TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 11, November 2011, Pages 5649–5661
S 0002-9947(2011)05123-8
Article electronically published on April 29, 2011

# INVARIANT CONFORMAL METRICS ON $\mathbb{S}^n$

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Abstract. In this paper we use the relationship between conformal metrics on the sphere and horospherically convex hypersurfaces in the hyperbolic space for giving sufficient conditions on a conformal metric to be radial under some constraints on the eigenvalues of its Schouten tensor. Also, we study conformal metrics on the sphere which are invariant by a $k-$parameter subgroup of conformal diffeomorphisms of the sphere, giving a bound on its maximum dimension.

Moreover, we classify conformal metrics on the sphere whose eigenvalues of the Shouten tensor are all constant (we call them *isoparametric conformal metrics*), and we use a classification result for radial conformal metrics which are solutions of some $\sigma_k-$Yamabe type problem for obtaining existence of rotational spheres and Delaunay-type hypersurfaces for some classes of Weingarten hypersurfaces in $\mathbb{H}^{n+1}$.

## 1. Introduction

In the last 30 years, the *Nirenberg Problem*, i.e., *which functions $S : \mathbb{S}^n \longrightarrow \mathbb{R}$ arise as the scalar curvature of some conformal metric on the sphere?*, has received an amazing number of contributions (see [1, 2, 3, 7, 11, 12, 13, 14, 15, 20, 23, 24, 25, 26, 34]), but sufficient and necessary conditions for the solvability are still unknown.

However, this problem opened the door of a rich subject in the last few years, *conformally invariant equations.* Let $\mathcal{F}(x_1, \dots, x_n)$ denote a smooth functional, and let $\Gamma \in C^\infty(\mathbb{S}^n)$. *Does there exist a conformal metric $g = e^{2\rho} g_0$ on $\mathbb{S}^n$ such that the eigenvalues $\lambda_i$ of its Schouten tensor satisfy*

$$\mathcal{F}(\lambda_1, \dots, \lambda_n) = \Gamma \quad \text{on } \mathbb{S}^n?$$

Given $(M, g)$ a Riemannian manifold, for $n \geq 3$, the Schouten tensor of $g$ is given by

$$\mathrm{Sch}_g := \frac{1}{n-2}\left(\mathrm{Ric}(g) - \frac{S(g)}{2(n-1)} g\right),$$

where $\mathrm{Ric}(g)$ and $S(g)$ are the Ricci tensor and the scalar curvature function of $g$, respectively.

Note that, when $\mathcal{F}(x_1, \dots, x_n) = x_1 + \cdots + x_n$ we have the Nirenberg Problem. Right now, the most developed topic for these equations occurs when we

Received by the editors November 17, 2008 and, in revised form, May 22, 2009.
2000 *Mathematics Subject Classification.* Primary 53A10; Secondary 49Q05, 53C42.
*Key words and phrases.* Conformal metric, $\sigma_k$ curvature, radial solution, Schouten tensor, hyperbolic Gauss map, rotational hypersurfaces, Weingarten hypersurfaces.
The author was partially supported by Spanish MEC-FEDER Grant MTM2007-65249, and Regional J. Andalucía Grants P06-FQM-01642 and FQM325.

consider $\mathcal{F}(\lambda_1, \dots, \lambda_n) \equiv \sigma_k(\lambda_i)$, as the $k-$th elementary symmetric polynomial of its arguments, to be equal to a constant, i.e.,

$$\sigma_k(\lambda_i) = \text{constant}. \tag{1.1}$$

Many deep results are known for these equations (see [7, 8, 9, 27, 28, 30, 31, 36] and the references therein). Most of these results are devoted to solutions either on $\mathbb{S}^n$ or $\mathbb{R}^n$, and little is known when we look for conformal metrics on a domain of the sphere (see [32, 33] and the references therein). Along this line, Chang, Han, and Yang [10] have classified all possible radial solutions to the equation (1.1) *"as guidance in studying the behavior of singular solutions in the general situation"*. This is natural since radial solutions are the simplest examples. Thus, the next step is: *under what (local) conditions can we know that the solution is radial?*

In a recent paper [21], the authors showed a correspondence between conformal metrics on the sphere and horospherically convex hypersurfaces in hyperbolic space. Here, they provide a back-and-forth construction which gives a hypersurface theory interpretation for the famous Nirenberg Problem, relating it with a natural formulation of the Christoffel problem in $\mathbb{H}^{n+1}$. Moreover, this correspondence is more general and it relates conformally invariant equations with Weingarten hypersurfaces horospherically convex. The main line in this paper is to use the deep theorems on conformal geometry to infer results in hypersurface theory, but, *how can the hypersurface theory help to get information on conformal geometry?*

We will see here that, using the hypersurface setting, we can obtain sufficient conditions under which a conformal metric is radial among others on invariant conformal metrics under a subgroup of conformal diffeomorphisms of the sphere. We should mention that the theorems included here are local results, besides the usual results in this direction that are from a global character.

In Section 2 we establish the necessary preliminaries on conformal geometry, and it is also devoted to summarizing the correspondence developed in [21] between conformal metrics and horospherically convex hypersurfaces; that is, given a conformal metric on the sphere, they construct a horospherically convex hypersurface in $\mathbb{H}^{n+1}$ and vice versa. In Section 3 we establish that if a conformal metric is invariant under a subgroup of conformal diffeomorphisms of the sphere, then its associated horospherically convex hypersurface is invariant under the subgroup of isometries induced by the subgroup of conformal diffeomorphism, and vice versa, i.e.,

**Lemma 3.2.** *Let $\phi : \Omega \subset \mathbb{S}^n \longrightarrow \mathbb{H}^{n+1}$ be a locally horospherically convex hypersurface with hyperbolic Gauss map $G(x) = x$, support function $e^\rho : \Omega \longrightarrow (0, +\infty)$, and let $g = e^{2\rho} g_0$ denote its horospherical metric. Let $T_{|\mathbb{H}^{n+1}} \in \mathcal{I}(\mathbb{H}^{n+1})$ be an isometry and $\Phi \in \mathcal{D}(\mathbb{S}^n)$ be its associated conformal diffeomorphism. Thus, if $\phi$ is $T_{|\mathbb{H}^{n+1}}-$invariant, then $g$ is $\Phi-$invariant.*

*Conversely, let $g = e^{2\rho} g_0$ be a conformal metric defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that the eigenvalues of its Schouten tensor, $\mathrm{Sch}_g$, satisfy*

$$\sup \{\lambda_i(x),\, x \in \Omega,\, i = 1, \dots, n\} < +\infty.$$

*Let $\Phi \in \mathcal{D}(\mathbb{S}^n)$ be a conformal diffeomorphism and $T_{|\mathbb{H}^{n+1}} \in \mathcal{I}(\mathbb{H}^{n+1})$ be its associated isometry. Thus, if $g$ is $\Phi-$invariant, then $\phi$, given by (2.3), is $T_{|\mathbb{H}^{n+1}}-$invariant.*

In Section 4 we classify the conformal metrics on the sphere whose eigenvalues of its Shouten tensor are all constant. We call these metrics *isoparametric conformal metrics*. Since the above classification has not been done before (as far as we know), we will include it here.

In Section 5 we state our main results. We give sufficient conditions under which a conformal metric is radial in terms of the eigenvalues of its Shouten tensor.

**Theorem 5.1.** *Let $g = e^{2\rho} g_0$ be a conformal metric defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that the eigenvalues, $\lambda_i$, for $i = 1, \dots, n$, of its Schouten tensor, $\mathrm{Sch}_g$, satisfy*

$$\sup \{\lambda_i(x),\, x \in \Omega,\, i = 1, \dots, n\} < +\infty.$$

*Furthermore, assume that the eigenvalues satisfy*

$$\begin{aligned} \lambda = \lambda_1 = \cdots = \lambda_{n-1}, \\ \nu = \nu(\lambda) = \lambda_n, \\ \lambda - \nu \neq 0. \end{aligned}$$

*Then $g$ is radial.*

Moreover, we study conformal metrics on the sphere which are invariant by a $k-$parameter subgroup of conformal diffeomorphisms of the sphere, giving a bound on its maximum dimension.

**Theorem 5.2.** *Let $g = e^{2\rho} g_0$ be a conformal metric defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that $g \notin \mathcal{C}(n)$ and the eigenvalues, $\lambda_i$, for $i = 1, \dots, n$, of its Schouten tensor, $\mathrm{Sch}_g$, satisfy*

$$\sup \{\lambda_i(x),\, x \in \Omega,\, i = 1, \dots, n\} < +\infty.$$

*Suppose that $g$ is invariant by a $k-$parameter subgroup of a conformal diffeomorphism $\mathcal{G} \leq \mathcal{D}(\mathbb{S}^n)$. Then the maximum value of $k$ is $k_{max} = \frac{n(n-1)}{2}$, and if $k = k_{max}$, the Schouten tensor of $g$, $\mathrm{Sch}_g$ has two eigenvalues $\lambda$ and $\nu$, where one of them, say $\lambda$, has multiplicity at least $n-1$. If, in addition, $\lambda \neq 0$, $\nu = \nu(\lambda)$ and $\nu - \lambda \neq 0$, then $g$ is radial.*

Finally, in Section 6, we give some existence results for some classes of Weingarten hypersurfaces which are rotationally invariant and horospherically convex, based on a result of Chang, Han, and Yang [10].

## 2. Preliminaries

2.1. **On conformal geometry.** Let $(M^n, g)$, $n \geq 3$, be a Riemannian manifold. The Riemann curvature tensor, Riem, can be decomposed as

$$\mathrm{Riem} = W_g + \mathrm{Sch}_g \odot g,$$

where $W_g$ is the Weyl tensor, $\odot$ is the Kulkarni-Nomizu product, and

$$\mathrm{Sch}_g := \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{S(g)}{2(n-1)} g\right)$$

is the *Schouten tensor*. Here $\mathrm{Ric}_g$ and $S(g)$ stand for the Ricci curvature and scalar curvature of $g$, respectively.

The eigenvalues of $\mathrm{Sch}_g$ are defined as the eigenvalues of the endomorphism $g^{-1}\mathrm{Sch}_g$, and we will denote them by $\lambda_i$, $i = 1, \dots, n$.

It is well known that the Schouten tensor encodes all the information on how curvature varies by a conformal change of metric. It is worth it to remark that the Weyl curvature tensor vanishes identically when $(M^n, g)$ is locally conformally flat since it is the situation of the present work. We will consider conformal metrics to be the standard metric on the $n-$sphere, $(\mathbb{S}^n, g_0)$, i.e.,

$$g = e^{2\rho} g_0.$$

**Definition 2.1.** Let us denote by $\mathcal{D}(\mathbb{S}^n)$ the group of conformal diffeomorphisms on the sphere and let $\Phi \in \mathcal{D}(\mathbb{S}^n)$ be a conformal diffeomorphism. Let $g = e^{2\rho} g_0$ be a conformal metric defined on a domain $\Omega \subset \mathbb{S}^n$. $g$ is $\Phi-$invariant if

$$g_x(u,v) = (\Phi^* g)_x(u,v),\ \forall x \in \Omega, \forall u, v \in T_x\mathbb{S}^n,\ \text{such that } \Phi(x) \in \Omega.$$

Moreover, given a continuous subgroup of conformal diffeomorphisms $\mathcal{G} \leq \mathcal{D}(\mathbb{S}^n)$, $g$ is $\mathcal{G}-$invariant if it is $\Phi-$invariant for all $\Phi \in \mathcal{G}$.

The basic example of a $\mathcal{G}-$invariant metric is that which is radially symmetric, i.e., when $\mathcal{G}$ is a subgroup of rotations. In this case, we say that $g$ is radial.

2.2. **On hypersurface theory.** First, let us establish the necessary notation that we will use throughout the work. Actually, here we will summarize the construction developed in [21] for the sake of completeness; that is, in order to prove our results, we will use the correspondence between conformal metrics on the sphere and locally horospherically convex hypersurfaces in $\mathbb{H}^{n+1}$. So, we will recall, briefly, how to construct a locally horospherically convex hypersurface from a conformal metric on the sphere.

Let us denote by $\mathbb{L}^{n+2}$ the $(n+2)-$dimensional Lorentz-Minkowski space, i.e., the vectorial space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric $\langle , \rangle$ given by

$$\langle \bar{x}, \bar{x} \rangle = -x_0^2 + \sum_{i=1}^{n+1} x_i^2,$$

where $\bar{x} \equiv (x_0, x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+2}$.

So, the $(n+1)-$dimensional hyperbolic space, de-Sitter space and null cone are given, respectively, by the hyperquadrics

$$\begin{aligned}
\mathbb{H}^{n+1} &= \left\{ \bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = -1,\ x_0 > 0 \right\}, \\
\mathbb{S}_1^{n+1} &= \left\{ \bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = 1 \right\}, \\
\mathbb{N}_+^{n+1} &= \left\{ \bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = 0,\ x_0 > 0 \right\}.
\end{aligned}$$

It is well known that $\mathbb{H}^{n+1}$ inherits from $(\mathbb{L}^{n+2}, \langle , \rangle)$ a Riemannian metric which makes it the standard model of Riemannian space of constant sectional curvature $-1$. Its ideal boundary at infinity, $\partial_\infty \mathbb{H}^{n+1}$, will be denoted by $\mathbb{S}^n_\infty$.

Horospheres will play an essential role in what follows, so, we go through describing their most important properties. In this model, horospheres in $\mathbb{H}^{n+1}$ are the intersection of affine degenerate hyperplanes of $\mathbb{L}^{n+2}$ with $\mathbb{H}^{n+1}$. Thus, it is clear that the boundary at infinity is a single point $x \in \mathbb{S}^n_\infty$. In this way, two horospheres are always congruent, and they are at a constant (hyperbolic) distance if their respective points at infinity agree. Moreover, given a point $x \in \mathbb{S}^n_\infty$, horospheres having $x$ as its point at infinity provide a foliation of $\mathbb{H}^{n+1}$.

From now on, $\phi : M^n \longrightarrow \mathbb{H}^{n+1}$ will denote an oriented immersed hypersurface and $\eta : M^n \longrightarrow \mathbb{S}_1^{n+1}$ its unit normal.

**Definition 2.2** ([17, 18, 4])**.** Let $\phi : M^n \longrightarrow \mathbb{H}^{n+1}$ denote an immersed oriented hypersurface in $\mathbb{H}^{n+1}$ with unit normal $\eta$. The *hyperbolic Gauss map*

$$G : M^n \longrightarrow \mathbb{S}^n_\infty \equiv \mathbb{S}^n$$

of $\phi$ is defined as follows: for every $p \in M^n$, $G(p) \in \mathbb{S}^n_\infty$ is the point at infinity of the unique horosphere in $\mathbb{H}^{n+1}$ passing through $\phi(p)$ and whose inner unit normal at $p$ agrees with $\eta(p)$.

Associated to $\phi$, let us consider the map

$$\psi := \phi + \eta : M^n \longrightarrow \mathbb{N}^{n+1}_+,$$

called the *associated light cone map.*

The map $\psi$ is strongly related to the hyperbolic Gauss map $G : M^n \longrightarrow \mathbb{S}^n_\infty$ of $\phi$. Indeed, the ideal boundary of $\mathbb{N}^{n+1}_+$ coincides with $\mathbb{S}^n_\infty$ and can be identified with the projective quotient space $\mathbb{N}^{n+1}_+/\mathbb{R}_+$. So, with all of this, we have $G = [\psi] : M^n \longrightarrow \mathbb{S}^n_\infty \equiv \mathbb{N}^{n+1}_+/\mathbb{R}_+$.

Note that $\psi_0 > 0$, given that $\psi = (\psi_0, \psi_1, \ldots, \psi_{n+1}) \in \mathbb{L}^{n+2}$. If we label $\psi_0 := e^\rho$, then we can interpret the *hyperbolic Gauss map* as the map

$$G : M^n \longrightarrow \mathbb{S}^n = \mathbb{N}^{n+1}_+ \cap \left\{x \in \mathbb{L}^{n+2} : x_0 = 1\right\}$$

given by

$$\psi = e^\rho(1, G). \tag{2.1}$$

Moreover, we call $e^\rho$ the *horospherical support function.* Also, if $\{e_1, \ldots, e_n\}$ denotes an orthonormal basis of principal directions of $\phi$ at $p$, and if $\kappa_1, \ldots, \kappa_n$ are the associated principal curvatures, it is immediate that

$$\langle (d\psi)_p(e_i), (d\psi)_p(e_j)\rangle = (1-\kappa_i)^2\delta_{ij} = e^{2\rho}\langle (dG)_p(e_i), (dG)_p(e_j)\rangle_{\mathbb{S}^n}. \tag{2.2}$$

Coming back to horospheres, we must remark that horospheres are the unique hypersurfaces such that, innerly oriented (i.e., when the unit normal points to the convex side), its associated light cone map is constant: $\phi + \eta = v \in \mathbb{N}^{n+1}_+$. Moreover, if we write $v = e^\rho(1, x)$, we see that $x \in \mathbb{S}^n$ is the point at infinity of the horosphere and $\rho$ is the signed hyperbolic distance of the horosphere to the point $\mathcal{O} = (1, 0, \ldots, 0) \in \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$.

In the hyperbolic setting we have a notion of convexity weaker than the usual geodesic convexity, i.e.,

**Definition 2.3** ([35])**.** Let $M^n \subset \mathbb{H}^{n+1}$ be an immersed oriented hypersurface, and let $\mathcal{H}_p$ denote the horosphere in $\mathbb{H}^{n+1}$ that is tangent to $M^n$ at $p$, and whose interior unit normal at $p$ agrees with the one of $M^n$. We will say that $M^n$ is *horospherically convex* at $p$ if there exists a neighborhood $V \subset M^n$ of $p$ so that $V \setminus \{p\}$ does not intersect $\mathcal{H}_p$, and in addition the distance function of the hypersurface to the horosphere does not vanish up to the second order at $p$ in any direction.

Thus, from (2.2), we have the following characterization of horospherically convex hypersurfaces:

**Lemma 2.4.** *Let $\phi : M^n \longrightarrow \mathbb{H}^{n+1}$ be an oriented hypersurface. The following conditions are equivalent at $p \in M^n$.*

(i) *All principal curvatures of $M^n$ at $p$ are simultaneously $< 1$ or $> 1$.*
(ii) *$M^n$ is horospherically convex at $p$.*

*In particular, if $\phi : M^n \longrightarrow \mathbb{H}^{n+1}$ is horospherically convex at $p$, then its Gauss map satisfies $dG_p \neq 0$.*

So, if $M^n$ is horospherically convex at $p$, then $dG_p \neq 0$ and there exist neighborhoods $U \subset M^n$ and $\Omega \subset \mathbb{S}^n$ such that $G : U \longrightarrow \Omega$ is a diffeomorphism, and

$$g = e^{2\rho}\langle dG, dG\rangle_{\mathbb{S}^n}$$

defines a conformally flat Riemannian metric on $M^n$, called the *horospherical metric*. Since $G$ is a diffeomorphism between $U$ and $\Omega$ we can use it as a parametrization of the hypersurface; i.e., we can assume that $\phi : \Omega \subset \mathbb{S}^n \longrightarrow \mathbb{H}^{n+1}$ and $G(x) = x$ on $\Omega \subset \mathbb{S}^n$.

Thus, if $G : M^n \longrightarrow \Omega \subseteq \mathbb{S}^n$ is a global diffeormorphism of the hypersurface onto a domain of the sphere, we can use the hyperbolic Gauss map as a global parametrization of $\phi$ as above; i.e., $\phi : \Omega \longrightarrow \mathbb{H}^{n+1}$ and $G(x) = x$. In this case, the horospherical metric is given by

$$g = e^{2\rho} g_0.$$

Now, we are ready to establish the mentioned relationship between conformal metrics on the sphere and horospherically convex hypersurfaces.

**Theorem 2.5** ([21]). *Let $\phi : \Omega \subset \mathbb{S}^n \longrightarrow \mathbb{H}^{n+1}$ be a horospherically convex hypersurface with hyperbolic Gauss map $G(x) = x$, support function $e^\rho : \Omega \longrightarrow (0, +\infty)$, and let $g = e^{2\rho} g_0$ denote its horospherical metric. Then it follows that*

$$\phi = \frac{e^\rho}{2}\left(1 + e^{-2\rho}\left(1 + ||\nabla^{g_0}\rho||^2_{g_0}\right)\right)(1, x) + e^{-\rho}(0, -x + \nabla^{g_0}\rho). \tag{2.3}$$

*Moreover, the eigenvalues, $\lambda_i$, of the Schouten tensor of $g$, $\mathrm{Sch}_g$, and the principal curvatures, $\kappa_i$, of $\phi$ are related by*

$$\lambda_i = \frac{1}{2} - \frac{1}{1 - \kappa_i}. \tag{2.4}$$

*Conversely, given a conformal metric $g = e^{2\rho} g_0$ defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that the eigenvalues of its Schouten tensor, $\mathrm{Sch}_g$, are less than $1/2$, then the map $\phi : \Omega \longrightarrow \mathbb{H}^{n+1}$ given by (2.3) defines a horospherically convex hypersurface in $\mathbb{H}^{n+1}$ whose hyperbolic Gauss map is given by $G(x) = x$, $x \in \Omega$.*

*Remark* 2.6. We must say that the condition on the eigenvalues of the Schouten tensor is easily removable; i.e., we only need to ask that

$$\sup\{\lambda_i(x),\, i = 1, \ldots, n,\, x \in \Omega\} < +\infty.$$

If this occurs, we can dilate the metric $g$ as $g_t = e^t g$ for $t > 0$. Then, the eigenvalues of $\mathrm{Sch}_{g_t}$ are given by

$$\lambda_i^t = e^{-t}\lambda_i.$$

Thus, for $t$ big enough, we can achieve $\lambda_i^t < 1/2$ for $i = 1, \ldots, n$.

## 3. Conformal diffeomorphisms and isometries

Let us denote by $\mathcal{I}(\mathbb{L}^{n+2})$, $\mathcal{I}(\mathbb{H}^{n+1})$ and $\mathcal{I}(\mathbb{N}^{n+1}_+)$ the group of isometries of $\mathbb{L}^{n+2}$, the $(n+1)-$dimensional hyperbolic space and the $(n+1)-$dimensional null cone, respectively.

It is well known (see [16]) that a conformal diffeomorphism $\Phi \in \mathcal{D}(\mathbb{S}^n)$ induces a unique isometry in $\mathbb{L}^{n+2}$, $T \in \mathcal{I}(\mathbb{L}^{n+2})$, such that restricted to $\mathbb{H}^{n+1}$ and $\mathbb{N}^{n+1}_+$

induces an isometry in these spaces and vice versa. The restrictions of $T \in \mathcal{I}(\mathbb{L}^{n+2})$ to $\mathbb{H}^{n+1}$ and $\mathbb{N}_+^{n+1}$ will be denoted by $T_{|\mathbb{H}^{n+1}}$ and $T_{|\mathbb{N}_+^{n+1}}$, respectively. Moreover, each isometry $T_{|\mathbb{H}^{n+1}} \in \mathcal{I}(\mathbb{H}^{n+1})$ induces a unique isometry $T_{|\mathbb{N}_+^{n+1}} \in \mathcal{I}(\mathbb{N}_+^{n+1})$ and vice versa.

**Definition 3.1.** Let $M^n \subset N^n$ be a domain of an $n-$manifold $N$. Let $\phi : M^n \subset N^n \longrightarrow \mathbb{H}^{n+1}$ be a hypersurface and $T_{|\mathbb{H}^{n+1}} \in \mathcal{I}(\mathbb{H}^{n+1})$ an isometry. $\phi$ is $T_{|\mathbb{H}^{n+1}}-$invariant if there exists $i_{T_{|\mathbb{H}^{n+1}}} : N^n \longrightarrow N^n$ a diffeomorphism such that

$$(T_{|\mathbb{H}^{n+1}} \circ \phi)(p) = \left(\phi \circ i_{T_{|\mathbb{H}^{n+1}}}\right)(p), \forall p \in M^n \text{ such that } i_{T_{|\mathbb{H}^{n+1}}}(p) \in M^n.$$

Moreover, given a continuous subgroup of isometries $\mathcal{T} \leq \mathcal{I}(\mathbb{H}^{n+1})$, $\phi$ is $\mathcal{T}-$invariant if it is $T_{|\mathbb{H}^{n+1}}-$invariant for all $T_{|\mathbb{H}^{n+1}} \in \mathcal{T}$.

The next result states the relationship between conformal metrics on the sphere which are invariant by a conformal diffeomorphism and horospherically convex hypersurfaces which are invariant by an isometry.

**Lemma 3.2.** *Let $\phi : \Omega \subset \mathbb{S}^n \longrightarrow \mathbb{H}^{n+1}$ be a locally horospherically convex hypersurface with hyperbolic Gauss map $G(x) = x$, support function $e^\rho : \Omega \longrightarrow (0, +\infty)$, and let $g = e^{2\rho} g_0$ denote its horospherical metric. Let $T_{|\mathbb{H}^{n+1}} \in \mathcal{I}(\mathbb{H}^{n+1})$ be an isometry and $\Phi \in \mathcal{D}(\mathbb{S}^n)$ its associated conformal diffeomorphism. Thus, if $\phi$ is $T_{|\mathbb{H}^{n+1}}-$invariant, then $g$ is $\Phi-$invariant.*

*Conversely, let $g = e^{2\rho} g_0$ be a conformal metric defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that the eigenvalues of its Schouten tensor, $\mathrm{Sch}_g$, are less than $1/2$. Let $\Phi \in \mathcal{D}(\mathbb{S}^n)$ be a conformal diffeomorphism and $T_{|\mathbb{H}^{n+1}} \in \mathcal{I}(\mathbb{H}^{n+1})$ its associated isometry. Thus, if $g$ is $\Phi-$invariant, then $\phi$, given by (2.3), is $T_{|\mathbb{H}^{n+1}}-$invariant.*

*Proof.* On one hand, if $\phi$ is horospherically convex, $\phi$ is $T_{|\mathbb{H}^{n+1}}-$invariant if and only if its associated light cone map $\psi$ is $T_{|\mathbb{N}_+^{n+1}}-$invariant, i.e., if

$$\left(T_{|\mathbb{N}_+^{n+1}} \circ \psi\right)(x) = (\psi \circ \Phi)(x),\ x \in \Omega \text{ such that } \Phi(x) \in \Omega, \tag{3.1}$$

$\Phi \in \mathcal{D}(\mathbb{S}^n)$ being the conformal diffeomorphism associated to $T_{|\mathbb{N}_+^{n+1}} \in \mathcal{I}(\mathbb{N}_+^{n+1})$.

On the other hand, we have an explicit correspondence between conformal diffeomorphisms on the sphere and isometries on $\mathbb{N}_+^{n+1}$ (see [16, Proposition 7.4]). Given an isometry $T_{|\mathbb{N}_+^{n+1}} \in \mathcal{I}(\mathbb{N}_+^{n+1})$, at points $(1, x) \in \mathbb{S}^n = \mathbb{N}_+^{n+1} \cap \left\{x \in \mathbb{L}^{n+2} : x_0 = 1\right\}$ we can see it as

$$T_{|\mathbb{N}_+^{n+1}}((1, x)) = e^{-\omega(x)}(1, \Phi(x)).$$

Then $\Phi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ defines a conformal diffeomorphism on the $n-$sphere with conformal factor $e^\omega$. Conversely, given a conformal diffeomorphism $\Phi \in \mathcal{D}(\mathbb{S}^n)$ with conformal factor $e^\omega$, at any point $e^t(1, x) \in \mathbb{N}_+^{n+1}$ define

$$T_{|\mathbb{N}_+^{n+1}}(e^t(1, x)) = e^t e^{-\omega(x)}(1, \Phi(x)).$$

Then $T_{|\mathbb{N}_+^{n+1}} \in \mathcal{I}(\mathbb{N}_+^{n+1})$.

We first prove the converse. By the previous considerations, we only need to prove (3.1). Thus, if $g = e^{2\rho} g_0$ is $\Phi-$invariant, hence by Definition 2.1 we have that

$$\rho(x) = \rho(\Phi(x)) + \omega(x), \text{ provided } \Phi(x) \in \Omega.$$

Let $T_{|\mathbb{N}^{n+1}_+}$ be the isometry of $\mathbb{N}^{n+1}_+$ associated to $\Phi$. Then

$$\begin{aligned}\left(T_{|\mathbb{N}^{n+1}_+}\circ\psi\right)(x) &= T_{|\mathbb{N}^{n+1}_+}\left(e^{\rho(x)}(1,x)\right) = e^{\rho(x)-\omega(x)}(1,\Phi(x))\\ &= e^{\rho(\Phi(x))}(1,\Phi(x)) = \psi(\Phi(x)) = (\psi\circ\Phi)(x).\end{aligned}$$

Now, if $\phi$ is $T_{|\mathbb{H}^{n+1}}$ −invariant, following the above computations, we can observe that

$$\rho(x) = \rho(\Phi(x)) + \omega(x),$$

$e^{\omega}:\Omega\to\mathbb{R}$ being the conformal factor of the conformal diffeomorphism, $\Phi$, associated to $T_{|\mathbb{H}^{n+1}}$. Thus, $g$ is $\Phi-$invariant. □

## 4. Isoparametric conformal metrics

Here, we will classify the class of conformal metrics on the sphere such that all the eigenvalues of its Schouten tensor are constant. We denote this class by $\mathcal{C}(n)$.

The local classification of conformal metrics on the class $g\in\mathcal{C}(n)$ can be done through a result of E. Cartan [5]. Suppose $g\in\mathcal{C}(n)$. Therefore, after possibly a dilation, the associated hypersurface given by Theorem 2.5 is an isoparametric hypersurface in $\mathbb{H}^{n+1}$; i.e., all its principal curvatures are constant. Thus, it is a piece of either a totally umbilical hypersurface (hypersphere, horosphere, totally geodesic hyperplane and equidistant) or a standard product $\mathbb{S}^k\times\mathbb{H}^{n-k}$ in $\mathbb{H}^{n+1}$. For this reason, we will call a metric in $\mathcal{C}(n)$ an *isoparametric conformal metric*.

It is known that

- solutions of
$$\sigma_k(\lambda_i) = 1 \text{ on } \mathbb{S}^n$$
are given by conformal diffeomorphisms of the standard metric on the sphere. Such a solution corresponds to a hypersphere via Theorem 2.5 (see [21]).
- Solutions of
$$\sigma_k(\lambda_i) = 0 \text{ on } \mathbb{R}^n$$
are explictly known (see [27]). Such a solution corresponds to a horosphere via Theorem 2.5 (see [21]).

Now, our task is to compute explicitly the horospherical support function associated to a totally geodesic hyperplane, an equidistant hypersurface and a standard product $\mathbb{S}^k\times\mathbb{H}^{n-k}$. To do so, we will give the parametrization of such a hypersurface and its unit normal vector field and, by means of equation (2.1), we will have an explicit formula for the horospherical support function and hyperbolic Gauss map. Thus, for an isoparametric hypersurface $\phi:\Omega\subset\mathbb{R}^n\longrightarrow\mathbb{H}^{n+1}\subset\mathbb{L}^{n+2}$ with unit normal $\eta:\Omega\subset\mathbb{R}^n\longrightarrow\mathbb{S}^{n+1}_1\subset\mathbb{L}^{n+2}$, we will have

$$\rho:\Omega\subset\mathbb{R}^n\longrightarrow\mathbb{R},$$

and

$$G:\Omega\subset\mathbb{R}^n\longrightarrow D\subset\mathbb{S}^n \text{ is a global diffeomorphism.}$$

Hence, the isoparametric conformal metric associated to that hypersurface is given by

$$g = e^{\rho(G^{-1}(y))}g_0,\ y\in D. \tag{4.1}$$

Let us describe the announced examples:

(1) **Totally geodesic hyperplanes:**

Set $\Omega = \{x \in \mathbb{R}^n : |x| < r\}$ and $D = \left\{y \in \mathbb{S}^n \subset \mathbb{R}^{n+1} : y_{n+1} > \frac{1-r^2}{1+r^2}\right\}$. Then,

$$\begin{aligned}\psi(x) &= \left(\frac{1+r^2}{2\sqrt{r^2-|x|}}, \frac{x}{\sqrt{r^2-|x|^2}}, \frac{1-r^2}{2\sqrt{r^2-|x|^2}}\right),\\ \eta(x) &= \left(\frac{1-r^2}{2r}, \mathbf{0}, \frac{1+r^2}{2r}\right).\end{aligned}$$

Thus, from (2.1), we get

$$\rho(x) = \ln\left(\frac{r(1+r^2)+(1-r^2)\sqrt{r^2-|x|^2}}{2r\sqrt{r^2-|x|^2}}\right), \tag{4.2}$$

$$G(x) = \left(\frac{2rx}{r(1+r^2)+(1-r^2)\sqrt{r^2-|x|^2}}, \frac{r(1-r^2)+(1+r^2)\sqrt{r^2-|x|^2}}{r(1+r^2)+(1-r^2)\sqrt{r^2-|x|^2}}\right). \tag{4.3}$$

In this case, the principal curvatures are all equal to zero, $k_i = 0$, $i = 1, \ldots, n$. Thus, the eigenvalues of the Shouten tensor associated to $g$ (given by (4.1)) are $\lambda_i = -1/2$, $i = 1, \ldots, n$.

(2) **Equidistant hypersurfaces:**

Set $t > 0$, $R^2 = t^2 + r^2$ and $\beta(s) = -t + \sqrt{R^2 - s}$ for $s < r$. Set $\Omega = \{x \in \mathbb{R}^n : |x| < r\}$ and $D = \left\{y \in \mathbb{S}^n \subset \mathbb{R}^{n+1} : y_{n+1} > (R+t)(1+r^2)\right\}$. Then

$$\psi(x) = \left(\frac{1+|x|^2+\beta(|x|^2)^2}{2\beta(|x|^2)}, \frac{x}{\beta(|x|^2)}, \frac{1-|x|^2-\beta(|x|^2)^2}{2\beta(|x|^2)}\right),$$

$$\eta(x) = \left(\frac{(1-t^2-R^2)\sqrt{R^2-|x|^2}+2tR^2}{2R\beta(|x|^2)}, \frac{tx}{R\beta(|x|^2)}, \frac{(1+t^2+R^2)\sqrt{R^2-|x|^2}-2tR^2}{2R\beta(|x|^2)}\right).$$

Thus, from (2.1), we get

$$\rho(x) = \ln\left(\frac{\alpha(|x|^2)}{2R\beta(|x|^2)}\right), \tag{4.4}$$

$$G(x) = \left(\frac{2(R-t)x}{\alpha(|x|^2)}, \frac{R+\sqrt{R^2-|x|^2}+(R+t)^2(R-\sqrt{R^2-|x|^2})}{\alpha(|x|^2)}\right), \tag{4.5}$$

where

$$\alpha(|x|^2) = R + \sqrt{R^2-|x|^2} + (R+t)^2(R-\sqrt{R^2-|x|^2}).$$

In this case, the principal curvatures are all equal to $-t/R$, $k_i = -t/R$, $i = 1, \ldots, n$. Thus, the eigenvalues of the Shouten tensor associated to $g$ (given by (4.1)) are $\lambda_i = -(R+t)/2(R-t)$, $i = 1, \ldots, n$.

(3) $\mathbb{H}^k\left(-\frac{1}{1+r^2}\right)\times\mathbb{S}^{n-k}\left(\frac{1}{r}\right)$**:**
For the sake of simplicity, we parametrize just a half of this hypersurface. Set $\Omega=\left\{x\in\mathbb{R}^k\right\}\times\left\{z\in\mathbb{R}^{n-k}:\,|z|<r\right\}\subset\mathbb{R}^n$, where $r>0$, and

$$D=\left\{\left(\frac{s}{\sqrt{1+r^2+s^2}}\theta_1,\frac{t\sqrt{1+r^2}}{r\sqrt{1+r^2+s^2}}\theta_2,\frac{\sqrt{1+r^2}\sqrt{r^2-t^2}}{r\sqrt{1+r^2+s^2}}\right):\begin{array}{ccc}\theta_1&\in&\mathbb{S}^{k-1}\\ \theta_2&\in&\mathbb{S}^{n-k-1}\\ s&\geq&0\\ t&<&r\end{array}\right\}$$
$\subset\mathbb{S}^n$.

Then,

$$\begin{aligned}\psi(x,z)&=\left(\sqrt{|x|^2+1+r^2},x,z,\sqrt{r^2-|z|^2}\right),\\ \eta(x,z)&=\left(\frac{r\sqrt{|x|^2+1+r^2}}{\sqrt{1+r^2}},\frac{rx}{\sqrt{1+r^2}},\frac{\sqrt{1+r^2}\,z}{r},\frac{\sqrt{1+r^2}\sqrt{r^2-|z|^2}}{r}\right).\end{aligned}$$

Thus, from (2.1), we get

$$\rho(x,z)=\ln\left(\frac{(r+\sqrt{1+r^2})\sqrt{|x|^2+1+r^2}}{\sqrt{1+r^2}}\right),\tag{4.6}$$

$$G(x,z)=\left(\frac{x}{\sqrt{|x|^2+1+r^2}},\frac{\sqrt{1+r^2}\,z}{r\sqrt{|x|^2+1+r^2}},\frac{\sqrt{1+r^2}\sqrt{r^2-|z|^2}}{r\sqrt{|x|^2+1+r^2}}\right).\tag{4.7}$$

In this case, the hypersuface has two principal curvatures given by $k_i=-\frac{r}{\sqrt{1+r^2}}$, for $i=1,\dots,k$, and $k_j=-\frac{\sqrt{1+r^2}}{r}$, for $j=k+1,\dots,n$. Thus, the eigenvalues of the Shouten tensor associated to $g$ (given by (4.1)) are $\lambda_i=-\frac{1}{2}-r^2+r\sqrt{1+r^2}$, for $i=1,\dots,k$, and $\lambda_j=\frac{1}{2}+r^2-r\sqrt{1+r^2}$, for $j=k+1,\dots,n$.

*Remark* 4.1. As we pointed out at the beginning of the section, hyperspheres and horospheres are the only solutions for $\sigma_k(\lambda_i)=1$ on $\mathbb{S}^n$ and $\sigma_k(\lambda_i)=0$ on $\mathbb{R}^n$. The other cases define complete metrics on a *subdomain* of the sphere. So, the natural question is: *Are these solutions the only solutions for such domains under the constraint* $\sigma_k(\lambda_i)=\text{constant}$*?*

## 5. Invariant conformal metrics on the sphere

In this section we will give sufficient conditions for a conformal metric on the sphere to be radial. The following local result is based on the correspondence given in Theorem 2.5, Lemma 2.1 and a deep result of Do Carmo-Dajzcer for hypersurfaces in hyperbolic space.

**Theorem 5.1.** *Let $g=e^{2\rho}g_0$ be a conformal metric defined on a domain of the sphere $\Omega\subset\mathbb{S}^n$ such that the eigenvalues, $\lambda_i$, for $i=1,\dots,n$, of its Schouten tensor,* $\mathrm{Sch}_g$*, satisfy*

$$\sup\left\{\lambda_i(x),\,x\in\Omega,\,i=1,\dots,n\right\}<+\infty.$$

*Furthermore, assume that the eigenvalues satisfy*

$$\lambda = \lambda_1 = \cdots = \lambda_{n-1},$$
$$\nu = \nu(\lambda) = \lambda_n,$$
$$\lambda - \nu \neq 0.$$

*Then $g$ is radial.*

*Proof.* Consider $t > 0$ large enough such that the eigenvalues of the Schouten tensor of $g_t = e^{2t} g$ are less than $1/2$ (see Remark 2.6). Consider the horospherically convex hypersurface, $\phi : \Omega \longrightarrow \mathbb{H}^{n+1}$, associated to $g_t$, given by (2.3) in Theorem 2.5. Hence, the principal curvatures of $\phi$ satisfy:

$$\tilde{\lambda} = \kappa_1 = \cdots = \kappa_{n-1},$$
$$\tilde{\nu} = \tilde{\nu}(\tilde{\lambda}) = \kappa_n,$$
$$\tilde{\lambda} - \tilde{\nu} \neq 0.$$

This follows from (2.4) and the assumptions on the eigenvalues of $\mathrm{Sch}_g$. Hence, using [6, Theorem 4.2], $\phi(\Omega)$ is contained in a rotational hypersurface, which means, via Lemma 3.2, that $g_t$ is radial, so $g$ is radial. □

The next result is about determining which conformal metrics on the sphere are invariant by a $k-$parameter subgroup of conformal diffeomorphisms of the sphere. We should remove the class of conformal metrics on the sphere such that all the eigenvalues of its Schouten tensor are constant, $\mathcal{C}(n)$, but this is not a significant problem, since there are not too many of them and we have to classify them. Again, the result is based on a theorem of M. Do Carmo and M. Dajczer.

**Theorem 5.2.** *Let $g = e^{2\rho} g_0$ be a conformal metric defined on a domain of the sphere $\Omega \subset \mathbb{S}^n$ such that $g \notin \mathcal{C}(n)$ and the eigenvalues, $\lambda_i$, for $i = 1, \ldots, n$, of its Schouten tensor,* $\mathrm{Sch}_g$*, satisfy*

$$\sup \{\lambda_i(x),\, x \in \Omega,\, i = 1, \ldots, n\} < +\infty.$$

*Suppose that $g$ is invariant by a $k-$parameter subgroup of a conformal diffeomorphism $\mathcal{G} \leq \mathcal{D}(\mathbb{S}^n)$. Then the maximum value of $k$ is $k_{max} = \frac{n(n-1)}{2}$, and if $k = k_{max}$, the Schouten tensor of $g$,* $\mathrm{Sch}_g$ *has two eigenvalues $\lambda$ and $\nu$, where one of them, say $\lambda$, has multiplicity at least $n-1$. If, in addition, $\lambda \neq 0$, $\nu = \nu(\lambda)$ and $\nu - \lambda \neq 0$, then $g$ is radial.*

*Proof.* As above, dilate $g$ until the eigenvalues of the Schouten tensor are less than $1/2$. Now, construct the horospherically convex hypersurface given by Theorem 2.5. The hypothesis on the $\mathcal{G}-$invariance of $g$ is translated into a $\mathcal{T}-$invariance of $\phi$ under a $k-$parameter subgroup $\mathcal{T} \leq \mathcal{I}(\mathbb{H}^{n+1})$. Thus, now applying [6, Theorem 4.7] we obtain the result. □

*Remark* 5.3. The above results hold for $n \geq 3$. It is clear that for $n = 2$ they are false.

## 6. A note on rotational hypersurfaces in $\mathbb{H}^{n+1}$

In a recent paper [10], the authors have classified all possible radial solutions to the equation

$$\sigma_k(\lambda_i) = c,\ c \equiv \text{constant};$$

that is, they consider conformal metrics $g = v(|x|)^{-2}|dx|^2$ on domains of the form

$$\{x \in \mathbb{R}^n,\, r_1 < |x| < r_2\},$$

$\sigma_k(\lambda_i)$ being the $k-$th elementary symmetric function of the eigenvalues of $\mathrm{Sch}_g$, and $0 \leq r_1 < r_2 \leq \infty$.

From the point of view of hypersurfaces in hyperbolic space, this classification result means (up to possibly a dilatation) that they have classified all rotational horospherically convex hypersurfaces verifying the Weingarten relationship

$$\sigma_k\left(\frac{1+\kappa_i}{2(1-\kappa_i)}\right) = \tilde{c},\ \tilde{c} \equiv \text{constant}.$$

It will take too long to describe all these solutions here, but we would like to mention two cases when $c > 0$: **Case I.1** and **Case I.3.a** in [10, Theorem 1] give the existence of hyperspheres (which was already known) and Delaunay-type hypersurfaces, respectively.

*Remark* 6.1. An interesting application of the above hypersurfaces could be to use them as barriers for the Plateau problem at infinity in the hyperbolic space for certain Weingarten functionals.

## Acknowledgement

The author wants to thank J.A. Gálvez, Y.Y. Li and R. Mazzeo for their interesting comments and help during the preparation of this work.

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