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INNER AND OUTER INEQUALITIES WITH APPLICATIONS TO APPROXIMATION PROPERTIES

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Abstract. Let X be a closed subspace of a Banach space W and let $\mathcal{F}$ be the operator ideal of finite-rank operators. If α is a tensor norm, $\mathcal{A}$ is a Banach operator ideal, and $\lambda > 0$, then we call the condition "$\|S\|_\alpha \leq \lambda\|S\|_{\mathcal{A}(X,W)}$ for all $S \in \mathcal{F}(X,X)$" an inner inequality and the condition "$\|T\|_\alpha \leq \lambda\|T\|_{\mathcal{A}(Y,W)}$ for all Banach spaces Y and for all $T \in \mathcal{F}(Y,X)$" an outer inequality. We describe cases when outer inequalities are determined by inner inequalities or by some subclasses of Banach spaces. This provides, among others, a unified approach to the study of approximation properties. We present various applications to Grothendieck's classical approximation properties, to the weak bounded approximation property, and to approximation properties of order p.

1. Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y and by $\mathcal{F}(X,Y)$ and $\mathcal{W}(X,Y)$ its subspaces of finite-rank and weakly compact operators. Let I_X denote the identity operator on X. Recall that X is said to have the *approximation property* (AP) if there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X. If (S_α) can be chosen with $\sup_\alpha \|S_\alpha\| \leq \lambda$ for some $\lambda \geq 1$, then X is said to have the *λ-bounded approximation property* (λ-BAP). If $\lambda = 1$, then X has the *metric approximation property* (MAP).

Recently, the weak bounded approximation property was introduced and studied in [21]. We say that X has the *weak λ-bounded approximation property* (weak λ-BAP) if for every Banach space Y and for every operator $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ such that $S_\alpha \to I_X$ uniformly on compact subsets of X. We say that X has the *weak metric approximation property* (weak MAP) if X has the weak 1-BAP.

In [27, Corollary 1], it is proven that the weak λ-BAP and the λ-BAP are equivalent for a Banach space X whenever X^* or X^{**} has the Radon-Nikodým

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property. It remains open whether the weak λ-BAP is strictly weaker than the λ-BAP (see Section 4.1 for the related Defant-Floret conjecture). If they were equivalent, then the answer to the long-standing famous open problem (Problem 3.8 in [2]), namely, whether the AP of a dual Banach space implies its MAP, would be "yes" (by [21]).

Since the BAP of X is defined in terms of the space X, it is natural to ask: could the weak BAP of X also be described in terms of the space X itself without having recourse to other Banach spaces Y? This question was the starting point of the present article.

We shall use the following observation. Since $\mathcal{F}(X,Y) = X^* \otimes Y$, for any tensor norm $\alpha = \|\cdot\|_\alpha$, we may consider the normed space $(\mathcal{F}(X,Y), \|\cdot\|_\alpha)$. If $\mathcal{A} = (\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach operator ideal, then $\mathcal{F}(X,Y) \subset \mathcal{A}(X,Y)$ and $(\mathcal{F}(X,Y), \|\cdot\|_{\mathcal{A}})$ is again a normed space.

In Section 3, Theorem 3.6, we shall give an affirmative answer to the above question by proving that X has the weak λ-BAP if and only if

$$\|S\|_\pi \le \lambda \|S\|_{\mathcal{N}(X,X^{**})} \text{ for all } S \in \mathcal{F}(X,X),$$

where π is the projective tensor norm and $\mathcal{N}$ is the ideal of nuclear operators. (The projective tensor norm is called the finite nuclear norm in [34].) This criterion is expressed in terms of finite-rank operators on X. It appears to be rather efficient, yielding numerous applications, in particular, easy or immediate proofs of main results on the weak BAP in [21], [23], [27] (see Sections 3.2 and 3.4).

To compare the weak BAP in its new reformulation with the BAP, let us recall the following theorem which is essentially due to Grothendieck [12, Chapter I, p. 179] (see, e.g., [4, p. 193] or [38, p. 80]). We state it in a clearly equivalent form using inequalities on finite-rank operators; $\mathcal{I}$ denotes the ideal of integral operators.

Theorem 1.1 (Grothendieck)**.** *Let X be a Banach space and let $1 \le \lambda < \infty$. The following statements are equivalent:*

(a) *X has the λ-BAP.*
(b) *$\|S\|_\pi \le \lambda \|S\|_{\mathcal{I}(X,X)}$ for all $S \in \mathcal{F}(X,X)$.*
(c) *$\|T\|_\pi \le \lambda \|T\|_{\mathcal{I}(Y,X)}$ for all Banach spaces Y and for all $T \in \mathcal{F}(Y,X)$.*

Definition 1.2. Let X be a closed subspace of a Banach space W. If α is a tensor norm, $\mathcal{A}$ is a Banach operator ideal, and $\lambda > 0$, then we shall call the condition

$$\|S\|_\alpha \le \lambda \|S\|_{\mathcal{A}(X,W)} \quad \text{for all} \quad S \in \mathcal{F}(X,X)$$

an *inner inequality* and the condition

$$\|T\|_\alpha \le \lambda \|T\|_{\mathcal{A}(Y,W)} \quad \text{for all Banach spaces } Y \text{ and for all} \quad T \in \mathcal{F}(Y,X)$$

an *outer inequality.*

An outer inequality is a rather common form to define approximation properties.

Definition 1.3 (Saphar)**.** Let $1 \le p \le \infty$ and let $1 \le \lambda < \infty$. A Banach space X is said to have the λ-BAP *of order* p (λ-BAP$_p$) if

$$\|T\|_{g_p} \le \lambda \|T\|_{\mathcal{I}_p(Y,X)} \text{ for all Banach spaces } Y \text{ and for all } T \in \mathcal{F}(Y,X).$$

The 1-BAP$_p$ is called the MAP$_p$.

Here, g_p is the Chevet-Saphar tensor norm (finite p-nuclear norm in [34]) and $\mathcal{I}_p$ is the ideal of p-integral operators (see, e.g., [38, pp. 135 and 173]).

The λ-BAP$_p$ was essentially introduced in 1970 by Saphar in his seminal paper [39] (we gave an equivalent reformulation via finite-rank operators). (In [4], the notion of the bounded g_p-approximation property with constant λ is used for the λ-BAP$_p$.) Since $\|\cdot\|_{g_1} = \|\cdot\|_\pi$ and $\mathcal{I}_1 = \mathcal{I}$, the λ-BAP$_1$ coincides with the λ-BAP and the MAP$_1$ with the MAP. In 1985, Bourgain and Reinov [1] (see, e.g., [4, 21.9]) proved that it is sufficient to check Definition 1.3 only for reflexive Banach spaces Y whenever $1 < p < \infty$. In other words, the outer inequality $\|\cdot\|_{g_p} \leq \lambda\|\cdot\|_{\mathcal{I}_p}$, $1 < p < \infty$, is determined by the class of reflexive Banach spaces.

In Section 2, we study two rather general questions. (1) When does an inner inequality $\|\cdot\|_\alpha \leq \|\cdot\|_{\mathcal{A}}$ imply the corresponding outer inequality (or just its restriction to all reflexive Banach spaces)? (2) When is an outer inequality determined by a subclass of Banach spaces (by reflexive spaces, for instance)? The results obtained in Section 2 are applied in Sections 3 and 4. Among others, we give a quick alternative proof of Grothendieck's result that X is finite-dimensional whenever $X^*\hat{\otimes}X = X^*\check{\otimes}X$ (i.e., the projective and injective tensor products coincide), we describe, with numerous applications, the weak λ-BAP via inner and outer inequalities, we show that the Bourgain-Reinov characterization of the λ-BAP$_p$ holds with separable reflexive spaces instead of reflexive spaces, and we exhibit the $\mathcal{Z}$-factorability (see Definition 1.6) as an essential reason for the BAP$_p$ to hold.

Our notation is standard. A Banach space X will be regarded as a subspace of its bidual X^{**} under the canonical embedding $j_X : X \to X^{**}$. The closed unit ball of X is denoted by B_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$. The tensor product $X \otimes Y$ with a tensor norm α is denoted by $X \otimes_\alpha Y$ and its completion by $X\hat{\otimes}_\alpha Y$.

Let $\mathcal{A}$ be a Banach operator ideal. Recall that $\mathcal{A}$ is *regular* if $T \in \mathcal{A}(X,Y)$ and $\|T\|_{\mathcal{A}} = \|j_Y T\|_{\mathcal{A}}$ whenever $j_Y T \in \mathcal{A}(X, Y^{**})$. We denote by $\mathcal{A}^*$ the *dual operator ideal of* $\mathcal{A}$. Its components are $\mathcal{A}^*(X,Y) = \{T \in \mathcal{L}(X,Y) : T^* \in \mathcal{A}(Y^*, X^*)\}$ with $\|T\|_{\mathcal{A}^*} = \|T^*\|_{\mathcal{A}}$. (The notation $\mathcal{A}^*$ is reserved for another concept in [34] where the dual operator ideal is denoted by $\mathcal{A}^{\text{dual}}$.) For Banach operator ideals $\mathcal{A}$ and $\mathcal{B}$, the inclusion $\mathcal{A} \subset \mathcal{B}$ means that $\mathcal{A}(X,Y) \subset \mathcal{B}(X,Y)$ and $\|T\|_{\mathcal{A}} \geq \|T\|_{\mathcal{B}}$ for all Banach spaces X and Y and for all $T \in \mathcal{A}(X,Y)$.

We denote by $\mathcal{L}$, $\mathcal{W}$, and $\mathcal{K}$ the Banach operator ideals of bounded, weakly compact, and compact linear operators, respectively. We also need the following Banach operator ideals: $\mathcal{N}_p-$ p-nuclear operators, $\mathcal{I}_p-$ p-integral operators, and $\mathcal{P}_p-$ absolutely p-summing operators (p-summing in [5]) (see [34] or [38]). If $1 \leq p \leq \infty$, then we write p^* for the conjugate index of p, that is, $1/p + 1/p^* = 1$.

We refer to the Diestel-Uhl and Ryan's books [6] and [38] for the classical approximation properties and to Pietsch's book [34] for operator ideals (see also [5] by Diestel, Jarchow, and Tonge for common operator ideals $\mathcal{N}_p, \mathcal{I}_p, \mathcal{P}_p$). Our main reference on the theory of tensor products and related operator ideals is the recent book of Ryan [38] (see also the encyclopedic monograph [4] by Defant and Floret which is the most accessible source for general approximation properties). In particular, by a *tensor norm* we mean, according to [38], a finitely generated uniform crossnorm. However, our exposition relies on basics from tensor products and Banach operator ideals, and we do not use the specific machinary of general tensor norms theory.

Finally, in the context of the present work, it is convenient to introduce the following two notions.

Definition 1.4. Let α be a tensor norm. We say that a Banach operator ideal $\mathcal{A}$ is the *dual operator ideal* of α provided

$$(X \otimes_\alpha Y)^* = \mathcal{A}(Y, X^*)$$

for all Banach spaces X and Y.

Remark 1.5. It is known that the dual operator ideal $\mathcal{A}$ exists for any tensor norm α (see, e.g., [38, pp. 187–190]). In fact, using notation from [38], $\mathcal{A} = \mathcal{L}_{\alpha^{t\prime}}$, the $\alpha^{t\prime}$-integral operators, where $\alpha^{t\prime}$ denotes the dual norm of the transpose of α.

Definition 1.6. Let $\mathcal{Z}$ be a set of Banach spaces. We say that a Banach operator ideal $\mathcal{A}$ is *$\mathcal{Z}$-factorable* if, for any Banach spaces X and Y, every $A \in \mathcal{A}(X, Y^*)$ admits a *nearly isometric factorization through some* $Z \in \mathcal{Z}$, meaning that, for every $\varepsilon > 0$, there exist $Z \in \mathcal{Z}, B \in \mathcal{A}(X, Z)$, and $C \in \mathcal{L}(Z, Y^*)$ such that $A = CB, \|B\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}} + \varepsilon$, and $\|C\| \leq 1$.

2. From inner inequalities towards outer inequalities

2.1. Our first result shows that the inner inequality $\|\cdot\|_\pi \leq \lambda\|\cdot\|_{\mathcal{A}}$ always implies the outer inequality with respect to the reflexive Banach spaces and, under some natural restrictions on $\mathcal{A}$, it implies the outer inequality with respect to all Banach spaces.

Proposition 2.1. *Let X be a closed subspace of a Banach space W and let $\mathcal{A}$ be a Banach operator ideal. If*

$$\|S\|_\pi \leq \lambda\|S\|_{\mathcal{A}(X,W)}$$

for all $S \in \mathcal{F}(X, X)$, then

$$\|T\|_\pi \leq \lambda\|T\|_{\mathcal{A}(Z,W)}$$

for all reflexive Banach spaces Z and for all $T \in \mathcal{F}(Z, X)$. If, moreover, $\mathcal{A}$ is regular and satisfies $\mathcal{A} \subset \mathcal{A}^{**}$, then

$$\|T\|_\pi \leq \lambda\|T\|_{\mathcal{A}(Y,W)}$$

for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$.

Proof. Let us consider a reflexive Banach space Z and $T \in \mathcal{F}(Z, X) = Z^* \otimes X$. Since $(Z^* \otimes_\pi X)^* = \mathcal{L}(X, Z)$, there exists $A \in \mathcal{L}(X, Z)$ with $\|A\| = 1$ such that

$$\begin{aligned} \|T\|_\pi = \text{trace } (TA) &\leq \|TA\|_\pi \leq \lambda\|TA\|_{\mathcal{A}(X,W)} \\ &\leq \lambda\|T\|_{\mathcal{A}(X,W)}\|A\| = \lambda\|T\|_{\mathcal{A}(X,W)}. \end{aligned}$$

For the "moreover" part, let $j : X \to W$ denote the identity embedding, let Y be a Banach space, and let $T = \sum_{i=1}^n y_i^* \otimes x_i \in \mathcal{F}(Y, X)$. Since $(Y^* \otimes_\pi X)^* = \mathcal{L}(X, Y^{**})$, there exists $A \in \mathcal{L}(X, Y^{**})$ with $\|A\| = 1$ such that

$$\|T\|_\pi = \sum_{i=1}^n (Ax_i)(y_i^*) = \text{trace } S,$$

where $S = \sum_{i=1}^n A^* y_i^* \otimes x_i \in \mathcal{F}(X, X)$. Since $T^{**}A = j_X S$, we have $j_W jS = j^{**} j_X S = j^{**} T^{**} A$, and therefore, by assumption,

$$\begin{aligned} \|T\|_\pi &= \text{trace } S \leq \|S\|_\pi \leq \lambda\|jS\|_{\mathcal{A}} = \lambda\|j_W jS\|_{\mathcal{A}} \\ &= \lambda\|j^{**}T^{**}A\|_{\mathcal{A}} \leq \lambda\|j^{**}T^{**}\|_{\mathcal{A}} = \lambda\|jT\|_{\mathcal{A}^{**}} \leq \lambda\|jT\|_{\mathcal{A}}, \end{aligned}$$

as desired. □

Remark 2.2. The finite nuclear norm (i.e., π) cannot be replaced by the nuclear norm in Proposition 2.1 (see Remark 3.2 below).

There are many important operator ideals, such as $\mathcal{I}_p$ and $\mathcal{P}_p$, $1 \le p \le \infty$, which satisfy the assumptions of the "moreover" part of Proposition 2.1 (see, e.g., [5, 5.14 and 2.5, together with 2.19]). In particular, taking $W = X$ and $\mathcal{A} = \mathcal{I}$ in Proposition 2.1 yields the implication (b)⇒(c) of Grothendieck's Theorem 1.1. This might be considered as an alternative direct proof of it since the proofs in the literature pass through condition (a) of Theorem 1.1 (see, e.g., [4, 16.3], [6, pp. 243–244], or [38, pp. 80–81]) as does the original proof in [12, Chapter I, pp. 179–180].

An inconvenience of Proposition 2.1 is that its "moreover" part does not apply to $\mathcal{N}_p$. Thus Proposition 2.1 enables us to pass from the inner inequality $\|\cdot\|_\pi \le \lambda\|\cdot\|_{\mathcal{N}_p}$ to the outer inequality only with respect to the reflexive Banach spaces. However, in this case we shall be able to go further and obtain the outer inequality with respect to all Banach spaces if we apply our next result. It shows that, in contrast with Proposition 2.1 which holds for the greatest tensor norm π, the outer inequality $\|\cdot\|_\alpha \le \lambda\|\cdot\|_{\mathcal{N}_p}$ is fully determined by the class of separable reflexive Banach spaces for any tensor norm α.

Theorem 2.3. *Let X be a closed subspace of a Banach space W. Let α be a tensor norm and let $1 \le p \le \infty$. If*

$$\|S\|_\alpha \le \lambda\|S\|_{\mathcal{N}_p(Z,W)}$$

for all separable reflexive Banach spaces Z and for all $S \in \mathcal{F}(Z,X)$, then

$$\|T\|_\alpha \le \lambda\|T\|_{\mathcal{N}_p(Y,W)}$$

for all Banach spaces Y and for all $T \in \mathcal{F}(Y,X)$.

Proof. Let Y be a Banach space and let $T = \sum_{i=1}^n y_i^* \otimes x_i \in \mathcal{F}(Y,X)$. We may assume that $\|y_i^*\| \le 1$. Since α is a tensor norm, we have $(Y^* \otimes_\alpha X)^* = \mathcal{A}(X, Y^{**})$ for its dual operator ideal $\mathcal{A}$ (see Remark 1.5). Let $A \in \mathcal{A}(X, Y^{**})$ be such that $\|A\|_{\mathcal{A}} = 1$ and

$$\|T\|_\alpha = \sum_{i=1}^n (Ax_i)(y_i^*).$$

Fix $\varepsilon > 0$ and, for $\|T\|_{\mathcal{N}_p(Y,W)}$, choose sequences $(\lambda_i) \in \ell_p$ (we denote by ℓ_∞ the space c_0), $(\varphi_i) \subset B_{Y^*}$, and $(w_i) \in \ell_{p^*}^w(W)$ with $\|(w_i)\|_{p^*}^w \le 1$ (we denote by $\ell_q^w(W)$ the Banach space of weakly q-summable W-valued sequences; see, e.g., [5, pp. 32–33] or [38, p. 134]) such that

$$Ty = \sum_{i=1}^\infty \lambda_i \varphi_i(y) w_i, \quad y \in Y,$$

and

$$\|T\|_{\mathcal{N}_p(Y,W)} + \varepsilon/2 > \|(\lambda_i)\|_p$$

(see, e.g., [5, p. 112] or [38, pp. 139–140]). Let (η_i) be a sequence of scalars tending to ∞ such that $\eta_i \ge 1$ for all i and

$$\|T\|_{\mathcal{N}_p(Y,W)} + \varepsilon > \|(\eta_i\lambda_i)\|_p.$$

Consider the closed absolutely convex hull in Y^* of the compact set

$$K := \{y_1^*, \ldots, y_n^*, 0, \varphi_1/\eta_1, \varphi_2/\eta_2, \ldots\}.$$

Since it is contained in B_{Y^*}, by the isometric version of the Davis-Figiel-Johnson-Pełczyński factorization lemma [3] due to [17], there exists a separable reflexive Banach space Z such that Z^* is a linear subspace of Y^*, the identity embedding $J : Z^* \to Y^*$ is compact and $\|J\| \leq 1$. Moreover, $K \subset J(B_{Z^*})$ and $Z = \overline{J^*(Y^{**})}$. Notice that actually $Z = \overline{J^*(Y)}$. Indeed, clearly, $(J^*|_Y)^* = J$. Hence

$$\overline{J^*(Y)} = \overline{\text{ran } \ J^*|_Y} = (\ker J)^\perp = \{0\}^\perp = Z,$$

as desired.

Choose $z_i^* \in B_{Z^*}, i = 1, \ldots, n$, and $\psi_i \in B_{Z^*}, i = 1, 2, \ldots$, such that

$$Jz_i^* = y_i^*, \quad J\psi_i = \varphi_i/\eta_i,$$

and consider $S = \sum_{i=1}^n z_i^* \otimes x_i \in \mathcal{F}(Z, X)$. Then, for all $y \in Y$,

$$SJ^*y = \sum_{i=1}^n z_i^*(J^*y)x_i = \sum_{i=1}^n (Jz_i^*)(y)x_i = \sum_{i=1}^n y_i^*(y)x_i = Ty.$$

Therefore

$$SJ^*y = \sum_{i=1}^\infty \lambda_i \varphi_i(y) w_i = \sum_{i=1}^\infty \lambda_i \eta_i \psi_i(J^*y) w_i.$$

Since $(\lambda_i \eta_i) \in \ell_p, (\psi_i) \subset B_{Z^*}, \|(w_i)\|_{p^*}^w \leq 1$, we see that

$$S' := \sum_{i=1}^\infty \lambda_i \eta_i \psi_i \otimes w_i \in \mathcal{N}_p(Z, W),$$

$$\|S'\|_{\mathcal{N}_p(Z,W)} \leq \|(\lambda_i \eta_i)\|_p < \|T\|_{\mathcal{N}_p(Y,W)} + \varepsilon,$$

and $SJ^*y = S'J^*y, y \in Y$, meaning that $S|_{J^*(Y)} = S'|_{J^*(Y)}$. Recalling that $\overline{J^*(Y)} = Z$, the last equality yields that $S = S'$, and therefore

$$\|S\|_{\mathcal{N}_p(Z,W)} < \|T\|_{\mathcal{N}_p(Y,W)} + \varepsilon.$$

Finally, by the assumption,

$$\begin{aligned} \|T\|_\alpha &= \sum_{i=1}^n (Ax_i)(Jz_i^*) = \sum_{i=1}^n z_i^*(J^*Ax_i) = \text{trace } (J^*AS) \\ &\leq \|J^*A\|_{\mathcal{A}} \|S\|_\alpha \leq \|S\|_\alpha \leq \lambda \|S\|_{\mathcal{N}_p(Z,W)} < \lambda \|T\|_{\mathcal{N}_p(Y,W)} + \lambda\varepsilon, \end{aligned}$$

and the desired inequality follows. □

2.2. Our third result enlightens an intrinsic general reason why the outer inequality $\|\cdot\|_\alpha \leq \lambda \|\cdot\|_{\mathcal{B}}$ for all Banach spaces would be determined by a given class of Banach spaces $\mathcal{Z}$. The reason resides in the $\mathcal{Z}$-factorability of the dual operator ideal of α (see Definitions 1.4 and 1.6).

Theorem 2.4. *Let $\mathcal{Z}$ be a set of Banach spaces. Let α be a tensor norm such that its dual operator ideal is $\mathcal{Z}$-factorable. Let $\mathcal{B}$ be a regular Banach operator ideal satisfying $\mathcal{B} \subset \mathcal{B}^{**}$. If X is a closed subspace of a Banach space W and*

$$\|S\|_\alpha \leq \lambda \|S\|_{\mathcal{B}(Z,W)}$$

for all $Z \in \mathcal{Z}$ and for all $S \in \mathcal{F}(Z, X)$, then

$$\|T\|_\alpha \leq \lambda \|T\|_{\mathcal{B}(Y,W)}$$

for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$.

Proof. Let Y be a Banach space and let $T = \sum_{i=1}^n y_i^* \otimes x_i \in \mathcal{F}(Y,X)$. Since $(Y^* \otimes_\alpha X)^* = \mathcal{A}(X, Y^{**})$, where $\mathcal{A}$ is the dual operator ideal of α, there exists $A \in \mathcal{A}(X, Y^{**})$ with $\|A\|_{\mathcal{A}} = 1$ such that

$$\|T\|_\alpha = \sum_{i=1}^n (Ax_i)(y_i^*) = \sum_{i=1}^n (A^* y_i^*)(x_i) = \text{trace } (T^{**}A).$$

Fix $\varepsilon > 0$ and choose $Z \in \mathcal{Z}, B \in \mathcal{A}(X,Z)$, and $C \in \mathcal{L}(Z, Y^{**})$ such that $A = CB, \|B\|_{\mathcal{A}} \le 1+\varepsilon$, and $\|C\| \le 1$. Define $S = \sum_{i=1}^n C^* y_i^* \otimes x_i \in \mathcal{F}(Z,X)$ and notice that

$$Sz = \sum_{i=1}^n (C^* y_i^*)(z) x_i = \sum_{i=1}^n (Cz)(y_i^*) x_i = T^{**}Cz, \quad z \in Z,$$

meaning that $T^{**}C = j_X S$. It may be useful to display the operators we need by the following commutative diagram, where $j : X \to W$ denotes the identity embedding:

$$\begin{tikzcd}
X \arrow[rr, "A"] \arrow[dr, "B"'] & & Y^{**} \arrow[r, "T^{**}"] & X^{**} \arrow[r, "j^{**}"] & W^{**} \\
& Z \arrow[ur, "C"] \arrow[rr, "S"] \arrow[d, "j_Z"'] & & X \arrow[ur, "j_X"] \arrow[rr, "j"] \arrow[d, "j_X"] & & W \arrow[ur, "j_W"'] \\
& Z^{**} \arrow[rr, "S^{**}"] & & X^{**}. & &
\end{tikzcd}$$

Since $T^{**}A = j_X SB = S^{**} j_Z B$, we have, by assumption,

$$\begin{aligned}
\|T\|_\alpha = \text{trace } (S^{**} j_Z B) &\le \|j_Z B\|_{\mathcal{A}} \|S\|_\alpha \le \|B\|_{\mathcal{A}} \|S\|_\alpha \\
&\le (1+\varepsilon)\lambda \|S\|_{\mathcal{B}(Z,W)}.
\end{aligned}$$

However,

$$\begin{aligned}
\|S\|_{\mathcal{B}(Z,W)} &= \|jS\|_{\mathcal{B}} = \|j_W jS\|_{\mathcal{B}} = \|j^{**} j_X S\|_{\mathcal{B}} \\
&= \|j^{**}T^{**}C\|_{\mathcal{B}} \le \|j^{**}T^{**}\|_{\mathcal{B}} = \|jT\|_{\mathcal{B}^{**}} \\
&\le \|jT\|_{\mathcal{B}} = \|T\|_{\mathcal{B}(Y,W)},
\end{aligned}$$

and the desired inequality follows. □

Example 2.5. It is a well-known result of Saphar [39] (see, e.g., [38, p. 142]) that the dual operator ideal of the Chevet-Saphar tensor norm g_p is $\mathcal{P}_{p^*}$. Using the Pietsch factorization theorem (see, e.g., [5, pp. 45–46]), $\mathcal{P}_{p^*}$ can easily be seen to be $\mathcal{Z}$-factorable, where $\mathcal{Z}$ is the set of closed subspaces of $L_{p^*}(\mu)$-spaces for probability measures μ.

We shall occasionally need the following fact which is due to Persson and Pietsch [33, proof of Lemma 7 on pp. 35–39] (see, e.g., [34, 19.1.11] or [38, p. 176], where the proof relies on theory of general tensor products). To be self-contained, we shall give it a quick proof.

Lemma 2.6 (Persson-Pietsch)**.** *Let X and Y be Banach spaces and let $1 \le p \le \infty$. If Y^* has the* MAP*, then all $T \in \mathcal{F}(Y,X)$ satisfy $\|T\|_{g_p} = \|T\|_{\mathcal{I}_p}$.*

Proof. Let $A \in \mathcal{P}_{p^*}(X, Y^{**})$ with $\|A\|_{\mathcal{P}_{p^*}} = 1$ be such that $\|T\|_{g_p} = \text{trace (AT)}$ (see Example 2.5). Since Y^* has the MAP, by Grothendieck's classics [12, Chapter I, p. 180] (see, e.g., [4, p. 193]), $\|S\|_\pi = \|S\|_{\mathcal{I}}$ for all $S \in \mathcal{F}(Y, Y^{**})$. Therefore, by a well-known Persson-Pietsch composition theorem (see, e.g., [5, 5.16]),

$$\|T\|_{g_p} \leq \|AT\|_\pi = \|AT\|_{\mathcal{I}} \leq \|A\|_{\mathcal{P}_{p^*}} \|T\|_{\mathcal{I}_p} = \|T\|_{\mathcal{I}_p},$$

as needed (recall that always $\|T\|_{\mathcal{I}_p} \leq \|T\|_{\mathcal{N}_p} \leq \|T\|_{g_p}$). □

Let us point out a version of Theorem 2.4 which shows that if the dual operator ideal of α is connected with $\mathcal{B}$ through some composition law involving an operator ideal $\mathcal{C}$, then the outer inequality $\|\cdot\|_\alpha \leq \lambda \|\cdot\|_{\mathcal{B}}$ follows from the inner inequality $\|\cdot\|_\pi \leq \lambda \|\cdot\|_{\mathcal{C}}$.

Theorem 2.7. *Let $\mathcal{Z}$ be a set of Banach spaces. Let α be a tensor norm such that its dual operator ideal $\mathcal{A}$ is $\mathcal{Z}$-factorable. Let $\mathcal{B}$ be a regular Banach operator ideal satisfying $\mathcal{B} \subset \mathcal{B}^{**}$. If X is a closed subspace of a Banach space W and there exists a Banach operator ideal $\mathcal{C}$ such that*

$$\|UA\|_{\mathcal{C}(X,W)} \leq \|U\|_{\mathcal{B}(Z,W)} \|A\|_{\mathcal{A}(X,Z)}$$

for all $Z \in \mathcal{Z}, A \in \mathcal{A}(X, Z)$ and $U \in \mathcal{F}(Z, X)$, then the condition

$$\|S\|_\pi \leq \lambda \|S\|_{\mathcal{C}(X,W)} \quad for\ all \quad S \in \mathcal{F}(X, X)$$

implies that

$$\|T\|_\alpha \leq \lambda \|T\|_{\mathcal{B}(Y,W)}$$

for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$.

Proof. The proof is almost the same as the proof of Theorem 2.4. Keeping its notation, the only modification in it would be that now

$$\begin{aligned} \|T\|_\alpha &= \text{trace } (j_X SB) = \text{trace } (SB) \leq \|SB\|_\pi \leq \lambda \|SB\|_{\mathcal{C}(X,W)} \\ &\leq \lambda \|S\|_{\mathcal{B}(Z,W)} \|B\|_{\mathcal{A}(X,Z)} \leq (1+\varepsilon)\lambda \|S\|_{\mathcal{B}(Z,W)}. \end{aligned}$$

□

By a result of Saphar [40, Proposition 3], the λ-BAP implies the λ-BAP$_p$ for all p. From Theorem 2.7 it is essentially clear why this is so (recall the well-known Persson-Pietsch composition theorem [33] (see, e.g., [5, 5.16]) asserting that $\|UA\|_{\mathcal{I}} \leq \|U\|_{\mathcal{I}_p} \|A\|_{\mathcal{P}_{p^*}}$). Actually Theorem 2.7 enables us to strengthen Saphar's result by showing that the λ-BAP$_p$, $1 < p < \infty$, is implied by the weak λ-BAP (see Proposition 4.4).

Bourgain and Reinov proved in [1, Theorem 2] that a Banach space X has the MAP$_p$, $1 < p < \infty$, whenever $\mathcal{P}_{p^*}(X, Y) = \mathcal{I}_{p^*}(X, Y)$ as sets for all reflexive Banach spaces Y; in particular, whenever X is the Hardy space H_∞ (of bounded analytic functions on the open unit disc). We conclude with a version of Theorem 2.4 which, among others, exposes a general reason why this is so (see Section 4.2 for more details).

Theorem 2.8. *Let X be a closed subspace of a Banach space W. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two sets of Banach spaces such that $\mathcal{Z} \subset \mathcal{Y}$ and $\mathcal{Y}$ contains countable ℓ_2 direct sums $(\sum_{n=1}^\infty Z_n)_2$ with $Z_n \in \mathcal{Z}$. Let α be a tensor norm such that, for its dual operator ideal $\mathcal{A}$ and for some $\mathcal{Z}$-factorable Banach operator ideal $\mathcal{C}$, one has the*

equality $\mathcal{A}(X, Y^{**}) = \mathcal{C}(X, Y^{**})$ *as sets whenever* $Y \in \mathcal{Y}$. *Let* $\mathcal{B}$ *be a regular Banach operator ideal satisfying* $\mathcal{B} \subset \mathcal{B}^{**}$. *If*

$$\|S\|_\alpha \leq \lambda \|S\|_{\mathcal{B}(Z,W)}$$

for all $Z \in \mathcal{Z}$ *and for all* $S \in \mathcal{F}(Z, X)$, *then for every* $Y \in \mathcal{Y}$ *there exists* $\mu_Y > 0$ *such that*

$$\|T\|_\alpha \leq \mu_Y \|T\|_{\mathcal{B}(Y,W)}$$

for all $T \in \mathcal{F}(Y, X)$.

Proof. Consider $Y \in \mathcal{Y}$. Notice that the norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{C}}$ are equivalent on $\mathcal{A}(X, Y^{**}) = \mathcal{C}(X, Y^{**})$ because, by the open mapping principle, both are equivalent to $\|\cdot\|_{\mathcal{A}} + \|\cdot\|_{\mathcal{C}}$. Let $a_Y > 0$ satisfy

$$\|A\|_{\mathcal{C}} \leq a_Y \|A\|_{\mathcal{A}} \quad \text{for all} \quad A \in \mathcal{A}(X, Y^{**}).$$

Let us show that there exists $b > 0$ such that

$$\|j_Z B\|_{\mathcal{A}} \leq b \|j_Z B\|_{\mathcal{C}} \quad \text{for all} \quad Z \in \mathcal{Z} \quad \text{and} \quad B \in \mathcal{C}(X, Z).$$

Indeed, if this were not the case, then there would exist $Z_n \in \mathcal{Z}$ and $B_n \in \mathcal{C}(X, Z_n)$ with $\|j_{Z_n} B_n\|_{\mathcal{A}} = n$ and $\|j_{Z_n} B_n\|_{\mathcal{C}} < 1/2^n$, $n \in \mathbb{N}$. Put

$$Z = (\sum_{n=1}^{\infty} Z_n)_2.$$

Then $Z^{**} = (\sum_{n=1}^{\infty} Z_n^{**})_2$. Let $j_n : Z_n^{**} \to Z^{**}$ and $p_n : Z^{**} \to Z_n^{**}$ denote, respectively, the natural norm one embedding and the natural norm one projection. Since

$$\|j_n j_{Z_n} B_n\|_{\mathcal{C}} \leq \|j_{Z_n} B_n\|_{\mathcal{C}} < \frac{1}{2^n},$$

the series $\sum_{n=1}^{\infty} j_n j_{Z_n} B_n$ converges (absolutely) in $\mathcal{C}(X, Z^{**})$. Let

$$D = \sum_{n=1}^{\infty} j_n j_{Z_n} B_n \in \mathcal{C}(X, Z^{**}) = \mathcal{A}(X, Z^{**})$$

(recall that $Z \in \mathcal{Y}$ by assumption). Since clearly $p_n D = j_{Z_n} B_n$, we would have

$$n = \|j_{Z_n} B_n\|_{\mathcal{A}} = \|p_n D\|_{\mathcal{A}} \leq \|D\|_{\mathcal{A}} \quad \text{for all} \quad n \in \mathbb{N},$$

which is impossible.

The proof continues essentially in the same way as the proof of Theorem 2.4. Keeping the notation of the proof of Theorem 2.4, the modifications in it would be that now $B \in \mathcal{C}(X, Z)$ with $\|B\|_{\mathcal{C}} \leq \|A\|_{\mathcal{C}} + \varepsilon \leq a_Y + \varepsilon$, and therefore

$$\|T\|_\alpha \leq \|j_Z B\|_{\mathcal{A}} \|S\|_\alpha \leq b \|j_Z B\|_{\mathcal{C}} \|S\|_\alpha \leq (a_Y + \varepsilon) b \lambda \|S\|_{\mathcal{B}(Z,W)}.$$

□

Remark 2.9. The above proof shows that if $\mathcal{Y}$ is invariant under countable ℓ_2 direct sums, then the conclusion of Theorem 2.8 is: *there exists* $\mu > 0$ *such that*

$$\|T\|_\alpha \leq \mu \|T\|_{\mathcal{B}(Y,W)}$$

for all $Y \in \mathcal{Y}$ *and for all* $T \in \mathcal{F}(Y, X)$.

3. Applications to Grothendieck's classics and to the weak bounded approximation property

3.1. Let X and Y be Banach spaces. Their (completed) projective and injective tensor products are denoted, as usual, by $X\hat{\otimes}Y$ and $X\check{\otimes}Y$. Since $\|u\|_\varepsilon \le \|u\|_\pi$ for all $u \in X \otimes Y$ (π and ε are projective and injective tensor norms, respectively), the identity embedding from $X \otimes_\pi Y$ to $X\check{\otimes}Y$ admits a unique bounded extension $j : X\hat{\otimes}Y \to X\check{\otimes}Y$. One writes $X\hat{\otimes}Y = X\check{\otimes}Y$, provided j is an isomorphism.

One of the most famous long-standing conjectures in functional analysis was the following Grothendieck conjecture (see, e.g., [12, Chapter I, p. 153]): if $X\hat{\otimes}Y = X\check{\otimes}Y$, then either X or Y must be finite dimensional. Recall that in 1981 Pisier [35] succeeded in constructing counterexamples to Grothendieck's conjecture. Here we would like to demonstrate a rather surprising application of Proposition 2.1 that yields an alternative easy proof for a case when Grothendieck's conjecture holds.

Corollary 3.1 (cf. [12, Chapter I, p. 153, Corollary 2])**.** *If $X^*\hat{\otimes}X = X^*\check{\otimes}X$ for a Banach space X, then X is finite dimensional.*

Proof. If $X^*\hat{\otimes}X = X^*\check{\otimes}X$, then there exists $\lambda \ge 1$ such that $\|S\|_\pi \le \lambda\|S\|$ for all $S \in \mathcal{F}(X,X)$. Then, by Proposition 2.1, $\|T\|_\pi \le \lambda\|T\|$ for all $T \in \mathcal{F}(c_0,X)$, or equivalently,

$$\|u\|_\pi \le \lambda\|u\|_\varepsilon \quad \forall u \in \ell_1 \otimes X.$$

This implies that $\ell_1\hat{\otimes}X = \ell_1\check{\otimes}X$. But it is well known that then $\dim X < \infty$. □

Remark 3.2. Notice that $X^*\hat{\otimes}X$ cannot be replaced by $\mathcal{N}(X,X)$ in Corollary 3.1. Indeed, Pisier's space P is an infinite-dimensional Banach space for which $\mathcal{N}(P,P) = P^*\check{\otimes}P$, or equivalently $\|S\|_{\mathcal{N}(P,P)} \le \lambda\|S\|$ for some $\lambda \ge 1$ and for all $S \in \mathcal{F}(P,P)$ (see [35, Theorem 3.7]). This also shows that the finite nuclear norm (i.e., π) cannot be replaced by the nuclear norm in Proposition 2.1. If it could, then we would have $\mathcal{N}(c_0,P) = \ell_1\check{\otimes}P$. However, $\mathcal{N}(c_0,P) = \ell_1\hat{\otimes}P$ (since $\ell_1 = c_0^*$ has the AP). Thus, as in the proof of Corollary 3.1, we would conclude that $\dim P < \infty$, which is not the case.

An open problem which goes back to Grothendieck's Résumé (see [5, p. 321]) is: does there exist an infinite-dimensional Banach space such that $\mathcal{N}(X,X) = \mathcal{K}(X,X)$? Another long-standing open problem is (see [25, Problem 1.e.9]): does the equality $X^*\check{\otimes}X = \mathcal{K}(X,X)$ imply that X has the AP? We shall see that a positive answer to the first problem would yield a negative answer to the second one (this seems to have passed unnoticed in the literature).

Corollary 3.3. *If $\mathcal{N}(X,X) = \mathcal{K}(X,X)$ for an infinite-dimensional Banach space X, then $X^*\check{\otimes}X = \mathcal{K}(X,X)$ but X does not have the* AP.

Proof. If X had the AP, then, as is well known, $X^*\hat{\otimes}X = \mathcal{N}(X,X)$. Hence we would have $X^*\hat{\otimes}X = X^*\check{\otimes}X$, implying $\dim X < \infty$, by Corollary 3.1. □

3.2. Applying Proposition 2.1 and then Theorem 2.3 yields the following result.

Corollary 3.4. *Let X be a closed subspace of a Banach space W and let $\lambda \ge 1$. Then the following assertions are equivalent:*

(a) $\|S\|_\pi \le \lambda\|S\|_{\mathcal{N}(X,W)}$ *for all* $S \in \mathcal{F}(X,X)$.

(b) $\|T\|_\pi \le \lambda \|T\|_{\mathcal{N}(Z,W)}$ *for all separable reflexive Banach spaces* Z *and for all* $T \in \mathcal{F}(Z,X)$.
(c) $\|T\|_\pi \le \lambda \|T\|_{\mathcal{N}(Y,W)}$ *for all Banach spaces* Y *and for all* $T \in \mathcal{F}(Y,X)$.

Let us consider two special cases $W = X$ and $W = X^{**}$ of Corollary 3.4.

The first case brings us to the AP. Characterizations (c) and (d) of the AP below are from Grothendieck's classics. The proof, relying on Corollary 3.4 of the implication (a)⇒(d), may be considered as an alternative to the known ones (see, e.g., [38, p. 76, Corollary 4.8]).

Proposition 3.5. *Let* X *be a Banach space. Then the following assertions are equivalent:*

(a) X *has the* AP.
(b) *There exists* $\lambda \ge 1$ *such that* $\|S\|_\pi \le \lambda \|S\|_{\mathcal{N}(X,X)}$ *for all* $S \in \mathcal{F}(X,X)$.
(c) $\|S\|_\pi = \|S\|_{\mathcal{N}(X,X)}$ *for all* $S \in \mathcal{F}(X,X)$.
(d) $\|T\|_\pi = \|T\|_{\mathcal{N}(Y,X)}$ *for all Banach spaces* Y *and for all* $T \in \mathcal{F}(Y,X)$.
(e) *For every separable reflexive Banach space* Z*, there exists* $\lambda \ge 1$ *such that* $\|T\|_\pi \le \lambda \|T\|_{\mathcal{N}(Z,X)}$ *for all* $T \in \mathcal{F}(Z,X)$.

Proof. It is easy to show (see, e.g., [25, p. 32]) that X has the AP if and only if the natural linear surjection j from $X^* \hat{\otimes} X$ onto $\mathcal{N}(X,X)$ is injective. But then j is an isometry (this is obvious from the definitions of the norms $\|\cdot\|_\pi$ and $\|\cdot\|_{\mathcal{N}}$). Therefore (b)⇒(a)⇒(c), while (c)⇒(b) is trivial. Corollary 3.4 yields that (c)⇒(d), while (d)⇒(c) and (d)⇒(e) are trivial. Finally, (e) clearly implies that the natural linear surjection from $Z^* \hat{\otimes} X$ onto $\mathcal{N}(Z,X)$ is injective, and therefore it is an isometry. In particular, (d) holds for all separable reflexive Banach spaces Y. Hence (d) holds in full generality by Corollary 3.4. □

The case $W = X^{**}$ in Corollary 3.4 will yield new characterizations of the weak λ-BAP (conditions (b), (c), and (d) below). The criterion (b) of the weak λ-BAP is one of the main results of the present paper: it expresses the weak λ-BAP of X in terms of X itself without using any other Banach space.

Theorem 3.6. *Let* X *be a Banach space and let* $\lambda \ge 1$. *Then the following assertions are equivalent:*

(a) X *has the weak* λ*-BAP.*
(b) $\|S\|_\pi \le \lambda \|S\|_{\mathcal{N}(X,X^{**})}$ *for all* $S \in \mathcal{F}(X,X)$.
(c) $\|T\|_\pi \le \lambda \|T\|_{\mathcal{N}(Y,X^{**})}$ *for all Banach spaces* Y *and for all* $T \in \mathcal{F}(Y,X)$.
(d) $\|T\|_\pi \le \lambda \|T\|_{\mathcal{N}(Z,X^{**})}$ *for all separable reflexive Banach spaces* Z *and for all* $T \in \mathcal{F}(Z,X)$.

Proof. The equivalence of conditions (b), (c), and (d) is immediate from Corollary 3.4. By [21, Theorem 3.2 and Remark 3.2], (a) holds if and only if, for every reflexive Banach space Z, the trace mapping $V : Z^* \hat{\otimes} X \to (\mathcal{F}(X,Z))^*$, $(Vu)(T) = \mathrm{trace}(Tu)$, $u \in Z^* \hat{\otimes} X$, $T \in \mathcal{F}(X,Z)$, satisfies $\|u\|_\pi \le \lambda \|Vu\|$ for all $u \in Z^* \otimes X$. It is known from Grothendieck's classics that $(\mathcal{F}(X,Z))^*$ can be canonically identified with $\mathcal{I}(Z,X^{**})$, and $\mathcal{I}(Z,X^{**}) = \mathcal{N}(Z,X^{**})$ isometrically whenever Z is reflexive (see, e.g., [6] or [38]). Hence (a) holds if and only if $\|T\|_\pi \le \lambda \|T\|_{\mathcal{N}(Z,X^{**})}$ for all reflexive Banach spaces Z and for all $T \in \mathcal{F}(Z,X)$. Therefore (a)⇒(d) and (c)⇒(a). □

Condition (b) of Theorem 3.6 was very recently applied to prove main results in [16]. This condition appears to be of essence of the weak BAP: it opens a way towards remarkably simple proofs of main results concerning the weak BAP, using, of course, some basics on classical tensor products (as, e.g., (a)⇔(c) in Proposition 3.5, (a)⇔(b) in Theorem 1.1, or the coincidence of $\mathcal{N}(X, Y^*)$ and $\mathcal{I}(X, Y^*)$, with the equality of norms, whenever X^* or Y^* has the Radon-Nikodým property). Some **examples** are as follows:

(1) *The weak* λ-BAP *and the* λ-BAP *are the same for* X *whenever* X^* *or* X^{**} *has the Radon-Nikodým property* [27], because all $S \in \mathcal{F}(X, X)$ satisfy

$$\|S\|_{\mathcal{N}(X,X^{**})} = \|S\|_{\mathcal{I}(X,X^{**})} = \|S\|_{\mathcal{I}(X,X)}.$$

(2) *If* X *has the* AP *and is complemented in* X^{**} *by a projection* P*, then* X *has the weak* $\|P\|$-BAP [21], because all $S \in \mathcal{F}(X, X)$ satisfy

$$\|S\|_\pi = \|S\|_{\mathcal{N}(X,X)} \leq \|P\|\|S\|_{\mathcal{N}(X,X^{**})}.$$

In particular, *the* AP *and the weak* MAP *are the same for dual Banach spaces.*

(3) The fact that *the* AP *of* X^* *implies the weak* MAP *of* X was proven in [21, Theorem 4.2] by relying on [30, Theorem 5] (a criterion of the AP for X^* in terms of approximability in the strong operator topology). Now this fact is obvious because all $S \in \mathcal{F}(X, X)$ satisfy

$$\|S\|_\pi = \|S\|_{X^*\otimes_\pi X} = \|S\|_{X^*\otimes_\pi X^{**}} = \|S^*\|_\pi = \|S^*\|_{\mathcal{N}(X^*,X^*)} = \|S\|_{\mathcal{N}(X,X^{**})}.$$

3.3. The proof of (1) suggests the following definition.

Definition 3.7. Let X be a Banach space and let $\mu \geq 1$. We say that X satisfies the *Radon-Nikodým* (RN) *μ-inequality* if

$$\|S\|_{\mathcal{N}(X,X^{**})} \leq \mu\|S\|_{\mathcal{I}(X,X)} \quad \text{for all} \quad S \in \mathcal{F}(X, X).$$

We say that X satisfies the RN *inequality* if X satisfies the RN μ-inequality for some μ. If $\mu = 1$, then we say that X satisfies the RN *equality.*

If $S \in \mathcal{F}(X^*, X^*)$, then $\|S\|_{\mathcal{N}(X^*,X^{***})} = \|S\|_{\mathcal{N}(X^*,X^*)}$ because of the canonical norm one projection from X^{***} onto X^*. Therefore the RN μ-inequality for X^* means that

$$\|S\|_{\mathcal{N}(X^*,X^*)} \leq \mu\|S\|_{\mathcal{I}(X^*,X^*)} \quad \text{for all} \quad S \in \mathcal{F}(X^*, X^*).$$

Proposition 3.8. *Let* X *be a Banach space and let* $\mu \geq 1$.

(a) *If* X^* *or* X^{**} *has the Radon-Nikodým property, then* X^* *satisfies the* RN *equality.*

(b) *If* X^* *satisfies the* RN *μ-inequality, then so does* X.

Proof. If X^* or X^{**} has the Radon-Nikodým property, then $\mathcal{N}(X^*, X^*) = \mathcal{I}(X^*, X^*)$ with the equality of norms. This proves (a). If $S \in \mathcal{F}(X, X)$, then $\|S\|_{\mathcal{N}(X,X^{**})} = \|S^*\|_{\mathcal{N}(X^*,X^*)}$ and $\|S\|_{\mathcal{I}(X,X)} = \|S^*\|_{\mathcal{I}(X^*,X^*)}$. This proves (b). □

If $X = L_1(0, 1)$, then $X, X^*, X^{**}, \ldots$ do not have the Radon-Nikodým property. However, they all satisfy the Radon-Nikodým equality (by Proposition 3.10 below, since they all have the MAP). Hence (a) in Proposition 3.8 is not invertible. We conjecture that (b) in Proposition 3.8 is not invertible. This is closely connected to the long-standing famous open problem that goes back to Grothendieck's memoir [12].

Problem 3.9 (cf., e.g., [2, p. 289])**.** Does the AP of the dual space X^* of a Banach space X imply the BAP or even the MAP of X^*?

The most far-reaching (positive) result (see [6, p. 246] and [36, Theorem 4]), which is essentially due to Grothendieck [12, Chapter I, proof of Theorem 15 on pp. 182–184], is that the AP of X^* implies the MAP whenever X^* or X^{**} has the Radon-Nikodým property. (There have been more recent different proofs of this fact, e.g., in [10], [17], [22], [27]. A recent new approach to Problem 3.9 has been suggested in [21] and [29].) This by now classical result will be at least formally improved in Corollary 3.11 below.

Proposition 3.10. *Let X be a Banach space. Then:*

(a) *If X has the weak λ-BAP and satisfies the* RN *μ-inequality, then X has the $\lambda\mu$-BAP.*

(b) *If X has the λ-BAP, then X satisfies the* RN *λ-inequality.*

Proof. (a) By Theorem 3.6, all $S \in \mathcal{F}(X,X)$ satisfy $\|S\|_\pi \le \lambda\|S\|_{\mathcal{N}(X,X^{**})} \le \lambda\mu\|S\|_{\mathcal{I}(X,X)}$. This means the $\lambda\mu$-BAP of X (see Theorem 1.1).

(b) By Theorem 1.1, all $S \in \mathcal{F}(X,X)$ satisfy $\|S\|_\pi \le \lambda\|S\|_{\mathcal{I}(X,X)}$. However, as is well known (and is clear from the definitions of the norms), $\|S\|_{\mathcal{N}(X,X^{**})} \le \|S\|_{\mathcal{N}(X,X)} \le \|S\|_\pi$. □

Corollary 3.11. *Assume that the dual space X^* of a Banach space X has the* AP*. Then X^* has the λ-*BAP *if and only if X^* satisfies the* RN *λ-inequality. In particular, X^* has the* MAP *if and only if X^* satisfies the* RN *equality.*

Proof. We know that X^* has the weak MAP (see [21, Corollary 3.4]; an easy proof was given above as example (2)). Therefore the claim is immediate from Proposition 3.10. Alternatively, the claim is also immediate from Proposition 3.5 and Theorem 1.1. □

3.4. Based on condition (b) of Theorem 3.6, we are now going to establish a new criterion of the weak BAP (see Corollary 3.14 below) which will be expressed in terms of extension operators. This was inspired by results of Vegard Lima [23] (see Remark 3.19).

Let X be a closed subspace of a Banach space W. An operator $\Phi \in \mathcal{L}(X^*, W^*)$ is called an *extension operator* if $(\Phi x^*)(x) = x^*(x)$ for all $x^* \in X^*$ and all $x \in X$. Pairs of Banach spaces W and their closed subspaces X for which there exists an extension operator $\Phi \in \mathcal{L}(X^*, W^*)$ were systematically studied by Fakhoury [7] and Kalton [14], and various examples are presented in [26, Section 5.5]. Remark that the existence of an extension operator Φ is equivalent to the annihilator $X^\perp$ of X being complemented in W^*. By Fakhoury [7] and Kalton [14], this also means that X is locally complemented in W ([31, Section 3] exposes some recent special features of the local complementation).

Since $\mathcal{N}(X,W)$ is canonically identified with a quotient space of $X^*\hat{\otimes}W$, its dual space $(\mathcal{N}(X,W))^*$ is canonically isometrically embedded in $(X^*\hat{\otimes}W)^* = \mathcal{L}(X^*, W^*)$. Therefore one writes

$$(\mathcal{N}(X,W))^* \subset \mathcal{L}(X^*, W^*).$$

This allows us to consider the question about the existence of extension operators inside of $(\mathcal{N}(X,W))^*$.

Theorem 3.12. *Let X be a closed subspace of a Banach space W and let $\lambda \geq 1$. Then the following assertions are equivalent:*

(a) $\|S\|_\pi \leq \lambda \|S\|_{\mathcal{N}(X,W)}$ *for all* $S \in \mathcal{F}(X,X)$.

(b) *There exists an extension operator* $\Phi \in \mathcal{L}(X^*, W^*)$ *such that* $\Phi \in (\mathcal{N}(X,W))^*$ *and* $\|\Phi\| \leq \lambda$.

Proof. Let $j : X \to W$ denote the identity embedding.

(a)⇒(b). Since

$$\|jS\|_{\mathcal{N}} \leq \|S\|_\pi \leq \lambda \|jS\|_{\mathcal{N}} \quad \text{for all} \quad S \in \mathcal{F}(X,X) = X^* \otimes X,$$

$S \mapsto jS$ yields an isomorphic embedding of $X^* \otimes_\pi X$ into $\mathcal{N}(X,W)$. Consider its (unique) bounded extension $J : X^* \hat{\otimes} X \to \mathcal{N}(X,W)$. Then J is an into isomorphism with its injection modulus $i(J) \geq 1/\lambda$. (Recall that if $A \in \mathcal{L}(Y,Z)$, then, by definition,

$$i(A) = \sup\{\tau > 0 : \tau\|y\| \leq \|Ay\| \quad \forall y \in Y\}.)$$

Since

$$J^* : (\mathcal{N}(X,W))^* \to (X^* \hat{\otimes} X)^* = \mathcal{L}(X^*, X^*)$$

and J is an into isomorphism, there exists a $\Phi \in (\mathcal{N}(X,W))^* \subset \mathcal{L}(X^*, W^*)$ so that $J^*\Phi = I_{X^*}$ and

$$\|\Phi\| \leq \frac{1}{i(J)} \|I_{X^*}\| \leq \lambda$$

(for a proof of this "folkloristic" fact, see, e.g., [13, Lemma 2.2]). Moreover, for all $x^* \in X^*$ and $x \in X$,

$$(\Phi x^*)(x) = \langle \Phi, x^* \otimes jx \rangle = \langle \Phi, J(x^* \otimes x) \rangle = \langle J^*\Phi, x^* \otimes x \rangle = \langle I_{X^*}, x^* \otimes x \rangle = x^*(x).$$

Hence Φ is an extension operator.

(b)⇒(a). It suffices to show that condition (b) of Corollary 3.4 holds. Thus, let Z be a reflexive Banach space and let $T = \sum_{i=1}^n z_i^* \otimes x_i \in \mathcal{F}(Z,X) = Z^* \otimes X$. Since $(Z^* \otimes_\pi X)^* = \mathcal{L}(X,Z)$, there exists $A \in \mathcal{L}(X,Z)$ with $\|A\| = 1$ such that

$$\begin{aligned} \|T\|_\pi &= \sum_{i=1}^n z_i^*(Ax_i) = \sum_{i=1}^n (A^* z_i^*)(x_i) = \sum_{i=1}^n (\Phi A^* z_i^*)(x_i) \\ &= \langle \Phi, \sum_{i=1}^n A^* z_i^* \otimes jx_i \rangle = \langle \Phi, jTA \rangle \leq \|\Phi\| \|jTA\|_{\mathcal{N}} \\ &\leq \lambda \|jT\|_{\mathcal{N}} \|A\| = \lambda \|T\|_{\mathcal{N}(Z,W)}, \end{aligned}$$

as needed. □

Two immediate corollaries follow (using Proposition 3.5 and Theorem 3.6).

Corollary 3.13. *A Banach space X has the* AP *if (and only if) $I_{X^*} \in (\mathcal{N}(X,X))^*$.*

(The "only if" part above is well known: if X has the AP, then $X^* \hat{\otimes} X = \mathcal{N}(X,X)$, and therefore $(\mathcal{N}(X,X))^* = \mathcal{L}(X^*, X^*)$.)

Corollary 3.14. *A Banach space X has the weak λ-BAP if and only if there exists an extension operator $\Phi \in \mathcal{L}(X^*, X^{***})$ such that $\Phi \in (\mathcal{N}(X, X^{**}))^*$ and $\|\Phi\| \leq \lambda$.*

The simpliest extension operator in $\mathcal{L}(X^*, X^{***})$ is of course j_{X^*}. This in mind, Corollary 3.14 could be compared with the following reformulation of Corollary 3.13 for X^*.

Corollary 3.15. *The dual X^* of a Banach space X has the* AP *if and only if* $j_{X^*} \in (\mathcal{N}(X, X^{**}))^*$.

Proof. Due to the canonical identification of $X^*\hat{\otimes}X^{**}$ and $X^{**}\hat{\otimes}X^*$, $\mathcal{N}(X, X^{**})$ and $\mathcal{N}(X^*, X^*)$ are also canonically identified. Therefore their dual spaces are canonically identified inside $\mathcal{L}(X^*, X^{***})$ and $\mathcal{L}(X^{**}, X^{**})$, respectively. But, in the latter identification, j_{X^*} corresponds to $I_{X^{**}}$. □

From Corollaries 3.14 and 3.15 we immediately get the following theorem due to Vegard Lima [23] which was proven in [23] by relying on rather deep characterizations of the AP and the weak MAP in terms of ideals (in the sense of [9]) from [23] and [20] (which, in turn, rely on results of [17], [18], [19]). Recall that a Banach space X is said to have the *unique extension property* (UEP) (cf. [10]) if j_{X^*} is the only norm one extension operator in $\mathcal{L}(X^*, X^{***})$.

Corollary 3.16 (see [23, Theorem 2.11])**.** *If a Banach space X has the* UEP *and the weak* MAP*, then X^* has the* AP.

Proof. By Corollary 3.14, there exists a norm one extension operator $\Phi \in (\mathcal{N}(X, X^{**}))^*$. Since $\Phi = j_{X^*}$, by Corollary 3.15, X^* has the AP. □

3.5. Let us now consider the dual system $\langle \mathcal{L}(X^*, W^*), X^*\hat{\otimes}W\rangle$. It is essentially known that

$$(\mathcal{N}(X, W))^* = (X \otimes W^*)^{\circ\circ}.$$

(In fact, let $q : X^*\hat{\otimes}W \to \mathcal{N}(X, W)$ be the canonical quotient mapping. It is straightforward to verify that $\ker q = (X \otimes W^*)^\circ \subset X^*\hat{\otimes}W$. Therefore

$$(\mathcal{N}(X, W))^* = ((X^*\hat{\otimes}W)/\ker q)^* = (\ker q)^\circ = (X \otimes W^*)^{\circ\circ}.)$$

Hence, by the bipolar theorem,

$$(\mathcal{N}(X, W))^* = \overline{X \otimes W^*}^{w^*},$$

the weak* closure of $X \otimes W^*$ in $\mathcal{L}(X^*, W^*)$.

This allows us to reformulate condition (b) of Theorem 3.12 as follows: *there exists an extension operator* $\Phi \in \overline{X \otimes W^*}^{w^*} \subset \mathcal{L}(X^*, W^*)$ *with* $\|\Phi\| \le \lambda$. Similarly, Corollary 3.13 asserts that X *has the AP if and only if* $I_{X^*} \in \overline{X \otimes X^*}^{w^*} \subset \mathcal{L}(X^*, X^*)$ (a direct proof of this fact can be done verbatim to [28, Lemma 2.2]). Corollary 3.18 below is immediate from Corollary 3.14 and the following lemma.

Lemma 3.17. *Let X and Y be Banach spaces. Then the weak* closures of $X \otimes Y$ and of $X \otimes Y^{**}$ coincide in $\mathcal{L}(X^*, Y^{**}) = (X^*\hat{\otimes}Y^*)^*$.*

Proof. A straightforward verification can be done similar to [28, Lemma 2.3]. □

Corollary 3.18. *A Banach space X has the weak λ-BAP if and only if there exists an extension operator* $\Phi \in \overline{X \otimes X^*}^{w^*} \subset \mathcal{L}(X^*, X^{***})$ *with* $\|\Phi\| \le \lambda$.

Remark 3.19. The special case of Corollary 3.18 with $\lambda = 1$ (i.e., the weak MAP case) was obtained by Vegard Lima in [23, Propositions 2.3 and 2.5]. His rather involved proof relied on Lindenstrauss's compactness argument [24], combined with results from [8] and [17], to construct the extension operator, and also on a main result from [10].

4. Applications to the approximation properties of order p

4.1. Let us recall the following notion which is also due to Saphar [39] as the notion of the λ-BAP$_p$ (see Definition 1.3).

Definition 4.1 (Saphar)**.** Let $1 \le p \le \infty$. A Banach space X is said to have the AP *of order* p (AP$_p$) if the natural surjection from $Y^* \hat{\otimes}_{g_p} X$ onto $\mathcal{N}_p(Y, X)$ is injective for all Banach spaces Y.

In [4], the notion of the g_p-approximation property is used for the AP$_p$. Since $\|\cdot\|_{g_1} = \|\cdot\|_\pi$ and $\mathcal{N}_1 = \mathcal{N}$, the AP$_1$ coincides with the AP.

It is clear from the definitions of the norms $\|\cdot\|_{g_p}$ and $\|\cdot\|_{\mathcal{N}_p}$ that the AP$_p$ of X is equivalent to the canonical identification $Y^* \hat{\otimes}_{g_p} X = \mathcal{N}_p(Y, X)$ for all Banach spaces Y. This in turn means that $\|T\|_{g_p} = \|T\|_{\mathcal{N}_p(Y,X)}$ for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$.

Bourgain and Reinov [1, Lemma 7] proved that it is enough to check the definitions for the AP$_p$ and λ-BAP$_p$ for the reflexive Banach spaces only, instead of all Banach spaces Y, provided $1 < p < \infty$. The same was also proved in [4, 21.9], strengthening the result on the AP$_p$ up to checking the definition for the *separable* reflexive Banach spaces only (here $1 \le p \le \infty$). The latter result is contained in Theorem 2.3 as the special case when $W = X$, $\alpha = g_p$, and $\lambda = 1$. Let us spell it out as follows (cf. Proposition 3.5 for the AP$_1$).

Proposition 4.2. *Let X be a Banach space and let $1 \le p \le \infty$. Then the following assertions are equivalent:*

(a) *X has the* AP$_p$.
(b) *$\|T\|_{g_p} = \|T\|_{\mathcal{N}_p(Y,X)}$ for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$.*
(c) *For every separable reflexive Banach space Z, there exists $\lambda \ge 1$ such that $\|T\|_{g_p} \le \lambda \|T\|_{\mathcal{N}_p(Z,X)}$ for all $T \in \mathcal{F}(Z, X)$.*

Our next result, which is an application of Theorems 2.3 and 2.4, shows that, for the λ-BAP$_p$ with $1 < p < \infty$, it is also enough to check the definition for the separable reflexive Banach spaces only. Defant and Floret [4, p. 283] **conjecture** that for $p = 1$, i.e., for the λ-BAP, the definition cannot be checked on the reflexive spaces only. As proved in [21] (and clear from Theorems 1.1 and 3.6), this conjecture means precisely that the weak λ-BAP is different from the λ-BAP. Results of [16], [21], [23], [29], and of the present paper seem to support this conjecture.

Theorem 4.3. *Let X be a Banach space, let $1 < p < \infty$, and let $\lambda \ge 1$. Then the following assertions are equivalent:*

(a) *X has the λ-BAP$_p$.*
(b) *$\|T\|_{g_p} \le \lambda \|T\|_{\mathcal{I}_p(Z,X)}$ for all separable reflexive Banach spaces Z and $T \in \mathcal{F}(Z, X)$.*
(c) *$\|T\|_{g_p} \le \lambda \|T\|_{\mathcal{N}_p(Y,X^{**})}$ for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$.*
(d) *$\|T\|_{g_p} \le \lambda \|T\|_{\mathcal{N}_p(Z,X^{**})}$ for all separable reflexive Banach spaces Z and for all $T \in \mathcal{F}(Z, X)$.*

Proof. The implications (a)⇒(b) and (b)⇒(d) are trivial (recall that $\|T\|_{\mathcal{I}_p(Z,X)} = \|T\|_{\mathcal{I}_p(Z,X^{**})} \le \|T\|_{\mathcal{N}_p(Z,X^{**})}$), and (d)⇒(c) is clear from Theorem 2.3. Finally, assume that (c) holds. By Persson's theorem [32], $\|T\|_{\mathcal{N}_p(Y,X^{**})} = \|T\|_{\mathcal{I}_p(Y,X^{**})}$ whenever Y is reflexive and $T \in \mathcal{F}(Y, X)$. Therefore $\|T\|_{g_p} \le \lambda \|T\|_{\mathcal{I}_p(Y,X^{**})}$ whenever Y is a closed subspace of an $L_{p^*}(\mu)$-space and, by Theorem 2.4 (see also

Example 2.5), the same inequality holds for all Banach spaces Y. This means (a). $\square$

For $p = 1$, condition (c) of Theorem 4.3 would be exactly the same as condition (c) of Theorem 3.6. This shows that, for $1 < p < \infty$, "the weak λ-BAP_p" would be exactly the λ-BAP_p. Therefore, it is not surprising, for instance, that (cf. example (2) after Theorem 3.6 for $p = 1$) *if X has the* AP_p, $1 < p < \infty$, *and is complemented in X^{**} by a projection P, then X has the* $\|P\|$-BAP_p [37, Theorem 4.2], [1, Lemma 8] (see, e.g., [4, 21.10]), because by Proposition 4.2 (b), for all Y, all $T \in \mathcal{F}(Y, X)$ satisfy

$$\|T\|_{g_p} = \|T\|_{\mathcal{N}_p(Y,X)} \leq \|P\|\|T\|_{\mathcal{N}_p(Y,X^{**})},$$

which means the $\|P\|$-BAP_p by Theorem 4.3 (c). In particular, *the* AP_p *and the* MAP_p, $1 < p < \infty$, *are the same for dual Banach spaces.*

4.2. The next result was established by Saphar [40, Proposition 3] under the assumption that X has the λ-BAP.

Proposition 4.4. *If a Banach space X has the weak* λ-BAP, *then X has the* λ-BAP_p, $1 < p < \infty$.

Proof. By assumption, $\|S\|_\pi \leq \lambda\|S\|_{\mathcal{N}(X,X^{**})}$ for all $S \in \mathcal{F}(X, X)$ (see Theorem 3.6 (b)). We are going to apply Theorem 2.7 with $\alpha = g_p$, $\mathcal{B} = \mathcal{I}_p$, $\mathcal{C} = \mathcal{N} = \mathcal{N}_1$, and $W = X^{**}$. We know (see Example 2.5) that $\mathcal{A} = \mathcal{P}_{p^*}$ and $\mathcal{Z}$ is contained in the set of all reflexive Banach spaces. By Persson's theorem [32], $\|U\|_{\mathcal{N}_p(Z,X^{**})} = \|U\|_{\mathcal{I}_p(Z,X^{**})}$ for all reflexive Banach spaces Z and for all $U \in \mathcal{F}(Z, X)$. Thus, from a well-known Persson-Pietsch composition theorem [33] (see, e.g., [5, 5.29]), we get that

$$\|UA\|_{\mathcal{N}(X,X^{**})} \leq \|U\|_{\mathcal{N}_p(Z,X^{**})}\|A\|_{\mathcal{P}_{p^*}} = \|U\|_{\mathcal{I}_p(Z,X^{**})}\|A\|_{\mathcal{P}_{p^*}}$$

for all $Z \in \mathcal{Z}$, $A \in \mathcal{P}_{p^*}(X, Z)$, and $U \in \mathcal{F}(Z, X)$. Therefore, by Theorem 2.7, for all Banach spaces Y and for all $T \in \mathcal{F}(Y, X)$,

$$\|T\|_{g_p} \leq \lambda\|T\|_{\mathcal{I}_p(Y,X^{**})} = \lambda\|T\|_{\mathcal{I}_p(Y,X)}.$$

This means that X has the λ-BAP_p. $\square$

If X^* has the AP, then it is well known that X^* has the AP_p, and therefore X^* has the MAP_p for $1 < p < \infty$. It is also well known that the latter fact does not imply the MAP_p for X, in general. It is interesting that, nevertheless, the MAP_p for X, $1 < p < \infty$, follows from the AP of X^*.

Corollary 4.5. *Let X be a Banach space. If X^* has the* AP, *then X has the* MAP_p, $1 < p < \infty$.

Proof. If X^* has the AP, then X has the weak MAP (see example (3) after Theorem 3.6), hence X has the MAP_p, $1 < p < \infty$, by Proposition 4.4. $\square$

We conclude with an application of Theorem 2.8.

Theorem 4.6. *Let X be a Banach space and let $1 < p < \infty$. Let $\mathcal{Z}$ be the set of reflexive Banach spaces having the* MAP *and let $\mathcal{C}$ be a $\mathcal{Z}$-factorable Banach operator ideal. If*

$$\mathcal{P}_{p^*}(X, Y) = \mathcal{C}(X, Y)$$

as sets for all reflexive Banach spaces Y, then X has the BAP_p.

Proof. Recall that $\mathcal{P}_{p^*}$ is the dual operator ideal of the tensor norm g_p (see Example 2.5). By Lemma 2.6, we have $\|S\|_{g_p} = \|S\|_{\mathcal{I}_p}$ for all $Z \in \mathcal{Z}$ and $S \in \mathcal{F}(Z, X)$. Since the set $\mathcal{Y}$ of all reflexive Banach spaces is clearly invariant under countable ℓ_2 direct sums, an immediate application of Theorem 2.8 and Remark 2.9 yields that $\|T\|_{g_p} \leq \mu \|T\|_{\mathcal{I}_p}$ for some $\mu \geq 1$ and for all $Y \in \mathcal{Y}$ and $T \in \mathcal{F}(Y, X)$. This means that X has the μ-BAP_p (see Theorem 4.3). □

In Theorem 4.6, one can take, for instance, $\mathcal{C} = \mathcal{I}_q, 1 < q < \infty$. In fact, $\mathcal{I}_q$ is clearly $L_q(\mu)$-spaces factorable (for probability measures μ) whenever $1 \leq q \leq \infty$ (see, e.g., [5, 5.6]). It is known that

$$\mathcal{P}_{p^*}(H_\infty, Y) = \mathcal{I}_{p^*}(H_\infty, Y)$$

for all Banach spaces Y whenever $1 < p < \infty$; this was independently proved by Kisliakov [15] (in 1981) and Gordon and Reisner [11] (in 1982). Therefore, by Theorem 4.6, H_∞ has the BAP_p. But H_∞ is well known to be a dual space. Hence H_∞ has the MAP_p, $1 < p < \infty$ (see the paragraph after Theorem 4.3). The latter result is due to Bourgain and Reinov [1]. More generally, they proved in [1] that all even duals $H_\infty^{(2n)}$ of H_∞ have the MAP_p, $1 < p < \infty$. Again, this (in a more general version [1, Theorem 2] which was discussed just before Theorem 2.8) follows from Theorem 4.6 since, by a result of Kisliakov [15, Corollary 3], if $1 < p < \infty$ and if X satisfies

$$\mathcal{P}_{p^*}(X, Y) = \mathcal{I}_{p^*}(X, Y)$$

for all (reflexive) Y, then the even duals $X^{(2n)}$ of X do the same (for a simple proof, relying on the fact that both $\mathcal{P}_{p^*}$ and $\mathcal{I}_{p^*}$ are dual operator ideals of some tensor norms, see [1, Theorem 5]; see also [4, 21.11] for an extended version). Recall that the related long-standing problem whether H_∞ has the AP (see, e.g., [25, Problem 1.e.10]) is still open.

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References

[1] J. Bourgain and O. Reinov, *On the approximation properties for the space H^∞*, Math. Nachr. **122** (1985) 19–27. MR871186 (88e:46022)

[2] P.G. Casazza, *Approximation properties*, in: W.B. Johnson and J. Lindenstrauss (eds.), Handbook of the Geometry of Banach Spaces, Vol. 1, Elsevier (2001) 271–316. MR1863695 (2003f:46012)

[3] W.J. Davis, T. Figiel, W.B. Johnson, and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974) 311–327. MR0355536 (50:8010)

[4] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies **176** (1993). MR1209438 (94e:46130)

[5] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge (1995). MR1342297 (96i:46001)

[6] J. Diestel and J.J. Uhl, Jr. *Vector Measures*, Mathematical Surveys **15**, Amer. Math. Soc., Providence, Rhode Island (1977). MR0453964 (56:12216)

[7] H. Fakhoury, *Sélections linéaires associées au théorème de Hahn-Banach,* J. Funct. Anal. **11** (1972) 436–452. MR0348457 (50:955)

[8] M. Feder and P.D. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. **21** (1975) 38–49. MR0377591 (51:13762)

[9] G. Godefroy, N.J. Kalton, and P.D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993) 13–59. MR1208038 (94k:46024)

[10] G. Godefroy and P.D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. **32** (1988) 672–695. MR955384 (89j:47026)

[11] Y. Gordon and S. Reisner, *Some geometrical properties of Banach spaces of polynomials*, Israel J. Math. **42** (1982) 99–116. MR687938 (84f:46035)

[12] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955). MR0075539 (17:763c)

[13] R. Haller, E. Oja, and E. Plewnia, *Quantitative versions of hereditary results on M-ideals of compact operators*, Math. Nachr. **246–247** (2002) 106–120. MR1944552 (2004a:46016)

[14] N.J. Kalton, *Locally complemented subspaces and $\mathcal{L}_p$-spaces for* $0 < p < 1$, Math. Nachr. **115** (1984) 71–97. MR755269 (86h:46006)

[15] S.V. Kisliakov, *Two remarks on the equality* $\Pi_p(X, \cdot) = I_p(X, \cdot)$, in: Investigations on Linear Operators and the Theory of Functions, XI, Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981) 135–148, 266. (In Russian. English summary.) MR629837 (83c:46009)

[16] Å. Lima, V. Lima, and E. Oja, *Bounded approximation properties via integral and nuclear operators*, Proc. Amer. Math. Soc. **138** (2010) 287–297. MR2550194 (2010k:46025)

[17] Å. Lima, O. Nygaard, and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. **119** (2000) 325–348. MR1802659 (2002b:46031)

[18] Å. Lima and E. Oja, *Hahn-Banach extension operators and spaces of operators*, Proc. Amer. Math. Soc. **130** (2002) 3631–3640. MR1920043 (2004f:46027)

[19] Å. Lima and E. Oja, *Ideals of compact operators*, J. Aust. Math. Soc. **77** (2004) 91–110. MR2069027 (2005f:46033)

[20] Å. Lima and E. Oja, *Ideals of operators, approximability in the strong operator topology, and the approximation property*, Michigan Math. J. **52** (2004) 253–265. MR2069799 (2005b:47162)

[21] Å. Lima and E. Oja, *The weak metric approximation property*, Math. Ann. **333** (2005) 471–484. MR2198796 (2006i:46025)

[22] Å. Lima and E. Oja, *Metric approximation properties and trace mappings*, Math. Nachr. **280** (2007) 571–580. MR2308485 (2008b:46031)

[23] V. Lima, *The weak metric approximation property and ideals of operators*, J. Math. Anal. Appl. **334** (2007) 593–603. MR2332578 (2008g:46022)

[24] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964). MR0179580 (31:3828)

[25] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Ergebnisse der Mathematik und ihrer Grenzgebiete **92**, Springer-Verlag, Berlin-Heidelberg-New York (1977). MR0500056 (58:17766)

[26] E. Oja, *Operators that are nuclear whenever they are nuclear for a larger range space*, Proc. Edinburgh Math. Soc. **47** (2004) 679–694. MR2097268 (2005g:46040)

[27] E. Oja, *The impact of the Radon-Nikodým property on the weak bounded approximation property*, Rev. R. Acad. Cien. Serie A. Mat. **100** (2006) 325–331. MR2267414 (2007h:46027)

[28] E. Oja, *Lifting of the approximation property from Banach spaces to their dual spaces*, Proc. Amer. Math. Soc. **135** (2007) 3581–3587. MR2336573 (2008f:46025)

[29] E. Oja *The strong approximation property*, J. Math. Anal. Appl. **338** (2008) 407–415. MR2386425 (2009i:46044)

[30] E. Oja and A. Pelander, *The approximation property in terms of the approximability of weak*-weak continuous operators*, J. Math. Anal. Appl. **286** (2003) 713–723. MR2008860 (2004h:46016)

[31] E. Oja and M. Põldvere, *Principle of local reflexivity revisited*, Proc. Amer. Math. Soc. **135** (2007) 1081–1088. MR2262909 (2007g:46017)

[32] A. Persson, *On some properties of p-nuclear and p-integral operators*, Studia Math. **33** (1969) 213–222. MR0247504 (40:769)

[33] A. Persson and A. Pietsch, *p-nucleare and p-integrale Abbildungen in Banachräumen*, Studia Math. **33** (1969) 19–62. MR0243323 (39:4645)

[34] A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, Amsterdam-New York-Oxford (1980). MR582655 (81j:47001)

[35] G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. **151** (1983) 181–208. MR723009 (85m:46017)

[36] O.I. Reinov, *Operators of RN type in Banach spaces*, Sibirsk. Mat. Zh. **19** (1978) 857–865. (In Russian.) MR503624 (80h:47028)

[37] O. Reinov, *Approximation properties of order p and the existence of non-p-nuclear operators with p-nuclear second adjoints*, Math. Nachr. **109** (1982) 125–134. MR705902 (84f:47025)

[38] R.A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer-Verlag, London (2002). MR1888309 (2003f:46030)

[39] P. Saphar, *Produits tensoriels d'espaces de Banach et classes d'applications linéaires*, Studia Math. **38** (1970) 71–100. MR0275121 (43:878)

[40] P. Saphar, *Hypothèse d'approximation à l'ordre p dans les espaces de Banach et approximation d'applications p absolument sommantes*, Israel J. Math. **13** (1972) 379–399. MR0333675 (48:12000)

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