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DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS ACTING FROM BLOCH SPACE TO H^∞

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Abstract. We study the boundedness and compactness of the differences of two weighted composition operators acting from the Bloch space $\mathcal{B}$ to the space H^∞ of bounded analytic functions on the open unit disk. Such a study has a relationship to the topological structure problem of composition operators on H^∞. Using this relation, we will estimate the operator norms and the essential norms of the differences of two composition operators acting from $\mathcal{B}$ to H^∞.

1. Introduction

Throughout this paper, let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\overline{\mathbb{D}}$ be its closure and $\partial\mathbb{D}$ its boundary. We denote by $\mathcal{H}(\mathbb{D})$ the set of analytic functions on $\mathbb{D}$ and by $\mathcal{S}(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Here we consider the operators induced by multiplying an analytic function and by the composition with an analytic self-map of $\mathbb{D}$. More precisely, for a function $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$, we define a *weighted composition operator* uC_φ by

$$uC_\varphi f = u \cdot (f \circ \varphi) \quad \text{for } f \in \mathcal{H}(\mathbb{D}).$$

It is clear that uC_φ is linear on $\mathcal{H}(\mathbb{D})$, and it is a natural generalization of multiplication and composition operators. Furthermore the isometries on many analytic function spaces are of the canonical forms of weighted composition operators.

Let $H^\infty = H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$. Then H^∞ is the Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

We recall that the Bloch space $\mathcal{B}$ consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$|||f||| = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Then $|||\cdot|||$ is a complete semi-norm on $\mathcal{B}$ and is Möbius invariant. Also, H^∞ is properly contained in $\mathcal{B}$. Let the little Bloch space $\mathcal{B}_o$ denote the subspace of $\mathcal{B}$ consisting of functions f with $\lim_{|z|\to 1}(1-|z|^2)f'(z) = 0$.

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It is well known that $\mathcal{B}$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \|\!|f|\!\|$$

and that $\mathcal{B}_o$ is a closed subspace of $\mathcal{B}$. In particular, $\mathcal{B}_o$ is the closure in $\mathcal{B}$ of the polynomials. See the books [1], [12] and [14] for a thorough treatment on such classical settings.

Originally Shapiro and Sundberg [13] investigated the topological structure of the space of composition operators on the Hardy space H^2. MacCluer, Zhao and the second author [9] studied the topological structure in the case of H^∞ and gave a relationship between such a problem and the boundedness and compactness of the difference $C_\varphi - C_\psi$ of two composition operators acting from the Bloch space to H^∞. Explicitly, the following are shown in [9]: that the compactness of $C_\varphi - C_\psi : H^\infty \to H^\infty$ is equivalent to the compactness of $C_\varphi - C_\psi$ acting from $\mathcal{B}$ to H^∞, and also that C_φ and C_ψ are in the same path component of the space of composition operators on H^∞ if and only if $C_\varphi - C_\psi : \mathcal{B} \to H^\infty$ is bounded. Moreover, it is proved that if

$$\int_0^{2\pi} \log(1 - |\varphi(e^{i\theta})|)d\theta > -\infty, \tag{1.1}$$

then C_φ is not isolated in the space of composition operators on H^∞. Izuchi, Zheng, and the first author [7] proved that the converse also holds. It is known as the de Leeuw and Rudin theorem ([2] or [3, p. 138]) that φ is not an extreme point of the closed unit ball of H^∞ if and only if φ satisfies (1.1). So the boundedness of the differences of two composition operators acting from $\mathcal{B}$ to H^∞ and the extremeness of φ play an interesting role in the study of the topological structure of the set of composition operators on H^∞. We [8] studied the topological structure of the set of composition operators on the Bloch spaces. In [6], Izuchi and both authors extended the problem of the topological structure to the space of weighted composition operators on H^∞. Here we continue such an investigation. In this paper we study the differences of two weighted composition operators acting from $\mathcal{B}$ to H^∞. In Section 2, we characterize the boundedness of the differences of two weighted composition operators acting from $\mathcal{B}$ to H^∞ and present a result using the results of [6] that there exist uC_φ and vC_φ whose difference is not bounded from $\mathcal{B}$ to H^∞ but which are in the same component. In Section 3, we will try to characterize the compactness of the differences. But such a characterization would be difficult for general weights $u, v \in \mathcal{H}(\mathbb{D})$. We can obtain the necessary and sufficient condition for the compactness of the differences in the case that u and v are bounded. In the unweighted case, the equivalence of the isolation of C_φ and an extremeness of φ holds. This will improve the estimation of the essential norm of $C_\varphi - C_\psi$ acting from $\mathcal{B}$ to H^∞. To determine the topological structure of the set of all composition operators, one of the important problems is to estimate the norms and the essential norms of differences of two composition operators. In Section 4, we will estimate the operator norms and the essential norms of the differences of two composition operators acting from $\mathcal{B}$ to H^∞ and add the improvement as stated above. At the end we attach the results on the estimations in the case of the differences of two weighted composition operators.

We shall need some basic properties of functions in $\mathcal{B}$ ([14]). For each $w \in \mathbb{D}$, let α_w be the Möbius transformation of $\mathbb{D}$ defined by

$$\alpha_w(z) = \frac{w - z}{1 - \overline{w}z}. \tag{1.2}$$

For w, z in $\mathbb{D}$, the pseudo-hyperbolic distance $\rho(w, z)$ between z and w is given by

$$\rho(w, z) = |\alpha_w(z)|,$$

and the hyperbolic metric $\beta(w, z)$ is given by

$$\beta(w, z) = \frac{1}{2} \log \frac{1 + \rho(w, z)}{1 - \rho(w, z)}.$$

It is known that, for $z, w \in \mathbb{D}$ and $f \in \mathcal{B}$,

$$\beta(z, w) = \sup_{|||f||| \le 1} |f(z) - f(w)|. \tag{1.3}$$

We present a growth condition for Bloch functions:

$$\begin{aligned} |f(w)| &\le |f(0)| + |||f|||\beta(0, w) \\ &= |f(0)| + |||f||| \frac{1}{2} \log \frac{1 + |w|}{1 - |w|}. \end{aligned}$$

Define

$$\widetilde{\beta}(z) = \max\{1, \beta(0, z)\}.$$

Hence we get the following estimate:

$$|f(w)| \le \widetilde{\beta}(w)\|f\|_{\mathcal{B}} \quad \text{for} \quad w \in \mathbb{D}.$$

In the rest of this paper, C_0, C_1, C_2 and C will stand for positive constants whose values may change from one occurrence to another.

2. Boundedness

In this section, we will characterize the boundedness of the differences of two weighted composition operators acting from $\mathcal{B}$ to H^∞ using results developed by the first author in [4].

Referring to the characterization of the boundedness of uC_φ in [11] and [6], we define the following notation to discuss the boundary behavior of the symbols u and φ.

Definition 2.1. For $u \in \mathcal{H}(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$, let Δ be the set of all convergent sequences $\{z_n\}$ in $\mathbb{D}$ and let $D_{u,\varphi}$ be the subset of Δ which consists of all sequences $\{z_n\}$ such that

$$\lim_{n\to\infty} |u(z_n)|\widetilde{\beta}(\varphi(z_n)) = \infty.$$

Then uC_φ is bounded from $\mathcal{B}$ to H^∞ if and only if $D_{u,\varphi} = \emptyset$.

We present the following lemma due to [4].

Lemma 2.2. *For z, w in $\mathbb{D}$, let $L(z, w)$ be the continuous function on $\mathbb{D} \times \mathbb{D}$ defined by*

$$L(z, w) = \frac{\frac{1}{2} \log \frac{(1+|z|)^2}{1-\overline{z}w}}{\widetilde{\beta}(z)}.$$

Then there exists a constant $C > 0$ such that $\sup_{z,w\in\mathbb{D}} |L(z, w)| \le C$.

We have a main result in this section.

Theorem 2.3. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Suppose that neither uC_φ nor vC_ψ is boundedly acting from $\mathcal{B}$ to H^∞. Then $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞ if and only if the following conditions hold:*

(i) $D_{u,\varphi} = D_{v,\psi}$.

(ii) $\sup_{\{z_n\}\in D_{u,\varphi}} \lim_{n\to\infty} |u(z_n) - v(z_n)|\, \widetilde{\beta}(\varphi(z_n)) < \infty$.

(iii) $\sup_{\{z_n\}\in D_{u,\varphi}} \lim_{n\to\infty} |v(z_n)|\, \beta(\varphi(z_n), \psi(z_n)) < \infty$.

It is possible to replace φ with ψ in condition (ii) and also v with u and $D_{u,\varphi}$ with $D_{v,\psi}$ in condition (iii).

Proof. We suppose that conditions (i)–(iii) hold. Let f be a function in $\mathcal{B}$ such that $\|f\|_{\mathcal{B}} = 1$. Let $\partial D_{u,\varphi}$ be the set of all cluster points of each $\{z_n\} \in D_{u,\varphi}$. Then $\partial D_{u,\varphi} \subset \partial\mathbb{D}$. For any $\{z_n\} \in D_{u,\varphi} = D_{v,\psi}$, we also have

$$\begin{aligned}
&|(uC_\varphi - vC_\psi)f(z_n)| \\
&\le |u(z_n) - v(z_n)||C_\varphi f(z_n)| + |v(z_n)||C_\varphi f(z_n) - C_\psi f(z_n)| \\
&\le |u(z_n) - v(z_n)|\widetilde{\beta}(\varphi(z_n)) + |v(z_n)|\beta(\varphi(z_n), \psi(z_n)).
\end{aligned}$$

Then there exists a constant $C_1 > 0$ and a subset K of $\mathbb{D}$ such that $\overline{K} \supset \partial D_{u,\varphi}$ and

$$\sup_{z\in K} |(uC_\varphi - vC_\psi)f(z)| \le C_1.$$

On the other hand, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}\setminus K} |(uC_\varphi - vC_\psi)f(z)| \\
&\le \sup_{z\in\mathbb{D}\setminus K} \big(|u(z)C_\varphi f(z)| + |v(z)C_\psi f(z)|\big) \\
&\le \sup_{z\in\mathbb{D}\setminus K} \big(|u(z)|\widetilde{\beta}(\varphi(z)) + |v(z)|\widetilde{\beta}(\psi(z))\big) \\
&\le C_2.
\end{aligned}$$

So, letting $C_0 = \max\{C_1, C_2\}$, we have that

$$\|uC_\varphi - vC_\psi\|_{\mathcal{B}\to H^\infty} \le C_0,$$

where $\|\cdot\|_{\mathcal{B}\to H^\infty}$ is the norm of an operator acting from $\mathcal{B}$ to H^∞.

Next we show the converse. Suppose $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞. We have that $u - v = (uC_\varphi - vC_\psi)1 \in H^\infty$. For $z \in \mathbb{D}$ and $f \in \mathcal{B}$,

$$\begin{aligned}
&\|uC_\varphi - vC_\psi\|_{\mathcal{B}\to H^\infty}\, \|f\|_{\mathcal{B}} \\
&\ge |(uC_\varphi - vC_\psi)f(z)| \\
&\ge |v(z)||C_\varphi f(z) - C_\psi f(z)| - |u(z) - v(z)||C_\varphi f(z)|.
\end{aligned} \tag{2.1}$$

Thus

$$|v(z)|\beta(\varphi(z), \psi(z)) \le \|uC_\varphi - vC_\psi\|_{\mathcal{B}\to H^\infty} + |u(z) - v(z)|\widetilde{\beta}(\varphi(z)).$$

This implies that condition (iii) follows from condition (ii).

We define subsets D_1 and D_2 of $D_{u,\varphi}$ as follows:

$$D_1 = \{\{z_n\} \in D_{u,\varphi} : \lim_{n\to\infty} \widetilde{\beta}(\varphi(z_n)) = 1\}$$

and

$$D_2 = \{\{z_n\} \in D_{u,\varphi} : \lim_{n\to\infty} \widetilde{\beta}(\varphi(z_n)) > 1\}.$$

Then $D_1 \cup D_2 = D_{u,\varphi}$ and $D_1 \cap D_2 = \emptyset$. For any $\{z_n\} \in D_1$, we have that

$$\lim_{n\to\infty} |u(z_n) - v(z_n)|\, \widetilde{\beta}(\varphi(z_n)) \le \|u - v\|_\infty.$$

This implies that

$$\sup_{\{z_n\}\in D_1} \lim_{n\to\infty} |u(z_n) - v(z_n)|\, \widetilde{\beta}(\varphi(z_n)) \le \|u - v\|_\infty. \tag{2.2}$$

By the triangle inequality, we also have that, for $\{z_n\} \in D_1$,

$$\begin{aligned} \lim_{n\to\infty} |v(z_n)| &= \lim_{n\to\infty} |v(z_n)|\widetilde{\beta}(\varphi(z_n)) \\ &\ge \lim_{n\to\infty} |u(z_n)|\widetilde{\beta}(\varphi(z_n)) - \|u - v\|_\infty \\ &= \infty. \end{aligned}$$

Hence we conclude that $D_1 \subset D_{v,\psi}$.

Let $\{z_n\} \in D_2$. By passing to a subsequence, we can suppose that $\widetilde{\beta}(\varphi(z_n)) > 1$ for any n. Put

$$f_n(z) = \frac{1}{2} \log \frac{(1 + |\varphi(z_n)|)^2}{1 - \overline{\varphi(z_n)} z}$$

and

$$g_n(z) = \left(\frac{1}{2} \log \frac{1 + |\varphi(z_n)|}{1 - |\varphi(z_n)|} \right)^{-1} \left(\frac{1}{2} \log \frac{(1 + |\varphi(z_n)|)^2}{1 - \overline{\varphi(z_n)} z} \right)^2.$$

Then $\|f_n\|_{\mathcal{B}} \le 2$ and Lemma 2.2 implies that there exists a constant $C > 0$ such that $\|g_n\|_{\mathcal{B}} \le C$ for all n. We remark that $f_n(\varphi(z_n)) = g_n(\varphi(z_n)) = \beta(0, \varphi(z_n)) = \widetilde{\beta}(\varphi(z_n))$. By the assumption of boundedness, there exists a constant $C > 0$ such that for any n,

$$\begin{aligned} C &\ge \|(uC_\varphi - vC_\psi) f_n\|_\infty \\ &\ge |(uC_\varphi - vC_\psi) f_n(z_n)| \\ &\ge |u(z_n) - v(z_n) L(\varphi(z_n), \psi(z_n))|\, \widetilde{\beta}(\varphi(z_n)), \end{aligned} \tag{2.3}$$

where L is the function defined in Lemma 2.2. Similarly, since $\|(uC_\varphi - vC_\psi) g_n\|_\infty \le C$, we get

$$\left| u(z_n) - v(z_n) L(\varphi(z_n), \psi(z_n))^2 \right| \widetilde{\beta}(\varphi(z_n)) \le C. \tag{2.4}$$

By multiplying (2.3) by $L(\varphi(z_n), \psi(z_n))$ and (2.4), Lemma 2.2 implies that

$$\begin{aligned} &|u(z_n)| \left| 1 - L(\varphi(z_n), \psi(z_n)) \right| \widetilde{\beta}(\varphi(z_n)) \\ &\quad = |u(z_n)| \left| \log \frac{1 + |\varphi(z_n)|}{1 - |\varphi(z_n)|} - \log \frac{(1 + |\varphi(z_n)|)^2}{1 - \overline{\varphi(z_n)} \psi(z_n)} \right| \\ &\quad \le C. \end{aligned} \tag{2.5}$$

Since $\{z_n\} \in D_2 \subset D_{u,\varphi}$, we obtain that

$$|u(z_n)| \left| \log \frac{(1 + |\varphi(z_n)|)^2}{1 - \overline{\varphi(z_n)} \psi(z_n)} \right| \to \infty. \tag{2.6}$$

We claim that for any $\{z_n\} \in D_2$, there exists a positive integer n_0 such that $\widetilde{\beta}(\psi(z_n)) > 1$ for any $n > n_0$. If $|u(z_n)| \to \infty$, then (2.5) implies that

$$\log \frac{1-\overline{\varphi(z_n)}\psi(z_n)}{1-|\varphi(z_n)|^2} = \log \frac{1}{1-\overline{\varphi(z_n)}\frac{\varphi(z_n)-\psi(z_n)}{1-\overline{\varphi(z_n)}\psi(z_n)}} \to 0.$$

Hence we get $\rho(\varphi(z_n), \psi(z_n)) \to 0$ and $\widetilde{\beta}(\psi(z_n)) > 1$ for n sufficiently large. If $|u(z_n)| \not\to \infty$, then (2.6) implies that $|\psi(z_n)| \to 1$, that is, $\widetilde{\beta}(\psi(z_n)) \to \infty$.

Here we interchange the roles of φ and ψ in the test functions and arguments above and obtain the following similar estimation to (2.5):

$$|v(z_n)| \left| \log \frac{1+|\psi(z_n)|}{1-|\psi(z_n)|} - \log \frac{(1+|\psi(z_n)|)^2}{1-\overline{\psi(z_n)}\varphi(z_n)} \right| \le C. \tag{2.7}$$

By (2.3) and (2.5),

$$|u(z_n) - v(z_n)| \left| \log \frac{(1+|\varphi(z_n)|)^2}{1-\overline{\varphi(z_n)}\psi(z_n)} \right| \le C. \tag{2.8}$$

So, by (2.6),

$$|v(z_n)| \left| \log \frac{(1+|\varphi(z_n)|)^2}{1-\overline{\varphi(z_n)}\psi(z_n)} \right| \to \infty. \tag{2.9}$$

By (2.7) and (2.9), we have

$$|v(z_n)| \left(\log \frac{1+|\psi(z_n)|}{1-|\psi(z_n)|} + \log \frac{(1+|\varphi(z_n)|)^2}{(1+|\psi(z_n)|)^2} \right) \to \infty.$$

This implies that

$$|v(z_n)|\widetilde{\beta}(\psi(z_n)) \to \infty \quad \text{as} \quad n \to \infty.$$

Now we get $D_2 \subset D_{v,\psi}$. Therefore $D_{u,\varphi} \subset D_{v,\psi}$. The converse inclusion also can be shown in the same way. Thus the equality $D_{u,\varphi} = D_{v,\psi}$ holds.

Next, by (2.3) and (2.4), we have that for $\{z_n\} \in D_2$,

$$|v(z_n)| \left| \log \frac{(1+|\varphi(z_n)|)^2}{1-\overline{\varphi(z_n)}\psi(z_n)} \right| \left|1 - L(\varphi(z_n), \psi(z_n))\right| \le C.$$

Thus (2.9) implies that $L(\varphi(z_n), \psi(z_n)) \to 1$. From (2.8) there exist a constant $C > 0$ and a subset K of $\mathbb{D}$ such that $\overline{K} \supset \partial D_2$ and

$$\begin{aligned} &\sup_{\{z_n\}\in D_2} \lim_{n\to\infty} \left|u(z_n) - v(z_n)\right| \beta(0, \varphi(z_n)) \\ &\quad = \sup_{\{z_n\}\in D_2} \lim_{n\to\infty} \left|u(z_n) - v(z_n)\right| \left| \frac{1}{2} \log \frac{(1+|\varphi(z_n)|)^2}{1-\overline{\varphi(z_n)}\psi(z_n)} \right| \frac{1}{|L(\varphi(z_n), \psi(z_n))|} \\ &\quad \le C. \end{aligned}$$

Combining (2.2), we obtain condition (ii). Hence we get all conditions. □

Remark 2.4. We define the subsets of $D_{u,\varphi}$ as follows:

$$E_{u,\varphi} = \{\{z_n\} \in \Delta : \lim_{n\to\infty} |u(z_n)| = \infty\}$$

and

$$F_{u,\varphi} = \{\{z_n\} \in \Delta : \lim_{n\to\infty} |u(z_n)| < \infty \text{ and } \lim_{n\to\infty} |u(z_n)|\widetilde{\beta}(\varphi(z_n)) = \infty\}.$$

Then $E_{u,\varphi} \cap F_{u,\varphi} = \emptyset$ and $E_{u,\varphi} \cup F_{u,\varphi} = D_{u,\varphi}$. Moreover, suppose that $uC_\varphi - vC_\psi$ is bounded from $\mathcal{B}$ to H^∞. Then, by the fact that $u - v \in H^\infty$ and conditions (i) and (ii) of Theorem 2.3, we conclude that $E_{u,\varphi} = E_{v,\psi}$ and $F_{u,\varphi} = F_{v,\psi}$.

Taking $\varphi = \psi$ and $w = u - v$, we obtain the characterization of boundedness of wC_φ in [11] and [6] as a corollary of Theorem 2.3.

Corollary 2.5. *Let w be in $\mathcal{H}(\mathbb{D})$ and φ be in $\mathcal{S}(\mathbb{D})$. Then wC_φ is boundedly acting from $\mathcal{B}$ to H^∞ if and only if*

$$\sup_{z\in\mathbb{D}} |w(z)|\widetilde{\beta}(\varphi(z)) < \infty.$$

We present an example for Theorem 2.3.

Example 2.6. Let $\sigma(z) = (1+z)/(1-z)$ and

$$\ell(z) = \frac{\sqrt{\sigma(z)} - 1}{\sqrt{\sigma(z)} + 1}$$

be a lens map. We put $\varphi(z) = (\ell(z) + 1)/2$. For sufficiently small $t > 0$, $\tau(z) = z + t(1-z)^2$ is an analytic self-map of $\mathbb{D}$. Here, define $\psi(z) = \tau(\varphi(z))$. Moreover, put

$$u(z) = \left(\log\log \frac{e^e}{1-\varphi(z)}\right)^{-1}$$

and $v(z) = u(z) - (1-\varphi(z))(1-\psi(z))$. Then neither uC_φ nor vC_ψ is bounded from $\mathcal{B}$ to H^∞, but $uC_\varphi - vC_\psi$ is bounded from $\mathcal{B}$ to H^∞.

Proof. We remark that $\overline{\mathbb{D}} \cap \overline{\varphi(\mathbb{D})} = \{1\}$. Let x be a real number in $(0,1)$. It is easy to check that

$$\lim_{x\to 1} |u(x)| \log \frac{e}{1-|\varphi(x)|^2} = \lim_{x\to 1} |v(x)| \log \frac{e}{1-|\psi(x)|^2} = \infty.$$

Thus neither uC_φ nor vC_ψ is bounded from $\mathcal{B}$ to H^∞.

Since $u(z) - v(z) = (1-\varphi(z))(1-\psi(z))$,

$$\lim_{z\to 1} |u(z) - v(z)| \log \frac{e}{1-|\varphi(z)|^2} = \lim_{z\to 1} |u(z) - v(x)| \log \frac{e}{1-|\psi(x)|^2} = 0.$$

Let $\varphi(z) = 1 + re^{i\theta}$ with $3\pi/4 \le \theta \le 5\pi/4$. Then

$$\begin{aligned} \rho(\varphi(z), \psi(z)) &= \left| \frac{t(1-\varphi(z))^2}{1 - |\varphi(z)|^2 - t\overline{\varphi(z)}(1-\varphi(z))^2} \right| \\ &= \frac{tr}{|r + 2\cos\theta + tr(1+re^{-i\theta})e^{2i\theta}|}. \end{aligned}$$

Since $3\pi/4 \le \theta \le 5\pi/4$, there exists a constant $C_1 > 0$ such that $\rho(\varphi(z), \psi(z)) \le C_1|1-\varphi(z)|$. Hence there exists a constant $C_2 > 0$ such that

$$\lim_{z\to 1} |u(z)|\, \beta(\varphi(z), \psi(z)) = \lim_{z\to 1} |v(z)|\, \beta(\varphi(z), \psi(z)) \le C_2.$$

Now we conclude that $uC_\varphi - vC_\psi$ is bounded from $\mathcal{B}$ to H^∞. $\square$

Proposition 2.7. *For $\varphi \in \mathcal{S}(\mathbb{D})$, let $\Gamma(\varphi) = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$. Suppose that $\Gamma(\varphi)$ has positive measure.*

(i) *If uC_φ is boundedly acting from $\mathcal{B}$ to H^∞, then u is the zero function.*
(ii) *If u is the nonzero function in $\mathcal{H}(\mathbb{D})$ and $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞, then $u = v$ and $\varphi = \psi$.*

Proof. (i) By Corollary 2.5, $u = 0$ on $\Gamma(\varphi)$. Thus we conclude that u is the zero function.

(ii) Since u is not the zero function, $\partial D_{u,\varphi}$ has positive measure. By (ii) in Theorem 2.3, $u - v = 0$ on $\partial D_{u,\varphi}$. This implies that $u = v$ on $\mathbb{D}$. Let F be the subset of $D_{u,\varphi}$ which consists of all sequences $\{z_n\}$ in $\mathbb{D}$ such that $u(z_n) \not\to 0$ and let ∂F be the set of all cluster points of each $\{z_n\} \in F$. Then ∂F also has positive measure. By (iii) in Theorem 2.3, we have that $\beta(\varphi(\zeta), \psi(\zeta)) = 0$ on ∂F. This implies that $\varphi = \psi$ on ∂F. We conclude that $\varphi = \psi$ on $\mathbb{D}$. □

We insert the characterization of the boundedness of the difference $uC_\varphi - vC_\psi : H^\infty \to H^\infty$. The compactness was given in [6] but the boundedness has been left.

Theorem 2.8. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Then $uC_\varphi - vC_\psi$ is boundedly acting from H^∞ to H^∞ if and only if the following conditions hold:*

(i) $u - v \in H^\infty$.
(ii) $\sup_{z \in D} |v(z)|\, \rho(\varphi(z), \psi(z)) < \infty$.

It is possible to replace v with u in condition (ii).

Example 2.9. Let $\varphi(z) = \ell(z)$ be the lens map in Example 2.5 and $\psi(z) = 1 - \sqrt{2(1-z)}$. Moreover, put

$$u(z) = (1+z) \log \frac{e}{1-z}$$

and $v(z) = u(z) - (1+z)$. Then neither uC_φ nor vC_ψ is bounded on H^∞, but $uC_\varphi - vC_\psi$ is bounded on H^∞.

Recall that MacCluer, Zhao and the second author [9] showed that the compactness of $C_\varphi - C_\psi : H^\infty \to H^\infty$ is equivalent to the compactness of $C_\varphi - C_\psi$ acting from $\mathcal{B}$ to H^∞ and moreover that C_φ and C_ψ are in the same path component of the space of composition operators on H^∞ if and only if $C_\varphi - C_\psi : \mathcal{B} \to H^\infty$ is bounded.

On the other hand, Izuchi and both authors [6] considered the topological structure of the set of weighted composition operators on H^∞. For example, they gave the following.

Theorem 2.10. *For $u, v \in H^\infty$ and $\varphi \in \mathcal{S}(\mathbb{D})$, uC_φ and vC_φ are in the same component in the set of weighted composition operators on H^∞.*

Now $uC_\varphi - vC_\varphi = (u - v)C_\varphi$ is not always boundedly mapping $\mathcal{B}$ to H^∞. So in the case of weighted composition operators, the boundedness from $\mathcal{B}$ to H^∞ is not equivalent to the same component.

3. Compactness

In this section, we will characterize the compactness of differences of two weighted composition operators. It is easy to prove the next lemma, called the Weak Convergence Theorem, by adapting the proof of [1, Proposition 3.11].

Lemma 3.1. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Suppose that $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞. Then $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is compact if and only if $\|(uC_\varphi - vC_\psi)f_n\|_\infty \to 0$ for any bounded sequence $\{f_n\}$ in $\mathcal{B}$ such that f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$.*

Definition 3.2. Let u be in $\mathcal{H}(\mathbb{D})$ and φ be in $\mathcal{S}(\mathbb{D})$. We denote by $G_{u,\varphi}$ the subset of Δ such that $|\varphi(z_n)| \to 1$ and

$$|u(z_n)|\beta(0, \varphi(z_n)) \not\to 0.$$

We put $\Delta_{u,\varphi} = D_{u,\varphi} \cup G_{u,\varphi}$, where $D_{u,\varphi}$ is the set of sequences defined in Definition 2.1.

Then we remark that $D_{u,\varphi} \subset \Delta_{u,\varphi}$ and that uC_φ is compact from $\mathcal{B}$ to H^∞ if and only if $\Delta_{u,\varphi} = \emptyset$. We also remark that $E_{u,\varphi}$ and $G_{u,\varphi}$ are not necessarily disjoint.

We give a sufficient condition for the compactness.

Proposition 3.3. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Suppose that $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞. Then $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is compact if the following conditions hold:*

(i) $\Delta_{u,\varphi} = \Delta_{v,\psi}$. *More precisely, $G_{u,\varphi} = G_{v,\psi}$.*

(ii) $\lim_{n\to\infty} |u(z_n) - v(z_n)|\, \beta(0, \varphi(z_n)) = 0$ *for any* $\{z_n\} \in \Delta_{u,\varphi}$.

(iii) $\lim_{n\to\infty} |v(z_n)|\, \beta(\varphi(z_n), \psi(z_n)) = 0$ *for any* $\{z_n\} \in \Delta_{u,\varphi}$.

It is possible to replace φ with ψ in condition (ii) and also v with u and $\Delta_{u,\varphi}$ with $\Delta_{v,\psi}$ in condition (iii).

Proof. Assume that conditions (i) – (iii) hold. Let $\{f_n\}$ be the sequence in $\mathcal{B}$ such that $\|f_n\|_\mathcal{B} \le 1$ and f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$. Since $uC_\varphi - vC_\psi$ is bounded from $\mathcal{B}$ to H^∞, we have that $u - v \in H^\infty$. Then, for any $\varepsilon > 0$, there exists a positive integer n_1 such that for any $n > n_1$ and any $z \in \mathbb{D}$,

$$|(u(z) - v(z))f_n(0)| \le \|u - v\|_\infty |f_n(0)| < \frac{\varepsilon}{2}. \tag{3.1}$$

Let $\partial\Delta_{u,\varphi}$ be the set of all cluster points of each $\{z_n\} \in \Delta_{u,\varphi}$. By conditions (i) and (ii), there exists a subset K of $\mathbb{D}$ such that $\overline{K} \supset \partial\Delta_{u,\varphi}$ and

$$|u(z) - v(z)|\beta(0, \varphi(z)) + |v(z)|\beta(\varphi(z), \psi(z)) < \frac{\varepsilon}{2}$$

for any $z \in K$. Since $\|f - f(0)\|_\mathcal{B} = |||f||| \le 1$ for $f \in \mathcal{B}$ such that $\|f\|_\mathcal{B} \le 1$, we get

$$\sup_{z\in K} |u(z)(f(\varphi(z)) - f(0)) - v(z)(f(\psi(z)) - f(0))| < \frac{\varepsilon}{2} \tag{3.2}$$

for any $f \in \mathcal{B}$ such that $\|f\|_\mathcal{B} \le 1$.

Since $G_{u,\varphi} = G_{v,\psi}$, there exists a constant $r \in (0, 1)$ such that $|u(z)|\beta(0, \varphi(z)) < \varepsilon/4$ if $z \in (\mathbb{D}\setminus K) \cap \{|\varphi(z)| > r\}$ and $|v(z)|\beta(0, \psi(z)) < \varepsilon/4$ if $z \in (\mathbb{D}\setminus K) \cap \{|\psi(z)| > r\}$. Since $E_{u,\varphi} = E_{v,\psi}$, both u and v are bounded on $\mathbb{D} \setminus K$. Put

$$M = \sup_{z\in\mathbb{D}\setminus K} \max\{|u(z)|, |v(z)|\}.$$

Since f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$, there exists a positive integer n_2 such that for any $n > n_2$,

$$\sup_{|z|\le r} |f_n(z) - f_n(0)| < \frac{\varepsilon}{4M}.$$

Hence we obtain that

$$\sup_{z\in\mathbb{D}\setminus K} |u(z)(f_n(\varphi(z)) - f_n(0)) - v(z)(f_n(\psi(z)) - f_n(0))| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{3.3}$$

Let $N = \max\{n_1, n_2\}$. By (3.1), (3.2) and (3.3), for any $n > N$,

$$\begin{aligned}
&\|(uC_\varphi - C_\psi)f_n\|_\infty \\
&\quad \le \|u - v\|_\infty |f_n(0)| + \sup_{z\in\mathbb{D}} |u(z)(f_n(\varphi(z)) - f_n(0)) - v(z)(f_n(\psi(z)) - f_n(0))| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

From Lemma 3.1, we conclude that $uC_\varphi - vC_\psi$ is compact from $\mathcal{B}$ to H^∞. □

Next we consider the converse implication.

Proposition 3.4. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. If $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is compact, then the following conditions hold:*

(i) $\Delta_{u,\varphi} = \Delta_{v,\psi}$. *More precisely,* $G_{u,\varphi} = G_{v,\psi}$.

(ii) $\lim_{n\to\infty} |u(z_n) - v(z_n)|\, \beta(0, \varphi(z_n)) = 0$ *for any* $\{z_n\} \in G_{u,\varphi}$.

(iii) $\lim_{n\to\infty} |v(z_n)|\, \beta(\varphi(z_n), \psi(z_n)) = 0$ *for any* $\{z_n\} \in G_{u,\varphi}$.

(iv) $\lim_{n\to\infty} |v(z_n)|\, \beta(\varphi(z_n), \psi(z_n))^2 = 0$ *for any* $\{z_n\} \in \Delta_{u,\varphi} \setminus G_{u,\varphi}$.

It is possible to replace φ with ψ in condition (ii) and also v with u and $\Delta_{u,\varphi}$ with $\Delta_{v,\psi}$ in condition (iii).

Proof. We suppose that $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is compact. Let $\{z_n\}$ be a sequence in $G_{u,\varphi}$. By passing to a subsequence, we may assume that $\beta(0, \varphi(z_n)) \ge 1$. For $k = 1, 2$, put

$$f_{n,k}(z) = \left(\frac{1}{2}\log\frac{1 + |\varphi(z_n)|}{1 - |\varphi(z_n)|}\right)^{-k} \left(\frac{1}{2}\log\frac{(1 + |\varphi(z_n)|)^2}{1 - \overline{\varphi(z_n)}z}\right)^{k+1}. \tag{3.4}$$

Then Lemma 2.2 implies that there exists a constant $C > 0$ such that $\|f_{n,k}\|_{\mathcal{B}} \le C$ for any n and $k = 1, 2$. Since $f_{n,k}$ converges to 0 uniformly on every compact subset of $\mathbb{D}$, Lemma 3.1 implies that $\|(uC_\varphi - vC_\psi)f_{n,k}\|_\infty \to 0$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \left|u(z_n) - v(z_n)L(\varphi(z_n), \psi(z_n))^{k+1}\right| \widetilde{\beta}(\varphi(z_n)) = 0. \tag{3.5}$$

Since $\{z_n\}$ is a sequence in $G_{u,\varphi}$, we obtain that

$$\lim_{n\to\infty} \left|v(z_n)L(\varphi(z_n), \psi(z_n))^{k+1}\right| \widetilde{\beta}(\varphi(z_n)) \ne 0. \tag{3.6}$$

By (3.5) for $k = 1$ and $k = 2$,

$$\lim_{n\to\infty} \left|v(z_n)L(\varphi(z_n), \psi(z_n))^2\big(1 - L(\varphi(z_n), \psi(z_n))\big)\right| \widetilde{\beta}(\varphi(z_n)) = 0.$$

Hence (3.6) implies that

$$\lim_{n\to\infty} L(\varphi(z_n), \psi(z_n)) = 1. \tag{3.7}$$

Then we obtain

$$\lim_{n\to\infty} \big|v(z_n)\big(1 - L(\varphi(z_n), \psi(z_n))\big)\big|\widetilde{\beta}(\varphi(z_n)) = 0. \tag{3.8}$$

From (3.6) again, we have that

$$\lim_{n\to\infty} |v(z_n)|\widetilde{\beta}(\varphi(z_n)) \neq 0.$$

Moreover (3.7) implies that $|\psi(z_n)| \to 1$. Using a test function

$$g_{n,k}(z) = \left(\frac{1}{2}\log\frac{1+|\psi(z_n)|}{1-|\psi(z_n)|}\right)^{-k}\left(\frac{1}{2}\log\frac{(1+|\psi(z_n)|)^2}{1-\overline{\psi(z_n)}z}\right)^{k+1},$$

we conclude that $L(\psi(z_n), \varphi(z_n)) \to 1$. Hence we get

$$\begin{aligned}
&\lim_{n\to\infty} |v(z_n)|\widetilde{\beta}(\psi(z_n)) \\
&\quad = \lim_{n\to\infty} |v(z_n)|\widetilde{\beta}(\varphi(z_n))\frac{1}{L(\psi(z_n), \varphi(z_n))}\left|\overline{L(\varphi(z_n), \psi(z_n))} + \frac{\log\frac{1+|\psi(z_n)|}{1+|\varphi(z_n)|}}{\widetilde{\beta}(\varphi(z_n))}\right| \\
&\quad \neq 0.
\end{aligned}$$

This means that $G_{u,\varphi} \subset G_{v,\psi}$. The converse inclusion can be shown by the same way. Since $uC_\varphi - vC_\psi$ is bounded from $\mathcal{B}$ to H^∞, Theorem 2.3 implies that $D_{u,\varphi} = D_{v,\psi}$. We get condition (i).

On the other hand, (3.5), (3.7), and (3.8) imply that

$$\begin{aligned}
&\lim_{n\to\infty} |u(z_n) - v(z_n)|\widetilde{\beta}(\varphi(z_n)) \\
&\quad = \lim_{n\to\infty} \Big(\big|u(z_n) - v(z_n)L(\varphi(z_n), \psi(z_n))^2\big|\widetilde{\beta}(\varphi(z_n)) \\
&\qquad + \big|v(z_n)(1 - L(\varphi(z_n), \psi(z_n))^2)\big|\widetilde{\beta}(\varphi(z_n))\Big) \\
&\quad = 0
\end{aligned}$$

for $\{z_n\} \in G_{u,\varphi}$. This is condition (ii).

Multiplying (3.5) for $k = 1$ by $L(\varphi(z_n), \psi(z_n))$ and combining (3.5) for $k = 2$,

$$\lim_{n\to\infty} |u(z_n)| \log\frac{1 - \overline{\varphi(z_n)}\psi(z_n)}{1 - |\varphi(z_n)|^2} = 0. \tag{3.9}$$

Similarly, we get

$$\lim_{n\to\infty} |v(z_n)| \log\frac{1 - \overline{\psi(z_n)}\varphi(z_n)}{1 - |\psi(z_n)|^2} = 0. \tag{3.10}$$

By (3.7) and condition (ii), we obtain

$$\lim_{n\to\infty} |u(z_n) - v(z_n)| \log\frac{1 - \overline{\varphi(z_n)}\psi(z_n)}{1 - |\varphi(z_n)|^2} = 0.$$

Therefore (3.9) implies that

$$\lim_{n\to\infty} |v(z_n)| \log\frac{1 - \overline{\varphi(z_n)}\psi(z_n)}{1 - |\varphi(z_n)|^2} = 0. \tag{3.11}$$

For $\{z_n\} \in G_{u,\varphi}$, let

$$h_n(z) = \frac{\varphi(z_n) - z}{1 - \overline{\varphi(z_n)}z} - \varphi(z_n).$$

Then $\|h_n\|_{\mathcal{B}} \leq 2$ and h_n converges to 0 uniformly on every compact subset of $\mathbb{D}$. Then

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)h_n\|_\infty \\
&\quad \geq |u(z_n)C_\varphi h_n(z_n) - v(z_n)C_\psi h_n(z_n)| \\
&\quad \geq \left| v(z_n)\frac{\varphi(z_n) - \psi(z_n)}{1 - \overline{\varphi(z_n)}\psi(z_n)} + (u(z_n) - v(z_n))\varphi(z_n) \right|.
\end{aligned}$$

By Lemma 3.1, we have that

$$\lim_{n\to\infty} \left| v(z_n)\frac{\varphi(z_n) - \psi(z_n)}{1 - \overline{\varphi(z_n)}\psi(z_n)} + (u(z_n) - v(z_n))\varphi(z_n) \right| = 0.$$

Since the condition (ii) implies $u(z_n) - v(z_n) \to 0$ for any $\{z_n\} \in G_{u,\varphi}$, we conclude that

$$\lim_{n\to\infty} |v(z_n)|\rho(\varphi(z_n), \psi(z_n)) = 0.$$

If $\rho(\varphi(z_n), \psi(z_n)) \to 0$, then we have that

$$\lim_{n\to\infty} |v(z_n)|\beta(\varphi(z_n), \psi(z_n)) = 0$$

because $\beta(z, w) \leq \rho(z, w)/(1 - \rho(z, w))$ for $z, w \in \mathbb{D}$. Here we suppose that $\rho(\varphi(z_n), \psi(z_n)) \not\to 0$. Then $v(z_n) \to 0$. Now, by (3.10) and (3.11), we get that

$$\begin{aligned}
&|v(z_n)| \log \frac{1}{1 - \rho(\varphi(z_n), \psi(z_n))^2} \\
&\quad \leq |v(z_n)| \log \frac{|1 - \overline{\varphi(z_n)}\psi(z_n)|^2}{(1 - |\varphi(z_n)|^2)(1 - |\psi(z_n)|^2)} \\
&\quad \leq |v(z_n)| \left(\log \frac{1 - \overline{\varphi(z_n)}\psi(z_n)}{1 - |\varphi(z_n)|^2} + \log \frac{1 - \overline{\psi(z_n)}\varphi(z_n)}{1 - |\psi(z_n)|^2} + 2\pi \right) \\
&\quad \to 0.
\end{aligned}$$

This implies condition (iii).

Let $\{z_n\} \in D_{u,\varphi} \setminus G_{u,\varphi}$. By the definition, we have that $|v(z_n)| \to \infty$ and

$$\lim_{n\to\infty} |\varphi(z_n)| < 1.$$

Since $uC_\varphi - vC_\psi$ is bounded,

$$\lim_{n\to\infty} |v(z_n)|\beta(\varphi(z_n), \psi(z_n)) < \infty.$$

Hence $\beta(\varphi(z_n), \psi(z_n)) \to 0$. Consequently we get condition (iv). □

If $u, v \in H^\infty$, that is, if $E_{u,\varphi} = E_{v,\psi} = \emptyset$, then we obtain the following:

Corollary 3.5. *Let u, v be in H^∞ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Suppose that $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is bounded. Then $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is compact if and only if the*

following conditions hold:

(i) $\Delta_{u,\varphi} = \Delta_{v,\psi}$. *More precisely,* $G_{u,\varphi} = G_{v,\psi}$.

(ii) $\lim_{n\to\infty} |u(z_n) - v(z_n)|\, \beta(0, \varphi(z_n)) = 0$ *for any* $\{z_n\} \in \Delta_{u,\varphi}$.

(iii) $\lim_{n\to\infty} |v(z_n)|\, \beta(\varphi(z_n), \psi(z_n)) = 0$ *for any* $\{z_n\} \in \Delta_{u,\varphi}$.

It is possible to replace φ with ψ in condition (ii) and also v with u and $\Delta_{u,\varphi}$ with $\Delta_{v,\psi}$ in condition (iii).

Izuchi and both authors [6] characterized the compactness of $uC_\varphi - vC_\psi$ on H^∞ under the assumption that $u, v \in H^\infty$. Here we add a result without boundedness of u and v. For that purpose, we need the following definition.

Definition 3.6. Let u be in $\mathcal{H}(\mathbb{D})$ and φ be in $\mathcal{S}(\mathbb{D})$. Denote by $\Gamma_{u,\varphi}$ the subset of Δ such that $|\varphi(z_n)| \to 1$ and $|u(z_n)| \not\to 0$ as $n \to \infty$.

Theorem 3.7. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Then $uC_\varphi - vC_\psi$ is compact on H^∞ if and only if the following conditions hold:*

(i) $\Gamma_{u,\varphi} = \Gamma_{v,\psi}$.

(ii) *For any* $\{z_n\} \in \Gamma_{u,\varphi}$, $\lim_{n\to\infty} |u(z_n) - v(z_n)| = 0$.

(iii) *For any* $\{z_n\} \in \Gamma_{u,\varphi}$, $\lim_{n\to\infty} |v(z_n)|\rho(\varphi(z_n), \psi(z_n)) = 0$.

It is possible to replace v with u in condition (iii).

Clearly, if $uC_\varphi - vC_\psi : \mathcal{B} \to H^\infty$ is bounded (compact), then $uC_\varphi - vC_\psi : H^\infty \to H^\infty$ is bounded (compact, respectively). But the inverse may not always hold for general weights u and v.

Here we present the characterization of the compactness of wC_φ in [11] and [6].

Corollary 3.8. *Let w be in $\mathcal{H}(\mathbb{D})$ and φ be in $\mathcal{S}(\mathbb{D})$. Suppose that wC_φ is boundedly acting from $\mathcal{B}$ to H^∞. Then wC_φ is compact from $\mathcal{B}$ to H^∞ if and only if*

$$\lim_{|\varphi(z)|\to 1} |w(z)|\beta(0, \varphi(z)) = 0.$$

Example 3.9. Let φ and ψ be in Example 2.6. Moreover, put

$$\mu(z) = \left(\log \frac{e}{1 - \varphi(z)}\right)^{-1}$$

and $\nu(z) = \mu(z) - (1 - \varphi(z))(1 - \psi(z))$. Then neither μC_φ nor νC_ψ is compact from $\mathcal{B}$ to H^∞, but $\mu C_\varphi - \nu C_\psi$ is compact from $\mathcal{B}$ to H^∞.

4. Norms and essential norms of $C_\varphi - C_\psi$

Finally, in this section, we will estimate the operator norms and the essential norms of the differences of two composition operators acting from $\mathcal{B}$ to H^∞. To determine the topological structure of the set of all composition operators, one of the important problems is to estimate the norms and the essential norms of differences of two composition operators.

Recall that the essential norm $\|T\|_{e,\, X\to Y}$ of a bounded linear operator T from X to Y is defined to be the distance from T to the closed ideal of compact operators, that is,

$$\|T\|_{e,\, X\to Y} = \inf\{\|T + K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\},$$

where $\|\cdot\|_{X\to Y}$ is the operator norm from X to Y. Notice that T is compact from X to Y if and only if $\|T\|_{e,\, X\to Y} = 0$.

Proposition 4.1. *Let $u \in \mathcal{H}(\mathbb{D})$ and $\varphi, \psi \in \mathcal{S}(\mathbb{D})$. Then*

$$\|u(C_\varphi - C_\psi)\|_{\mathcal{B}\to H^\infty} = \sup_{z\in\mathbb{D}} |u(z)|\, \beta(\varphi(z), \psi(z)).$$

Proof. By the identity (1.3) of the induced distance on $\mathcal{B}$, we have that

$$\begin{aligned} \|u(C_\varphi - C_\psi)\|_{\mathcal{B}\to H^\infty} &= \sup_{\|f\|_{\mathcal{B}}\le 1} \sup_{z\in\mathbb{D}} |(u(C_\varphi - C_\psi)f(z))| \\ &= \sup_{z\in\mathbb{D}} |u(z)| \sup_{\|f\|_{\mathcal{B}}\le 1} |f(\varphi(z)) - f(\psi(z))| \\ &= \sup_{z\in\mathbb{D}} |u(z)|\, \beta(\varphi(z), \psi(z)). \end{aligned}$$

□

We can obtain a result in [9] as a corollary of Proposition 4.1.

Corollary 4.2. *Let $\varphi, \psi \in S(\mathbb{D})$. Then $C_\varphi - C_\psi$ is bounded from $\mathcal{B}$ to H^∞ if and only if*

$$\sup_{z\in\mathbb{D}} \rho(\varphi(z), \psi(z)) < 1.$$

Next we estimate the essential norm of $C_\varphi - C_\psi$.

Proposition 4.3. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{D})$. Suppose that $C_\varphi - C_\psi$ is bounded from $\mathcal{B}$ to H^∞. Then we have that*

$$\limsup_{|\varphi(z)|\to 1} \beta(\varphi(z), \psi(z)) \le \|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} \le 2 \limsup_{|\varphi(z)|\to 1} \beta(\varphi(z), \psi(z)).$$

Proof. First, we suppose that $\|\varphi\|_\infty < 1$. By Corollary 4.2, we have that $\|\psi\|_\infty < 1$. Then both C_φ and C_ψ are compact from $\mathcal{B}$ to H^∞ (see [11]). So we have that $\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} = 0$.

Next we suppose that $\|\varphi\|_\infty = 1$. Fix a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let $\rho_n = \rho(\varphi(z_n), \psi(z_n))$ and $\alpha_n = \alpha_{\varphi(z_n)}$, where $\alpha_{\varphi(z_n)}$ is the automorphism of $\mathbb{D}$ defined by (1.2). Moreover we may assume that $\lim_{|\varphi(z_n)|\to 1} \rho_n > 0$. Put $\lambda_n = \alpha_n(\psi(z_n))/\rho_n$ and

$$f_{r,n}(z) = \frac{1}{2} \log \frac{1 + r\overline{\lambda_n}\alpha_n(z)}{1 - r\overline{\lambda_n}\alpha_n(z)} - \frac{1}{2} \log \frac{1 + r\overline{\lambda_n}\varphi(z_n)}{1 - r\overline{\lambda_n}\varphi(z_n)} \tag{4.1}$$

for $r, 0 < r < 1$. Then we have that $f_{r,n}(0) = 0$ and

$$\begin{aligned} (1 - |z|^2)\, |f'_{r,n}(z)| &= \left| \frac{r\overline{\lambda_n}(1 - |z|^2)(1 - |\varphi(z_n)|^2)}{(1 - \overline{\varphi(z_n)}z)^2(1 - r^2\overline{\lambda_n}^2\alpha_n(z)^2)} \right| \\ &\le r\, \frac{1 - |\alpha_n(z)|^2}{1 - r^2|\alpha_n(z)|^2} \le r. \end{aligned} \tag{4.2}$$

So $\|f_{r,n}\|_{\mathcal{B}} \le r$ for any n. From the first line of (4.2), we conclude that $\{f_{r,n}\}$ is in the little Bloch space $\mathcal{B}_o$.

We notice that if a bounded sequence in $\mathcal{B}_o$ converges to zero uniformly on compact subsets of $\mathbb{D}$, then it also converges weakly to zero in $\mathcal{B}$ (for example, refer to [10]). For any compact operator $K : \mathcal{B} \to H^\infty$, $\|Kf_{r,n}\|_\infty \to 0$ as $n \to \infty$. Then

we obtain that

$$\begin{aligned}\|C_\varphi - C_\psi - K\|_{\mathcal{B}\to H^\infty} &\geq \frac{1}{r}\limsup_{n\to\infty}\left(\|(C_\varphi - C_\psi)f_{r,n}\|_\infty - \|Kf_{r,n}\|_\infty\right)\\ &= \frac{1}{r}\limsup_{n\to\infty}\|f_{r,n}(\varphi(z)) - f_{r,n}(\psi(z))\|_\infty\\ &\geq \frac{1}{r}\limsup_{n\to\infty}|f_{r,n}(\varphi(z_n)) - f_{r,n}(\psi(z_n))|\\ &= \frac{1}{r}\limsup_{n\to\infty}\frac{1}{2}\log\frac{1+r\rho_n}{1-r\rho_n}.\end{aligned}$$

Taking the supremum over all sequences $\{z_n\}$ such that $|\varphi(z_n)| \to 1$, we get that

$$\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} \geq \frac{1}{r}\limsup_{|\varphi(z)|\to 1}\frac{1}{2}\log\frac{1+r\rho(\varphi(z),\psi(z))}{1-r\rho(\varphi(z),\psi(z))}.$$

Letting $r \to 1$, we obtain that

$$\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} \geq \limsup_{|\varphi(z)|\to 1}\beta(\varphi(z),\psi(z)).$$

To estimate the upper bound of $\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty}$, let $\varphi_r(z) = r\varphi(z)$ and $\psi_r(z) = r\psi(z)$ for $r \in (0,1)$. Put $K_r = C_{\varphi_r} - C_{\psi_r}$. Since K_r is compact from $\mathcal{B}$ to H^∞, we have that

$$\begin{aligned}\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} &\leq \|C_\varphi - C_\psi - K_r\|_{\mathcal{B}\to H^\infty}\\ &= \sup_{\|f\|_{\mathcal{B}}\leq 1}\sup_{z\in\mathbb{D}}\|(C_\varphi - C_\psi - K_r)f\|_\infty.\end{aligned}$$

Let s be a positive number less than 1 and put $D_s = \{z \in \mathbb{D} : |\varphi(z)| \leq s\}$. Since D_s is a compact subset of $\mathbb{D}$, φ_r converges uniformly to φ on D_s. We obtain that

$$\lim_{r\to 1}\sup_{z\in D_s}\beta(\varphi(z),\varphi_r(z)) = 0.$$

On the other hand, Corollary 4.2 implies that there exists $t \in (0,1)$ such that $|\psi(z)| \leq t$ on D_s. Then we get

$$\lim_{r\to 1}\sup_{z\in D_s}\beta(\psi(z),\psi_r(z)) = 0.$$

So it follows that

$$\begin{aligned}&\lim_{r\to 1}\sup_{\|f\|_{\mathcal{B}}\leq 1}\sup_{z\in D_s}\big|\big((C_\varphi - C_\psi - K_r)f\big)(z)\big|\\ &\quad\leq \lim_{r\to 1}\sup_{\|f\|_{\mathcal{B}}\leq 1}\sup_{z\in D_s}\big(|f(\varphi(z)) - f(\varphi_r(z))| + |f(\psi(z)) - f(\psi_r(z))|\big)\\ &\quad\leq \lim_{r\to 1}\sup_{z\in D_s}\big(\beta(\varphi(z),\varphi_r(z)) + \beta(\psi(z),\psi_r(z))\big)\\ &\quad= 0.\end{aligned}$$

This implies that

$$\begin{aligned}\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} &\leq \lim_{r\to 1}\sup_{\|f\|_{\mathcal{B}}\leq 1}\sup_{z\in\mathbb{D}\setminus D_s}\big|\big((C_\varphi - C_\psi - K_r)f\big)(z)\big|\\ &\leq \lim_{r\to 1}\sup_{z\in\mathbb{D}\setminus D_s}\big(\beta(\varphi(z),\psi(z)) + \beta(\varphi_r(z),\psi_r(z))\big).\end{aligned}$$

Here we have that

$$\begin{aligned}\rho(\varphi_r(z),\psi_r(z)) &= \left|\frac{r(\varphi(z)-\psi(z))}{1-r^2\overline{\varphi(z)}\psi(z)}\right| \\ &\le \frac{1}{r}\,\rho(\varphi(z),\psi(z)).\end{aligned}$$

For r close enough to 1, we get

$$\frac{1}{r}\sup_{z\in\mathbb{D}\setminus D_s}\rho(\varphi(z),\psi(z))<1.$$

Then we have that

$$\beta(\varphi_r(z),\psi_r(z))\le\frac{1}{2}\,\log\frac{r+\rho(\varphi(z),\psi(z))}{r-\rho(\varphi(z),\psi(z))}.$$

Hence we obtain that

$$\|C_\varphi-C_\psi\|_{e,\,\mathcal{B}\to H^\infty}\le 2\sup_{z\in\mathbb{D}\setminus D_s}\beta(\varphi(z),\psi(z)).$$

Letting $s\to 1$, we conclude that

$$\|C_\varphi-C_\psi\|_{e,\,\mathcal{B}\to H^\infty}\le 2\limsup_{|\varphi(z)|\to 1}\beta(\varphi(z),\psi(z)).$$

The proof is complete. □

As we mentioned, C_φ is not an isolated point of the space of composition operators on H^∞ if and only if φ holds (1.1). If φ is not an extreme point of the closed unit ball of H^∞, the corresponding function ω defined by

$$\omega(z)=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1-|\varphi(e^{i\theta})|)d\theta\right) \tag{4.3}$$

is an outer function in H^∞ such that $|\varphi|+|\omega|\le 1$ on $\mathbb{D}$ and $|\varphi(e^{i\theta})|+|\omega(e^{i\theta})|=1$ a.e. Refer to [3] for more information on the Hardy space theory. Thus we have that $|\omega(z)|\to 0$ as $|\varphi(z)|\to 1$. Here we give the following.

Theorem 4.4. *Let φ and ψ be in $\mathcal{S}(\mathbb{D})$. Suppose that $C_\varphi-C_\psi$ acts boundedly from $\mathcal{B}$ to H^∞. Then*

$$\limsup_{|\varphi(z)|\to 1}\beta(\varphi(z),\psi(z))\le\|C_\varphi-C_\psi\|_{e,\,\mathcal{B}\to H^\infty}\le\limsup_{|\omega(z)|\to 0}\beta(\varphi(z),\psi(z)),$$

where ω is the outer function of (4.3).

Proof. By the assumption, it follows that C_φ and C_ψ are in the same path component of the space of all composition operators on H^∞. Then C_φ is not an isolated point of that space, which equals that φ is not an extreme point of the closed unit ball of H^∞. Therefore we get the outer function ω defined by (4.3). Since ω has no zero on $\mathbb{D}$, the m-th root of ω can be defined as a function in H^∞. Put $K_m=\omega^{1/m}(C_\varphi-C_\psi)$ for every positive integer m. Then Theorem 3.3 implies that K is compact from $\mathcal{B}$ to H^∞. So we have that

$$\begin{aligned}\|C_\varphi-C_\psi\|_{e,\,\mathcal{B}\to H^\infty} &\le \|C_\varphi-C_\psi-K_m\|_{\mathcal{B}\to H^\infty} \\ &= \sup_{\|f\|_\infty\le 1}\sup_{z\in\mathbb{D}}\left|(1-\omega(z)^{1/m})(f\circ\varphi(z)-f\circ\psi(z))\right| \\ &\le \sup_{z\in\mathbb{D}}|1-\omega(z)^{1/m}|\,\beta(\varphi(z),\psi(z)).\end{aligned}$$

Letting $m \to \infty$, we have that

$$\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} \le \lim_{n\to\infty} \sup_{z\in\mathbb{D}} |1-\omega(z)^{1/m}|\,\beta(\varphi(z),\psi(z)).$$

For any sequence $\{z_n\} \subset \mathbb{D}$ such that $\limsup |\omega(z_n)| > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} |1-\omega(z_n)^{1/m}|\,\beta(\varphi(z_n),\psi(z_n)) = 0.$$

Thus we obtain that

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,\,\mathcal{B}\to H^\infty} \\
&\quad \le \lim_{m\to\infty} \sup\Big\{ \limsup_{n\to\infty} |1-\omega(z_n)^{1/m}|\beta(\varphi(z_n),\psi(z_n)) : \{z_n\}\subset\mathbb{D},\ |\omega(z_n)|\to 0\Big\} \\
&\quad = \sup\Big\{ \limsup_{n\to\infty} \beta(\varphi(z_n),\psi(z_n)) : \{z_n\}\subset\mathbb{D},\ |\omega(z_n)|\to 0\Big\} \\
&\quad = \limsup_{|\omega(z)|\to 0} \beta(\varphi(z_n),\psi(z_n)).
\end{aligned}$$

Our proof is complete. □

At the end of this section we add the results on the estimations of the operator norms and the essential norms in the case of the differences of two weighted composition operators.

We check the proof of Theorem 2.3 and can roughly estimate the operator norms. Put

$$\begin{aligned}
M(z) \quad = \quad \min\Big\{ &|u(z)\widetilde{\beta}(\varphi(z))| + |v(z)\widetilde{\beta}(\psi(z))|, \\
&|\big(u(z)-v(z)\big)\widetilde{\beta}(\varphi(z))| + |v(z)|\,\beta(\varphi(z),\psi(z)), \\
&|\big(u(z)-v(z)\big)\widetilde{\beta}(\psi(z))| + |u(z)|\,\beta(\varphi(z),\psi(z))\Big\}.
\end{aligned}$$

Proposition 4.5. *Let u, v be in $\mathcal{H}(\mathbb{D})$ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Then*

$$\begin{aligned}
C \max\Bigg\{ &\sup_{\{z_n\}\in D_{u,\varphi}} \lim_{n\to\infty} |u(z_n)-v(z_n)|\widetilde{\beta}(\varphi(z_n)), \\
&\sup_{\{z_n\}\in D_{u,\varphi}} \lim_{n\to\infty} |v(z_n)|\beta(\varphi(z_n),\psi(z_n))\Bigg\} \\
&\le \|uC_\varphi - vC_\psi\|_{\mathcal{B}\to H^\infty} \\
&\le \sup_{z\in\mathbb{D}} M(z).
\end{aligned}$$

Also we can estimate the essential norms. We use the following lemma (see [5]).

Lemma 4.6. *For z, w in $\mathbb{D}$, let $L(z,w)$ be the continuous function on $\mathbb{D}\times\mathbb{D}$ defined in Lemma 2.2. Then*

$$\lim_{|z|\to 1} \sup_{w\in\mathbb{D}} |L(z,w)| = 1.$$

Theorem 4.7. *Let u, v be in H^∞ and φ, ψ be in $\mathcal{S}(\mathbb{D})$. Suppose that $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞. Then*

$$\begin{aligned}
&\max\left\{\frac{1}{2}\limsup_{|\varphi(z)|\to 1}|u(z)-v(z)|\widetilde{\beta}(\varphi(z)),\ \limsup_{|\varphi(z)|\to 1}|v(z)|\beta(\varphi(z),\psi(z))\right\}\\
&\quad\le \|uC_\varphi - vC_\psi\|_{e,\mathcal{B}\to H^\infty}\\
&\quad\le 2\limsup_{|\varphi(z)|\to 1}\left\{|u(z)-v(z)|\widetilde{\beta}(\varphi(z))+|v(z)|\beta(\varphi(z),\psi(z))\right\}.
\end{aligned}$$

Proof. First of all, we will estimate the upper bound. For $r \in (0,1)$, let $\varphi_r(z) = \varphi(rz)$ and $\psi_r(z) = \psi(rz)$. We can easily check that $K_r = uC_{\varphi_r} - vC_{\psi_r}$ is bounded and compact from $\mathcal{B}$ to H^∞. Then we have

$$\|uC_\varphi - vC_\psi\|_{e,\,\mathcal{B}\to H^\infty} \le \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}\|(uC_\varphi - vC_\psi - K_r)f\|_\infty.$$

Let $s \in (0,1)$ and $D_s = \{z \in \mathbb{D} : |\varphi(z)| \le s\}$. So it follows that

$$\begin{aligned}
&\lim_{r\to 1}\sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in D_s}\left|(C_\varphi - C_\psi - K_r)f(z)\right|\\
&\quad\le \lim_{r\to 1}\sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in D_s}\left(\|u\|_\infty\beta(\varphi(z),\varphi_r(z)) + \|v\|_\infty\beta(\psi(z),\psi_r(z))\right)\\
&\quad= 0.
\end{aligned}$$

We use a similar way as in the latter part of the proof of Proposition 4.3 and can obtain that

$$\begin{aligned}
&\|uC_\varphi - vC_\psi\|_{e,\,\mathcal{B}\to H^\infty}\\
&\quad\le \|uC_\varphi - vC_\psi - K_r\|_{\mathcal{B}\to H^\infty}\\
&\quad\le 2\limsup_{|\varphi(z)|\to 1}\left\{|u(z)-v(z)|\widetilde{\beta}(\varphi(z))+|v(z)|\beta(\varphi(z),\psi(z))\right\}.
\end{aligned}$$

Next, we will consider the lower bound. Fix a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Take the same test function as (4.1):

$$f_{r,n}(z) = \frac{1}{2}\log\frac{1+r\overline{\lambda_n}\alpha_n(z)}{1-r\overline{\lambda_n}\alpha_n(z)} - \frac{1}{2}\log\frac{1+r\overline{\lambda_n}\varphi(z_n)}{1-r\overline{\lambda_n}\varphi(z_n)},$$

where $r, 0 < r < 1$, $\rho_n = \rho(\varphi(z_n), \psi(z_n))$, $\alpha_n = \alpha_{\varphi(z_n)}$, and $\lambda_n = \alpha_n(\psi(z_n))/\rho_n$. Then $\{f_{r,n}\}$ is in the little Bloch space $\mathcal{B}_o$ and converges to 0 uniformly on every compact subset of $\mathbb{D}$. As we discussed in the proof of Proposition 4.3, $f_{r,n}$ converges to 0 weakly in $\mathcal{B}$. For any compact operator $K : \mathcal{B} \to H^\infty$, $\|Kf_{r,n}\|_\infty \to 0$ as $n \to \infty$. Then we obtain that

$$\begin{aligned}
&\|uC_\varphi - vC_\psi - K\|_{\mathcal{B}\to H^\infty}\\
&\quad\ge \frac{1}{r}\left(|u(z_n)f_{r,n}(\varphi(z_n)) - v(z_n)f_{r,n}(\psi(z_n))| - \|Kf_{r,n}\|_\infty\right)\\
&\quad\ge \frac{1}{r}\left(\frac{1}{2}|v(z_n)|\log\frac{1+r\rho_n}{1-r\rho_n}\right.\\
&\qquad\left. -|u(z_n)-v(z_n)|\frac{1}{2}\log\frac{1+r\overline{\lambda_n}\varphi(z_n)}{1-r\overline{\lambda_n}\varphi(z_n)} - \|Kf_{r,n}\|_\infty\right).
\end{aligned}$$

Since $uC_\varphi - vC_\psi$ is boundedly acting from $\mathcal{B}$ to H^∞, condition (ii) in Theorem 2.3 implies that if $\{z_n\} \in D_{u,\varphi}$ and $|\varphi(z_n)| \to 1$, then $|u(z_n) - v(z_n)| \to 0$. If $\{z_n\} \notin D_{u,\varphi}$ and $|\varphi(z_n)| \to 1$, then $|u(z_n)| \to 0$ and similarly, $|v(z_n)| \to 0$. Consequently,

$$\lim_{|\varphi(z_n)|\to 1} |u(z_n) - v(z_n)| = 0.$$

Taking the supremum over all sequences $\{z_n\}$ such that $|\varphi(z_n)| \to 1$, we get that

$$\|uC_\varphi - vC_\psi\|_{e,\,\mathcal{B}\to H^\infty} \geq \frac{1}{r} \limsup_{|\varphi(z)|\to 1} \frac{1}{2} |v(z)| \log \frac{1 + r\rho(\varphi(z), \psi(z))}{1 - r\rho(\varphi(z), \psi(z))}.$$

Letting $r \to 1$, we obtain that

$$\|uC_\varphi - vC_\psi\|_{e,\,\mathcal{B}\to H^\infty} \geq \limsup_{|\varphi(z)|\to 1} |v(z)|\beta(\varphi(z), \psi(z)). \tag{4.4}$$

By the similar estimate as (2.1),

$$\begin{aligned}
&\|uC_\varphi - vC_\psi - K\|_{\mathcal{B}\to H^\infty} \|g_n\|_{\mathcal{B}} \\
&\quad \geq \|(uC_\varphi - vC_\psi - K)g_n\|_\infty \\
&\quad \geq |u(z) - v(z)||g_n(\varphi(z))| \\
&\qquad - |v(z)||g_n(\varphi(z)) - g_n(\psi(z))| - \|Kg_n\|_\infty
\end{aligned}$$

for any compact operator $K : \mathcal{B} \to H^\infty$ and any bounded sequence $\{g_n\} \subset \mathcal{B}$ which converges to 0 weakly in $\mathcal{B}$.

By passing to a subsequence, we may assume that $1 \leq \beta(0, \varphi(z_n)) = \widetilde{\beta}(\varphi(z_n))$. For $p > 0$, take the test function similar to (3.4):

$$g_{n,p}(z) = \left(\frac{1}{2} \log \frac{1 + |\varphi(z_n)|}{1 - |\varphi(z_n)|}\right)^{-p} \left(\frac{1}{2} \log \frac{(1 + |\varphi(z_n)|)^2}{1 - \overline{\varphi(z_n)}z}\right)^{p+1}.$$

Then $|g_{n,p}(0)| \to 0$ as $n \to \infty$ and $\|g_{n,p}\|_{\mathcal{B}} \leq C_p$. Moreover, Lemma 4.6 implies that $\|g_{n,p}\|_{\mathcal{B}} \to p + 1$. Since $\|Kh_n\|_\infty \to 0$ as $n \to \infty$, we get

$$\begin{aligned}
&(p+1)\|uC_\varphi - vC_\psi - K\|_{\mathcal{B}\to H^\infty} \\
&\quad \geq \lim_{n\to\infty} |u(z_n) - v(z_n)|\widetilde{\beta}(\varphi(z_n)) - (p+1) \lim_{n\to\infty} |v(z_n)|\beta(\varphi(z_n), \psi(z_n))
\end{aligned}$$

for any $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n)| \to 1$. By letting $p \to 0$, we obtain from (4.4) that

$$\limsup_{|\varphi(z)|\to 1} |u(z) - v(z)|\widetilde{\beta}(\varphi(z)) \leq 2\|uC_\varphi - vC_\psi\|_{e,\mathcal{B}\to H^\infty}.$$

Our proof is complete. □

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