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THE SPACE OF LORENTZIAN FLAT TORI IN ANTI-DE SITTER 3-SPACE

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Abstract. We describe the space of isometric immersions from the Lorentz plane $\mathbb{L}^2$ into the anti-de Sitter 3-space $\mathbb{H}_1^3$, and solve several open problems in this context raised by M. Dajczer and K. Nomizu in 1981. We also obtain from the above result a description of the space of Lorentzian flat tori isometrically immersed in $\mathbb{H}_1^3$ in terms of pairs of closed curves with wave-front singularities in the hyperbolic plane $\mathbb{H}^2$ satisfying some compatibility conditions.

1. Introduction

A classical problem in Lorentzian geometry is the description of the isometric immersions between Lorentzian spaces of constant curvature. In this paper we investigate the specific problem of classifying the isometric immersion from the Lorentz plane $\mathbb{L}^2$ into the 3-dimensional anti-de Sitter space $\mathbb{H}_1^3$.

The study of isometric immersions from $\mathbb{L}^2$ into $\mathbb{H}_1^3$ starts from a pioneering work by M. Dajczer and K. Nomizu [DaNo] in 1981. There, these authors gave a local description of such surfaces in terms of the Lie group structure of $\mathbb{H}_1^3$, using a classical idea by L. Bianchi [Bia] to describe the flat surfaces of the Riemannian unit sphere $\mathbb{S}^3$. Nevertheless, the global problem of finding all isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}_1^3$ turned out to be more subtle than its Euclidean counterpart and remained open in that paper. Moreover, Dajczer and Nomizu proposed in [DaNo] several specific open problems on the structure of the space of such isometric immersions from $\mathbb{L}^2$ into $\mathbb{H}_1^3$ that still remain unanswered.

In this paper we provide a general description of all isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}_1^3$ in terms of pairs of curves with singularities (*wave fronts*) in the hyperbolic plane $\mathbb{H}^2$. In particular, we give an answer to the open problems proposed in [DaNo]. In order to do so, we adapt to the Lorentzian setting an important idea by Y. Kitagawa [Kit1] used to describe complete flat surfaces in $\mathbb{S}^3$ via the Hopf fibration. The main difficulty in such an adaptation is that, in the Lorentzian case, the asymptotic curves of a timelike flat surface have varying causal character. This is a substantial complication in proving that an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}_1^3$

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can be globally parametrized by asymptotic curves, which is the key idea of the Riemannian case.

An important fact in the context we are working is that, among all isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, some of them are actually universal coverings of immersed (and sometimes embedded) Lorentzian tori in $\mathbb{H}^3_1$. The basic examples in this sense are the *Hopf tori* constructed in [BFLM, BFLM2] by means of the Hopf fibration of $\mathbb{H}^3_1$ over $\mathbb{H}^2$.

The existence of Lorentzian flat tori in $\mathbb{H}^3_1$ is a very remarkable fact since, in the Lorentzian context, there are very severe restrictions for the existence of compact immersed Lorentzian surfaces in an ambient Lorentzian 3-manifold. Indeed:

(1) Even intrinsically, any compact orientable surface that admits a Lorentzian metric must be homeomorphic to a torus (by the Poincaré-Hopf index theorem; see for instance [ONe]).
(2) If in a Lorentzian 3-manifold there exists an immersed compact Lorentzian surface, then such a 3-manifold cannot be *chronological* (i.e. it has to admit closed timelike curves). The reason is that any compact Lorentzian surface must have closed timelike curves; see [MiSa, Theorem 3.6]. In particular, there are no compact Lorentzian surfaces in the universal covering of $\mathbb{H}^3_1$ (which is the unique complete simply connected Lorentzian 3-manifold of constant curvature -1).

These results show that, in fact, the case of Lorentzian flat tori in $\mathbb{H}^3_1$ can be seen as one of the most geometrically simple situations in which compact Lorentzian surfaces exist inside a Lorentzian 3-manifold.

Our second main objective here is to describe the space of Lorentzian flat tori in $\mathbb{H}^3_1$, as an application of our previous description of all isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$. In particular, we prove that any such torus can be recovered in terms of two closed curves in $\mathbb{H}^2$, one of them regular and the other one possibly having wave-front singularities. This result can be seen as an extension to the Lorentzian setting of Kitagawa's classification of (Riemannian) flat tori of the unit sphere $\mathbb{S}^3$ [Kit1], although there are several technical differences in the proof and the final classification theorem. For results about complete flat surfaces in $\mathbb{S}^3$, we may refer the reader to [Kit1, Kit2, Wei, GaMi, AGM2, DaSh] and the references therein.

We have organized this paper as follows. In Section 2 we give some preliminaries on the geometry of $\mathbb{H}^3_1$ as a Lie group by means of a pseudo-quaternionic structure, and we introduce the different Hopf fibrations existing on $\mathbb{H}^3_1$. In Section 3 we prove that any isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ admits a global parametrization by asymptotic curves. The resulting coordinates are not *Tchebyshev coordinates* in the Euclidean sense, since the asymptotic curves in this Lorentzian context cannot be parametrized by arc-length (indeed, they have varying causal character). This detail is one of the main sources of complication of the paper.

In Section 4 we improve the classical Dajczer-Nomizu theorem in [DaNo] on the construction of timelike flat surfaces in $\mathbb{H}^3_1$ as a product of two curves. More specifically, we use the asymptotic coordinates constructed in Section 3 to prove that every isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ can be recovered as the pseudo-quaternionic product of two regular curves in $\mathbb{H}^3_1$, both in general with varying causal character, that satisfy some compatibility conditions.

In Section 5 we show a general method to construct regular curves in $\mathbb{H}^3_1$ that satisfy the hypotheses required by the classification theorem of Section 4. This method is an extension to the Lorentzian setting of Kitagawa's theory for studying complete flat surfaces in $\mathbb{S}^3$. Here, we use the Hopf fibration of $\mathbb{H}^3_1$ over $\mathbb{H}^2$ and we prove that such regular curves in $\mathbb{H}^3_1$ can be obtained as *asymptotic lifts* of curves with wave-front singularities in $\mathbb{H}^2$. With this, we obtain our main result (Theorem 22), which parametrizes the space of isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ in terms of the space of curves with wave-front singularities in $\mathbb{H}^2$.

Also in Section 5, we apply an idea by Kitagawa [Kit1] and Dadok-Sha [DaSh] to prove that all Lorentzian flat tori of $\mathbb{H}^3_1$ are exactly obtained when in Theorem 22 one starts with closed curves in $\mathbb{H}^2$, possibly with wave-front singularities, but with a well-defined unit normal at every point. Again, this provides a parametrization of the space of Lorentzian flat tori in $\mathbb{H}^3_1$. We conclude this section by analyzing in detail the examples of Lorentzian Hopf cylinders and Lorentzian Hopf tori.

Finally, in Section 6 we give an answer to the Dajczer-Nomizu open questions regarding the construction of isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$.

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2. The geometry of $\mathbb{H}^3_1$

Let $\mathbb{R}^4_2$ be the vector space $\mathbb{R}^4$ endowed with the semi-Riemannian metric

$$\langle\ ,\ \rangle = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2.$$

The hypersurface $\mathbb{H}^3_1 = \{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\}$ is then a model for the anti-de Sitter space of dimension 3. In this way, the induced metric on $\mathbb{H}^3_1$ is a Lorentzian metric of constant curvature -1. The space $\mathbb{H}^3_1$ is topologically a cylinder. Moreover, it is an $\mathbb{S}^1$-fibration over the hyperbolic plane $\mathbb{H}^2$ with timelike fibers, and its universal covering $\widetilde{\mathbb{H}^3_1}$ is the unique Lorentzian space-form of constant curvature -1.

Following the construction of [BFLM2], we will identify $\mathbb{R}^4_2$ with a certain set of maps $\mathbb{R}^4_2 \longrightarrow \mathbb{R}^4_2$, and $\mathbb{H}^3_1$ with a subset of it. The composition induces a natural product structure on $\mathbb{R}^4_2$ and $\mathbb{H}^3_1$, which will be seen then as Lie groups.

Let us consider $1 = \mathrm{Id}_{\mathbb{R}^4_2}$, and $i, j, k : \mathbb{R}^4_2 \longrightarrow \mathbb{R}^4_2$ given by:

$$\begin{array}{rcl} i(x_0, x_1, x_2, x_3) & = & (x_1, -x_0, x_3, -x_2), \\ j(x_0, x_1, x_2, x_3) & = & (x_2, -x_3, x_0, -x_1), \\ k(x_0, x_1, x_2, x_3) & = & (x_3, x_2, x_1, x_0). \end{array}$$

These maps satisfy:

$$\begin{array}{ccccccccccccccc} i^2 & = & i\circ i & = & -1, & ji & = & i\circ j & = & -k, & ki & = & i\circ k & = & j, \\ ij & = & j\circ i & = & k, & j^2 & = & j\circ j & = & 1, & kj & = & j\circ k & = & i, \\ ik & = & k\circ i & = & -j, & jk & = & k\circ j & = & -i, & k^2 & = & k\circ k & = & 1. \end{array}$$

Note that we are using the letters i, j, k, as is usual for quaternions, but here the product structure is a different one.

We now consider the vector space $\mathcal{F} = \mathrm{span}\{1, i, j, k\}$, and the isomorphism $\varphi : \mathcal{F} \longrightarrow \mathbb{R}^4_2$ defined by

$$\varphi(1) = \frac{\partial}{\partial x_0}, \quad \varphi(i) = \frac{\partial}{\partial x_1}, \quad \varphi(j) = \frac{\partial}{\partial x_2} \quad \text{and } \varphi(k) = \frac{\partial}{\partial x_3}.$$

In this way, $\mathbb{R}^4_2$ can be identified with the Lie group $\mathcal{F} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ endowed with the semi-Riemannian metric $\varphi^*(\langle\ ,\ \rangle)$ and we will denote its metric simply by $\langle\ ,\ \rangle$.

For $z = a + bi + cj + dk$ we use the notation $\mathrm{R}e(z) = a$. We say that z is *real* (resp. *pure imaginary*) if $b = c = d = 0$ (resp. $a = 0$). Finally, we define the *conjugate* of z as $\overline{z} = a - bi - cj - dk$. It is easy to check that, given $z_1, z_2 \in \mathbb{R}^4_2$, $\overline{z_1 z_2} = \overline{z}_2\,\overline{z}_1$. One can easily prove:

Proposition 1. *The following properties hold:*

i) *For* $z \in \mathbb{R}^4_2$, $\langle z, z\rangle = -z\overline{z} = -\overline{z}z = \langle \overline{z}, \overline{z}\rangle$.
ii) *In general, for* $z_1, z_2 \in \mathbb{R}^4_2$, $\langle z_1, z_2\rangle = -\mathrm{Re}(z_1\overline{z}_2)$.
iii) $z \in \mathbb{H}^3_1$ *if, and only if,* $z^{-1} = \overline{z}$.
iv) $\langle\ ,\ \rangle$ *is bi-invariant under multiplication by elements of* $\mathbb{H}^3_1$, *i.e., if* $z_1, z_2 \in \mathbb{H}^3_1$, *then* $\langle z_1\,\eta\,z_2, z_1\,\rho\,z_2\rangle = \langle \eta, \rho\rangle$.

Property iv) tells us that the Lie group structure induced on $\mathbb{H}^3_1$ by this quaternion-like product is its canonical Lie group structure, that is, the one for which its metric is bi-invariant. Besides, we have the identities:

$$\mathbb{H}^3_1 = \{z \in \mathbb{R}^4_2 : \langle z, z\rangle = -1\} = \{z \in \mathbb{R}^4_2 : z\overline{z} = 1\} = \{z \in \mathbb{R}^4_2 : \overline{z} = z^{-1}\}.$$

Observe also that $1 \in \mathbb{H}^3_1$ and that the vectors $\{i, j, k\}$ form an orthonormal basis of $T_1\mathbb{H}^3_1$, i.e. $\langle i, i\rangle = -1, \langle j, j\rangle = \langle k, k\rangle = 1$, and $\langle i, k\rangle = \langle j, k\rangle = \langle i, j\rangle = 0$. This basis can be extended to a global left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ on $\mathbb{H}^3_1$ as:

$$E_1(z) = zi, \qquad E_2(z) = zj, \qquad E_3(z) = zk \qquad \forall z \in \mathbb{H}^3_1.$$

Taking into account that we are thinking of $\mathbb{H}^3_1$ as a hypersurface of $\mathbb{R}^4_2$, there is a natural way to define a cross product on each tangent space $T_z\mathbb{H}^3_1$. For $u, v \in T_z\mathbb{H}^3_1$, $u \times v$ is the unique vector in $T_z\mathbb{H}^3_1$ such that $\langle u \times v, w\rangle = \det(z, u, v, w)$ for all $w \in T_z\mathbb{H}^3_1$. In particular, $i \times j = k$, $j \times k = i$ and $k \times i = j$.

For a curve $a : I \longrightarrow \mathbb{H}^3_1$ with $a(0) = 1$, and a vector field X along a, we say that X is *left (resp. right) invariant* along a if, for all $t \in I$, $X(t) = a(t)X(0)$ (resp. $X(t) = X(0)a(t)$). Let ∇ denote the Levi-Civita connection of $\mathbb{H}^3_1$. The next lemma is similar to the analogous result in the sphere $\mathbb{S}^3$ (see [Kit1], [Spi]). Hence, we will omit the proof.

Lemma 2. *Let* $a : I \longrightarrow \mathbb{H}^3_1$ *be a curve with* $a(0) = 1$ *and* X *a vector field along* a. *Then:*

i) X *is left invariant along* a *if, and only if,* $\nabla_{a'}X = a' \times X$.
ii) X *is right invariant along* a *if, and only if,* $\nabla_{a'}X = X \times a'$.

To close this section, let us now define the family of *Hopf fibrations* on $\mathbb{H}^3_1$. For each nonzero purely imaginary $\rho \in \mathbb{R}^4_2$, we define the map $h_\rho : \mathbb{H}^3_1 \to \mathbb{R}^4_2$ as

$$h_\rho(z) = z\,\rho\,\overline{z} \qquad \forall\, z \in \mathbb{H}^3_1. \tag{2.1}$$

Proposition 3. *For every nonzero purely imaginary* $\rho, \eta \in \mathbb{R}^4_2$ *and every* $z \in \mathbb{H}^3_1$ *we have:*

i) $\langle h_\rho(z), 1\rangle = 0$.
ii) $\langle h_\rho(z), h_\eta(z)\rangle = \langle \rho, \eta\rangle$. *In particular,* $\langle h_\rho(z), h_\rho(z)\rangle = \langle \rho, \rho\rangle$.
iii) *If* $\langle \rho, \rho\rangle \le 0$, *then* $\langle \rho, i\rangle$ *and* $\langle h_\rho(z), i\rangle$ *have the same sign.*

Proof: i) and ii) are consequences of the bi-invariance of the metric.
To prove iii) we set $\varphi(z) = \langle h_\rho(z), i\rangle$. Obviously, φ is a continuous function over $\mathbb{H}^3_1$ with $\varphi(1) = \langle \rho, i\rangle$. If φ changed sign, there would exist some $z_0 \in \mathbb{H}^3_1$ such

that $\varphi(z_0) = 0$. But this is impossible because $\varphi(z_0) = 0$ means that $h_\rho(z_0)$ has no part on i and, by i) and ii) we know that $h_\rho(z_0)$ is purely imaginary with $\langle h_\rho(z_0), h_\rho(z_0)\rangle = \langle \rho, \rho\rangle \leq 0$. □

After Proposition 3 we can distinguish three fundamental types of maps h_ρ by looking at their images:

$$\begin{aligned} &h_+ : \mathbb{H}^3_1 \longrightarrow \mathbb{S}^2_1(r) && \text{if } \langle \rho, \rho\rangle = r^2, \\ &h_- : \mathbb{H}^3_1 \longrightarrow \big(\mathbb{H}^2(r)\big)^{\pm} && \text{if } \langle \rho, \rho\rangle = -r^2 \text{ and } \langle \rho, i\rangle \lessgtr 0, \\ &h_0 : \mathbb{H}^3_1 \longrightarrow \big(\Lambda^2\big)^{\pm} && \text{if } \langle \rho, \rho\rangle = 0 \text{ and } \langle \rho, i\rangle \lessgtr 0. \end{aligned}$$

Here,

$$\begin{aligned} \mathbb{S}^2_1(r) &= \{z \in \mathbb{R}^4_2 : \langle z, 1\rangle = 0, \langle z, z\rangle = r^2\}, \\ \big(\mathbb{H}^2(r)\big)^{\pm} &= \{z \in \mathbb{R}^4_2 : \langle z, 1\rangle = 0, \langle z, z\rangle = -r^2, \langle z, i\rangle \lessgtr 0\}, \\ \big(\Lambda^2\big)^{\pm} &= \{z \in \mathbb{R}^4_2 : \langle z, 1\rangle = 0, \langle z, z\rangle = 0, \langle z, i\rangle \lessgtr 0\}; \end{aligned}$$

i.e. $\big(\mathbb{H}^2(r)\big)^+$, $\big(\mathbb{H}^2(r)\big)^-$, $\big(\Lambda^2\big)^+$ and $\big(\Lambda^2\big)^-$ denote each of the connected components of $\mathbb{H}^2(r)$ and $\Lambda^2\backslash\{0\}$, respectively.

All the maps h_ρ are fibrations over their corresponding base manifolds and, since their definition is similar to that of the classical Hopf fibration $h : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$, we will also call them *Hopf fibrations*.

Now, we are going to focus on the fibrations h_ρ with $\langle \rho, \rho\rangle = 1$, $\langle \rho, \rho\rangle = -1$ or $\langle \rho, \rho\rangle = 0$. In those cases we will denote simply by $\mathbb{S}^2_1$, $\big(\mathbb{H}^2\big)^{\pm}$ or $\big(\Lambda^2\big)^{\pm}$ their base manifold. Moreover, when no confusion can arise, we will also omit the reference to the connected component, using simply $\mathbb{H}^2$ or Λ^2. It is not difficult to show that $h_\rho(z_1) = h_\rho(z_2)$ if and only if $z_2 = \pm\, z_1 e^{t\rho}$, where

$$\begin{aligned} e^{t\rho} &:= \cosh(t)1 + \sinh(t)\rho && \text{if } \langle \rho, \rho\rangle = 1, \\ e^{t\rho} &:= \cos(t)1 + \sin(t)\rho && \text{if } \langle \rho, \rho\rangle = -1, \\ e^{t\rho} &:= 1 + t\rho && \text{if } \langle \rho, \rho\rangle = 0. \end{aligned} \tag{2.2}$$

3. Isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$

Consider an isometric immersion $f : \mathbb{L}^2 \to \mathbb{H}^3_1$ from the Lorentz plane $\mathbb{L}^2$ into the anti-de Sitter 3-space $\mathbb{H}^3_1$. Here, $\mathbb{L}^2$ will be viewed as the vector space $\mathbb{R}^2$ endowed with the Lorentzian metric $ds^2 = -dx^2 + dy^2$ in canonical coordinates (x, y). Before starting, let us remark that most of what follows can be adapted for (not necessarily complete) simply connected Lorentzian flat surfaces in $\mathbb{H}^3_1$; see the remark at the end of this section.

Let $N(x, y) : \mathbb{R}^2 \to \mathbb{S}^3_2 := \{p \in \mathbb{R}^4_2 : \langle p, p\rangle = 1\}$ denote the unit normal of the immersion f, chosen so that the frame $\{f, f_x, f_y, N\}$ is a positively oriented orthonormal frame in the manifold $\mathbb{R}^4_2$. Then, the first and second fundamental forms of the immersion are given, respectively, by

$$\left\{ \begin{array}{lcl} I &=& \langle df, df\rangle = -dx^2 + dy^2, \\ II &=& -\langle df, dN\rangle = a dx^2 + 2b dx dy + c dy^2, \end{array} \right. \tag{3.1}$$

where $a := -\langle f_x, N_x\rangle, b := -\langle f_x, N_y\rangle$ and $c := -\langle f_y, N_y\rangle$ satisfy the Gauss-Codazzi equations

$$a_y = b_x, \qquad c_x = b_y, \qquad ac - b^2 = -1.$$

Thus, there is some $\phi(x,y) \in C^\infty(\mathbb{R}^2)$ that is a solution to the hyperbolic Monge-Ampère equation $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1$ such that $a = \phi_{xx}$, $b = \phi_{xy}$ and $c = \phi_{yy}$. Hence,

$$II = \phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2, \qquad \phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1. \tag{3.2}$$

This implies that, associated to f, there exists a Euclidean isometric immersion $\widetilde{f}(x,y) : \mathbb{R}^2 \to \mathbb{S}^3$ of the Euclidean plane into the unit 3-sphere $\mathbb{S}^3$ with first and second fundamental forms given, respectively, by

$$\widetilde{I} = dx^2 + dy^2, \quad II = \phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2. \tag{3.3}$$

This is just a consequence of the classical fact that $(\widetilde{I}, II)$ as in (3.3) satisfy the Gauss-Codazzi equations for surfaces in $\mathbb{S}^3$. This correspondence was observed with a different formulation by Dajczer and Nomizu [DaNo]. It must be emphasized that this correspondence is not *geometric*, in the sense that it depends on the specific coordinates (x,y) in $\mathbb{L}^2$ that we choose. In other words, two different global Lorentzian coordinates (x,y) and (x',y') in $\mathbb{L}^2$ differing by an isometry generate, in general, two noncongruent flat surfaces in $\mathbb{S}^3$.

Now, since $\widetilde{f}$ is a complete flat surface in $\mathbb{S}^3$, it is classically known (see [Spi] for instance) that there exist globally defined *Tchebyshev coordinates* (u,v) on the surface. In other words, we may parametrize the surface as $\widetilde{f}(u,v) : \mathbb{R}^2 \to \mathbb{S}^3$ so that

$$\widetilde{I} = du^2 + 2\text{cos}\omega dudv + dv^2, \quad II = 2\text{sin}\omega dudv, \tag{3.4}$$

where $\omega(u,v) \in C^\infty(\mathbb{R}^2)$ satisfies $0 < \omega(u,v) < \pi$ and $\omega_{uv} = 0$. Note that from the expression of II in (3.4) it is clear that the u-curves and the v-curves are the asymptotic curves of the immersion $\widetilde{f}$.

Let us now find the explicit formula of the global diffeomorphism of $\mathbb{R}^2$ given by the change of coordinates $(u,v) \mapsto (x(u,v), y(u,v))$. By comparing $\widetilde{I}$ in (3.3) and (3.4) we get

$$\left\{ \begin{array}{ccc} x_u^2 + y_u^2 & = & 1, \\ x_u x_v + y_u y_v & = & \cos\omega(u,v), \\ x_v^2 + y_v^2 & = & 1. \end{array} \right. \tag{3.5}$$

Any solution to (3.5) must be of the form

$$\begin{aligned} x_u &= \text{cos}\omega_1, \quad y_u = \text{sin}\omega_1, \\ x_v &= \text{cos}\omega_2, \quad y_v = -\text{sin}\omega_2, \end{aligned} \tag{3.6}$$

where $\omega_i \in C^\infty(\mathbb{R}^2)$ satisfy $\omega_1 + \omega_2 = \omega$ (these functions are uniquely determined up to changes of the form $\omega_1 \mapsto \omega_1 + 2k\pi$, $\omega_2 \mapsto \omega_2 - 2k\pi$, with $k \in \mathbb{Z}$). Now using that $(x_u)_v = (x_v)_u$ and $(y_u)_v = (y_v)_u$ we get

$$-(\omega_1)_v \sin\omega_1 = -(\omega_2)_u \sin\omega_2, \qquad (\omega_1)_v \cos\omega_1 = -(\omega_2)_u \cos\omega_2;$$

i.e. either $(\omega_1)_v = (\omega_2)_u = 0$ or $\sin(\omega_1 + \omega_2) = 0$, the latter not being possible since $\sin\omega(u,v) \in (0,\pi)$. Thus, the function $\omega(u,v)$ appearing in (3.4) can be put in the form

$$\omega(u,v) = \omega_1(u) + \omega_2(v), \qquad \omega_i \in C^\infty(\mathbb{R}). \tag{3.7}$$

From here, the coordinates (x, y) are given in terms of (u, v) by

$$\begin{cases} x(u,v) = \int \cos\omega_1 du + \int \cos\omega_2 dv + c_1, \\ y(u,v) = \int \sin\omega_1 du - \int \sin\omega_2 dv + c_2, \end{cases} \tag{3.8}$$

where c_1 and c_2 are integration constants that can be chosen to be zero, up to a translation in the (x, y)-plane. In particular, the map given by (3.8) is a global diffeomorphism of $\mathbb{R}^2$, whenever we start with a complete flat surface in $\mathbb{S}^3$.

Remark 4. Let us point out that the map $(x(u, v), y(u, v)) : \mathbb{R}^2 \to \mathbb{R}^2$ given by (3.8) is a global diffeomorphism if and only if the Riemannian metric

$$\widetilde{I} = du^2 + 2\cos(\omega_1(u) + \omega_2(v))\, dudv + dv^2 \tag{3.9}$$

is complete.

Once here, we can use (3.8) to express the Lorentzian metric $I = -dx^2 + dy^2$ in terms of the global (u, v)-coordinates. First of all, let us observe that

$$\begin{array}{rcl} dx^2 & = & x_u^2 du^2 + 2x_u x_v dudv + x_v^2 dv^2 \\ & = & \cos^2\omega_1 du^2 + 2\cos\omega_1 \cos\omega_2 dudv + \cos^2\omega_2 dv^2. \end{array}$$

Then, the Lorentzian metric can be expressed as

$$\begin{aligned} -dx^2 + dy^2 &= -2dx^2 + dx^2 + dy^2 \\ &= -2dx^2 + du^2 + 2\cos\omega dudv + dv^2 \\ &= -\cos(2\omega_1)du^2 - 2\cos(\omega_1 - \omega_2)dudv - \cos(2\omega_2)dv^2. \end{aligned}$$

From the above discussion we have the following result.

Proposition 5. *Let $\omega_1(u), \omega_2(v) \in C^\infty(\mathbb{R})$ such that $\omega_1(u) + \omega_2(v) \in (0, \pi)$ for all $(u, v) \in \mathbb{R}^2$. Then, there exists an immersion $f(u, v) : \mathbb{R}^2 \to \mathbb{H}_1^3$ whose first, second and third fundamental forms are given by*

$$\begin{cases} I & = & -\cos(2\omega_1)du^2 & -2\cos(\omega_1 - \omega_2)dudv & -\cos(2\omega_2)dv^2, \\ II & = & & 2\sin(\omega_1 + \omega_2)dudv, & \\ III & = & \cos(2\omega_1)du^2 & -2\cos(\omega_1 - \omega_2)dudv & +\cos(2\omega_2)dv^2. \end{cases} \tag{3.10}$$

In this way, f describes a flat timelike surface in $\mathbb{H}_1^3$ whose asymptotic curves are the images of the coordinate curves in the (u, v)-plane. Moreover, f represents an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}_1^3$ exactly when the local diffeomorphism $(x(u, v), y(u, v))$ of $\mathbb{R}^2$ given by (3.8) *is actually a global diffeomorphism. A sufficient condition for* (3.8) *to be a global diffeomorphism is that*

$$0 < c_1 \leq \omega_1(u) + \omega_2(v) \leq c_2 < \pi \qquad \forall (u, v) \in \mathbb{R}^2. \tag{3.11}$$

Conversely, any isometric immersion $f(x, y) : \mathbb{L}^2 \to \mathbb{H}_1^3$ admits a parametrization $f(u, v) : \mathbb{R}^2 \to \mathbb{H}_1^3$ such that (3.10) *holds for some $\omega_1(u), \omega_2(v) \in C^\infty(\mathbb{R})$ satisfying $\omega_1(u) + \omega_2(v) \in (0, \pi)$ for all $(u, v) \in \mathbb{R}^2$. In that situation, the change of coordinates $(u, v) \mapsto (x(u, v), y(u, v))$ is given by* (3.8).

Proof. All the statements of the converse part follow from the previous discussion, except for the expression of the third fundamental form $III = \langle dN, dN\rangle$. In this sense, a standard derivation of the Gauss-Weingarten formulas of the immersion $f(u,v):\mathbb{R}^2\to\mathbb{H}^3_1$ yields

$$\left\{\begin{array}{lcl} f_{uu} & = & \dfrac{\omega_1'\cos(\omega_1+\omega_2)}{\sin(\omega_1+\omega_2)}f_u-\dfrac{\omega_1'}{\sin(\omega_1+\omega_2)}f_v-\cos(2\omega_1)f,\\ f_{uv} & = & \sin(\omega_1+\omega_2)N-\cos(\omega_1-\omega_2)f,\\ f_{vv} & = & -\dfrac{\omega_2'}{\sin(\omega_1+\omega_2)}f_u+\dfrac{\omega_2'\cos(\omega_1+\omega_2)}{\sin(\omega_1+\omega_2)}f_v-\cos(2\omega_2)f \end{array}\right. \tag{3.12}$$

and

$$\left\{\begin{array}{lcl} N_u & = & \dfrac{\cos(\omega_1-\omega_2)}{\sin(\omega_1+\omega_2)}f_u-\dfrac{\cos(2\omega_1)}{\sin(\omega_1+\omega_2)}f_v,\\ N_v & = & -\dfrac{\cos(2\omega_2)}{\sin(\omega_1+\omega_2)}f_u+\dfrac{\cos(\omega_1-\omega_2)}{\sin(\omega_1+\omega_2)}f_v. \end{array}\right. \tag{3.13}$$

From (3.13) and the expression of I in (3.10) we get that

$$III=\langle dN,dN\rangle=\cos(2\omega_1)du^2-2\cos(\omega_1-\omega_2)dudv+\cos(2\omega_2)dv^2,$$

as wished.

Now, assume that we are given $\omega_1(u),\omega_2(v)\in C^\infty(\mathbb{R})$ with $\omega_1+\omega_2\in(0,\pi)$. Then, the existence of the immersion $f(u,v):\mathbb{R}^2\to\mathbb{H}^3_1$ such that (3.10) holds follows from the Gauss-Codazzi equations and (3.12), (3.13). The metric I is flat and timelike, and if we use the local coordinates (x,y) given by (3.8), we have

$$I=-dx^2+dy^2. \tag{3.14}$$

So, clearly, f will describe an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ with canonical coordinates (x,y) if (3.8) is a global diffeomorphism. Conversely, assume that f describes an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ with canonical coordinates (x',y'). Then $I=-dx^2+dy^2=-dx'^2+dy'^2$ by (3.14). But this implies that (x,y) and (x',y') differ by an isometry of $\mathbb{L}^2$, and hence (3.8) is a global diffeomorphism.

Finally, if (3.11) holds and we denote $\Phi(u,v)=(x(u,v),y(u,v)):\mathbb{R}^2\to\mathbb{R}^2$, then we immediately get from (3.8) that the gradient of Φ^{-1} has bounded norm around any point; i.e. $||D(\Phi^{-1})||\leq M<\infty$ for some $M>0$. So, Φ is a global diffeomorphism by the Hadamard-Plastock inversion theorem. (Alternatively, if (3.11) holds, then $|\cos\omega|\leq c_0<1$ for some c_0, and so $\widetilde{I}\geq(1-c_0^2)(du^2+dv^2)$ for the Riemannian metric in (3.9); thus $\widetilde{I}$ is complete, and by Remark 4, Φ is a global diffeomorphism.) $\square$

Remark 6. In the previous arguments, the only place where completeness plays a role is in the existence of the global parameters (u,v). Nonetheless, these parameters always exist locally, as can be deduced from (3.8) and the fact that the *flat* coordinates (x,y) always exist locally for any (abstract) Lorentzian flat surface. Moreover, if we start with a simply connected Lorentzian flat surface Σ, then one can still choose a *coordinate immersion* $(x,y):\Sigma\to\mathbb{L}^2$ into the Lorentz plane that serves as a substitute to the one-to-one coordinates (x,y) that exist locally or for complete Lorentzian flat surfaces (see [AGM1]).

It then becomes clear from these comments that the whole previous process can be readily formulated for arbitrary simply connected Lorentzian flat surfaces isometrically immersed in $\mathbb{H}^3_1$. Obviously, in that case, we should not impose that (3.8) is a global diffeomorphism.

Remark 7. In the Euclidean case, the functions ω_i in (3.7) are uniquely determined up to the change

$$\omega_1(u) \mapsto \omega_1(u) + c, \qquad \omega_2(v) \mapsto \omega_2(v) - c, \qquad c \in \mathbb{R}; \tag{3.15}$$

In the present Lorentzian case, this ambiguity does not hold anymore; i.e. a change such as (3.15) also changes the resulting Lorentzian flat surface in $\mathbb{H}^3_1$.

4. A representation formula

Our aim in this section is to prove that any isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ can be represented, with respect to the characteristic parameters (u, v) provided by Proposition 5, as the product of two adequate curves in $\mathbb{H}^3_1$. We split this result into two separate theorems.

The first one is:

Theorem 8. *Let $f(u, v) : \mathbb{R}^2 \longrightarrow \mathbb{H}^3_1$ be an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ where (u, v) are the global characteristic parameters given in Proposition* 5*. Let $N(u, v) : \mathbb{R}^2 \longrightarrow \mathbb{S}^3_2$ denote its unit normal, and assume without loss of generality that $f(0, 0) = 1$ and $N(0, 0) = j$. Then, we have*

$$f(u, v) = a_1(u)a_2(v), \qquad N(u, v) = a_1(u)ja_2(v), \tag{4.1}$$

for $a_1(u) := f(u, 0)$ and $a_2(v) := f(0, v)$. Moreover, these two asymptotic curves satisfy

$$\langle a_1'(u), a_1(u)j\rangle = 0 = \langle a_2'(v), ja_2(v)\rangle. \tag{4.2}$$

To prove Theorem 8, we will use the following result.

Lemma 9. *Under the hypotheses of Theorem* 8*, we have:*

i) *N, f_v and N_v are left-invariant along $a_1(u)$.*
ii) *N, f_u and N_u are right-invariant along $a_2(v)$.*

Proof. By (3.10) we see that N_u is orthogonal to N, f and f_u and that $\langle N_u, N_u\rangle = \cos(2\omega_1) = -\langle f_u, f_u\rangle$. Therefore, we have $N_u = \pm f_u \times N$.

To determine this sign we take into account that, since $\omega_1 + \omega_2 \in (0, \pi)$,

$$\begin{aligned} 0 > -\sin(\omega_1 + \omega_2) &= \langle N_u, f_v\rangle = \langle \pm f_u \times N, f_v\rangle \\ &= \mp\langle f_u \times f_v, N\rangle = \mp\left\langle f_u \times f_v, \frac{f_u \times f_v}{||f_u \times f_v||}\right\rangle = \mp||f_u \times f_v||. \end{aligned}$$

Then, we deduce that

$$||f_u \times f_v|| = \sin(\omega_1 + \omega_2) \tag{4.3}$$

and that $N_u = f_u \times N$. In particular,

$$\nabla_{a_1'} N = N_u(u, 0) = f_u(u, 0) \times N(u, 0) = a_1' \times N.$$

So, by Lemma 2 we conclude that N is left-invariant along a_1. A similar argument yields

$$N_v(u, v) = N(u, v) \times f_v(u, v). \tag{4.4}$$

Thus, particularizing at points of the form $(0, v)$ we can apply Lemma 2 to deduce that N is right-invariant along a_2.

Now, we consider the vector field $f_{uv}(u, v)$. Using (3.12) and (4.3) we get

$$f_{uv}(u, v) = f_u(u, v) \times f_v(u, v) - \cos(\omega_1 - \omega_2)f(u, v).$$

At points of the form $(u,0)$, this equality provides

$$\nabla_{a_1'} f_v = \big(f_{uv}(u,0)\big)^{\top} = f_u(u,0)\times f_v(u,0) = a_1'\times f_v.$$

Again, Lemma 2 gives the desired conclusion. The fact that f_u is right-invariant along a_2 is obtained in the same way.

Finally, if we use left-invariancy along a_1 of N and f_v in (4.4), we obtain

$$\begin{aligned} N_v(u,0) &= N(u,0)\times f_v(u,0) = \big(a_1(u)N(0,0)\big)\times\big(a_1(u)f_v(0,0)\big)\\ &= a_1(u)\big(N(0,0)\times f_v(0,0)\big) = a_1(u)N_v(0,0); \end{aligned}$$

that is, N_v is left-invariant along a_1. Also in this case, we can use similar arguments to prove that N_u is right-invariant along a_2. This finishes the proof of Lemma 9. □

Proof of Theorem 8*:* First of all, let us observe that (4.2) follows from $\langle f_u, N\rangle = \langle f_v, N\rangle = 0$ at points of the form $(u,0)$ or $(0,v)$, and from the left-right-invariance of N given by Lemma 9.

In order to prove (4.1), we start by combining the structure equations (3.12) and (3.13) with basic trigonometric laws to obtain

$$\omega_1'\big(N_u - \sin(2\omega_1)f_u\big) = \cos(2\omega_1)\big(f_{uu}+\cos(2\omega_1)f\big) \tag{4.5}$$

and

$$N_u - \sin(2\omega_1)f_u = \frac{\cos(2\omega_1)}{\sin(\omega_1+\omega_2)}\left(\cos(\omega_1+\omega_2)f_u - f_v\right). \tag{4.6}$$

Now, for a fixed v_0 we define the curves $\Gamma_1,\Gamma_2:\mathbb{R}\longrightarrow\mathbb{H}^3_1$ as

$$\Gamma_1(u) = f(u,v_0), \qquad \Gamma_2(u) = a_1(u)a_2(v_0).$$

Next, let us construct frames along each of these two curves. It is important to observe that Γ_1 and Γ_2 do not have constant causal character. Thus, the frame that we introduce here is not the Frenet frame of the curve, and has to be constructed *ad hoc*. So, consider:

$$\left\{\begin{array}{rcl} \vec{t}_1(u) &=& \Gamma_1'(u) = f_u(u,v_0),\\ \vec{n}_1(u) &=& \dfrac{\cos(\omega_1(u)+\omega_2(v_0))}{\sin(\omega_1(u)+\omega_2(v_0))}f_u(u,v_0) - \dfrac{1}{\sin(\omega_1(u)+\omega_2(v_0))}f_v(u,v_0),\\ \vec{b}_1(u) &=& N(u,v_0), \end{array}\right.$$

$$\left\{\begin{array}{rcl} \vec{t}_2(u) &=& \Gamma_2'(u) = a_1'(u)a_2(v_0),\\ \vec{n}_2(u) &=& \dfrac{1}{\cos(2\omega_1(u))}\Big(a_1'(u)ja_2(v_0) - \sin(2\omega_1(u))a_1'(u)a_2(v_0)\Big),\\ \vec{b}_2(u) &=& a_1(u)ja_2(v_0). \end{array}\right.$$

Note that the definition of $\vec{n}_2$ is valid, at first, only at points with $\cos(2\omega_1(u)) \neq 0$. However, using the left-invariance of N along a_1 and (4.6), we get

$$\begin{aligned}\vec{n}_2(u) &= \frac{1}{\cos(2\omega_1(u))}\Big(a_1'(u)j - \sin(2\omega_1(u))a_1'(u)\Big)a_2(v_0)\\ &= \frac{1}{\cos(2\omega_1(u))}\Big(N_u(u,0) - \sin(2\omega_1(u))f_u(u,0)\Big)a_2(v_0)\\ &= \left(\frac{\cos(\omega_1(u)+\omega_2(0))}{\sin(\omega_1(u)+\omega_2(0))}f_u(u,0) - \frac{1}{\sin(\omega_1(u)+\omega_2(0))}f_v(u,0)\right)a_2(v_0).\end{aligned} \tag{4.7}$$

Thus, $\vec{n}_2(u)$ is actually well defined for all u.

We now claim that the references $\{\vec{t}_1, \vec{n}_1, \vec{b}_1\}$ and $\{\vec{t}_2, \vec{n}_2, \vec{b}_2\}$ coincide at $u = 0$. Indeed, for $\vec{t}_i$ and $\vec{b}_i$, this is a direct consequence of the fact that f_u and N are right-invariant along a_2. For $\vec{n}_i$, the result follows from the second identity in (4.7) evaluated at $u = 0$, and the right-invariance of N_u along a_2:

$$\begin{aligned}\vec{n}_2(0) &= \frac{1}{\cos(2\omega_1(0))}\left(N_u(0,0)a_2(v_0) - \sin(2\omega_1(0))f_u(0,0)a_2(v_0)\right)\\ &= \frac{1}{\cos(2\omega_1(0))}\left(N_u(0,v_0) - \sin(2\omega_1(0))f_u(0,v_0)\right)\\ &= \left(\frac{\cos(\omega_1(0)+\omega_2(v_0))}{\sin(\omega_1(0)+\omega_2(v_0))}f_u(0,v_0) - \frac{1}{\sin(\omega_1(0)+\omega_2(v_0))}f_v(0,v_0)\right)\\ &= \vec{n}_1(0).\end{aligned}$$

Finally, we are going to show that both references satisfy the same system of differential equations. In the case of $\{\vec{t}_1, \vec{n}_1, \vec{b}_1\}$ we just have to apply (3.12) and (3.13) to deduce that

(4.8)

$$\begin{aligned}\nabla_{\Gamma_1'}\vec{t}_1 &= \big(f_{uu}(u,v_0)\big)^\top = \omega_1'(u)\vec{n}_1(u),\\ \nabla_{\Gamma_1'}\vec{n}_1 &= \frac{\partial}{\partial u}\left(\frac{\cos(\omega_1(u)+\omega_2(v_0))}{\sin(\omega_1(u)+\omega_2(v_0))}\right)f_u(u,v_0) + \frac{\cos(\omega_1(u)+\omega_2(v_0))}{\sin(\omega_1(u)+\omega_2(v_0))}\big(f_{uu}(u,v_0)\big)^\top\\ &\quad - \frac{\partial}{\partial u}\left(\frac{1}{\sin(\omega_1(u)+\omega_2(v_0))}\right)f_v(u,v_0) - \frac{1}{\sin(\omega_1(u)+\omega_2(v_0))}\big(f_{vu}(u,v_0)\big)^\top\\ &= -\omega_1'(u)f_u(u,v_0) - N(u,v_0)\\ &= -\omega_1'(u)\vec{t}_1(u) - \vec{b}_1(u),\\ \nabla_{\Gamma_1'}\vec{b}_1 &= \big(N_u(u,v_0)\big)^\top\\ &= \frac{\cos(\omega_1(u)-\omega_2(v_0))}{\sin(\omega_1(u)+\omega_2(v_0))}f_u(u,v_0) - \frac{\cos(2\omega_1(u))}{\sin(\omega_1(u)+\omega_2(v_0))}f_v(u,v_0)\\ &= \sin(2\omega_1(u))f_u(u,v_0)\\ &\quad + \cos(2\omega_1(u))\left(\frac{\cos(\omega_1(u)+\omega_2(v_0))}{\sin(\omega_1(u)+\omega_2(v_0))}f_u(u,v_0) - \frac{1}{\sin(\omega_1(u)+\omega_2(v_0))}f_v(u,v_0)\right)\\ &= \sin(2\omega_1(u))\vec{t}_1(u) + \cos(2\omega_1(u))\vec{n}_1(u).\end{aligned}$$

To obtain the differential equations of the reference $\{\vec{t}_2, \vec{n}_2, \vec{b}_2\}$, the main idea is to use that $a_1''(u) = f_{uu}(u,0)$ whenever we find the terms $\big(a_1''(u)\big)^\top a_2(v_0)$ or

$\big(a_1''(u)\big)^\top ja_2(v_0)$. Then, we can apply (4.5) and express $\big(f_{uu}(u,0)\big)^\top$ in terms of $N_u(u,0)$ and $f_u(u,0)$. The last step is to use that $f_u(u,0)=a_1'(u)$ and $N_u(u,0)=a_1'(u)j$ (which follows from the left-invariance of N along a_1) and rewrite all terms as products of the curves a_1 and a_2 and their derivatives.

Using the above scheme, we have

$$\begin{aligned}\nabla_{\Gamma_2'}\vec{t}_2 &= \big(a_1''(u)\big)^\top a_2(v_0)=\big(f_{uu}(u,0)\big)^\top a_2(v_0)\\ &= \left(\frac{\omega_1'(u)}{\cos(2\omega_1(u))}N_u(u,0)-\frac{\omega_1'(u)\sin(2\omega_1(u))}{\cos(2\omega_1(u))}f_u(u,0)\right)a_2(v_0)\\ &= \frac{\omega_1'(u)}{\cos(2\omega_1(u))}\big(a_1'(u)ja_2(v_0)-\sin(2\omega_1(u))a_1'(u)a_2(v_0)\big)\\ &= \omega_1'(u)\vec{n}_2(u).\end{aligned}$$

Next, using (4.7) and performing basically the same computation as in (4.8), we get

$$\nabla_{\Gamma_2'}\vec{n}_2=-\omega_1'(u)a_1'(u)a_2(v_0)-a_1(u)ja_2(v_0)=-\omega_1'(u)\vec{t}_2(u)-\vec{b}_2(u).$$

At last,

$$\begin{aligned}\nabla_{\Gamma_2'}\vec{b}_2 &= a_1'(u)ja_2(v_0)\\ &= \cos(2\omega_1(u))\left(\frac{1}{\cos(2\omega_1(u))}\big(a_1'(u)ja_2(v_0)-\sin(2\omega_1(u))a_1'(u)a_2(v_0)\big)\right)\\ &\quad+\sin(2\omega_1(u))a_1'(u)a_2(v_0)\\ &= \sin(2\omega_1(u))\vec{t}_2(u)+\cos(2\omega_1(u))\vec{n}_2(u).\end{aligned}$$

Therefore, we have proved that $\{\vec{t}_1,\vec{n}_1,\vec{b}_1\}$ and $\{\vec{t}_2,\vec{n}_2,\vec{b}_2\}$ agree at $u=0$ and satisfy the same system of differential equations. Hence, we can conclude that these two references coincide along $\mathbb{R}$. In particular, we deduce that $\Gamma_1\equiv\Gamma_2$. Since this can be done for any v_0, we obtain that $f(u,v)=a_1(u)a_2(v)$ and also that $N(u,v)=a_1(u)ja_2(v)$. This concludes the proof of Theorem 8. □

Our objective now is to study the converse of Theorem 8. In other words, we wish to obtain a method to construct flat surfaces from two given curves in $\mathbb{H}_1^3$, satisfying some conditions.

Let us first note that if a curve $a:\mathbb{R}\to\mathbb{H}_1^3$ satisfies $\langle a',aj\rangle=0$, then, for the curve $\bar{a}$ we have $\langle\bar{a}',j\bar{a}\rangle=0$. This provides a simplification of condition (4.2). Namely, after this observation, we are left with the problem of finding out if two given curves $a_1(u),a_2(v):\mathbb{R}\to\mathbb{H}_1^3$, both satisfying $\langle a_i',a_ij\rangle=0$, always describe an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}_1^3$ such that $f(u,0)=a_1(u)$ and $f(0,v)=\overline{a_2(v)}$ for the characteristic parameters given in Proposition 5. In order to do this, we introduce the following terminology.

Let $a:\mathbb{R}\longrightarrow\mathbb{H}_1^3$ be a regular curve such that $\langle a'(s),a(s)j\rangle=0$ for all $s\in\mathbb{R}$. Then, we can write $\overline{a(s)}a'(s)=\lambda(s)i+\mu(s)k$ for $\lambda,\mu\in C^\infty(\mathbb{R})$.

Definition 10. In the above situation, we say that s is the *asymptotic parameter* of the curve a if $\lambda(s)^2+\mu(s)^2=1$. In that case, we can write

$$\overline{a(s)}a'(s)=\cos(\omega^a(s))i+\sin(\omega^a(s))k \tag{4.9}$$

for some $\omega^a \in \mathcal{C}^\infty(\mathbb{R})$, which is uniquely determined up to translations of the form $\omega^a \mapsto \omega^a + 2k\pi$, with $k \in \mathbb{Z}$.

Obviously, any curve in $\mathbb{H}^3_1$ with $\langle a', aj\rangle = 0$ can be reparametrized by its asymptotic parameter. With this, the following result is a converse to Theorem 8 and completes the desired representation theorem.

Theorem 11. *Let $a_1(u), a_2(v) : \mathbb{R} \longrightarrow \mathbb{H}^3_1$ be two regular curves, with $a_1(0) = 1 = a_2(0)$, satisfying:*

i) *$\langle a_i', a_i j\rangle = 0$, for $i = 1, 2$.*
ii) *u and v are the asymptotic parameters of a_1 and a_2, respectively.*
iii) *The functions $\omega_1 = \omega^{a_1}$ and $\omega_2 = \pi - \omega^{a_2}$ (ω^{a_i} as in Definition 10) satisfy*

$$\sin(\omega_1(u) + \omega_2(v)) > 0 \qquad \forall (u, v) \in \mathbb{R}^2. \tag{4.10}$$

iv) *The map $(x(u, v), y(u, v))$ given by (3.8) is a global diffeomorphism.*

Then, $f : \mathbb{R}^2 \longrightarrow \mathbb{H}^3_1$ defined by

$$f(u, v) = a_1(u)\overline{a_2(v)} \tag{4.11}$$

describes an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, and (u, v) are the global characteristic parameters given in Proposition 5.

Proof. By definition of $\omega_1(u), \omega_2(v)$, the condition (4.10) and the ambiguity in Definition 10, it is clear that we can suppose that $\omega_1(u)+\omega_2(v) \in (0, \pi)$. Moreover, we have

$$\overline{a_1(u)}a_1'(u) \;=\; \cos(\omega_1(u))i + \sin(\omega_1(u))k, \tag{4.12}$$

$$\overline{a_2(v)}a_2'(v) \;=\; -\cos(\omega_2(v))i + \sin(\omega_2(v))k, \tag{4.13}$$

and, conjugating the last expression,

$$\overline{a_2'(v)}a_2(v) = \cos(\omega_2(v))i - \sin(\omega_2(v))k. \tag{4.14}$$

Hence, from (4.11), (4.12) and (4.14) we obtain

$$\langle f_u(u, v), f_u(u, v)\rangle = \langle a_1'(u), a_1'(u)\rangle = -\cos(2\omega_1(u)),$$
$$\langle f_v(u, v), f_v(u, v)\rangle = \langle \overline{a_2'(v)}, \overline{a_2'(v)}\rangle = -\cos(2\omega_2(v))$$

and

$$\begin{aligned}\big\langle f_u(u, v), f_v(u, v)\big\rangle &= \big\langle \cos(\omega_1(u))i + \sin(\omega_1(u))k\,,\ \cos(\omega_2(v))i - \sin(\omega_2(v))k\big\rangle\\ &= -\cos(\omega_1(u))\cos(\omega_2(v)) - \sin(\omega_1(u))\sin(\omega_2(v))\\ &= -\cos(\omega_1(u) - \omega_2(v)).\end{aligned} \tag{4.15}$$

Now, to find the expression of the second fundamental form in coordinates (u, v), we take into account that, by (4.12) and (4.14),

$$\begin{aligned}&f_u(u, v) \times f_v(u, v)\\ &= \Big(a_1(u)\big(\cos(\omega_1(u))i+\sin(\omega_1(u))k\big)\overline{a_2(v)}\Big)\\ &\quad\times \Big(a_1(u)\big(\cos(\omega_2(v))i - \sin(\omega_2(v))k\big)\,\overline{a_2(v)}\Big)\\ &= \sin(\omega_1(u) + \omega_2(v))\ a_1(u)\,j\,\overline{a_2(v)}\end{aligned}$$

and so,

$$N(u,v) = \frac{f_u(u,v) \times f_v(u,v)}{\|f_u(u,v) \times f_v(u,v)\|} = a_1(u)\, j\, \overline{a_2(v)}.$$

After that, we obviously get

$$\langle f_u(u,v), N_u(u,v)\rangle = 0 = \langle f_v(u,v), N_v(u,v)\rangle, \tag{4.16}$$

and we also deduce

$$\langle N_u(u,v), N_u(u,v)\rangle = \cos(2\,\omega_1(u)), \tag{4.17}$$

$$\langle N_v(u,v), N_v(u,v)\rangle = \cos(2\,\omega_2(v)). \tag{4.18}$$

Besides, it follows immediately from (4.12), (4.14) that $\langle f_v, N_u\rangle = -\sin(\omega_1+\omega_2)$. This completes the proof, using Proposition 5. □

5. The classification results

In this section we will improve the representation formula for flat surfaces in $\mathbb{H}^3_1$ in Theorem 11, by presenting a geometric method to describe the curves in $\mathbb{H}^3_1$ satisfying the condition $\langle a', a\, j\rangle = 0$. As a consequence, we will obtain the main classification results of this paper.

Let us start by considering the unit tangent bundle to $\mathbb{H}^2$,

$$TU(\mathbb{H}^2) = \{(x,y) : x \in \mathbb{H}^2, y \in \mathbb{S}^2_1, \langle x,y\rangle = 0\},$$

where we are viewing here $\mathbb{H}^2 = \mathbb{H}^3_1 \cap \{x_0 = 0\}$ and $\mathbb{S}^2_1 = \mathbb{S}^3_2 \cap \{x_0 = 0\}$. With this, we can consider the map $\pi : \mathbb{H}^3_1 \to TU(\mathbb{H}^2)$ given by

$$\pi(x) = (x\, i\, \bar{x}, x\, k\, \bar{x}) = (h_i(x), h_k(x)). \tag{5.1}$$

This map is a double covering map with $\pi(-x) = \pi(x)$ for every $x \in \mathbb{H}^3_1$. From now on, let us use the notation

$$h(x) := h_i(x) = x\, i\, \bar{x} : \mathbb{H}^3_1 \to \mathbb{H}^2.$$

Definition 12. A *Legendrian curve* in $TU(\mathbb{H}^2)$ is an immersion $\alpha = (\gamma, \nu) : I \subset \mathbb{R} \to TU(\mathbb{H}^2)$ such that $\langle \gamma', \nu\rangle = 0$.

Associated to such a Legendrian curve we may define the metric

$$\langle d\alpha, d\alpha\rangle_{\mathcal{S}} := \langle d\gamma, d\gamma\rangle + \langle d\nu, d\nu\rangle.$$

As $\langle \gamma', \gamma'\rangle \geq 0$ and $\langle \nu', \nu'\rangle \geq 0$, and α is an immersion, we have that $\langle \alpha', \alpha'\rangle_{\mathcal{S}} > 0$ everywhere. In particular, we may parametrize α by its arc-length parameter with respect to $\langle , \rangle_{\mathcal{S}}$.

In what follows, let $p_{\mathbb{H}^2} : TU(\mathbb{H}^2) \to \mathbb{H}^2$ denote the canonical projection of $TU(\mathbb{H}^2)$ onto $\mathbb{H}^2$.

Definition 13. A *wave front* (or simply a *front*) in $\mathbb{H}^2$ is a smooth map $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2$ that lifts to a Legendrian curve; i.e. there exists a Legendrian curve $\alpha : I \subset \mathbb{R} \to TU(\mathbb{H}^2)$ such that $p_{\mathbb{H}^2}(\alpha) = \gamma$. Under these conditions, we call the map $\nu : I \subset \mathbb{R} \to \mathbb{S}^2_1$ such that $\alpha = (\gamma, \nu)$ the *unit normal* of the front.

A *closed front* in $\mathbb{H}^2$ is defined similarly as the projection of a closed Legendrian curve $\alpha : \mathbb{S}^1 \to TU(\mathbb{H}^2)$.

It is clear that any regular curve in $\mathbb{H}^2$ is a front, but the converse is not true in general. For instance, the parallel curves of a regular curve in $\mathbb{H}^2$ are fronts which have singularities, in general. Besides, there are periodic curves in $\mathbb{H}^2$ with singularities that are not closed fronts with the above definition, since they do not have a globally well-defined unit normal (e.g., a closed curve with exactly one cusp). For more details about fronts, see [SUY, MuUm, KUY, KRSUY].

The next lemma provides an important simplification to the equation $\langle a', a\, j\rangle = 0$.

Lemma 14. *Let $a(u) : \mathbb{R} \to \mathbb{H}^3_1$ be a regular curve. The following statements are equivalent:*

(1) $\langle a'(u), a(u)\, j\rangle = 0$.
(2) $\pi(a(u)) : \mathbb{R} \to TU(\mathbb{H}^2)$ *is a Legendrian curve (π as in (5.1)).*
(3) $\gamma(u) := h(a(u))$ *is a front in $\mathbb{H}^2$ with unit normal $\nu(u) = a(u)k\overline{a(u)}$.*

Proof. It is immediate from the definition of front in $\mathbb{H}^2$ and (5.1) that (2) and (3) are equivalent. To prove that (2) $\Rightarrow$ (1), assume that

$$\pi(a(u)) = (a(u)i\overline{a(u)}, a(u)k\overline{a(u)})$$

is Legendrian. Then $a'(u) \neq 0$ and $\langle a'(u)i\overline{a(u)} + a(u)i\overline{a'(u)}, a(u)k\overline{a(u)}\rangle = 0$. Using that $k\, i = j$ and the left-right-invariance, this equation gives

$$\langle a'(u), a(u)\, j\rangle = \langle i\overline{a'(u)}, k\overline{a(u)}\rangle = \langle a'(u)\, i, a(u)\, k\rangle = -\langle a'(u), a(u)\, j\rangle;$$

i.e. (1) holds.

To prove that (1) $\Rightarrow$ (2), we define $\alpha(u) := \pi(a(u)) = (\gamma(u), \nu(u))$, i.e. $\gamma(u) = a(u)i\overline{a(u)}$ and $\nu(u) = a(u)k\overline{a(u)}$. Let us assume that u is the *asymptotic parameter* of $a(u)$ as in Definition 10. Then, by (4.9) we have (omitting the parameter u for clarity)

$$\bar{a}a' = \cos(\omega^a)\, i + \sin(\omega^a)\, k. \tag{5.2}$$

Hence, $\langle \gamma', \nu\rangle = \langle \bar{a}a'\, i + i(\overline{\bar{a}a'}), k\rangle = 0$. So, to prove (2) we only have left to check that $\pi(a(u))$ is an immersion, i.e. that $\langle \gamma', \gamma'\rangle + \langle \nu', \nu'\rangle > 0$ everywhere. We compute

$$\begin{aligned} \langle \gamma', \gamma'\rangle &= 2\langle a', a'\rangle + 2\langle \bar{a}a'i, i(\overline{\bar{a}a'})\rangle \qquad \text{(by (5.2))} \\ &= -2\cos(2\omega^a) + 2\langle (-\cos(\omega^a)\, 1 + \sin(\omega^a)\, j, \cos(\omega^a)1 + \sin(\omega^a)j\rangle \\ &= 4\sin^2(\omega^a). \end{aligned} \tag{5.3}$$

A similar computation using the general relations $\langle xk, xk\rangle = -\langle x, x\rangle = \langle kx, kx\rangle$ gives $\langle \nu', \nu'\rangle = 4\cos^2(\omega^a)$, and consequently

$$\langle \alpha'(u), \alpha'(u)\rangle_S = \langle \gamma'(u), \gamma'(u)\rangle + \langle \nu'(u), \nu'(u)\rangle = 4. \tag{5.4}$$

This yields (2) and completes the proof. $\square$

Using this result, we may give the following definition.

Definition 15. Let $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2$ be a front in $\mathbb{H}^2$ with Legendrian lift $\alpha : I \subset \mathbb{R} \to TU(\mathbb{H}^2)$. An *asymptotic lift* of γ is a regular curve $a : I \subset \mathbb{R} \to \mathbb{H}^3_1$ such that $\pi \circ a = \alpha$, where $\pi : \mathbb{H}^3_1 \to TU(\mathbb{H}^2)$ is the double cover (5.1).

It is obvious that any front has an asymptotic lift, which is unique up to sign once we fix the Legendrian lift α (since π is a double covering with $\pi(x) = \pi(-x)$). Also, by Lemma 14, the asymptotic lift of $\gamma(u)$ satisfies $h(a(u)) = \gamma(u)$ and $\langle a'(u), a(u)\, j\rangle = 0$.

Let us also observe that if we substitute the unit normal ν of the front γ by $-\nu$, then the asymptotic lift $a(u)$ switches to $a(u)\, i$.

Remark 16. By (5.4), we see that the asymptotic parameter u of the asymptotic lift $a(u)$ according to Definition 10 is one half of the arc-length parameter w.r.t. the metric $\langle , \rangle_S$ of the Legendrian lift $\alpha(u)$ of $\gamma(u)$.

Let $\gamma(u) : I \subset \mathbb{R} \to \mathbb{H}^2$ be a front with unit normal $\nu : I \subset \mathbb{R} \to \mathbb{S}^2_1$. If $\gamma'(u_0) \neq 0$, its geodesic curvature at that point is

$$k_g(u_0) = \frac{\langle \gamma''(u_0), \nu(u_0)\rangle}{||\gamma'(u_0)||^2}.$$

Now, if $\gamma'(u_0) = 0$, then $\nu'(u_0) \neq 0$ around u_0, and we have $\gamma'(u) = \lambda(u)\nu'(u)$ for some smooth function $\lambda(u)$ defined in a neighborhood of u_0. Clearly, $\lambda(u_0) = 0$ and $\lambda = -1/k_g$ at regular points of γ. This justifies the following definition:

Definition 17. Let $\gamma(u) : I \subset \mathbb{R} \to \mathbb{H}^2$ be a front with unit normal $\nu : I \subset \mathbb{R} \to \mathbb{S}^2_1$. The *geodesic curvature* of γ is the smooth map $k_g : I \subset \mathbb{R} \to \mathbb{R} \cup \{\infty\} \equiv \mathbb{R}P^1$ given by

$$\begin{cases} k_g(u) = \dfrac{\langle \gamma''(u), \nu(u)\rangle}{||\gamma'(u)||^2} & \text{if } \gamma'(u) \neq 0, \\ \qquad \infty & \text{if } \gamma'(u) = 0. \end{cases}$$

The geodesic curvature of a front in $\mathbb{H}^2$ and the angle function of its asymptotic lift in $\mathbb{H}^3_1$ are related by the following simple formula:

Lemma 18. *Let $a(u) : I \subset \mathbb{R} \to \mathbb{H}^3_1$ be a regular curve in $\mathbb{H}^3_1$ with $\langle a'(u), a(u)\, j\rangle = 0$, where u is its asymptotic parameter. Then, the geodesic curvature $k_g(u)$ of the front $\gamma(u) = h(a(u)) : I \subset \mathbb{R} \to \mathbb{H}^2$ is given by*

$$k_g(u) = \cot(\omega^a(u)), \tag{5.5}$$

where $\omega^a(u)$ is the angle function of $a(u)$ (see Definition 10*).*

Proof. We know by (5.3) that $\gamma'(u_0) = 0$ if and only if $\sin \omega^a(u_0) = 0$. Thus, (5.5) holds trivially at the singular points of $\gamma(u)$.

For the rest of the points, we use (5.3), the equivalence (1) $\Leftrightarrow$ (3) in Lemma 14, and the left-right-invariance to compute

$$k_g = -\frac{\langle \gamma', \nu'\rangle}{||\gamma'||^2} = \frac{-1}{4\sin^2(\omega^a)} \langle \bar{a}a'\, i + i\, (\overline{\bar{a}a'}), \bar{a}a'\, k + k\, (\overline{\bar{a}a'})\rangle.$$

Now using (5.2) and the relations $i\,k = -k\,i = -j$ and $k^2 = -i^2 = 1$, we have

$$k_g = \frac{-1}{4\sin^2(\omega^a)}\,\langle 2\sin(\omega^a)\,j, -2\cos(\omega^a)\,j\rangle = \cot(\omega^a),$$

as desired. $\square$

Remark 19. Let us note that the function $\cot : \mathbb{R} \to \mathbb{R}P^1$ is a continuous, surjective, π-periodic covering map. This allows us to choose, on every subset $A \subsetneq \mathbb{R}P^1$, a continuous determination of $\cot^{-1}$ such that:

$$\begin{array}{lcl} \cot^{-1}(A) & \subset & (0,\pi) \quad \text{if } \infty \notin A, \\ \cot^{-1}(A) & \subset & (\pi - c, 2\pi - c) \text{ for some } c \in (0,\pi) \quad \text{if } \infty \in A. \end{array}$$

From now on, by $\cot^{-1}$ we shall mean this specific continuous determination.

Lemma 20. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2$ be a front with unit normal $\nu : I \subset \mathbb{R} \to \mathbb{S}^2_1$, whose geodesic curvature function $k_g : I \subset \mathbb{R} \to \mathbb{R}P^1$ is not surjective onto $\mathbb{R}P^1$ (this holds, for instance, if γ is regular). Then, γ admits an asymptotic lift $a : I \subset \mathbb{R} \to \mathbb{H}^3_1$ such that:*

1. *If γ is regular, then $\sin(\omega^a) > 0$. This happens with respect to the unit normal ν such that $\{\gamma', \nu\}$ is always a positively oriented basis of $T_\gamma\mathbb{H}^2$.*
2. *If γ is not regular, then $\overline{a(u_0)}\,a'(u_0) = -i$ (i.e. $\cos(\omega^a(u_0)) = -1$) for all points with $\gamma'(u_0) = 0$, possibly by reversing the sense of the parametrization of γ.*

If this holds for the unit normal ν, then ν will be called the positive unit normal *of the front γ.*

Proof. If γ is regular, then by (5.3) we have $\sin(\omega^a) \neq 0$ everywhere.

Now, let ν denote the unit normal of γ for which $\sin(\omega^a) > 0$. Then, by Lemma 14, $\{\gamma', \nu\}$ will be a positively oriented basis of $T_\gamma\mathbb{H}^2$ if and only if $\langle \gamma', a\,j\,\bar{a}\rangle > 0$ at every point (observe that $\{ai\bar{a}, aj\bar{a}, ak\bar{a}\}$ is always positively oriented). Now, from (5.2) we get

$$\begin{array}{rcl} \langle \gamma', a\,j\,\bar{a}\rangle & = & \langle a'\,i\,\bar{a} + a\,i\,\overline{a'}, a\,j\,\bar{a}\rangle = \langle \bar{a}a'\,i, j\rangle + \langle i\,(\overline{\bar{a}a'}), j\rangle \\ & = & -2\langle \bar{a}a', ji\rangle = 2\sin(\omega^a) > 0, \end{array}$$

which proves the claim.

Now, assume that $\gamma'(u_0) = 0$ for some u_0. Then $\sin(\omega^a(u_0)) = 0$ and, reversing the sense of the parametrization of γ if necessary, we may assume that $\cos(\omega^a(u_0)) = -1$. Now, by Remark 19 and the hypothesis that k_g is not surjective onto $\mathbb{R}P^1$, the claim that $\cos(\omega^a(u)) = -1$ actually holds at every singular point of the front γ. This concludes the proof. $\square$

Definition 21. An *admissible front pair* in $\mathbb{H}^2$ is a pair of fronts $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{H}^2$ with $\gamma_1(0) = \gamma_2(0) = i$ and $\nu_1(0) = \nu_2(0) = k$, such that

i) γ_1 is actually a regular curve in $\mathbb{H}^2$.

ii) If $k_1, k_2 : \mathbb{R} \to \mathbb{R}P^1$ denote the geodesic curvatures of γ_1 and γ_2, respectively, with respect to their positive unit normals, then

$$k_1(u) \neq k_2(v) \qquad \forall (u,v) \in \mathbb{R}^2,$$

and actually $k_1(u) > k_2(v)$ holds if γ_2 is also a regular curve.

We observe that if γ_1, γ_2 satisfy $k_1(\mathbb{R}) \cap k_2(\mathbb{R}) = \emptyset$, then by switching the roles of γ_1 and γ_2 if necessary, $\{\gamma_1, \gamma_2\}$ is an admissible front pair in $\mathbb{H}^2$.

These elements will let us describe in a very precise way the moduli space of isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ in terms of suitable pairs of curves with front-like singularities in $\mathbb{H}^2$. Indeed, we have

Theorem 22 (Classification of complete examples)**.** *Let $\gamma_1(u), \gamma_2(v) : \mathbb{R} \to \mathbb{H}^2$ be an admissible front pair in $\mathbb{H}^2$, where $u/2$ (resp. $v/2$) is the arc-length parameter of γ_1 (resp. γ_2) with respect to the metric $\langle,\rangle_{\mathcal{S}}$.*

Let $k_1(u), k_2(v) : \mathbb{R} \to \mathbb{R}P^1$ and $a_1(u), a_2(v) : \mathbb{R} \to \mathbb{H}^3_1$ denote, respectively, the geodesic curvatures and asymptotic lifts of γ_1 and γ_2 with respect to their positive unit normals. Assume that:

- *For $\omega_1(u) := \cot^{-1}(k_1(u))$ and $\omega_2(v) := \pi - \cot^{-1}(k_2(v))$, the map $(x(u,v), y(u,v))$ defined in (3.8) is a global diffeomorphism.*

Then, $f : \mathbb{R}^2 \longrightarrow \mathbb{H}^3_1$ given by $f(u,v) = a_1(u)\overline{a_2(v)}$ is an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, and (u,v) are the global characteristic parameters given in Proposition 5.

Conversely, every isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ can be recovered by this process from an admissible front pair in $\mathbb{H}^2$.

Proof. For the direct part, we just have to show that $a_1(u)$ and $a_2(v)$ satisfy the hypotheses of Theorem 11. Since they are the asymptotic lifts of the curves γ_i, Lemma 14 tells us that $\langle a_1', a_1 j\rangle = \langle a_2', a_2 j\rangle = 0$ and, by Remark 16, we know that u and v are the asymptotic parameters of a_1 and a_2. Also by Lemma 14 and the sign ambiguity of the asymptotic lift, we may assume that $a_1(0) = a_2(0) = 1$.

Now, observe that condition ii) in Definition 21 implies, in particular, that both $k_1(\mathbb{R}), k_2(\mathbb{R}) \subsetneq \mathbb{R}P^1$. So, by Remark 19, the functions $\cot^{-1}(k_1(u))$ and $\cot^{-1}(k_2(v))$ make sense.

Let ω^{a_1} (resp. ω^{a_2}) denote the angle function associated to a_1 (resp. a_2). As γ_1 is regular, by Lemma 20 and the $2\pi k$-ambiguity in defining ω^{a_i}, we may assume that $\omega^{a_1}(\mathbb{R}) \subset (0,\pi)$. Thus, by Lemma 18 and the above comments we have

$$\omega^{a_1}(u) = \cot^{-1}(k_1(u)) \in (0,\pi),$$

and similarly, $\omega^{a_2}(v) = \cot^{-1}(k_2(v))$, where $\omega^{a_2}(\mathbb{R}) \subset (0,\pi)$ if γ_2 is regular, and $\omega^{a_2}(\mathbb{R}) \subset (\pi - c, 2\pi - c)$ for some $c > 0$ if γ_2 has some singular point.

Now define $\omega_1(u) = \cot^{-1}(k_1(u))$ and $\omega_2(v) = \pi - \cot^{-1}(k_2(v))$. If we prove that $\omega_1(u) + \omega_2(v) \in (0,\pi)$ for all $(u,v) \in \mathbb{R}^2$, all conditions of Theorem 8 will be fulfilled, as we wished.

In the case that γ_2 is regular, we clearly have $\omega_1(u) + \omega_2(v) > 0$, and as $k_1(u) > k_2(v)$ for every (u,v), we conclude that $\cot^{-1}(k_1(u)) - \cot^{-1}(k_2(v)) < 0$, i.e. $\omega_1(u) + \omega_2(v) < \pi$, as desired.

In the case that γ_2 has some singular point, it is clear that $\omega_1(u_0) + \omega_2(v_0) \in (0,\pi)$ for some adequate $(u_0, v_0) \in \mathbb{R}^2$. Once we know that, it is also clear that $\omega_1(u) + \omega_2(v) \neq \{0,\pi\}$ at every point, since otherwise the condition $k_1(u) \neq k_2(v)$ would not hold everywhere. So, again, $\omega_1(u) + \omega_2(v) \in (0,\pi)$ for all $(u,v) \in \mathbb{R}^2$. This finishes the first part of the proof.

For proving the converse part of the theorem we recall that, from Theorem 8, we already know that every isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ can be put in the form $f(u,v) = a_1(u)a_2(v)$. Thus, taking $\gamma_1(u) = p_{\mathbb{H}^2} \circ \pi(a_1(u))$, $\gamma_2(v) = p_{\mathbb{H}^2} \circ \pi\big(\overline{a_2(v)}\big)$, we can recover the immersion f by applying the direct part to the curves γ_1 and γ_2. $\square$

Let us now consider a Lorentzian flat surface Σ which is compact and orientable. Then Σ is a torus and its universal covering $\widetilde{\Sigma}$ is a plane. The next classification result establishes which isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ in Theorem 22 are the universal covering of some Lorentzian flat torus in $\mathbb{H}^3_1$. This then provides a description of the moduli space of Lorentzian flat tori in $\mathbb{H}^3_1$.

Theorem 23 (Classification of flat tori)**.** *Let $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{H}^2$ be two closed fronts in $\mathbb{H}^2$ with $\gamma_1(p_0) = \gamma_2(p_0) = i$ and $\nu_1(p_0) = \nu_2(p_0) = k$ for some $p_0 \in \mathbb{S}^1$ (here ν_i is the positive unit normal of γ_i). Assume that*

$$k_1(\mathbb{S}_1) \cap k_2(\mathbb{S}_1) = \emptyset, \tag{5.6}$$

where k_i is the geodesic curvature of γ_i in $\mathbb{H}^2$. Then, after permuting γ_1 and γ_2 if necessary, the Lorentzian flat surface in $\mathbb{H}^3_1$ that they generate via Theorem 22 has compact image, and describes therefore a Lorentzian flat torus isometrically immersed in $\mathbb{H}^3_1$.

Conversely, every Lorentzian flat torus of $\mathbb{H}^3_1$ can be constructed following the process described in Theorem 22, starting with a pair of closed fronts γ_1, γ_2 in $\mathbb{H}^2$ satisfying the regularity condition (5.6)*.*

Proof. The first part is immediate, taking into account that if γ_i is a closed front with unit normal ν_i, then $\alpha_i := (\gamma_i, \nu_i)$ is regular and closed in $TU(\mathbb{H}^2)$, and as π in (5.1) is a double covering, it follows that $a_i := \pi^{-1}(\alpha_i)$ will be a closed curve in $\mathbb{H}^3_1$. With this, $f = a_1 \overline{a_2}$ is the product of two closed curves in $\mathbb{H}^3_1$, and thereby it is compact with the topology of a torus.

Conversely, let Σ denote a flat Lorentzian torus in $\mathbb{H}^3_1$, let $\widetilde{\Sigma} \equiv \mathbb{L}^2$ denote its universal covering, and $p : \widetilde{\Sigma} \to \Sigma$ the canonical covering map. So, we shall regard $\widetilde{\Sigma}$ in the obvious way as a complete Lorentzian flat surface isometrically immersed in $\mathbb{H}^3_1$, with second fundamental form $\widetilde{II}$ given by $p^*(II) = \widetilde{II}$, where II stands for the second fundamental form of the torus Σ. In these conditions, by Theorem 22, we can parametrize $\widetilde{\Sigma}$ as an immersion $f(u,v) : \mathbb{R}^2 \to \mathbb{H}^3_1$ such that

$$f(u,v) = a(u)\,\overline{b(v)}, \qquad N(u,v) = a(u)\,j\,\overline{b(v)}.$$

Here we have assumed that, up to a rigid motion, $f(0,0) = 1$ and $N(0,0) = j$.

Next let us consider the map $N\,\bar{f} : \widetilde{\Sigma} \to \mathbb{S}^2_1$. It is obvious that $N\,\bar{f}$ is well defined in Σ, and thus $N\,\bar{f}(\Sigma) = N\,\bar{f}(\widetilde{\Sigma})$ is compact in $\mathbb{H}^2$. Moreover, in terms of the parameters (u,v) we have

$$(N\,\bar{f})(u,v) = a(u)\,j\,\overline{a(u)},$$

and hence $N\,\bar{f}(\widetilde{\Sigma})$ is a closed curve in $\mathbb{S}^2_1$. Next we prove that it is also regular.

Let us denote $\beta_1(u) := a(u)\,j\,\overline{a(u)} : \mathbb{R} \to \mathbb{S}^2_1$. Then, using the basic properties of the pseudo-quaternionic model for $\mathbb{H}^3_1$, we have

$$\begin{aligned}
\overline{\beta_1}\,\beta_1' &= -a\,j\,\bar{a}(a'\,j\,\bar{a} + a\,j\,\overline{a'}) = -a\,j\,\bar{a}\,a'\,j\,\bar{a} - a\,\overline{a'} \\
&= -a\,j\,\bar{a}\,a\,j\,\overline{a'} - a\,\overline{a'} \qquad (\text{since } \langle a', a\,j\rangle = 0 \Rightarrow \mathrm{Re}(a'\,j\,\bar{a}) = 0) \\
&= -2a\,\overline{a'} \neq 0.
\end{aligned}$$

Therefore, $\beta_1(u)$ is a regular curve, which is also closed.

In the same way, we can define $-\bar{N}\,f : \widetilde{\Sigma} \to \mathbb{S}^2_1$, and the process above shows that the curve $\beta_2(v) := b(v)\,j\,\overline{b(v)} : \mathbb{R} \to \mathbb{S}^2_1$ is a closed regular curve.

It is important to remark that, by the way they were constructed, the curves β_i may be seen as defined on the flat torus Σ. Next consider the map

$$G = (\beta_1, \beta_2) : \Sigma \to \mathbb{S}^2_1 \times \mathbb{S}^2_1.$$

It is obvious that G is a local diffeomorphism, and $G(\Sigma) \equiv \beta_1 \times \beta_2 \subset \mathbb{S}^2_1 \times \mathbb{S}^2_1$ is a (flat) torus. Thus, by compactness, G is a finite folded covering map. In this way, the lift to Σ of each curve of the form $\Gamma := \beta_1 \times \{p\}$ or $\Gamma := \{p\} \times \beta_2$ of the torus $\beta_1 \times \beta_2$ is a closed curve in Σ.

In addition, it is clear from the definition of β_1, β_2 that a curve $\widetilde{\alpha}$ is an asymptotic curve on $\widetilde{\Sigma}$ (if and only if $\alpha = p \circ \widetilde{\alpha}$ is an asymptotic curve on Σ) if and only if $\widetilde{G_i} \circ \widetilde{\alpha}$ is constant for some $i = 1, 2$, where by definition $\widetilde{G_i} = G_i \circ p$. Thus, α is an asymptotic curve on Σ if and only if $G_i \circ \alpha$ is constant for some $i = 1, 2$, i.e. if and only if α is the lift via the finite fold covering G of a curve of the form $\Gamma := \beta_1 \times \{p\}$ or $\Gamma := \{p\} \times \beta_2$ on $\beta_1 \times \beta_2$.

To sum up, we have proved the fundamental fact that *the asymptotic curves of a Lorentzian flat torus in $\mathbb{H}^3_1$ are closed*. In particular, the Hopf projection into $\mathbb{H}^2$ of such an asymptotic curve is a closed front. This fact together with the converse part of Theorem 22 proves that every Lorentzian flat torus in $\mathbb{H}^3_1$ can be reconstructed by means of two closed fronts in $\mathbb{H}^2$ verifying the regularity condition (5.6). This completes the proof. $\square$

Remark 24. Theorem 23 constitutes the extension to the Lorentzian setting of the classification of Riemannian flat tori in the 3-sphere $\mathbb{S}^3$ by Kitagawa [Kit1]. Let us remark that Theorem 23 follows from our main result (Theorem 22) and a reformulation of the proof of Kitagawa's theorem given by Dadok and Sha in [DaSh].

Hopf cylinders

The most simple examples of isometric immersions from $\mathbb{L}^2$ into $\mathbb{H}^3_1$ are provided by Hopf cylinders. Next, we will analyze these Hopf cylinders from the viewpoint developed in this paper.

Let us denote by Λ^2 the positive light cone $\Lambda^2 = \{x \in \mathbb{L}^3 : \langle x, x\rangle = 0, x_0 > 0\}$.

Definition 25. Let σ be a spacelike regular or timelike regular curve in $\mathbb{S}^2_1$ (resp. $\mathbb{H}^2$, Λ^2) and $\rho \in \mathbb{R}^4_2$ be pure imaginary and nonzero with $\langle \rho, \rho\rangle = 1$ (resp. $\langle \rho, \rho\rangle = -1$, $\langle \rho, \rho\rangle = 0$). Then the flat surface in $\mathbb{H}^3_1$ given by $M_\rho(\sigma) = h_\rho^{-1}(\sigma)$ is called a *Hopf cylinder*.

The Hopf cylinders $M_\rho(\sigma)$ with $\langle \rho, \rho\rangle = -1$ or $\langle \rho, \rho\rangle = 0$ are always timelike, whereas those with $\langle \rho, \rho\rangle = 1$ can be both spacelike or timelike, depending on the causal character of the curve σ. Moreover, if σ is a closed curve in $\mathbb{H}^2$, the Hopf cylinder $M_\rho(\sigma)$ is actually compact, and is called a *Lorentzian Hopf torus*.

Since complete Lorentzian Hopf cylinders are particular cases of isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, Theorem 22 tells us that they can be obtained from two curves γ_1, γ_2 with front singularities in $\mathbb{H}^2$. In this situation one may ask whether there exists any condition on the curves γ_i which characterizes Lorentzian Hopf cylinders among all isometric immersions of $\mathbb{L}^2$ into $\mathbb{H}^3_1$.

Theorem 26. *Let $f : \mathbb{L}^2 \longrightarrow \mathbb{H}^3_1$ be an isometric immersion that is a Lorentzian Hopf cylinder $M_\rho(\sigma)$. We assume that $f(0,0) = 1$ and $N(0,0) = j$ (this forces*

$\langle \rho, j\rangle = 0$). Then, f can be recovered following the process described in Theorem 22 from two fronts γ_1, γ_2 in $\mathbb{H}^2$ such that at least one of them has constant geodesic curvature k_i. Moreover,

$$\begin{aligned} |k_i| > 1 &\iff \langle \rho, \rho\rangle = -1, \\ |k_i| = 1 &\iff \langle \rho, \rho\rangle = 0, \\ |k_i| < 1 &\iff \langle \rho, \rho\rangle = 1. \end{aligned} \tag{5.7}$$

Proof. Let $c(t)$ be the fiber of h_ρ passing through $1 = f(0,0)$. It is a geodesic of $\mathbb{H}^3_1$ and, hence, an asymptotic curve of the immersion.

After (2.2) we know that this curve is given by $c(t) = e^{t\rho}$ and it is easy to check that

$$\overline{c(t)}c'(t) = c'(t)\overline{c(t)} = \rho.$$

If we reparametrize this curve by its asymptotic parameter s (see Definition 10), then, for some constant $\omega_0 \in \mathbb{R}$, we can write

$$\overline{c(s)}c'(s) = c'(s)\overline{c(s)} = \cos(\omega_0)i + \sin(\omega_0)k. \tag{5.8}$$

From this expression it is clear that

$$\rho = \lambda\big(\cos(\omega_0)i + \sin(\omega_0)k\big), \quad \text{with } \lambda > 0. \tag{5.9}$$

On the other hand, if we now consider the global characteristic parameters (u, v) of the immersion f described in Proposition 5, we can apply Theorem 8 and conclude that $f(u, v) = a_1(u)a_2(v)$ with $a_1 = f(u, 0)$ and $a_2 = f(0, v)$. Using the terminology of Theorem 22, we get that the immersion f can be recovered from the fronts $\gamma_1 = h(a_1)$ and $\gamma_2 = h(\overline{a_2})$ in $\mathbb{H}^2$.

The fact that $c(s)$ is an asymptotic curve of f passing through $f(0,0)$ implies that it is a reparametrization of one of the curves a_i. In this situation, the corresponding γ_i would have constant geodesic curvature if and only if the front $h(c(s))$ (or $h(\overline{c(s)})$) does. But, applying Lemma 18, we deduce from (5.8) that the geodesic curvatures of $h(c(s))$ and $h(\overline{c(s)})$ are both given by $k_g = \cot(\omega_0)$. Finally, if we recall (5.9), we can relate the different possibilities for $\langle \rho, \rho\rangle$ with $|\cot(\omega_0)|$. Namely, we get

$$\begin{aligned} \langle \rho, \rho\rangle = -1 &\Leftrightarrow |\cos(\omega_0)| > |\sin(\omega_0)| \Leftrightarrow |\cot(\omega_0)| > 1, \\ \langle \rho, \rho\rangle = 0 &\Leftrightarrow |\cos(\omega_0)| = |\sin(\omega_0)| \Leftrightarrow |\cot(\omega_0)| = 1, \\ \langle \rho, \rho\rangle = 1 &\Leftrightarrow |\cos(\omega_0)| < |\sin(\omega_0)| \Leftrightarrow |\cot(\omega_0)| < 1. \end{aligned}$$

Therefore, (5.7) is established. □

6. The Dajczer-Nomizu questions

The global study of isometric immersions from $\mathbb{L}^2$ into $\mathbb{H}^3_1$ was probably initiated by M. Dajczer and K. Nomizu [DaNo] in 1981. In Theorem 7.6 of that paper, the authors presented a method to construct timelike flat surfaces in $\mathbb{H}^3_1$ by multiplying two regular curves $b_1(s) : \mathbb{R} \to \mathbb{H}^3_1$ and $b_2(t) : \mathbb{R} \to \mathbb{H}^3_1$ of $\mathbb{H}^3_1$ satisfying some additional hypotheses. Translating their notation to our context, these hypotheses on the curves are:

i) $\langle b_1', b_1'\rangle \equiv -1$, $\langle b_2', b_2'\rangle \equiv 1$; i.e. one curve is timelike and the other one is spacelike.
ii) $b_1(0) = 1 = b_2(0)$.
iii) $\langle b_1', b_1\xi_0\rangle \equiv 0 \equiv \langle b_2', \xi_0 b_2\rangle$ (we may assume $\xi_0 = j$).

Under these assumptions, they conclude that the surface $f : \mathbb{R}^2 \longrightarrow \mathbb{H}^3_1$ given by $f(s,t) = b_1(s)b_2(t)$ is a timelike flat surface for some domain $D \subset \mathbb{R}^2$ containing the origin. ($D \subset \mathbb{R}^2$ is the connected component of the origin of all points $(s,t) \in \mathbb{R}^2$ at which f is an immersion.)

Moreover, the curves b_1 and b_2 are the asymptotic curves of f.

After proving this result, they proposed the following global open problems related to the above construction:

Q1: *Does $D = \mathbb{R}^2$ if the curves b_1 and b_2 are defined on all $\mathbb{R}$?*

Q2: *If $D = \mathbb{R}^2$, is the surface (geodesically) complete?*

Q3: *Can every isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ be obtained as a product of two appropriate curves?*

Taking into account Theorem 7.6 of [DaNo], it is reasonable to think that Question 3 was formulated as a problem restricted to spacelike or timelike curves, although this was not explicitly stated there. So, the following problem should also be considered:

Q4: *Can every isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ be obtained as a product of two curves so that one of them is everywhere timelike and the other is everywhere spacelike?*

Note that we have already given a positive answer to Q3 in Theorem 8. Moreover, in Theorem 22 we have seen that those two curves a_1 and a_2 can be obtained as asymptotic lifts of two fronts, γ_1 and γ_2, in $\mathbb{H}^2$. We also know that, if k_i represents the geodesic curvature of γ_i and we take $\omega_1(u) := \cot^{-1}(k_1)$ and $\omega_2(v) := \pi - \cot^{-1}(k_2)$, then $\langle a_i', a_i' \rangle = -\cos(2\omega_i)$. Therefore, since

$$-\cos(2\omega_i) = \frac{1-\cot^2(\omega_i)}{1+\cot^2(\omega_i)},$$

we conclude that

$$\begin{aligned} a_i' \text{ is timelike} &\iff |k_i| > 1, \\ a_i' \text{ is null} &\iff |k_i| = 1, \\ a_i' \text{ is spacelike} &\iff |k_i| < 1. \end{aligned} \tag{6.1}$$

This remark gives us an easy way to find a counterexample to Q4. We just have to take fronts γ_1 and γ_2 in $\mathbb{H}^2$ satisfying the hypotheses of Theorem 22, but both with $|k_i| > 1$ or both with $|k_i| < 1$. In that case, their asymptotic lifts a_1 and a_2 would generate an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ and, according to (6.1), they would have the same causal character.

Theorem 22 is also the key to providing a positive answer to Q1. First, let us consider two curves $b_1(s)$ and $b_2(t)$ as in Theorem 7.6 of [DaNo]. We can reparametrize b_1 and $\overline{b_2}$ by taking

$$a_1(u) = b_1(s(u)), \qquad a_2(v) = \overline{b_2(t(v))}$$

with u, v the asymptotic parameters according to Definition 10. In this way, we obtain two curves satisfying $a_1(0) = a_2(0) = 1$, $\langle a_1', a_1 j \rangle = \langle a_2', a_2 j \rangle = 0$ and so that a_1 is everywhere timelike and a_2 is everywhere spacelike.

Now, consider the fronts $\gamma_1 = h(a_1)$, $\gamma_2 = h(a_2)$ in $\mathbb{H}^2$ (see Lemma 14). Then, by changing the order of γ_1 and γ_2 if necessary (which simply means conjugation in the product $a_1(u)\overline{a_2(v)}$), we see that $\{\gamma_1, \gamma_2\}$ is an admissible front pair. Therefore, the Lorentzian flat surface (not necessarily complete) obtained via Theorem 22 has no singular points. That is, the immersion

$$f(s,t) = b_1(s)b_2(t) = a_1(u(s))\overline{a_2(v(t))}$$

is defined over all $\mathbb{R}^2$. Thus, we have answered Q1 affirmatively.

Finally, the following theorem shows that Q2 has, in general, a negative answer. Besides, it also shows that it is still possible to give some sufficient conditions in the sense of [Cec] and [Sas] to ensure completeness in the Lorentzian case. This was exactly the way the problem was formulated in [DaNo], where it is claimed: *This [Q2] seems to be a much more difficult problem than the question of completeness of flat surfaces in $\mathbb{S}^3$ treated in* [Cec] *and* [Sas].

Theorem 27. *Let $b_1, b_2 : \mathbb{R} \to \mathbb{H}^3_1$ be two regular curves with $b_1(0) = b_2(0) = 1$, and such that $-\langle b_1', b_1'\rangle = \langle b_2', b_2'\rangle = 1$ and $\langle b_1', b_1 j\rangle = \langle b_2', b_2 j\rangle = 0$. Consider the timelike flat surface*

$$f(s,t) = b_1(s)\,\overline{b_2(t)} : \mathbb{R}^2 \to \mathbb{H}^3_1,$$

which has no singular points by the above explanation. Then:

1. *The asymptotic parameters (u,v) of f are globally defined on $\mathbb{R}^2$.*
2. *The surface f is geodesically complete if the angle function $\omega(u,v) = \omega_1(u)+\omega_2(v)$ associated to the asymptotic parameters (u,v) satisfies $0 < c \leq \sin\omega(u,v)$ for some $c > 0$.*
3. *There exist curves b_1, b_2 as before so that the resulting timelike flat surface is not geodesically complete.*

Proof. Let us first prove that the parameters (u,v) are globally defined on $\mathbb{R}^2$. We shall only prove that $u = u(s)$ is globally defined on $\mathbb{R}$ (the case of $v = v(t)$ is analogous). As $\langle b_1'(s), b_1(s)j\rangle \equiv 0$ and $\langle b_1'(s), b_1'(s)\rangle \equiv -1$, we can write

$$\overline{b(s)}b'(s) \;=\; \pm\,\cosh\big(\theta(s)\big)\,i \;\pm\; \sinh\big(\theta(s)\big)\,k \quad \text{for some } \theta \in \mathcal{C}^\infty(\mathbb{R}).$$

So, the asymptotic parameter of b_1 is given by

$$u(s) = \int_0^s \sqrt{\cosh^2(\theta(r)) + \sinh^2(\theta(r))}\,dr.$$

Hence,

$$|u(s)| = \left|\int_0^s \sqrt{\cosh^2(\theta(r)) + \sinh^2(\theta(r))}\,dr\right| \geq \left|\int_0^s 1\,dr\right| = |s|.$$

As s is globally defined on $\mathbb{R}$, so is u. This proves the first claim. Besides, the second claim follows directly from Proposition 5.

Finally, to prove the third claim we need to find a timelike flat surface $f(u,v) : \mathbb{R}^2 \to \mathbb{H}^3_1$ with globally defined asymptotic coordinates (u,v), such that $\langle f_u, f_u\rangle = -1$, $\langle f_v, f_v\rangle = 1$, the curves $f(u,0)$ and $f(0,v)$ are globally defined on $\mathbb{R}$ when parametrized by arc-length, but such that the surface is not geodesically complete.

For that, let us consider a smooth function $\omega_1(u) : \mathbb{R} \to (0, \pi/4)$ satisfying:

- $\displaystyle\int_0^\infty \sqrt{\cos(2\omega_1(u))}du = \infty\ ,\ \int_{-\infty}^0 \sqrt{\cos(2\omega_1(u))}du = \infty,$
- $\displaystyle\int_0^\infty \cos(2\omega_1(u))du < \infty.$

Now define $\omega_2 := \pi/2 + \omega_1 : \mathbb{R} \to (\pi/2, 3\pi/4)$. By Proposition 5, ω_1 and ω_2 define a timelike flat surface $f(u,v) : \mathbb{R}^2 \to \mathbb{H}^3_1$ with globally defined asymptotic parameters. Besides, by (3.10), the curves $f(u,0)$ and $f(0,v)$ are globally defined on $\mathbb{R}$ when parametrized by arc-length. Now, to prove that the surface is not geodesically complete, we need to ensure that the map $(x(u,v), y(u,v))$ in (3.8) is not a global diffeomorphism of $\mathbb{R}^2$. But by Remark 4, we just need to prove that the Riemannian flat metric $\widetilde{I} := dx^2 + dy^2$ is noncomplete.

Now, consider the divergent line $\gamma(t) = (t,t) : [0,\infty) \to \mathbb{R}^2$ in the (u,v)-plane. Then, by (3.8), we have

$$\widetilde{I}(\gamma'(u), \gamma'(u)) = 2(1 - \sin(2\omega_1(u))).$$

So, noting that

$$\sqrt{1 - \sin(2\omega_1)} = \frac{\cos(2\omega_1)}{\sqrt{1 + \sin(2\omega_1)}},$$

we have by the condition imposed on ω_1 from the start that

$$\int_0^\infty \sqrt{\widetilde{I}(\gamma', \gamma')}du < \infty;$$

i.e. γ is a divergent curve of finite length. Thus, the map (3.8) is not a global diffeomorphism, and the timelike flat surface $f(u,v)$ is not (geodesically) complete. $\square$

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