$$A_I = \bigcup_{s=1}^{k} A_{i_s}$$

with $A_\emptyset = A$. We now check that A_I is a $K(\pi, 1)$-space for $I \subsetneq \{1, 2, \ldots, n\}$. Let

$$J = \{j_1, j_2, \ldots, j_t\} = \{1, 2, \ldots, m\} \smallsetminus \left(\bigcup_{s=1}^{k} \Lambda_{i_s} \right)$$

with $j_1 < j_2 < \cdots < j_t$. Since $I \neq \{1, 2, \ldots, n\}$, J is not empty. Let $(z_1, \ldots, z_m) \in F(S^2, m)$. Observe that $(z_1, \ldots, z_m) \in A_I$ if and only if $z_{j_s} \neq \infty$ for $1 \leq s \leq t$. Consider the Fadell-Neuwirth fibration [9]

$$d_I \colon F(S^2, m) \longrightarrow F(S^2, t).$$

There is a pull-back diagram

$$\begin{tikzcd} A_I \arrow[r, hook] \arrow[d, "d_I|_{A_I}"'] \arrow[dr, phantom, "\text{pull}"] & F(S^2, m) \arrow[d, "d_I"] \\ F(\mathbb{R}^2, t) \arrow[r, hook] & F(S^2, t) \end{tikzcd}$$

with a fibration

$$F(S^2 \smallsetminus Q_t, m - t) \to A_I \to F(\mathbb{R}^2, t),$$

where Q_t is a set of t distinct points in S^2. Since $t \geq 1$,

$$F(S^2 \smallsetminus Q_t, m - t) \cong F(\mathbb{R}^2 \smallsetminus Q_{t-1}, m - t)$$

is a $K(\pi, 1)$-space. Together with the fact that the base space $F(\mathbb{R}^2, t)$ is a $K(\pi, 1)$-space, the total space A_I is a $K(\pi, 1)$-space for any $I \subsetneq \{1, 2, \ldots, n\}$.

By Lemma 5.3, the inclusion $f \colon A = F(\mathbb{R}^2, m) \hookrightarrow A_i$ induces an epimorphism

$$f_* \colon P_m = \pi_1(A) \twoheadrightarrow \pi_1(A_i)$$

whose kernel

$$R_i = \mathrm{Ker}(f_*) = \langle\langle A_{0,j} \mid j \in \Lambda_i \rangle\rangle^{P_m}.$$

The assertion follows from Theorem 1.3. □

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[3]Similar to footnote 2, one needs to do some modifications such that the cofibrant hypothesis holds: Take a triangulation on $F(S^2, m)$ and deform linearly A and each A_i into certain subcomplexes with necessary subdivisions for having a cofibrant n-partition of $F(S^2, m)$.

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