QUIVER VARIETIES AND PATH REALIZATIONS
ARISING FROM ADJOINT CRYSTALS OF TYPE $A_n^{(1)}$

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ABSTRACT. Let $B(\Lambda_0)$ be the level 1 highest weight crystal of the quantum affine algebra $U_q(A_n^{(1)})$. We construct an explicit crystal isomorphism between the geometric realization $B(\Lambda_0)$ of $B(\Lambda_0)$ via quiver varieties and the path realization $P^{ad}(\Lambda_0)$ of $B(\Lambda_0)$ arising from the adjoint crystal $B^{ad}$.

INTRODUCTION

The theory of perfect crystals developed in [7] has many important and interesting applications to the representation theory of quantum affine algebras and the theory of vertex models in mathematical physics. In particular, the crystal $B(\lambda)$ of an integrable highest weight module over a quantum affine algebra can be realized as the crystal $P^B(\lambda)$ consisting of $\lambda$-paths arising from a given perfect crystal $B$. In [1], Benkart, Frenkel, Kang and Lee gave a uniform construction of level 1 perfect crystals for all quantum affine algebras. These perfect crystals are called the adjoint crystals because, when forgetting 0-arrows, they coincide with the direct sums of the trivial crystals and the crystals of adjoint or little adjoint representations of finite-dimensional simple Lie algebras.

On the other hand, for a symmetric Kac-Moody algebra $g$, Lusztig gave a geometric construction of $U_q^{-}(g)$ in terms of perverse sheaves on quiver varieties and introduced the notion of canonical basis which yields natural bases for all integrable highest weight modules as well [13, 14]. In [12], Kashiwara and Saito defined a crystal structure on the set $B(\infty)$ of irreducible components of Lusztig’s quiver varieties and showed that $B(\infty)$ is isomorphic to the crystal $B(\infty)$ of $U_q^{-}(g)$. Moreover, in [16, 17], Nakajima defined a new family of quiver varieties associated with a dominant integral weight $\lambda$ and gave a geometric realization of the integrable highest weight $g$-module $V(\lambda)$. In [12], Saito defined a crystal structure on the set $B(\lambda)$ of irreducible components of certain Lagrangian subvarieties of Nakajima’s quiver varieties and showed that $B(\lambda)$ is isomorphic to the crystal $B(\lambda)$ of $V(\lambda)$.

Therefore, for quantum affine algebras it is natural to investigate the crystal isomorphism between the geometric realization $B(\lambda)$ and the path realization $P^B(\lambda)$.

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1. THE QUANTUM AFFINE ALGEBRA $U_q(A_n^{(1)})$

Let $I = \mathbb{Z}/(n+1)\mathbb{Z}$ be the index set. The affine Cartan datum of $A_n^{(1)}$-type consists of

(i) the affine Cartan matrix

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

(ii) the dual weight lattice $P^\vee = \bigoplus_{i=0}^n \mathbb{Z}h_i \oplus \mathbb{Z}d,$

(iii) the affine weight lattice $P = \bigoplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}d \subset \mathfrak{h}^*,$ where

$$\mathfrak{h} = \mathbb{C} \otimes P^\vee,$$ $\Lambda_i(h_j) = \delta_{ij},$ $\Lambda_i(d) = 0,$ $\delta(h_i) = 0,$ $\delta(d) = 1$ $(i, j \in I),$ 

(iv) the set of simple coroots $\Pi^\vee = \{h_i| i \in I\},$

(v) the set of simple roots $\Pi = \{\alpha_i| i \in I\}$ given by

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d) = \delta_{0,j} \quad (i, j \in I).$$

The free abelian group $Q = \bigoplus_{i=0}^n \mathbb{Z}\alpha_i$ is called the root lattice, and the semigroup $Q^+ = \sum_{i=0}^n \mathbb{Z}_{\geq 0}\alpha_i$ is called the positive root lattice. For $\alpha = \sum_{i \in I} k_i \alpha_i \in Q^+,$ the number $ht(\alpha) = \sum_{i \in I} k_i$ is called the height of $\alpha.$ For $\lambda, \mu \in \mathfrak{h}^*,$ we define $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q^+.$ The elements in $P^+ = \{\lambda \in P| \lambda(h_i) \geq 0, i \in I\}$ are called the dominant integral weights. Note that the minimal imaginary root is given by $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n \in Q^+.$ The element $c = h_0 + h_1 + \cdots + h_n \in P^\vee$ is called the canonical central element.

Given $n \in \mathbb{Z}$ and any symbol $q,$ we define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and set $[0]_q! = 1,$ $[n]_q! = [n]_q[n-1]_q \cdots [1]_q.$ For $m \geq n \geq 0,$ let

$$\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

**Definition 1.1.** The quantum affine algebra $U_q(\mathfrak{g}) = U_q(A_n^{(1)})$ is an associative algebra over $\mathbb{C}(q)$ with 1 generated by $e_i, f_i$ $(i \in I)$ and $q^h$ $(h \in P^\vee)$ satisfying the...
An abstract crystals. For convenience, we recall some of the basic definitions and properties of abstract crystals.

Example 1.3. Let \( C \) be the crystal of their reducible highest weight module \( V(\lambda) \) with highest weight \( \lambda \in P^+ \).

The definition of category \( O^g \), Kashiwara operators and crystal bases can be found in [10, 4]. It was shown in [10] that every \( U_q(g) \)-module in the category \( O^g \) has a crystal basis. The notion of abstract crystals was introduced in [11]. For convenience, we recall some of the basic definitions and properties of abstract crystals.

Definition 1.2. An abstract crystal associated with the Cartan datum \((A, \Pi, \Pi', P_+, P^-)\) is a set \( B \) together with the maps \( \omega : B \to P, \, \varepsilon_i, \, \tilde{f}_i : B \to B \cup \{0\}, \) and \( \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\} \) satisfying the following properties:

1. \( \varepsilon_i(b) = (\varepsilon_i + (h_i, wt(b))) \) for all \( i \in I \),
2. \( \omega(\varepsilon_i b) = wt(b) + \alpha_i \) if \( \varepsilon_i b \in B \),
3. \( \omega(\tilde{f}_i b) = wt(b) - \alpha_i \) if \( \tilde{f}_i b \in B \),
4. \( \varepsilon_i(\varepsilon_i b) = \varepsilon_i(b) - 1, \varphi_i(\varepsilon_i b) = \varphi_i(b) + 1 \) if \( \varepsilon_i b \in B \),
5. \( \varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1 \) if \( \tilde{f}_i b \in B \),
6. \( \tilde{f}_i b = b' \) if and only if \( b = \varepsilon_i b' \) for \( b, b' \in B, \, i \in I \),
7. \( \varepsilon_i(b) = -\infty \) for \( b \in B \), then \( \varepsilon_i b = \tilde{f}_i b = 0 \).

We often say that \( B \) is a \( U_q(g) \)-crystal. We denote \( B_\lambda = \{b \in B | wt(b) = \lambda\} \) so that \( B = \bigsqcup_{\lambda \in P} B_\lambda \).

Example 1.3.

1. Let \( (L, B) \) be a crystal basis of \( M \in O^g \). Then \( B \) has a crystal structure, where the maps \( \varepsilon_i, \varphi_i \) are given by
   \[ \varepsilon_i(b) = \max\{k \geq 0 | \varepsilon_i b \neq 0\}, \quad \varphi_i(b) = \max\{k \geq 0 | \tilde{f}_i b \neq 0\}. \]

In particular, we denote by \( B(\lambda) \) the crystal of the irreducible highest weight module \( V(\lambda) \) with highest weight \( \lambda \in P^+ \).

2. Let \( (L(\infty), B(\infty)) \) be a crystal basis of \( U_q^{-}(g) \). Then \( B(\infty) \) has a crystal structure, where the maps \( \varepsilon_i, \varphi_i \) are given by
   \[ \varepsilon_i(b) = \max\{k \geq 0 | \varepsilon_i b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + (h_i, wt(b)). \]

3. For \( \lambda \in P \), let us consider \( T_\lambda = \{t_\lambda\} \) with the maps
   \[ \omega(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \tilde{f}_i t_\lambda = 0 \text{ for } i \in I, \]
   \[ \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \text{ for } i \in I. \]

Then \( T_\lambda \) is a crystal.

4. Let \( C = \{c\} \). We define the maps
   \[ \omega(c) = 0, \quad \varepsilon_i c = \tilde{f}_i c = 0, \quad \varepsilon_i(c) = \varphi_i(c) = 0 \text{ (i \in I)}. \]

Then \( C \) is a crystal.
Definition 1.4. Let $B_1$ and $B_2$ be crystals.
(1) A map $\psi : B_1 \to B_2$ is a crystal morphism if it satisfies the following properties:
(a) for $b \in B_1$, we have
\[ wt(\psi(b)) = wt(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \text{ for all } i \in I, \]
(b) for $b \in B_1$ and $i \in I$ with $\tilde{f}_i b \in B_1$, we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.
(2) A crystal morphism $\psi : B_1 \to B_2$ is called strict if
\[ \psi(\tilde{e}_i b) = \tilde{e}_i \psi(b), \quad \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for all } i \in I \text{ and } b \in B_1. \]
Here, we understand $\psi(0) = 0$.
(3) $\psi$ is called an embedding if the underlying map $\psi : B_1 \to B_2$ is injective.

Let $B_1$ and $B_2$ be crystals. The tensor product $B_1 \otimes B_2$ is defined to be the set $B_1 \times B_2$ together with the following maps:
(a) $wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$,
(b) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle\}$,
(c) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, wt(b_2) \rangle\}$,
(d) $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$
(e) $\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$

It was shown in [11] that there is a unique strict crystal embedding
\[ B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C \]
sending $u_\lambda$ to $1 \otimes t_\lambda \otimes c$. Here, $u_\lambda$ is the highest weight element of $B(\lambda)$. We denote by $\iota_\lambda$ the composition of the strict embedding and the natural projection:
\[ (1.1) \quad \iota_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda \otimes C \twoheadrightarrow B(\infty). \]
Note that $\iota_\lambda$ is injective but not a crystal morphism.

2. Path realization
Let $U'_q(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, q^{\pm h_i}$ $(i \in I)$, and we set $\overline{P}^\vee = \bigoplus_{i=0}^{n} \mathbb{Z}h_i$, $\overline{P} = \mathbb{C} \otimes_{\mathbb{Z}} \overline{P}^\vee$, $\overline{P} = \bigoplus_{i=0}^{n} \mathbb{Z} \Lambda_i$ and $\overline{P}^+ = \sum_{i=0}^{n} \mathbb{Z}_{\geq 0} \Lambda_i$. Denote by $cl : P \to \overline{P}$ the natural projection from $P$ to $\overline{P}$. An abstract crystal $B$ associated with $U'_q(\mathfrak{g})$ is called a classical crystal. For $b \in B$, we define
\[ \varepsilon(b) = \sum_{i=0}^{n} \varepsilon_i(b) \Lambda_i, \quad \varphi(b) = \sum_{i=0}^{n} \varphi_i(b) \Lambda_i. \]

Definition 2.1. A perfect crystal of level $\ell$ is a finite classical crystal $B$ satisfying the following conditions:
(a) there exists a finite-dimensional $U'_q(\mathfrak{g})$-module with a crystal basis whose crystal graph is isomorphic to $B$,
(b) $B \otimes B$ is connected,
(c) there exists a classical weight $\lambda_0 \in \overline{P}$ such that
\[ wt(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i, \quad \#(B_{\lambda_0}) = 1, \]
(d) for any \(b \in B\), we have \(\langle c, \varepsilon(b) \rangle \geq \ell\),

(e) for each \(\lambda \in \overline{B}^\ell := \{\mu \in \overline{B}^\ell | \langle c, \mu \rangle = \ell\}\), there exist unique vectors \(b^\lambda\) and \(b_\lambda\) in \(B\) such that \(\varepsilon(b^\lambda) = \lambda\), \(\phi(b_\lambda) = \lambda\).

Given a dominant integral weight \(\lambda\) with \(\lambda(c) = \ell\) and a perfect crystal \(B\) of level \(\ell\), it was shown in [7] that there exists a unique crystal isomorphism, called the fundamental isomorphism for perfect crystals,

\[
\psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes B,
\]

sending \(u_\lambda\) to \(u_{\varepsilon(b_\lambda)} \otimes b_\lambda\). By applying this crystal isomorphism repeatedly, we get a sequence of crystal isomorphisms

\[
B(\lambda) \xrightarrow{\sim} B(\lambda_1) \otimes B \xrightarrow{\sim} B(\lambda_2) \otimes B \otimes B \xrightarrow{\sim} \cdots,
\]

where \(\lambda_0 = \lambda\), \(b_0 = b_\lambda\), \(\lambda_{k+1} = \varepsilon(b_k)\), \(b_{k+1} = b_{\lambda_{k+1}}\) (\(k \geq 0\)). The sequence \(p_\lambda := (b_k)_{k=0}^\infty\) is called the ground-state path of weight \(\lambda\), and a sequence \(p = (p_k)_{k=0}^\infty\) of elements \(p_k \in B\) is called a \(\lambda\)-path in \(B\) if \(p_k = b_k\) for all \(k \gg 0\). We denote by \(P^B(\lambda)\) the set of \(\lambda\)-paths in \(B\), which gives rise to the path realization of \(B(\lambda)\).

**Theorem 2.2** ([7]). There exists a unique crystal isomorphism \(B(\lambda) \xrightarrow{\sim} P^B(\lambda)\) which maps \(u_\lambda\) to \(p_\lambda\).

We list some examples of perfect crystals of level 1 and the corresponding ground-state paths (see [1] [8], etc).

**Example 2.3.**

1. The crystal \(B^1\) and its ground-state \(p^1_{\Lambda_i}\) of weight \(\Lambda_i\) \((i \in I)\) are given as

\[
\begin{array}{cccccccccccc}
& & b_1 & \xrightarrow{1} & b_2 & \xrightarrow{2} & \cdots & \xrightarrow{n-1} & b_n & \xrightarrow{n} & b_{n+1} & ,
\end{array}
\]

\[
p^1_{\Lambda_i} = (\ldots, b_1, b_2, \ldots, b_n, b_{n+1}, b_1, b_2, \ldots, b_{i-1}, b_i).
\]

We denote by \(P^1(\Lambda_i)\) the set of all \(\Lambda_i\)-paths in \(B^1\).

2. The crystal \(B^n\) and its ground-state \(p^n_{\Lambda_i}\) of weight \(\Lambda_i\) \((i \in I)\) are given as

\[
\begin{array}{cccccccccccc}
& & & & b_{n+1} & \xrightarrow{n} & b_n & \xrightarrow{n-1} & \cdots & \xrightarrow{2} & b_2 & \xrightarrow{1} & b_1 ,
\end{array}
\]

\[
p^n_{\Lambda_i} = (\ldots, b_{n+1}, b_n, \ldots, b_2, b_1, b_{n+1}, b_n, \ldots, b_{i+2}, b_{i+1}).
\]

We denote by \(P^n(\Lambda_i)\) the set of all \(\Lambda_i\)-paths in \(B^n\).

3. The adjoint crystal \(B^{ad}\) is given as follows.

Let

\[
B^{ad} = \{\emptyset\} \cup \{b_{\pm \alpha_{ij}} \mid 1 \leq i \leq j \leq n\} \cup \{h_i \mid i = 1, \ldots, n\},
\]

where \(\alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j\) for \(1 \leq i \leq j \leq n\). We define the \(i\)-arrows (\(i \in I\)) by

\[
(i \neq 0) \quad b_\alpha \xrightarrow{i} b_\beta \iff \alpha - \alpha_i = \beta,
\]

\[
\begin{array}{cccccccccccc}
b_\alpha & \xrightarrow{i} & h_i & \xrightarrow{i} & b_{-\alpha_i},
\end{array}
\]

\[
(i = 0) \quad b_\alpha \xrightarrow{0} b_\beta \iff \alpha + \theta = \beta \ (\alpha, \beta \neq \pm \theta),
\]

\[
\begin{array}{cccccccccccc}
b_{-\theta} & \xrightarrow{0} & \emptyset & \xrightarrow{0} & b_\theta,
\end{array}
\]
where \( \theta = \alpha_1 + \cdots + \alpha_n \). The crystal \( B^{\text{ad}} \) is a perfect crystal of level 1 with the ground-state path of weight \( \Lambda_0 \),
\[
P^{\text{ad}}_{\Lambda_0} = (\ldots, \emptyset, \ldots, \emptyset, \emptyset).
\]
There is a crystal isomorphism \( p^{\text{ad}} : B^n \otimes B^1 \to B^{\text{ad}} \) given by
\[
p^{\text{ad}}(b_j \otimes b_i) = \begin{cases} 
b_{\text{wt}(b_j \otimes b_i)} & \text{if } \text{wt}(b_j \otimes b_i) \neq 0, \\
h_i & \text{if } \text{wt}(b_j \otimes b_i) = 0, i \neq n + 1, \\
\emptyset & \text{otherwise.} \end{cases}
\]
(2.2)

We denote by \( P^{\text{ad}}(\Lambda_0) \) the set of all \( \Lambda_0 \)-paths in \( B^{\text{ad}} \).

3. Combinatorics of Young walls

In \cite{4, 6}, Kang gave a combinatorial realization of crystal graphs for basic representations of quantum affine algebras of type \( A_n^{(1)} (n \geq 1) \), \( A_{2n-1}^{(2)} (n \geq 3) \), \( D_n^{(1)} (n \geq 4) \), \( A_{2n}^{(2)} \), \( D_{2n+1}^{(2)} (n \geq 2) \), and \( B_n^{(1)} (n \geq 3) \) by using new combinatorial objects called Young walls, which are a generalization of colored Young diagrams used in \cite{2, 5} \cite{15}. In this work, we focus on the quantum affine algebra \( U_q(A_n^{(1)}) \).

The Young wall \( Y^1 \) (resp. \( Y^n \)) is a wall consisting of colored blocks stacked by the following rules:

(a) the colored blocks should be stacked in the pattern \( P^1 \) (resp. \( P^n \)) of weight \( \Lambda_k \) given below,

(b) except for the rightmost column, there should be no free space to the right of any block.

The patterns are given as follows:

\[
\begin{align*}
\begin{array}{ccccccc}
\hline
  & 1 & 2 & 3 & \cdots & k & k+1 \\
\hline
 k & \emptyset & \emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
 k-1 & \emptyset & \emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & \emptyset & \emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
 0 & 1 & 2 & 3 & \cdots & k & k+1 \\
\hline
\end{array}
\end{align*}
\]

the pattern \( P^1 \) of weight \( \Lambda_k \)

\[
\begin{align*}
\begin{array}{ccccccc}
\hline
  & 1 & 2 & 3 & \cdots & k & k+1 \\
\hline
 k & \emptyset & \emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
 k-1 & \emptyset & \emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 1 & \emptyset & \emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
 0 & 1 & 2 & 3 & \cdots & k & k+1 \\
\hline
\end{array}
\end{align*}
\]

the pattern \( P^n \) of weight \( \Lambda_k \)

Note that the height of the columns of a Young wall \( Y \) are weakly decreasing from right to left, so we denote it by \( Y = (y_i)_{i \geq 0} \), where \( y_i \) is the \( i \)-th column of \( Y \).

**Definition 3.1.** Let \( Y \) be a Young wall corresponding to the pattern \( P^1 \) (resp. \( P^n \)).

1. An \( i \)-block in \( Y \) is called a **removable \( i \)-block** if \( Y \) remains a Young wall after removing the block.

2. A place in \( Y \) is called an **admissible \( i \)-slot** if one may add an \( i \)-block to obtain another Young wall.

3. A column in \( Y \) is said to be **\( i \)-removable** (resp. **\( i \)-admissible**) if the column has a removable \( i \)-block (resp. an admissible \( i \)-slot).
Now we define the action of Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) on Young walls. Let $Y = (y_k)_{k \geq 0}$ be a Young wall corresponding to the pattern $P^1$ (resp. $P^n$).

(a) To each column $y_k$ of $Y$, we assign
\[
\begin{cases}
- & \text{if } y_k \text{ is } i\text{-removable}, \\
+ & \text{if } y_k \text{ is } i\text{-admissible}.
\end{cases}
\]

(b) From this sequence of +’s and −’s, cancel out all (+, −) pairs to obtain a finite sequence of −’s followed by +’s. This sequence (−, ..., −, +, ...) is called the $i$-signature of $Y$.

c) We define $\tilde{e}_iY$ to be the Young wall obtained from $Y$ by removing the $i$-block corresponding to the rightmost − in the $i$-signature of $Y$. If there is no − in the $i$-signature, we set $\tilde{e}_iY = 0$.

d) We define $\tilde{f}_iY$ to be the Young wall obtained from $Y$ by adding an $i$-block to the column corresponding to the leftmost + in the $i$-signature of $Y$. If there is no + in the $i$-signature, we set $\tilde{f}_iY = 0$.

We also define
\[
\begin{align*}
\text{wt}(y_j) &= \sum_{i \in I} k_{ij} \alpha_i \ (j \in \mathbb{Z}_{\geq 0}), \\
\text{wt}(Y) &= \Lambda_k - \sum_{j \geq 0} \text{wt}(y_j), \\
\varepsilon_i(Y) &= \text{the number of } -'s \text{ in the } i\text{-signature of } Y, \\
\varphi_i(Y) &= \text{the number of } +'s \text{ in the } i\text{-signature of } Y,
\end{align*}
\]

where $k_{ij}$ is the number of $i$-blocks in the $j$-th column $y_j$ of $Y$. Note that the height of $y_j$ is $\text{ht}(\text{wt}(y_j))$. Let $Y^1(\Lambda_k)$ (resp. $Y^n(\Lambda_k)$) be the set of all Young walls $Y^1$ (resp. $Y^n$) whose shapes are $(n+1)$-reduced Young diagrams; i.e., $Y^1 = (y_j)_{j \geq 0} \in Y^1(\Lambda_k)$ (resp. $Y^n = (\psi_j)_{j \geq 0} \in Y^n(\Lambda_k)$) if and only if
\[
\text{ht}(\text{wt}(y_j)) - \text{ht}(\text{wt}(y_{j+1})) < n + 1 \quad \text{(resp. } \text{ht}(\text{wt}(\psi_j)) - \text{ht}(\text{wt}(\psi_{j+1})) < n + 1)\]
\]

for $j \geq 0$. Then $Y^1(\Lambda_k)$ (resp. $Y^n(\Lambda_k)$) has a $U_q(\mathfrak{g})$-crystal structure, and we have the following theorem.

**Theorem 3.2 [15].** There is a unique crystal isomorphism $B(\Lambda_k) \xrightarrow{\sim} Y^1(\Lambda_k)$ (resp. $Y^n(\Lambda_k)$) which maps the highest weight element $u_\Lambda$ to the empty Young wall $\emptyset$.

Let $p^1 = (\ldots, b_{i_2}, b_{i_1}, b_{i_0})$ be a $\Lambda_k$-path in $P^1$. Consider a Young wall $Y^1_k(p^1) = (y_j(p^1))_{j \geq 0}$ such that the $j$-th column $y_j(p^1)$ is $\emptyset$ if $j > \text{ht}(\Lambda_k - \text{wt}(p^1))$. Otherwise $y_j(p^1)$ is the smallest $j$-th column in $P^1$ satisfying the following conditions:

(a) the top color of $y_j(p^1)$ is $i_j - 1$,

(b) $y_{j+1}(p^1) \leq y_j(p^1)$.

One can prove that the Young wall $Y^1_k(p^1)$ is contained in $Y^1(\Lambda_k)$, and the map
\[
Y^1_k : P^1(\Lambda_k) \rightarrow Y^1(\Lambda_k)
\]

is a crystal isomorphism. If we set $Y^1 = (y_j)_{j \geq 0} \in Y^1(\Lambda_k)$, then the inverse image $p^1$ of $Y^1$ under the crystal isomorphism $Y^1_k$ is
\[
p^1 = (\ldots, b_{a_1}, \ldots, b_{a_1}, b_{a_0}),
\]

where $a_j \equiv \text{ht}(\text{wt}(y_j)) - j + k \pmod{n+1}$ for all $j \geq 0$. 

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In a similar manner, given a \( \Lambda_k \)-path \( p^n = (\ldots, \mathbf{b}_{i_2}, \mathbf{b}_{i_1}, \mathbf{b}_0) \) in \( B^n \), we have a Young wall \( Y^n_k(p^n) = (\mathbf{y}_j(p^n))_{j \geq 0} \) such that the \( j \)-th column \( \mathbf{y}_j(p^n) \) is \( \emptyset \) if \( j > \text{ht}(\Lambda_k - \text{wt}(p^n)) \). Otherwise \( \mathbf{y}_j(p^n) \) is the smallest \( j \)-th column in \( P^n \) satisfying the following conditions:

1. the top color of \( \mathbf{y}_j(p^n) \) is \( i_j \),
2. \( \mathbf{y}_{j+1}(p^n) \leq \mathbf{y}_j(p^n) \).

One can prove that the Young wall \( Y^n_k(p^n) \) is contained in \( \mathcal{Y}^n(\Lambda_k) \) and the map

\[
Y^n_k : P^n(\Lambda_k) \rightarrow \mathcal{Y}^n(\Lambda_k)
\]

is a crystal isomorphism. If we set \( Y^n = (\mathbf{y}_j)_{j \geq 0} \in \mathcal{Y}^n(\Lambda_k) \), then the inverse image \( p^n \) of \( Y^n \) under the crystal isomorphism \( Y^n_k \) is

\[
p^n = (\ldots, \mathbf{b}_{i_2}, \mathbf{b}_{i_1}, \mathbf{b}_0),
\]

where \( b_j = 1 - \text{ht}(\text{wt}(\mathbf{y}_j)) + j + k (\text{mod} \ n + 1) \) for all \( j \geq 0 \).

**Example 3.3.** Let \( g \) be the Kac-Moody algebra of type \( A_3^{(1)} \). Fix an element

\[
b = f_0 f_2 f_1 (f_1 f_2 f_3 f_0)^3 u_{\Lambda_0} \in B(\Lambda_0),
\]

where \( u_{\Lambda_0} \) is the highest weight element of \( B(\Lambda_0) \). Then the \( \Lambda_0 \)-paths \( p^1 \in P^1(\Lambda_0) \), \( p^n \in P^n(\Lambda_0) \) and \( p^{ad} \in P^{ad}(\Lambda_0) \) corresponding to \( b \) are given as

\[
p^1 = (\ldots, b_3, b_1, b_2, b_3, b_0, 0, 1, b_2, b_3, b_0, b_1, b_2, b_3, b_0, b_1),
\]
\[
p^n = (\ldots, b_3, b_2, b_0, b_1, b_2, b_3, b_0, b_1, b_2, b_3, b_0, b_1),
\]
\[
p^{ad} = (\ldots, 0, 0, b_{\alpha_1 + \alpha_2 + \alpha_3}, h_1, h_1, b_{-\alpha_1 - \alpha_2}),
\]

which yield the Young walls \( Y^1 := Y^n_0(p^1) \) and \( Y^n := Y^n_0(p^n) \) as follows:

\[
Y^1 = \begin{array}{ccccccc}
2 & 1 & 1 & 1 & 2 & 3 & 0
\end{array},
Y^n = \begin{array}{ccccccc}
2 & 1 & 3 & 2 & 1 & 0 & 3
\end{array}.
\]

4. **Geometric constructions of crystal graphs**

In this section, we review geometric constructions of crystal bases via quiver varieties. See [3] [12] [14] [17] [18] [19] for more details.

Let \( I = \mathbb{Z}/(n+1)\mathbb{Z} \) and \( H \) be the set of the arrows such that \( i \rightarrow j \) with \( i, j \in I, \; i - j = \pm 1 \). For \( h \in H \), we denote by \( \text{in}(h) \) (resp. \( \text{out}(h) \)) the incoming (resp. outgoing) vertex of \( h \). Define an involution \( - : H \rightarrow H \) to be the map interchanging \( i \rightarrow j \) and \( j \rightarrow i \). Let

\[
\Omega = \{ h \in H | \text{in}(h) - \text{out}(h) = 1 \}
\]

so that \( H = \Omega \sqcup \bar{\Omega} \); i.e.,

\[
(I, \Omega) = \begin{array}{cccc}
1 & 2 & \cdots & n-1 & n
\end{array},
(I, \bar{\Omega}) = \begin{array}{cccc}
1 & 2 & \cdots & n-1 & n
\end{array}.
\]
Note that our graph is an affine Dynkin graph of type $A_n^{(1)}$. We take the map $\epsilon : H \to \{-1, 1\}$ given by

$$\epsilon(h) = \begin{cases} 1 & \text{if } h \in \Omega, \\ -1 & \text{if } h \in \overline{\Omega}. \end{cases}$$

For $\alpha = \sum_{i=0}^{n} k_i \alpha_i \in Q^+$, we define the $I$-graded vector space

$$V(\alpha) = \bigoplus_{i=0}^{n} V_i(\alpha),$$

where $V_i(\alpha)$ is the $\mathbb{C}$-vector space with an ordered basis $v^i(\alpha) = \{v^i_0, v^i_1, \ldots, v^i_{k_i-1}\}$ for all $i$. Fix an ordered basis $v(\alpha) = \{v_0^0, \ldots, v_{k_0-1}^0, v_1^0, \ldots, v_{k_1-1}^1, \ldots, v_0^n, \ldots, v_{k_n-1}^n\}$ for $V(\alpha)$ and set

$$\dim V(\alpha) = \sum_{i=0}^{n} k_i \alpha_i = \alpha.$$

In a similar manner, for $\lambda = \sum_{i=0}^{n} w_i \Lambda_i \in P^+$ we define the $I$-graded vector space

$$W(\lambda) = \bigoplus_{i=0}^{n} W_i(\lambda),$$

where $W_i(\lambda)$ is a $\mathbb{C}$-vector space of dimension $w_i$.

Given $\alpha \in Q^+$, we set $V = V(\alpha)$ (resp. $V_i = V_i(\alpha)$ ($i = I$)) and let

$$E(\alpha) = E_\Omega(\alpha) \oplus E_\overline{\Omega}(\alpha),$$

where

$$E_\Omega(\alpha) = \bigoplus_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) = \bigoplus_{i \in I} \text{Hom}(V_{i-1}, V_i),$$

$$E_\overline{\Omega}(\alpha) = \bigoplus_{h \in \overline{\Omega}} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) = \bigoplus_{i \in I} \text{Hom}(V_i, V_{i-1}).$$

Let us denote by $\pi_{\Omega}$ (resp. $\pi_{\overline{\Omega}}$) the natural projection from $E(\alpha)$ to $E_\Omega(\alpha)$ (resp. $E_\overline{\Omega}(\alpha)$). For $\chi \in E(\alpha)$, if there is no danger of confusion, we write $x = (x_i \in \text{Hom}(V_{i-1}, V_i))_{i \in I}$ (resp. $\overline{x} = (\overline{x}_i \in \text{Hom}(V_i, V_{i-1}))_{i \in I}$) for $\pi_{\Omega}(\chi)$ (resp. $\pi_{\overline{\Omega}}(\chi)$).

The matrix representation of $x \in E_\Omega(\alpha)$ in the ordered basis $v(\alpha)$ is given as

$$x = \begin{pmatrix} 0 & \cdots & 0 & x_0 \\ x_1 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & x_m \end{pmatrix},$$

where $x_i$ is the matrix representation of $x|_{V_{i-1}} : V_{i-1} \to V_i$ in the ordered bases $v^{i-1}(\alpha)$ and $v^i(\alpha)$. We may also consider the matrix representation of $\overline{x} \in E_{\overline{\Omega}}(\alpha)$...
We denote by $\text{Irr} \Lambda(\chi)$ for a representation of the quiver $(\Lambda, \chi)$ the vector space with basis $x$.

Here, $\alpha$ is a composition map.

The algebraic group $G(\alpha) := \prod_{i \in I} \text{Aut}(V_i) \subset \text{Aut}(V)$ acts on $E(\alpha)$ by $(g, \chi) = g\chi g^{-1}$ for $g \in G(\alpha)$, $\chi \in E(\alpha)$. Let $\langle \cdot, \cdot \rangle$ be the nondegenerate, sympletic form on $E(\alpha)$ defined by

$$\langle \chi, \chi' \rangle = \sum_{h \in H} \epsilon(h) \text{tr}(\chi h \chi' h^{-1})$$

for $\chi, \chi' \in E(\alpha)$. Note that $E(\alpha)$ may be viewed as the cotangent bundle of $E^s(\alpha)$ (resp. $E^\Lambda(\alpha)$) under this form. The moment map $\mu = (\mu_i : E(\alpha) \to \text{End}(V_i))_{i \in I}$ is given by

$$\mu_i(\chi) = \sum_{h \in H, \text{in}(h) = i} \epsilon(h) \chi h \chi^{-1} = x_i \tau_i - \tau_i x_{i+1}$$

for $\chi = x + \tau \in E(\alpha)$. Note that

$$\mu_i(\chi) = 0 \text{ for all } i \in I \text{ if and only if } [x, \tau] = x \tau - \tau x = 0.$$  

Here, $x \tau = (x_i \tau_i : V_i \to V_{i+1})_{i \in I}$ and $\tau x = (\tau_{i+1} x_{i+1} : V_i \to V_{i+1})_{i \in I}$.

An element $\chi \in E(\alpha)$ is nilpotent if there exists an $N \geq 2$ such that for any sequence $h_1, \ldots, h_N \in H$ satisfying $\text{in}(h_i) = \text{out}(h_{i+1})$ ($i = 1, \ldots, N - 1$), the composition map $\chi h_N \cdots \chi h_1$ is zero. We define Lusztig’s quiver variety to be

$$\Lambda(\alpha) = \{ \chi \in E(\alpha) | \chi : \text{nilpotent}, \mu_i(\chi) = 0 \text{ for all } i \in I \}.$$  

We denote by $\text{Irr} \Lambda(\alpha)$ the set of all irreducible components of $\Lambda(\alpha)$.

For a pair $(k', k) \leq (k)$ of integers, let $V(k', k) = \bigoplus_{i \in I} V_i(k', k)$ be the $I$-graded vector space with basis $\{ e_j | k' \leq j \leq k \}$ such that

$$V_i(k', k) = \text{span}_C \{ e_j \mid j \equiv i \pmod{n + 1} \}$$

for $i \in I$. Consider the $C$-linear map $x(k', k) : V(k', k) \to V(k', k)$ sending $e_i$ to $e_{i+1}$, where $e_{k+1} = 0$. Then it is clear that the representation $(V(k', k), x(k', k))$ of the quiver $(I, \Omega)$ is indecomposable and nilpotent. Note that the isomorphism class of $(V(k', k), x(k', k))$ does not change when $k'$ and $k$ are simultaneously translated by a multiple $n + 1$. Moreover, any indecomposable nilpotent finite-dimensional representation of the quiver $(I, \Omega)$ is isomorphic to $(V(k', k), x(k', k))$ for some pair.
(k' ≤ k). Let $Z$ be the set of all pairs (k' ≤ k) of integers defined up to simultaneous translation by a multiple of n + 1 and let $\tilde{Z}$ be the set of all functions from $Z$ to $\mathbb{Z}_{≥0}$ with finite support. Note that $\tilde{Z}$ naturally corresponds to isomorphism classes of nilpotent finite-dimensional representations of the quiver $(\Gamma, \Omega)$. The set of $G(\alpha)$-orbits on the set of nilpotent elements in $E_{\Omega}(\alpha)$ is naturally indexed by the subset $\tilde{Z}(\alpha)$ of $\tilde{Z}$ such that, for $f \in \tilde{Z}(\alpha)$, $$\sum_{k' \leq k} f(k', k) \cdot \# \{j | k' \leq j \leq k, \ j \equiv i \ (mod \ n + 1)\} = \dim V_i \ (i \in I).$$

Here the sum is taken over all k' ≤ k up to simultaneous translation by a multiple of n + 1. An element $f \in \tilde{Z}(\alpha)$ is aperiodic if, for any k' ≤ k, not all integers $f(k', k), f(k' + 1, k + 1), \ldots, f(k' + n, k + n)$ are greater than zero. For any $f \in \tilde{Z}(\alpha)$, let $C_f$ be the conormal bundle of the $G(\cdot)$-orbit corresponding to $f$, and let $\mathcal{C}_f$ be the closure of $C_f$. Then we have

**Theorem 4.1** ([13]). The map $f \mapsto \mathcal{C}_f$ is a 1-1 correspondence between the set of aperiodic elements in $\tilde{Z}(\alpha)$ and $\text{Irr}\Lambda(\alpha)$.

In a similar manner, for a pair (k ≥ k') of integers, let $\pi(k, k') : V(k', k) \rightarrow V(k', k)$ be the $\mathbb{C}$-linear map sending $e_i$ to $e_{i-1}$, where $e_{k'-1} = 0$. Then the representation $(V(k', k), \pi(k, k'))$ of the quiver $(\Gamma, \Omega)$ is indecomposable and nilpotent, and the isomorphism class of $(V(k', k), \pi(k, k'))$ does not change when $k$ and $k'$ are simultaneously translated by a multiple n + 1. Any indecomposable nilpotent finite-dimensional representation of the quiver $(\Gamma, \Omega)$ is isomorphic to $(V(k', k), \pi(k, k'))$ for some pair (k ≥ k'). Let $\tilde{Z}$ be the set of all pairs (k ≥ k') of integers defined up to simultaneous translation by a multiple of n + 1 and let $\tilde{Z}$ be the set of all functions from $Z$ to $\mathbb{Z}_{≥0}$ with finite support. Then the set of $G(\alpha)$-orbits on the set of nilpotent elements in $E_{\pi}(\alpha)$ is naturally indexed by the subset $\tilde{Z}(\alpha)$ of $\tilde{Z}$ such that, for $f \in \tilde{Z}(\alpha)$, $$\sum_{k \geq k'} f(k, k') \cdot \# \{j | k \geq j \geq k', \ j \equiv i \ (mod \ n + 1)\} = \dim V_i \ (i \in I).$$

Here the sum is taken over all k ≥ k' up to simultaneous translation by a multiple of n + 1. An element $f \in \tilde{Z}(\alpha)$ is aperiodic if, for any k ≥ k', not all integers $f(k, k'), f(k + 1, k' + 1), \ldots, f(k + n, k' + n)$ are greater than zero. Then one can show that there is a 1-1 correspondence between the set of aperiodic elements in $\tilde{Z}(\alpha)$ and $\text{Irr}\Lambda(\alpha)$.

Moreover, Kashiwara and Saito [12] gave a crystal structure on $\mathbb{B}(\infty) := \bigsqcup_{\alpha \in Q^+} \text{Irr}\Lambda(\alpha)$ and proved the following theorem.

**Theorem 4.2** ([12]). There is a unique crystal isomorphism $\mathbb{B}(\infty) \cong \mathcal{B}(\infty)$.

Now we introduce a description of Nakajima’s quiver varieties presented in [16]. Given $\alpha \in Q^+$ and $\lambda \in P^+$, we set $W = W(\lambda)$ (resp. $W_i = W_i(\lambda)$) and let $$E(\lambda, \alpha) = \Lambda(\alpha) \times \sum_{i \in I} \text{Hom}(V_i, W_i).$$
The group $G(\alpha)$ acts on $E(\lambda, \alpha)$ by $(g, (\chi, t)) = (g\chi g^{-1}, t g^{-1})$. For $\chi \in \Lambda(\alpha)$, an $I$-graded subspace $S$ of $V(\alpha)$ is $\chi$-stable if $\chi_h(S_{\text{out}(h)}) \subset S_{\text{in}(h)}$ for all $h \in H$. An element $(\chi, t) \in E(\lambda, \alpha)$ is called a stable point of $E(\lambda, \alpha)$ if it satisfies the following conditions: if $S$ is a $\chi$-stable subspace of $V$ with $t_i(S_i) = 0$ ($i \in I$), then $S = 0$. Let $E(\lambda, \alpha)^{st}$ be the set of all stable points of $E(\lambda, \alpha)$ and define

$$A(\lambda, \alpha) = E(\lambda, \alpha)^{st}/G(\alpha).$$

Let $\text{Irr} \Lambda(\lambda, \alpha)$ (resp. $\text{Irr} E(\lambda, \alpha)$) be the set of all irreducible components of $\Lambda(\lambda, \alpha)$ (resp. $E(\lambda, \alpha)$). Since $\text{Irr} \Lambda(\lambda, \alpha)$ can be identified with

$$\{Z \in \text{Irr} E(\lambda, \alpha) | Z \cap E(\lambda, \alpha)^{st} \neq \emptyset\},$$

each irreducible component $X$ in $\text{Irr} \Lambda(\lambda, \alpha)$ can be written as

$$X = \left(\left(X_0 \times \sum_{i \in I} \text{Hom}(V_i, W_i)\right) \cap E(\lambda, \alpha)^{st}\right)/G(\alpha)$$

for some irreducible component $X_0$ in $\text{Irr} \Lambda(\alpha)$.

In [18], Saito gave a crystal structure on $\text{Irr} \Lambda(\lambda, \alpha)$ and proved the following theorem.

**Theorem 4.3** ([18]). There is a unique crystal isomorphism $\mathbb{B}(\lambda) \cong B(\lambda)$.

In [3], Frenkel and Savage gave an enumeration of $\text{Irr} \Lambda(\lambda, \alpha)$ in terms of Young and Maya diagrams for type $A_n^{(1)}$. Combining Theorem 4.2 and Theorem 4.3 with (1.1), we obtain an injective map

$$\iota_\lambda: \mathbb{B}(\lambda) \hookrightarrow \mathbb{B}(\infty).$$

For each irreducible component $X_0 \in \iota_\Lambda(\mathbb{B}(\Lambda_k))$, Frenkel and Savage constructed a special point in $X_0 \times \sum_{i \in I} \text{Hom}(V_i, W_i)$ which is not killed by the stability condition, and they showed that there is a 1-1 correspondence between the set of such special points and the set of $(n + 1)$-reduced colored Young diagrams. Savage later established a crystal isomorphism between $\mathbb{B}(\lambda)$ and Young walls for quantum affine algebras of type $A_n^{(1)}$ and $D_n^{(1)}$ in [19].

We briefly recall the result of [3] for type $A_n^{(1)}$ in terms of Young walls. Note the orientation that appeared in [3] is $\Omega$. Take a Young wall $Y^n \in Y^n(\Lambda_k)$ such that $\text{wt}(Y^n) = \alpha$. Let $l_i$ be the length of the $i$-th row of the Young wall $Y^n$ ($i \geq 1$) and let $N$ be the height of $Y^n$. Set

$$A_{Y^n} := \{(l_i - i + k, 1 - i + k) | 1 \leq i \leq N\} \subset \mathbb{Z}$$

and consider the function $f \in \overline{Z}(\alpha)$ given by

$$f(s, s') = \begin{cases} 1 & \text{if } (s, s') \in A_{Y^n}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $f$ is aperiodic. Let $\overline{O}_f$ be the closure of the conormal bundle of the $G(\alpha)$-orbit $O_f$ in $E_{\text{TV}}(\alpha)$ corresponding to $f$ and define the irreducible component

$$X_{Y^n} := \left(\left(\overline{O}_f \times \sum_{i \in I} \text{Hom}(V_i, W_i)\right) \cap E(\Lambda_k, \alpha)^{st}\right)/G(\alpha) \in \text{Irr} \Lambda(\lambda, \alpha).$$
By [3, Theorem 5.5], the map \( Y^n \to X_{Y^n} \) is a 1-1 correspondence between
\[ \{ Y^n \in \mathcal{Y}^n(\Lambda_k) \mid \text{wt}(Y^n) = \alpha \} \quad \text{and} \quad \text{Irr}(\Lambda_k, \alpha). \]
Moreover, it is proved in [19, Theorem 8.4] that the map \( Y^n \to X_{Y^n} \) from \( \mathcal{Y}^n(\Lambda_k) \) to \( \mathcal{B}(\Lambda_k) \) is a crystal isomorphism. We would like to point out that \( \mathcal{C}_f = \iota_{\Lambda_k}(X_{Y^n}) \).

Now we construct an element in the \( \mathcal{G}(\alpha) \)-orbit \( \mathcal{O}_f \) in \( E_{\mathcal{H}}(\alpha) \) from the Young wall \( Y^n \). Let \( \overline{b}_{ij} \) be the \( i \)-th block from the bottom in the \( j \)-th column of \( Y^n \). Let \( \text{Color}(\overline{b}_{ij}) \) be the color of \( \overline{b}_{ij} \), which is an element in \( I \). Define
\[
\alpha(\overline{b}_{ij}) := \# \{ \overline{b}_{rs} \in Y^n \mid \text{Color}(\overline{b}_{rs}) = \text{Color}(\overline{b}_{ij}), \ (r, s) \prec (i, j) \},
\]
where \( \prec \) is the lexicographical order; i.e., \( (r, s) \prec (i, j) \) if and only if \( r < i \) or \( r = i \) and \( s < j \). We define
\[
\varpi(Y^n) := \sum_{\overline{b}_{ij} \in Y^n, \ j > 0} E_{\alpha(\overline{b}_{ij})}^\text{Color}(\overline{b}_{ij}), \ \alpha(\overline{b}_{ij-1}) \in E_{\mathcal{H}}(\alpha).
\]
For \( 1 \leq i \leq N \), we denote by \( J_i \) the subspace of \( V(\alpha) \) generated by
\[
\{ v_{\text{Color}(\overline{b}_{ij})} \mid 0 \leq j < l_i \}.
\]
By construction, one can show that \( J_i \) is invariant under \( \varpi(Y^n) \) and the representation \( (J_i, \varpi(Y^n)\vert_{J_i}) \) of the quiver \( (I, \overline{\Omega}) \) is isomorphic to the representation \( (V(1 - i + k, l_i - i + k), \varpi(l_i - i + k, 1 - i + k)) \) of the quiver \( (I, \overline{\Omega}) \) for \( 1 \leq i < N \).
Here, \( \varpi(Y^n)\vert_{J_i} \) is the restriction of \( \varpi(Y^n) \) on the invariant subspace \( J_i \). Hence \( \varpi(Y^n) \) is contained in the \( \mathcal{G}(\alpha) \)-orbit \( \mathcal{O}_f \) corresponding \( f \), which yields
\[
\iota_{\Lambda_k}(X_{Y^n}) = \text{the closure of the conormal bundle of the } \mathcal{G}(\alpha) \text{-orbit of } \varpi(Y^n).
\]
By a direct computation, for \( t \in \mathbb{Z}_{\geq 0} \), we have
\[
\ker(\varpi(Y^n))^t = \bigoplus_{i=1}^N \ker(\varpi(Y^n)\vert_{J_i})^t
\]
\[
= \bigoplus_{i=1}^N \text{span}_C \{ v_{\text{Color}(\overline{b}_{ij})} \mid \overline{b}_{ij} \in \text{the } i \text{-th row of } Y^n, \ j < t \}
\]
\[
= \text{span}_C \{ v_{\text{Color}(\overline{b}_{ij})} \mid \overline{b}_{ij} \in Y^n, \ j < t \}.
\]
In the same manner, we take a Young wall \( Y^1 \in \mathcal{Y}^1(\Lambda_k) \) such that \( \text{wt}(Y^1) = \alpha \). Denote by \( X_{Y^1} \) the image of \( Y^1 \) under the crystal isomorphism
\[
\mathcal{Y}^1(\Lambda_k) \xrightarrow{\sim} \mathcal{B}(\Lambda_k).
\]
Let \( b_{ij} \) be the \( i \)-th block from the bottom in the \( j \)-th column of \( Y^1 \), and let \( \text{Color}(b_{ij}) \) be the color of \( b_{ij} \). Set
\[
\alpha(b_{ij}) := \# \{ b_{rs} \in Y^1 \mid \text{Color}(b_{rs}) = \text{Color}(b_{ij}), \ (r, s) \prec (i, j) \},
\]
where \( \prec \) is the lexicographical order, and define
\[
\varpi(Y^1) := \sum_{b_{ij} \in Y^1, \ j > 0} E_{\alpha(b_{ij})}^\text{Color}(b_{i,j-1}), \ \alpha(b_{i,j-1}) \in E_{\mathcal{H}}(\alpha).
\]
Then we have
\[ (4.7) \quad \iota_{\Lambda_x}(X_{Y^1}) = \text{the closure of the conormal bundle of the } G(\alpha)\text{-orbit of } x(Y^1). \]
Moreover, we obtain
\[ (4.8) \quad \ker(x(Y^1))^t = \text{span}_\mathbb{C}\{v^{\text{Color}(b_{ij})}_x \mid b_{ij} \in Y^1, \ j < t \} \]
for \( t \in \mathbb{Z}_{\geq 0} \).

**Example 4.4.** We use the same notation as in Example 3.3. Set
\[ \alpha := \Lambda_0 - \text{wt}(b) = 4\alpha_0 + 4\alpha_1 + 4\alpha_2 + 3\alpha_3, \]
and let \( X \) be the irreducible component in \( \mathcal{B}(\Lambda_0) \) corresponding to \( b \) via the crystal isomorphism given in Theorem 4.3. Then we have
\[ x(Y^1) = (E_{00}^0 + E_{11}^0 + E_{22}^0) + (E_{10}^1 + E_{11}^1 + E_{32}^1) + (E_{02}^2 + E_{11}^2 + E_{22}^2) + (E_{00}^3 + E_{11}^3 + E_{22}^3), \]
\[ \pi(Y^n) = (E_{10}^0 + E_{21}^0 + E_{32}^0) + (E_{00}^1 + E_{12}^1 + E_{23}^1) + (E_{02}^2 + E_{11}^2 + E_{33}^2) + (E_{00}^3 + E_{11}^3), \]
and \( \iota_{\Lambda_0}(X) \) is the closure of the conormal bundle of the \( G(\alpha)\)-orbit of \( x(Y^1) \) (resp. \( \pi(Y^n) \)). However, we note that
\[ x(Y^1) + \pi(Y^n) \notin E(\alpha) \]
since \([x(Y^1), \pi(Y^n)] \neq 0\).

5. Quiver varieties and the perfect crystals \( B^1, B^n \)

In this section, we give an explanation of the 1-1 correspondence between the geometric realization \( \mathcal{B}(\Lambda_k) \) and the path realization of the crystal \( B(\Lambda_k) \) associated with the perfect crystals \( B^1 \) and \( B^n \), and we give a geometric interpretation of the fundamental theorem of perfect crystals in the case of the perfect crystals \( B^1 \) and \( B^n \). Let \( \alpha \in Q^+ \) and let \( \lambda \) be a dominant integral weight of level 1. Choose an irreducible component \( X \) in \( \text{Irr}(\Lambda, \lambda) \). For a generic point \( \chi = x + \pi \in \iota(\lambda)(X) \), we will give an explicit description of the \( \Lambda \)-path in \( B^1 \) (resp. \( B^n \)) corresponding to \( X \) using the dimensions of the spaces \( \ker x^{i+1}/\ker x^i \) (resp. \( \ker \pi^{i+1}/\ker \pi^i \)) for \( i \geq 0 \). For this purpose, we need a few lemmas.

**Lemma 5.1.** Let \( X_0 \) be an irreducible component in \( \text{Irr}(\alpha) \). Then, for any \( \chi = x + \pi \in X_0 \) and \( k \in \mathbb{Z}_{\geq 0} \), we have

- (a) \( \ker(x\pi)^k = \ker(\pi x)^k \),
- (b) \( \ker x^k \) and \( \ker \pi^k \) are \( \chi \)-stable,
- (c) \( \ker(x\pi)^k \) is \( \chi \)-stable.

**Proof.** Let \( \chi = x + \pi \in X_0 \) and \( k \in \mathbb{Z}_{\geq 0} \). By (4.11) we have \([x, \pi] = 0\), which yields
\[ \chi(\ker x^k) \subset \ker x^k, \quad \chi(\ker \pi^k) \subset \ker \pi^k \quad \text{and} \quad \chi(\ker(x\pi)^k) \subset \ker(x\pi)^k. \]
Our assertion follows from the fact that \( \ker(x\pi)^k, \ker x^k \) and \( \ker \pi^k \) are \( I \)-graded vector spaces. \( \Box \)

**Lemma 5.2.** For each \( X_0 \in \text{Irr}(\alpha) \), there exists an open subset \( U \subset X_0 \) such that
\[ (5.1) \quad \ker x^k \cong \ker x'^k \quad \text{and} \quad \ker \pi^k \cong \ker \pi'^k \]
for any \( \chi = x + \pi, \chi' = x' + \pi' \in U \) and \( k \in \mathbb{Z}_{\geq 0} \).
Proof. By Theorem 4.1, there is an open subset $U_1 \subset X_0$ such that $\pi_\Omega(U_1)$ is contained in the $G(\alpha)$-orbit of some element in $E_\Omega(\alpha)$. In the same manner, there is an open subset $U_2 \subset X_0$ such that $\pi_{\Omega}(U_2)$ is contained in the $G(\alpha)$-orbit of some element in $E_{\Omega}(\alpha)$. Set $U = U_1 \cap U_2 \subset X_0$. Then, by construction, for any $x = x + \pi, x' = x' + \pi' \in U$ there exist $g, g' \in G(\alpha)$ such that

$$x = gx'g^{-1} \quad \text{and} \quad \pi = g\pi'g^{-1},$$

which yield, for any $k \in \mathbb{Z}_{\geq 0}$,

$$\ker x^k = \ker (gx'g^{-1})^k = g(\ker x^k) \quad \text{and} \quad \ker x^k = \ker (g\pi'g^{-1})^k = g(\ker \pi^k).$$

□

An element $\chi \in X_0$ in the open subset $U \subset X_0$ in Lemma 5.2 will be called a generic point. Thanks to Lemma 5.2, we may consider

$$\dim(\ker x^k) \quad \text{and} \quad \dim(\ker x^{k+1}/\ker x^k)$$

(resp. $\dim(\ker \pi^k)$ and $\dim(\ker \pi^{k+1}/\ker \pi^k)$)

for a generic map $\chi = x + \pi$ in an irreducible component $X_0 \in \text{Irr}(\alpha)$. Recall the injective map given in (4.2)

$$\iota_{\Lambda_k} : \mathbb{B}(\Lambda_k) \hookrightarrow \mathbb{B}(\infty)$$

for $0 \leq k \leq n$. Applying Lemma 5.1 and Lemma 5.2 to (4.6) and (4.3), we obtain the following theorem.

**Theorem 5.3 (cf. [3]).** Let

$$p_k^1 : \mathbb{B}(\Lambda_k) \rightarrow \mathcal{P}_1(\Lambda_k) \quad \text{(resp. } p_k^n : \mathbb{B}(\Lambda_k) \rightarrow \mathcal{P}_n(\Lambda_k))$$

be the unique crystal isomorphism given by Theorem 2.2 and Theorem 4.3 and take an irreducible component $X \in \mathbb{B}(\Lambda_k)$. Then, for a generic point $\chi = x + \pi \in \iota_{\Lambda_k}(X)$, we have

(a)

$$p_k^1(X) = (\ldots, b_{a_i}, \ldots, b_{a_1}, b_{a_0}),$$

where $a_i \equiv \dim(\ker x^{i+1}/\ker x^i) - i + k \pmod{n + 1}$ for all $i \geq 0$,

(b)

$$p_k^n(X) = (\ldots, \overline{b}_b, \ldots, \overline{b}_{b_i}, \overline{b}_{b_0}),$$

where $b_i \equiv 1 - \dim(\ker \pi^{i+1}/\ker \pi^i) + i + k \pmod{n + 1}$ for all $i \geq 0$.

Proof. Let $U$ be an open subset of $\iota_{\Lambda_k}(X)$ as in Lemma 5.2. By Lemma 5.2 it suffices to show that (a) and (b) hold for some $\chi = x + \pi \in U$.

Let $Y^1$ be the Young wall in $\mathcal{Y}^1(\Lambda_k)$ corresponding to $X$ under the crystal isomorphism $\mathcal{Y}^1(\Lambda_k) \cong \mathbb{B}(\Lambda_k)$, and let $b_{ij}$ be the $i$-th block from the bottom in the $j$-th column of $Y^1$. By (4.7), there exists $\chi = x + \pi \in U$ such that

$$x = gx(Y^1)g^{-1}$$

for some $g \in G(\alpha)$. Then, by the equation (4.8), for $t \in \mathbb{Z}_{\geq 0}$, we have

$$\ker x^t = \ker (gx(Y^1)g^{-1})^t = g(\ker (x(Y^1)^t))$$

$$= g\left(\text{span}_{C}\{ v_{\alpha(b_{ij})} \mid b_{ij} \in Y^1, \ j < t\}\right).$$
Let $y_t$ be the $t$-th column of $Y^1$ for $t \in \mathbb{Z}_{\geq 0}$. Note that $y_t = \{b_{it} \in Y^1 | i \geq 1 \}$. Then, we have

\[
\text{wt}(y_t) = \sum_{b_{it} \in y_t} \alpha_{\text{Color}(b_{it})} = \dim \left( \text{span}_C \{ v_{\alpha(b_{it})} | b_{it} \in y_t \} \right)
= \dim \left( \text{span}_C \{ v_{\alpha(b_{ij})} | b_{ij} \in Y^1, j < t + 1 \} \right) - \dim \left( \text{span}_C \{ v_{\alpha(b_{ij})} | b_{ij} \in Y^1, j < t \} \right)
= \dim(\ker(x(Y^1))^{t+1}) - \dim(\ker(x(Y^1))^t)
= \dim(\ker x^{t+1}) - \dim(\ker x^t)
= \dim(\ker x^{t+1}/\ker x^t),
\]

which implies that the height of $y_t$ is

\[
\text{ht} \left( \text{wt}(y_t) \right) = \dim(\ker x^{t+1}/\ker x^t).
\]

Consequently, assertion (a) follows from \eqref{eq:height} and \eqref{eq:dimension}.

The remaining assertion (b) can be proved in the same manner. \hfill \Box

Combining the crystal isomorphisms \eqref{eq:isomorphism1} and \eqref{eq:isomorphism2},

\[
Y^1_k : \mathcal{P}^1(\Lambda_k) \longrightarrow \mathcal{Y}^1(\Lambda_k) \quad \text{and} \quad Y^n_k : \mathcal{P}^n(\Lambda_k) \longrightarrow \mathcal{Y}^n(\Lambda_k),
\]

with \eqref{eq:isomorphism3} and \eqref{eq:isomorphism4}, we have the following proposition, which, together with Theorem \ref{thm:isomorphism}, yields an explicit 1-1 correspondence between $\mathcal{B}(\Lambda_k)$ and $\mathcal{P}^1(\Lambda_k)$ (resp. $\mathcal{P}^n(\Lambda_k)$).

**Proposition 5.4.** Let

\[
q^1_k : \mathcal{P}^1(\Lambda_k) \longrightarrow \mathcal{B}(\Lambda_k) \quad \text{(resp. $q^n_k : \mathcal{P}^n(\Lambda_k) \longrightarrow \mathcal{B}(\Lambda_k)$)}
\]

be the unique crystal isomorphism given by Theorem \ref{thm:isomorphism} and Theorem \ref{thm:isomorphism2} and take a $\Lambda_k$-path $p^1 \in \mathcal{P}^1(\Lambda_k)$ (resp. $p^n \in \mathcal{P}^n(\Lambda_k)$). Let

$\alpha = \Lambda_k - \text{wt}(p^1)$ and $X_1 = q^1_k(p^1)$ (resp. $\beta = \Lambda_k - \text{wt}(p^n)$ and $X_n = q^n_k(p^n)$).

Then

(a) $\iota_{\Lambda_k}(X_1)$ is the closure of the conormal bundle of the $G(\alpha)$-orbit of $x(Y_k^1(p^1));$

(b) $\iota_{\Lambda_k}(X_n)$ is the closure of the conormal bundle of the $G(\beta)$-orbit of $x(Y_k^n(p^n)).$

Recall the fundamental isomorphism theorem of perfect crystals \eqref{eq:isomorphism}. From Theorem \ref{thm:isomorphism2} we have the following crystal isomorphisms:

\[
\psi^1_k : \mathcal{B}(\Lambda_k) \xrightarrow{\sim} \mathcal{B}(\Lambda_{k-1}) \otimes \mathcal{B}^1,
\]

\[
\psi^n_k : \mathcal{B}(\Lambda_k) \xrightarrow{\sim} \mathcal{B}(\Lambda_{k+1}) \otimes \mathcal{B}^n
\]

for $0 \leq k \leq n$. We would like to give a geometric interpretation to the crystal isomorphisms $\psi^1_k$, $\psi^n_k$ in terms of quiver varieties. To do that, we need a few lemmas.

Let $V$ be an $I$-graded vector space and $\chi$ an element of $\text{Hom}(V,V)$. If $W$ is a $\chi$-invariant $I$-graded subspace of $V$, then $\chi$ can be viewed as an element in $\text{Hom}(V/W,V/W)$ (resp. $\text{Hom}(W,W)$), which is denoted by $\chi|_{V/W}$ (resp. $\chi|_W$).
Lemma 5.5. Let $x \in \bigoplus_{i \in I} \text{Hom}(V_{i-1}, V_i)$ for an $I$-graded vector space $V := \bigoplus_{i \in I} V_i$ and set
\[ W := \ker x \quad \text{and} \quad y := x|_{V/W}. \]
Take an element
\[ \overline{y} \in \bigoplus_{i \in I} \text{Hom}(V_i/W_i, V_{i-1}/W_{i-1}) \quad \text{with} \quad [y, \overline{y}] = 0, \]
where $W_i$ is the $i$-subspace of $W$ for $i \in I$. Then there exists an element
\[ \tau \in \bigoplus_{i \in I} \text{Hom}(V_i, V_{i-1}) \]
such that
\[ [x, \tau] = 0 \quad \text{and} \quad \tau|_{V/W} = \overline{y}. \]

Proof. Let $r = \dim V - \dim W$ and $s = \dim W$. Take an ordered basis for $W$ and extend it to be an ordered basis for $V$ so that the matrix representations of $x, y$ and $\overline{y}$ are given as follows:
\[ x = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}, \quad y = B \quad \text{and} \quad \overline{y} = C \]
for some $r \times r$ matrices $B$ and $C$, and an $s \times r$ matrix $A$. Note that $[B, C] = 0$. Since the matrix
\[ \begin{pmatrix} A \\ B \end{pmatrix} \]
has full rank, the equation
\[ \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix} \]
has a solution. Since $x(V_i) \subset V_{i+1}$ and $\begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix}$ maps $V_i$ to $V_i$, we can choose an $s \times s$ matrix $X$ and an $s \times r$ matrix $Y$ such that $\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$ is a solution of equation (5.3) and maps $V_i$ to $V_{i-1}$ for $i \in I$. Let
\[ \tau = \begin{pmatrix} X & Y \\ 0 & C \end{pmatrix}. \]
By construction, we have
\[ \tau(V_i) \subset V_{i-1} \quad (i \in I), \quad \tau|_{V/W} = \overline{y} \]
and
\[ [x, \tau] = x\tau - \tau x = \begin{pmatrix} 0 & AC \\ 0 & BC \end{pmatrix} - \begin{pmatrix} 0 & XA + YB \\ 0 & CB \end{pmatrix} = 0. \]

Lemma 5.6. Let $U$ be an open subset of $X_0 \in \text{Irr} \Lambda(\alpha)$ as in Lemma 5.2. Set
\[ \beta = \dim(\ker x) \quad \text{resp.} \quad \gamma = \dim(\ker \tau)(\beta) \]
for $\chi = x + \tau \in U$.

(a) There exists an irreducible component $X'_0 \in \text{Irr} \Lambda(\alpha - \beta)$ such that, for $\chi = x + \tau \in U$,
\[ \phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in X'_0, \]
where $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism.
(b) There exists an irreducible component $X''_0 \in \text{Irr}(\Lambda(\alpha - \gamma))$ such that, for $\chi = x + \varpi \in U$,

$$\phi \circ (\chi|_{V(\alpha)/\ker\varpi}) \circ \phi^{-1} \in X''_0,$$

(5.5)

where $\phi : V(\alpha)/\ker\varpi \to V(\alpha - \gamma)$ is an $I$-graded vector space isomorphism.

Proof. Note that $\beta$ and $\gamma$ are well-defined by Lemma 5.2. We first deal with case (a). For an element $\chi = x + \varpi \in U$, let

$$\chi_{\phi} := \phi \circ (\chi|_{V(\alpha)/\ker\varpi}) \circ \phi^{-1} \in \text{End}(V(\alpha - \beta)),$$

where $\phi : V(\alpha)/\ker\varpi \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism. Since $\chi \in \Lambda(\alpha)$, we have $\chi_{\phi} \in \Lambda(\alpha - \beta)$. Take two elements $\chi = x + \varpi, \chi' = x' + \varpi' \in U$, and choose two $I$-graded vector space isomorphisms $\phi : V(\alpha)/\ker\varpi \to V(\alpha - \beta)$ and $\phi' : V(\alpha)/\ker\varpi' \to V(\alpha - \beta)$. From the properties of $U$ described in the proof of Lemma 5.2, we have

$$x = g\varphi' g^{-1}$$

for some $g \in G(\alpha)$, which yields that $\pi_{\Omega}(\chi_{\phi})$ and $\pi_{\Omega}(\chi'_{\phi'})$ are in the same $G(\alpha - \beta)$-orbit. Therefore, there exists an irreducible component $X'_0 \in \text{Irr}(\Lambda(\alpha - \beta))$ such that

$$\chi_{\phi}, \chi'_{\phi'} \in X'_0.$$

Since $\chi, \chi'$ are arbitrary, our assertion follows.

The remaining case (b) can be proved in the same manner. $\square$

Theorem 5.7. Let $X_0 = \im_{\Lambda_k}(X)$ for an irreducible component $X \in \text{Irr}(\Lambda_k, \alpha)$. Set

$$d = \dim(\ker x) \quad \text{and} \quad \beta = \dim(\ker x) \quad (\text{resp.} \quad d' = \dim(\ker\varpi) \quad \text{and} \quad \gamma = \dim(\ker\varpi))$$

for a generic point $\chi = x + \varpi \in X_0$.

(a) There exists a unique irreducible component $X' \in \text{Irr}(\Lambda_{k-1}, \alpha - \beta)$ satisfying the following conditions:

(i) there is an open subset $U \subset X_0$ such that, for $\chi = x + \varpi \in U$,

$$\phi \circ (\chi|_{V(\alpha)/\ker\varpi}) \circ \phi^{-1} \in \im_{\Lambda_{k-1}}(X'),$$

where $\phi : V(\alpha)/\ker\varpi \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism,

(ii) there is an open subset $U' \subset \im_{\Lambda_{k-1}}(X')$ such that any element $\chi' \in U'$ can be written as

$$\chi' = \phi \circ (\chi|_{V(\alpha)/\ker\varpi}) \circ \phi^{-1}$$

for some $\chi = x + \varpi \in X_0$ and some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker\varpi \to V(\alpha - \beta)$,

(iii) moreover, we have

$$\psi^1_k(X) = X' \otimes b_{\alpha} \quad \text{and} \quad \wt(b_{\alpha}) = \Lambda_k - \Lambda_{k-1} - \cl(\beta),$$

where $a \equiv d + k \pmod{n + 1}$. 

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(b) There exists a unique irreducible component $X'' \in \text{Irr} \Lambda(\Lambda_{k+1}, \alpha - \gamma)$ satisfying the following conditions:

(i) there is an open subset $U \subset X_0$ such that, for $\chi = x + \tau \in U$,
\[
\phi \circ (\chi|_{V(\alpha)/\ker \tau}) \circ \phi^{-1} \in \iota_{\Lambda_{k+1}}(X''),
\]
where $\phi : V(\alpha)/\ker \tau \to V(\alpha - \gamma)$ is an I-graded vector space isomorphism,

(ii) there is an open subset $U'' \subset \iota_{\Lambda_{k+1}}(X'')$ such that any element $\chi'' \in U''$ can be written as
\[
\chi'' = \phi \circ (\chi|_{V(\alpha)/\ker \tau}) \circ \phi^{-1}
\]
for some $\chi = x + \tau \in X_0$ and some I-graded vector space isomorphism $\phi : V(\alpha)/\ker \tau \to V(\alpha - \gamma)$,

(iii) moreover, we have
\[
\psi_b^n(X) = X'' \otimes b_b \quad \text{and} \quad \text{wt}(b_b) = \Lambda_k - \Lambda_{k+1} - \text{cl}(\gamma),
\]
where $b \equiv 1 - d' + k \pmod{n + 1}$.

Proof. We first deal with case (a) of the crystal isomorphism $\psi_b^1 : \mathcal{B}(\Lambda_k) \to \mathcal{B}(\Lambda_{k-1}) \otimes B^1$. Let $Y$ be the Young wall in $Y^1(\Lambda_k)$ corresponding to $X$ and let $Y'$ be the Young wall obtained by removing the 0-th column from $Y$. Then $Y'$ can be viewed as an element in $Y^1(\Lambda_{k-1})$. Take the irreducible component $X'$ in $\mathcal{B}(\Lambda_{k-1})$ corresponding to $Y'$. By Theorem 5.3 and (5.2), we have
\[
X' \in \text{Irr} \Lambda(\Lambda_{k-1}, \alpha - \beta) \quad \text{and} \quad \psi_b^1(X) = X' \otimes b_a,
\]
where $a \equiv d + k \pmod{n + 1}$ and $\text{wt}(b_a) = \Lambda_k - \Lambda_{k-1} - \text{cl}(\beta)$.

Let $U$ be an open subset of $X_0$ as in Lemma 5.6 and take an element $\chi = x + \tau \in U$. Since $x(Y)|_{V(\alpha)/\ker(x(Y))}$ is naturally identified with $x(Y')$ and $x$ is contained in the $G(\alpha)$-orbit of $x(Y)$, by Lemma 5.6 we have
\[
\phi \circ (\chi|_{V(\alpha)/\ker \chi}) \circ \phi^{-1} \in \iota_{\Lambda_{k-1}}(X')
\]
for an I-graded vector space isomorphism $\phi : V(\alpha)/\ker \chi \to V(\alpha - \beta)$.

Take an element $\chi' = x' + \tau'$ in an open subset $U'$ of $\iota_{\Lambda_{k-1}}(X')$ given as in Lemma 5.2. Then $x'$ can be written as
\[
x' = gx(Y')g^{-1}
\]
for some $g \in G(\alpha - \beta)$, which yields that there is $x$ in the $G(\alpha)$-orbit of $x(Y)$ such that
\[
\phi \circ (x|_{V(\alpha)/\ker x}) \circ \phi^{-1} = x'
\]
for some I-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$. Assertion (ii) follows from Lemma 5.5.

The remaining case (b), $\psi_b^n : \mathcal{B}(\Lambda_k) \to \mathcal{B}(\Lambda_{k+1}) \otimes B^n$, can be proved in the same manner. \hfill \square

Example 5.8. We use the same notation as in Example 4.4. Let $X_0 = \iota_{\Lambda_0}(X) \in \text{Irr} \Lambda(\alpha)$. By Theorem 4.1 it suffices to consider a generic point in the fiber $\pi_{\Omega}^{-1}(\pi(Y^n)) \subset X_0$. By 14 Section 12.8, Proposition 15.5, we have
\[
\pi_{\Omega}^{-1}(\pi(Y^n)) = \{ \chi \in X_0 \mid \pi_{\Omega}(\chi) = \pi(Y^n) \}
\]
\[
= \{ x + \pi(Y^n) \mid x \in E_\Omega(\alpha), \ [x, \pi(Y^n)] = 0 \}
\]
\[
= \{ x + \pi(Y^n) \mid x = x(a_1, \ldots, a_{13}), \ a_1, \ldots, a_{13} \in \mathbb{C} \}. \]
Here,
\[
x(a_1, \ldots, a_{13}) = \begin{pmatrix} 0 & 0 & 0 & x_0 \\ x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \end{pmatrix} \in E_{13}(\alpha),
\]
for \(a_1, \ldots, a_{13} \in \mathbb{C}\). Let
\[
x := x(a_1, \ldots, a_{13}), \quad \overline{x} := \overline{x}(Y^n),
\]
and consider \(a_1, \ldots, a_{13}\) as indeterminates. Then we have
\[
\dim(\ker x^k) = \begin{cases} 0 & \text{if } k = 0, \\ 2 + k & \text{if } 1 \leq k \leq 12, \\ 15 & \text{otherwise}, \end{cases}
\]
and
\[
\dim(\ker \overline{x}^k) = \begin{cases} 0 & \text{if } k = 0, \\ 4 & \text{if } k = 1, \\ 8 & \text{if } k = 2, \\ 11 & \text{if } k = 3, \\ 14 & \text{if } k = 4, \\ 15 & \text{otherwise}. \end{cases}
\]
Hence we obtain
\[
p_0^\prime(X) = (\ldots, b_3, b_1, b_2, b_3, b_0, b_1, b_2, b_3, b_0, b_1, b_2, b_3, b_4),
p_0^{\prime\prime}(X) = (\ldots, \overline{b}_3, \overline{b}_2, \overline{b}_0, \overline{b}_1, \overline{b}_2, \overline{b}_1).
\]

6. Quiver Varieties and Adjoint Crystals

In this section, we will prove the main theorem of this paper, Theorem 6.3 which shows that there exists an explicit crystal isomorphism between the geometric realization \(B(\Lambda_0)\) and the path realization \(\mathcal{P}^{ad}(\Lambda_0)\) of \(B(\Lambda_0)\) arising from the adjoint crystal \(B^{ad}\). By Theorem 2.2 and Theorem 4.3, we have the crystal isomorphism
\[
p^{ad} : \mathcal{B}(\Lambda_0) \to \mathcal{P}^{ad}(\Lambda_0).
\]
Let \(\alpha \in Q^+\) and let \(X\) be an irreducible component of \(\text{Irr}(\Lambda_0, \alpha)\). For a generic point \(\chi = x + \overline{x} \in \ell_{\alpha_0}(X)\), we will give an explicit description of the \(\Lambda_0\)-path \(p^{ad}(X)\) in terms of dimension vectors of \(\ker(x\overline{x})^{k+1} / \ker(x\overline{x})^k\) for \(k \geq 0\).

Let \(\alpha, \beta \in Q^+\) with \(\beta \leq \alpha\). Consider the diagram given in \([14]\):
\[
(6.1) \quad \Lambda(\alpha - \beta) \xrightarrow{\overline{\pi}} \Lambda(\alpha - \beta) \times \Lambda(\beta) \xrightarrow{p_1} F' \xrightarrow{p_2} F'' \xrightarrow{p_3} \Lambda(\alpha),
\]
where \(F''\) is the variety of all pairs \((\chi, W)\) such that
(a) \(\chi \in \Lambda(\alpha)\),
(b) \(W\) is a \(\chi\)-stable subspace of \(V(\alpha)\) with \(\dim W = \beta\),
and $F'$ is the variety of all quadruples $(\chi, W, f, g)$ such that

(a) $(\chi, W) \in F''$;
(b) $f = (f_i)_{i \in I}, g = (g_i)_{i \in I}$ give an exact sequence

$$0 \to V_i(\beta) \xrightarrow{f_i} V_i(\alpha) \xrightarrow{g_i} V_i(\alpha - \beta) \to 0 \quad (i \in I)$$

such that $\text{im } f = W$.

Then we have

$$p_1(\chi, W, f, g) = (\tilde{g} \circ (\chi|_{V(\alpha)/W}) \circ \tilde{g}^{-1}, \ f^{-1} \circ (\chi|W) \circ f),$$

where $\tilde{g} : V(\alpha)/W \to V(\alpha - \beta)$ is the $I$-graded vector space isomorphism induced by $g$,

$$p_2(\chi, W, f, g) = (\chi, W), \quad p_3(\chi, W) = \chi,$$

and $\pi$ is the natural first projection. Note that $p_2$ is a $G(\alpha - \beta) \times G(\beta)$-principal bundle and an open map.

Let $U$ be an open subset of $X_0 \in \text{Irr} \Lambda(\alpha)$ as in Lemma 5.2 and $\beta = \text{dim} (\ker x)$ for $\chi = x + \pi \in U$. Define the map $\iota : U \to F''$ by

$$\iota(\chi) = (\chi, \ker x)$$

for $\chi = x + \pi \in U$. Note that $p_1 \circ \iota = \text{id}|_U$. By Lemma 5.6 there exists an irreducible component $X'_0 \in \text{Irr} \Lambda(\alpha - \beta)$ such that, for any $\chi = x + \pi \in U$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in X'_0,$$

where $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$ is an $I$-graded vector space isomorphism. Given an open subset $U' \subset \Lambda(\alpha - \beta)$ with $U' \cap X'_0 \neq \emptyset$, by Lemma 5.6

$$\tilde{U} := \iota^{-1} \circ p_2 \circ p_1^{-1} \circ \pi^{-1}(U')$$

is a nonempty open subset of $X_0$. Therefore, given an open subset $U' \subset X'_0$, there exists an open subset $\tilde{U} \subset X_0$ such that, for any element $\chi = x + \pi \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker x}) \circ \phi^{-1} \in U'$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker x \to V(\alpha - \beta)$.

In the same manner, let $\gamma = \text{dim} (\ker \pi)$, and consider the diagram

$$\Lambda(\alpha - \gamma) \xleftarrow{\pi} \Lambda(\alpha - \gamma) \times \Lambda(\gamma) \xrightarrow{p_1, p_2} F' \xrightarrow{p_3} F'' \xrightarrow{p_4} \Lambda(\alpha).$$

Define the map $\overline{\iota} : U \to F''$ by

$$\overline{\iota}(\chi) = (\chi, \ker \pi)$$

for $\chi = x + \pi \in U$, and let $X''_0$ be an irreducible component as in Lemma 5.6 Then one can deduce that, given an open subset $U' \subset X''_0$, there exists an open subset $\tilde{U} \subset X_0$ such that, for any element $\chi = x + \pi \in \tilde{U}$,

$$\phi \circ (\chi|_{V(\alpha)/\ker \pi}) \circ \phi^{-1} \in U'$$

for some $I$-graded vector space isomorphism $\phi : V(\alpha)/\ker \pi \to V(\alpha - \beta)$. Consequently, we have the following lemma.
Lemma 6.1. With the same notation as in Lemma 5.6, we have the following:

(a) Given an open subset \( U' \subset X'_0 \), there exists an open subset \( \check{U} \subset X_0 \) such that, for any element \( \chi = x + \varpi \in \check{U} \),

\[
\phi \circ \left( \chi|_{V(\alpha)/\ker x} \right) \circ \phi^{-1} \in U'
\]

for some I-graded vector space isomorphism \( \phi : V(\alpha)/\ker x \to V(\alpha - \beta) \).

(b) Given an open subset \( U'' \subset X''_0 \), there exists an open subset \( \check{U} \subset X_0 \) such that, for any element \( \chi = x + \varpi \in \check{U} \),

\[
\phi \circ \left( \chi|_{V(\alpha)/\ker \varpi} \right) \circ \phi^{-1} \in U''
\]

for some I-graded vector space isomorphism \( \phi : V(\alpha)/\ker \varpi \to V(\alpha - \gamma) \).

Combining Lemma 6.1 with Lemma 5.6, we have the following lemma.

Lemma 6.2. Let \( \alpha \in Q^+ \). For each \( X_0 \in \text{Irr} \Lambda(\alpha) \), there exists an open subset \( U \subset X_0 \) such that

\[
\ker(x\varpi)^k \cong \ker(x'\varpi')^k \quad \text{and} \quad \ker x(x\varpi)^k \cong \ker x'(x'\varpi')^k
\]

for any \( \chi = x + \varpi, \chi' = x' + \varpi' \in U \) and \( k \in \mathbb{Z}_{\geq 0} \).

Proof. Since the case that \( \text{ht}(\alpha) = 0 \) is trivial, we may assume \( \text{ht}(\alpha) > 0 \). Let \( U_0 \) be an open subset of \( X_0 \) as in Lemma 5.2 and \( \beta = \dim(\ker x) \) for \( \chi = x + \varpi \in U_0 \). Take the irreducible component \( X'_0 \in \text{Irr} \Lambda(\alpha - \beta) \) given in Lemma 5.6 and choose an open subset \( U'_0 \) of \( X'_0 \) satisfying the conditions of Lemma 5.2. Let \( \gamma = \dim(\ker \varpi) \) for \( \chi = x + \varpi \in U'_0 \). By Lemma 6.1 there exists an open subset \( \check{U}_0 \subset X_0 \) such that, for any element \( \chi = x + \varpi \in \check{U}_0 \),

\[
\phi \circ \left( \chi|_{V(\alpha)/\ker x} \right) \circ \phi^{-1} \in U'_0
\]

for some I-graded vector space isomorphism \( \phi : V(\alpha)/\ker x \to V(\alpha - \beta) \). Set \( \check{U}_0 = U_0 \cap \check{U}_0 \). Then, for any \( \chi = x + \varpi \in \check{U}_0 \), we have

\[
\dim \ker x\varpi = \dim(\ker x) + \dim(\ker x\varpi/\ker x) = \dim(\ker x) + \dim \{ v \in V(\alpha)/\ker x \mid \varpi|_{V(\alpha)/\ker x}(v) = 0 \} = \dim(\ker x) + \dim(\ker \varpi|_{V(\alpha)/\ker x}) = \beta + \gamma.
\]

Let us take the irreducible component \( X''_0 \in \text{Irr} \Lambda(\alpha - \beta - \gamma) \) associated with \( X'_0 \), which is given as in Lemma 5.6. By the induction hypothesis, there exists an open subset \( U'' \subset X''_0 \) satisfying (6.4). Applying Lemma 6.1 to \( X''_0, X'_0 \), and \( X_0 \), there exists an open subset \( \check{U} \subset X_0 \) such that, for any \( \chi = x + \varpi \in \check{U} \),

\[
\phi \circ \left( \chi|_{V(\alpha)/\ker x\varpi} \right) \circ \phi^{-1} \in U''
\]

for some I-graded vector space isomorphism \( \phi : V(\alpha)/\ker x\varpi \to V(\alpha - \beta - \gamma) \). Let \( U = \check{U} \cap \check{U}_0 \); then, by construction, \( U \) holds condition (6.4). \( \square \)

An element \( \chi \in X_0 \) in the open subset \( U \subset X_0 \) satisfying (5.1) and (6.4) is called a generic point. Note that, for \( \chi = x + \varpi \) in an irreducible component \( X_0 \in \text{Irr} \Lambda(\alpha) \), since \( [x, \varpi] = 0 \), we have

\[
\dim(\ker x|_{\ker(x\varpi)^k+1/\ker(x\varpi)^k}) = \dim(\ker x(x\varpi)^k) - \dim(\ker(x\varpi)^k),
\]
where $x|_{\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k}}$ is the linear map in $\text{End}(\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k})$ induced by $x$. Thanks to Lemma 5.6 one can talk about
\[
\dim \ker(x\mathfrak{F})^{k}, \quad \dim \ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k} \quad \text{and} \quad \dim(\ker(x|_{\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k}}))
\]
for a generic point $\chi = x + \mathfrak{F}$ in an irreducible component $X_0 \in \text{Irr}(\Lambda)$ and $k \in \mathbb{Z}_{\geq 0}$.

Finally, we are ready to state the main theorem in this paper.

**Theorem 6.3.** Let
\[
P^{\text{ad}} : \mathbb{B}(\Lambda_0) \longrightarrow \mathbb{P}^{\text{ad}}(\Lambda_0)
\]
be the unique crystal isomorphism given by Theorem 2.2 and Theorem 4.3 and take an irreducible component $X \in \mathbb{B}(\Lambda_0)$. For a generic point $\chi = x + \mathfrak{F} \in \iota_{\Lambda_0}(X)$ and $k \in \mathbb{Z}_{\geq 0}$, let
\[
\theta_k = \dim(\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k}), \\
c_k \equiv \dim(\ker(x|_{\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k}})) \pmod{n+1},
\]
where $0 \leq c_k \leq n$. Then we have
\[
(6.6) \quad P^{\text{ad}}(X) = (\ldots, p_k, \ldots, p_1, p_0),
\]
where
\[
p_k = \begin{cases} 
\begin{array}{ll}
b_{-\text{cl}(\theta_k)} & \text{if cl}(\theta_k) \neq 0, \\
h_{c_k} & \text{if cl}(\theta_k) = 0 \text{ and } c_k \neq 0, \\
\emptyset & \text{otherwise}.
\end{array}
\end{cases}
\]

**Proof.** Let $X_0 = \iota_{\Lambda_0}(X)$ with $\text{wt}(X) = \Lambda_0 - \alpha$ for some $\alpha \in Q^+$. We will use induction on $\text{ht}(\alpha)$. Since the case that $\text{ht}(\alpha) = 0$ is trivial, we may assume $\alpha \neq 0$. Note that
\[
\theta_k = \dim(\ker(x\mathfrak{F})^{k+1}) - \dim(\ker(x\mathfrak{F})^{k}), \\
c_k \equiv \dim(\ker(x|_{\ker(x\mathfrak{F})^{k+1}/\ker(x\mathfrak{F})^{k}})) \pmod{n+1}
\]
for a generic point $\chi = x + \mathfrak{F} \in X_0$ and $k \in \mathbb{Z}_{\geq 0}$.

Let $\beta = \dim(\ker(x))$ for a generic point $\chi = x + \mathfrak{F} \in X_0$, and choose the irreducible component $X_0' \in \text{Irr}(\Lambda_0 - \alpha - \beta)$ associated with $X_0$ as in Lemma 5.6. Similarly, let $\gamma = \dim(\ker(x\mathfrak{F}))$ for a generic point $\chi = x + \mathfrak{F} \in X_0'$, and take the irreducible component $X_0'' \in \text{Irr}(\Lambda_0 - \alpha - \beta - \gamma)$ associated with $X_0'$ as in Lemma 5.6. By Theorem 5.7, we have
\[
\psi^{1}_0(X_0) = X_0' \otimes b_{a} \quad \text{and} \quad \psi^{n}_0(X_0') = X_0'' \otimes \overline{b}_{a}
\]
for some $b_{a} \in B^{1}$, $\overline{b}_{a} \in B^{n}$. From (2.1) and Theorem 4.2, we have
\[
\psi^{0}_0 : \mathbb{B}(\Lambda_0) \sim \mathbb{B}(\Lambda_0) \otimes B^{\text{ad}}.
\]

Then, it follows from the crystal isomorphism (2.2) and Theorem 6.7 that
\[
\psi^{0}_0(X_0) = X_0'' \otimes P^{\text{ad}}(\overline{b}_{a} \otimes b_{a}).
\]

By the induction hypothesis, there is an open subset $U'' \subset X_0''$ satisfying (6.6). By Lemma 6.1 and (6.5), there is an open subset $\hat{U} \subset X_0$ such that, for any $\chi = x + \mathfrak{F} \in \hat{U}$,
\[
\phi \circ (\chi|_{\ker(x\mathfrak{F})}) \circ \phi^{-1} \in U''
\]
for some isomorphism $\phi : V(\alpha)/\ker(x\mathfrak{F}) \rightarrow V(\alpha - \beta - \gamma)$.

On the other hand, by Lemma 6.2 there exists an open subset $\hat{U} \subset X_0$ satisfying (6.4). Set
\[
U = \hat{U} \cap \hat{U}
\]
and choose an element \( \chi = x + \mathfrak{p} \in U \). Suppose that \( \text{wt}(B_b \otimes b_a) \neq 0 \). Then, by Theorem 5.7 and (2.2), since
\[
\text{wt}(B_b \otimes b_a) = \text{wt}(B_b) + \text{wt}(b_a)
= \Lambda_n - \Lambda_0 - \text{cl}(\gamma) + \Lambda_0 - \text{cl}(\beta)
= -\text{cl}(\beta + \gamma)
= -\text{cl}(\dim(\ker x\mathfrak{p}))
= -\text{cl}(\theta_0),
\]
we obtain
\[
p_{\text{ad}}(B_b \otimes b_a) = b_{\text{wt}(B_b \otimes b_a)} = b_{-\text{cl}(\theta_0)}.
\]
Suppose \( \text{wt}(B_b \otimes b_a) = 0 \) and \( a \neq n + 1 \). Then, by Theorem 5.7 and (2.2), we have
\[
p_{\text{ad}}(B_b \otimes b_a) = h_a
\]
and \( a \equiv \dim x \mod n + 1 \), which implies that \( a = c_0 \). In the same manner, if \( \text{wt}(B_b \otimes b_a) = 0 \) and \( a = n + 1 \), we have
\[
p_{\text{ad}}(B_b \otimes b_a) = \emptyset.
\]
Since, for an arbitrary isomorphism \( \phi : V(\alpha)/\ker x\mathfrak{p} \to V(\alpha - \beta - \gamma) \),
\[
\dim(\ker(x\mathfrak{p})^{k+1}) = \dim(\ker(x\mathfrak{p})^{k+1}/\ker(x\mathfrak{p})) + \dim(\ker(x\mathfrak{p}))
= \dim(\ker(x\mathfrak{p})|_{V(\alpha)/\ker(x\mathfrak{p})})^k + \dim(\ker(x\mathfrak{p}))
= \dim \ker((\phi \circ x|V(\alpha)/\ker(x\mathfrak{p}) \circ \phi^{-1})^k + \dim(\ker(x\mathfrak{p}))
= \dim(\ker(x\mathfrak{p})),
\]
our assertion follows from a standard induction argument. \( \square \)

The following corollary, which is an immediate consequence of Theorem 5.7 and Theorem 6.3, can be regarded as a geometric interpretation of the fundamental isomorphism theorem for perfect crystals,
\[
\psi_0^{\text{ad}} : B(\Lambda_0) \xrightarrow{\sim} B(\Lambda_0) \otimes B^{\text{ad}}.
\]

**Corollary 6.4.** Let \( X_0 = \iota_{\Lambda_0}(X) \) for some \( X \in \text{Irr}\Lambda(\Lambda_0, \alpha) \). For a generic point \( \chi = x + \mathfrak{p} \in X_0 \), set
\[
\theta = \dim(\ker(x\mathfrak{p})) \quad \text{and} \quad c = \dim(\ker x).
\]
Then there exists a unique irreducible component \( X' \in \text{Irr}\Lambda(\Lambda_0, \alpha - \theta) \) satisfying the following conditions:

(a) there is an open subset \( U \subset X_0 \) such that, for \( \chi = x + \mathfrak{p} \in U \),
\[
\phi \circ (\chi|_{V(\alpha)/\ker x\mathfrak{p}}) \circ \phi^{-1} \in \iota_{\Lambda_0}(X'),
\]
where \( \phi : V(\alpha - \theta) \to V(\alpha)/\ker x\mathfrak{p} \) is an \( I \)-graded vector space isomorphism,

(b) there is an open subset \( U' \subset \iota_{\Lambda_0}(X') \) such that any element \( \chi' \in U' \) can be written as
\[
\chi' = \phi \circ (\chi|_{V(\alpha)/\ker x\mathfrak{p}}) \circ \phi^{-1}
\]
for some \( \chi = x + \mathfrak{p} \in X_0 \) and some \( I \)-graded vector space isomorphism
\[
\phi : V(\alpha)/\ker x\mathfrak{p} \to V(\alpha - \theta),
\]

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(c) moreover, we have
\[ \psi_{0}^{\text{ad}}(X) = X' \otimes p, \]
where
\[ p = \begin{cases} 
  b_{-\text{cl}(\theta)} & \text{if } \text{cl}(\theta) \neq 0, \\
  h_{c} & \text{if } \text{cl}(\theta) = 0 \text{ and } c \neq 0, \\
  \emptyset & \text{otherwise}.
\end{cases} \]

Example 6.5. We use the same notation as in Example 5.8. Let \( W_{i} = \ker(x_{x}^{i}) \) for \( i \in \mathbb{Z}_{\geq 0} \). Then we have
\[ \dim(W_{i}) = \begin{cases} 
  0 & \text{if } i = 0, \\
  \alpha_{0} + 2\alpha_{1} + 2\alpha_{2} + \alpha_{3} & \text{if } i = 1, \\
  2\alpha_{0} + 3\alpha_{1} + 3\alpha_{2} + 2\alpha_{3} & \text{if } i = 2, \\
  3\alpha_{0} + 4\alpha_{1} + 4\alpha_{2} + 3\alpha_{3} & \text{if } i = 3, \\
  4\alpha_{0} + 4\alpha_{1} + 4\alpha_{2} + 3\alpha_{3} & \text{otherwise}
\end{cases} \]
and
\[ \dim(\ker(x_{W_{i+1}/W_{i}})) = \dim(\ker(x(x\bar{x})^{i+1})) - \dim(\ker(x\bar{x})^{i}) = \begin{cases} 
  3 & \text{if } i = 0, \\
  1 & \text{if } i = 1, \\
  1 & \text{if } i = 2, \\
  0 & \text{otherwise}.
\end{cases} \]

By Theorem 6.3, we have
\[ p^{\text{ad}}(X) = (\ldots, \emptyset, \emptyset, b_{\alpha_{1}+\alpha_{2}+\alpha_{3}}, h_{1}, h_{1}, b_{-\alpha_{1}-\alpha_{2}}). \]

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