

SOME 4-POINT HURWITZ NUMBERS IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this paper, we compute the number of covers of curves with given branch behavior in characteristic p for one class of examples of genus 0, with four branch points and degree p . Our techniques involve related computations in the case of three branch points, and allow us to conclude in many cases that for a particular choice of degeneration, all the covers we consider degenerate to separable (admissible) covers. Starting from a good understanding of the complex case, the proof is centered on the theory of stable reduction of Galois covers.

1. INTRODUCTION

This paper considers the question of determining the number of covers between genus-0 curves with fixed ramification in positive characteristic. More concretely, we consider covers $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched at r ordered points Q_1, \dots, Q_r of fixed *ramification type* $(d; C_1, \dots, C_r)$, where d is the degree of f and $C_i = e_1(i) \cdots e_{s_i}(i)$ is a conjugacy class in S_d . This notation indicates that there are s_i ramification points in the fiber $f^{-1}(Q_i)$, with ramification indices $e_j(i)$. The *Hurwitz number* $h(d; C_1, \dots, C_r)$ is the number of covers of fixed ramification type over \mathbb{C} , up to isomorphism. This number does not depend on the position of the branch points. If p is a prime not dividing any of the ramification indices $e_j(i)$, the *p -Hurwitz number* $h_p(d; C_1, \dots, C_r)$ is the number of covers of fixed ramification type whose branch points are generic over an algebraically closed field k of characteristic p . The genericity hypothesis is necessary because in positive characteristic the number of covers often depends on the position of the branch points.

The only general result on p -Hurwitz numbers is that they are always less than or equal to the Hurwitz number, with equality when the degree of the Galois closure is prime to p . This is because every tame cover in characteristic p can be lifted to characteristic 0, and in the prime-to- p case, every cover in characteristic 0 specializes to a cover in characteristic p with the same ramification type (see Corollaire 2.12 of Exposé XIII in [11]). We say a cover has *good reduction* when such a specialization exists. However, in the general case, some covers in characteristic 0 specialize to inseparable covers in characteristic p ; these covers are said to have *bad reduction*. Thus, the difference $h(d; C_1, \dots, C_r) - h_p(d; C_1, \dots, C_r)$ is the number of covers in characteristic 0 with generic branch points and bad reduction.

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We will always consider covers which have one or two ramified points (of arbitrary ramification index) over each branch point. Accordingly, in place of C_i we will often write e_i , which indicates the class of a single cycle of length e_i and corresponds to having a single ramified point of index e_i (along with $d - e_i$ unramified points) in the fiber $f^{-1}(Q_i)$. Similarly, e_1-e_2 indicates the class of a product of disjoint cycles of lengths e_1 and e_2 , corresponding to two ramified points in the fiber. In [18] and [19], the value $h_p(d; e_1, e_2, e_3)$ is computed for genus-0 covers and any e_i prime to p using linear series techniques. In this paper, we treat the considerably more difficult case of genus-0 covers of type $(p; e_1, e_2, e_3, e_4)$. Our main result is the following.

Theorem 1.1. *Given e_1, e_2, e_3, e_4 strictly between 1 and p , with $\sum_i e_i = 2p + 2$, we have*

$$\begin{aligned} h_p(p; e_1, e_2, e_3, e_4) &= h(p; e_1, e_2, e_3, e_4) - p \\ &= \min_i (e_i(d + 1 - e_i)) - p. \end{aligned}$$

Note that our hypotheses implicitly exclude the case $p = 2$, and the condition on the sum of the e_i is equivalent to imposing that the cover have genus 0. The characteristic-0 Hurwitz number is computed in [15]; see Section 2 below. It is not surprising that the case $r = 4$ is more delicate than the case $r = 3$, as the $r = 3$ case is rigid and amenable to analysis from a fundamentally combinatorial point of view, while the geometry in the case $r = 4$ is already nontrivial. On the other hand, the fact that it only takes a single degeneration to get from $r = 4$ to $r = 3$ places some bounds on the complexity of the problem, making the case that $r = 4$ more approachable than cases with larger r .

An important auxiliary result is the computation of the p -Hurwitz number $h_p(p; e_1-e_2, e_3, e_4)$.

Theorem 1.2. *Given odd integers e_1, e_2, e_3, e_4 strictly between 1 and p , with $e_1 + e_2 \leq p$ and $\sum_i e_i = 2p + 2$, we have that*

$$h_p(p; e_1-e_2, e_3, e_4) = \begin{cases} h(p; e_1-e_2, e_3, e_4) - (p + 1 - e_1 - e_2) : & e_1 \neq e_2, \\ h(p; e_1-e_2, e_3, e_4) - (p + 1 - e_1 - e_2)/2 : & e_1 = e_2. \end{cases}$$

Corollary 6.9 gives a more general result including the case that some of the e_i are even, but in some cases we compute the p -Hurwitz number only up to a factor 2. Note that there are explicit formulas for $h(p; e_1, e_2, e_3, e_4)$ and $h(p; e_1-e_2, e_3, e_4)$; see Theorem 2.1 and Lemma 2.2 below.

Our technique involves the use of “admissible covers”, which are certain covers between degenerate curves (see Section 2). Admissible covers provide a compactification of the space of covers of smooth curves in characteristic 0, but in positive characteristic this is not the case, and it is an interesting question when, under a given degeneration of the base, a cover of smooth curves does in fact have an admissible cover as a limit. In this case we say the smooth cover has *good degeneration*. In [2] one finds examples of covers with generic branch points without good degeneration.

In contrast, our technique for proving Theorem 1.1 simultaneously shows that many of the examples we consider have good degeneration.

Theorem 1.3. *Given odd integers $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$ with $\sum_i e_i = 2p + 2$, every cover of type $(p; e_1, e_2, e_3, e_4)$ with generic branch points $(0, 1, \lambda, \infty)$ has good degeneration under the degeneration sending λ to ∞ .*

As with Theorem 1.2, our methods do not give a complete answer in some cases with even e_i , but we do prove a more general result in Theorem 8.2.

Building on the work of Raynaud [20], Wewers uses the theory of stable reduction in [26] to give formulas for the number of covers with three branch points and having Galois closure of degree strictly divisible by p which have bad reduction to characteristic p . In [6], some p -Hurwitz numbers are calculated using the existence portion of Wewers' theorems, but these are in cases which are rigid (meaning the classical Hurwitz number is 1) or very close to rigid, so one does not have to carry out calculations with Wewers' formulas. In [21], Selander uses the full statement of Wewers' formulas to compute some examples in small degree. Our result in Theorem 1.2 is the first explicit calculation of an infinite family of p -Hurwitz numbers which fully uses Wewers' formulas, and its proof occupies the bulk of the present paper.

We begin in Sections 2 and 3 by reviewing the situation in characteristic 0 and some group-theoretic background. We then recall the theory of stable reduction in Section 4. In order to apply Wewers' formulas, in Section 5 we analyze the possible structures of the stable reductions which arise, and then in Section 6 we apply Wewers' formulas to compute the number of smooth covers with a given stable reduction. Here we are forced to use a trick comparing the number of covers in the case of interest to the number in a related case where we know all covers have bad reduction. In Section 7 we then apply Corollary 6.9 as well as the formulas for $h_p(d; e_1, e_2, e_3)$ of [18] and the classical Hurwitz number calculations in [15] to estimate the number of admissible covers in characteristic p . This provides a sufficient lower bound on $h_p(p; e_1, e_2, e_3, e_4)$. Finally, we use the techniques of [3], again based on stable reduction, to directly prove in Section 8 that $h_p(p; e_1, e_2, e_3, e_4)$ is bounded above by $h(p; e_1, e_2, e_3, e_4) - p$. We thus conclude Theorems 1.1 and 1.3.

2. THE CHARACTERISTIC-0 SITUATION

In this paper, we consider covers $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ branched at r ordered points Q_1, \dots, Q_r of fixed *ramification type* $(d; C_1, \dots, C_r)$, where d is the degree of f and $C_i = e_1(i) \cdots e_{s_i}(i)$ is a conjugacy class in S_d . This means that there are s_i ramification points in the fiber $f^{-1}(Q_i)$, with ramification indices $e_j(i)$. The *Hurwitz number* $h(d; C_1, \dots, C_r)$ is the number of covers of fixed ramification type over \mathbb{C} , up to isomorphism. This number does not depend on the position of the branch points.

Riemann's Existence Theorem implies that the Hurwitz number $h(d; C_1, \dots, C_r)$ is the cardinality of the set of *Hurwitz factorizations* defined as

$$\{(g_1, \dots, g_r) \in C_1 \times \cdots \times C_r \mid \langle g_i \rangle \subset S_d \text{ transitive, } \prod_i g_i = 1\} / \sim,$$

where \sim denotes uniform conjugacy by S_d .

The group $\langle g_i \rangle$ is called the *monodromy group* of the corresponding cover. For a fixed monodromy group G , a variant equivalence relation is given by *G -Galois covers*, where we work with Galois covers together with a fixed isomorphism of the Galois group to G . The group-theoretic interpretation is then that the g_i are in

G (with the action on a fiber recovered by considering G as a subgroup of $S_{|G|}$), and the equivalence relation \sim_G is uniform conjugacy by G . To contrast with the G -Galois case, we sometimes emphasize that we are working up to S_d -conjugacy by referring to the corresponding covers as *mere covers*.

In this paper, we are mainly interested in the *pure-cycle* case, where every C_i is the conjugacy class in S_d of a single cycle. In this case, we write $C_i = e_i$, where e_i is the length of the cycle. Also occurring (as a degeneration from the pure-cycle case) is the case that C_i consists of two disjoint cycles; we write e_1-e_2 for the corresponding class in this case. These cases correspond geometrically to having a single ramified point over each branch point, or two ramified points over the given branch point, respectively. A cover $f : Y \rightarrow \mathbb{P}^1$ over \mathbb{C} of ramification type $(d; e_1, e_2, \dots, e_r)$ has genus $g(Y) = 0$ if and only if $\sum_{i=1}^r e_i = 2d - 2 + r$.

Giving closed formulas for Hurwitz numbers may get very complicated, even in characteristic zero. The following result from [15] illustrates that the genus-0 pure-cycle case is more tractable than the general case, as one may give closed formulas for the Hurwitz numbers, at least if the number r of branch points is at most 4.

Theorem 2.1. *Under the hypothesis $\sum_{i=1}^r e_i = 2d - 2 + r$, we have the following.*

- (a) $h(d; e_1, e_2, e_3) = 1$.
- (b) $h(d; e_1, e_2, e_3, e_4) = \min_i(e_i(d + 1 - e_i))$.
- (c) *Let $f : \mathbb{P}^1_{\mathbb{C}} \rightarrow \mathbb{P}^1_{\mathbb{C}}$ be a cover of ramification type $(d; e_1, e_2, \dots, e_r)$ with $r \geq 3$. The Galois group of the Galois closure of f is either S_d or A_d unless $(d; e_1, e_2, \dots, e_r) = (6; 4, 4, 5)$ in which case the Galois group is S_5 acting transitively on 6 letters.*

These statements are Lemma 2.1, Theorem 4.2, and Theorem 5.3 of [15]. We mention that Boccara ([1]) proves a partial generalization of Theorem 2.1 (a). He gives a necessary and sufficient condition for $h(d; C_1, C_2, \ell)$ to be nonzero in the case that C_1, C_2 are arbitrary conjugacy classes of S_d and only $C_3 = \ell$ is assumed to be the conjugacy class of a single cycle.

Later in our analysis we will be required to study covers of type $(d; e_1-e_2, e_3, e_4)$, so we mention a result which is not stated explicitly in [15], but which follows easily from the arguments therein. We will only use the case that $e_4 = d$, but we state the result in general since the argument is the same.

Lemma 2.2. *Given e_1, e_2, e_3, e_4 and d with $2d + 2 = \sum_i e_i$ and $e_1 + e_2 \leq d$, if $e_1 \neq e_2$ we have*

$$h(d; e_1-e_2, e_3, e_4) = (d + 1 - e_1 - e_2) \min(e_1, e_2, d + 1 - e_3, d + 1 - e_4),$$

and if $e_1 = e_2$ we have

$$h(d; e_1-e_2, e_3, e_4) = \lceil \frac{1}{2}(d + 1 - e_1 - e_2) \min(d + 1 - e_3, d + 1 - e_4) \rceil.$$

Note that this number is always positive. In particular, when $e_4 = d$ we have

$$h(d; e_1-e_2, e_3, d) = \begin{cases} d + 1 - e_1 - e_2 & \text{if } e_1 \neq e_2, \\ (d + 2 - e_1 - e_2)/2 & \text{if } e_1 = e_2, d \text{ even,} \\ (d + 1 - e_1 - e_2)/2 & \text{if } e_1 = e_2, d \text{ odd.} \end{cases}$$

Proof. Without loss of generality, we may assume that $e_1 \leq e_2$ and $e_3 \leq e_4$. Thus, we want to prove that $h(d; e_1-e_2, e_3, e_4)$ is given by the smaller of $e_1(d + 1 - e_1 - e_2)$ and $(d + 1 - e_4)(d + 1 - e_1 - e_2)$ when $e_1 \neq e_2$, by $((d + 1 - e_4)(d + 1 - e_1 - e_2) + 1)/2$

when $e_1 = e_2$ and all of d, e_3, e_4 are even, and by $(d + 1 - e_4)(d + 1 - e_1 - e_2)/2$ otherwise. Even though we do not assume $e_2 \leq e_3$, this formula still follows from the argument of Theorem 4.2 (ii) of [15]. The first observations to make are that since $e_1 + e_2 \leq d$, we have $e_3 + e_4 \geq d + 2$, and it follows that although we may not have $e_2 \leq e_3$, we have $e_1 < e_4$. Moreover, we have $e_1 + e_3 \leq d + 1$ and $e_2 + e_4 \geq d + 1$. We are then able to check that the Hurwitz factorizations $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ described in case (ii) of *loc. cit.* still give valid Hurwitz factorizations (g_1, g_2, g_3) by setting $g_1 = \sigma_1\sigma_2$, just as in Proposition 4.7 of *loc. cit.* Moreover, just as in Proposition 4.9 of *loc. cit.* we find that every Hurwitz factorization must be one of the enumerated ones.

It remains to consider when two of the described possibilities yield equivalent Hurwitz factorizations. If $e_1 \neq e_2$, we can extract σ_1 and σ_2 as the disjoint cycles (of distinct orders) in g_1 , so we easily see that the proof of Proposition 4.8 of *loc. cit.* is still valid. Thus the Hurwitz number is simply the number of possibilities enumerated in Theorem 4.2 (ii) of [15], which is the minimum of $e_1(d + 1 - e_1 - e_2)$ and $(d + 1 - e_4)(d + 1 - e_1 - e_2)$, as desired.

Now suppose $e_1 = e_2$. We then easily check that $e_1 + e_4 \geq d + 1$, so that the number of enumerated possibilities is $(d + 1 - e_4)(d + 1 - e_1 - e_2)$. Here, we see that we potentially have a given Hurwitz factorization (g_1, g_2, g_3) that is simultaneously equivalent to two of the enumerated possibilities, since σ_1 and σ_2 can be switched. Theorem 4.2 (ii) of *loc. cit.* enumerates Hurwitz factorizations via a pair of parameters m, k , which are described intrinsically in the argument of Proposition 4.8 of *loc. cit.* as follows: $e_4 + 1 - m$ is the size of the intersection of the supports of the cycles σ_1, σ_4 , while the intersection of the supports of σ_3 and σ_4 consists of two contiguous words, and k is the length of the word with nonempty overlap with the support of σ_2 . This lets us compute how m, k change under switching σ_1 and σ_2 , and we find that the pair (m, k) is sent to $(e_3 + 2e_4 - d - m, e_3 + e_4 - d - k)$. We thus find that each Hurwitz factorization is equivalent to two distinct enumerated possibilities, with the exception that if d and e_4 (and therefore necessarily e_3) are even, the Hurwitz factorization corresponding to $(m, k) = ((e_3 + 2e_4 - d)/2, (e_3 + e_4 - d)/2)$ is not equivalent to any other. We therefore conclude the desired statement. \square

We now explain how Theorem 4.2 of [15] can be understood in terms of degenerations. Harris and Mumford [12] developed the theory of *admissible covers*, giving a description of the behavior of branched covers under degeneration. Admissible covers in the case we are interested in may be described geometrically as follows: let X_0 be the reducible curve consisting of two smooth rational components X_0^1 and X_0^2 joined at a single node Q . We suppose we have points Q_1, Q_2 on X_0^1 , and Q_3, Q_4 on X_0^2 , all distinct from Q . An *admissible cover* of type $(d; C_1, C_2, *, C_3, C_4)$ is then a connected, finite flat cover $f_0 : Y_0 \rightarrow X_0$ which is étale away from the preimage of Q and the Q_i , and if we denote by $Y_0^1 \rightarrow X_0^1$ and $Y_0^2 \rightarrow X_0^2$ the (possibly disconnected) covers of X_0^1 and X_0^2 induced by f_0 , we also require that $Y_0^1 \rightarrow X_0^1$ has ramification type $(d; C_1, C_2, C)$ for Q_1, Q_2, Q and $Y_0^2 \rightarrow X_0^2$ has ramification type $(d; C, C_3, C_4)$ for Q, Q_3, Q_4 , for some conjugacy class C in S_d , and furthermore that for $P \in f_0^{-1}(Q)$, the ramification index of f_0 at P is the same on Y_0^1 and Y_0^2 . In characteristic p , we further have to require that ramification above the node is tame. We refer to $Y_0^1 \rightarrow X_0^1$ and $Y_0^2 \rightarrow X_0^2$ as the *component covers* determining

f_0 . When we wish to specify the class C , we say the admissible cover is of type $(d, C_1, C_2, *C, C_3, C_4)$.

The two basic theorems on admissible covers concern degeneration and smoothing. First, in characteristic 0, or when the monodromy group has order prime to p , if a family of smooth covers of type $(d; C_1, C_2, C_3, C_4)$ with branch points (Q_1, Q_2, Q_3, Q_4) is degenerated by sending Q_3 to Q_4 , the limit is an admissible cover of type $(d; C_1, C_2, *, C_3, C_4)$. On the other hand, given an admissible cover of type $(d; C_1, C_2, *, C_3, C_4)$, irrespective of characteristic there is a deformation to a cover of smooth curves, which then has type $(d; C_1, C_2, C_3, C_4)$. Such a deformation is not unique in general; we call the number of smooth covers arising as smoothings of a given admissible cover (for a fixed smoothing of the base) the *multiplicity* of the admissible cover.

Suppose we have a family of covers $f : Y \rightarrow X$, with smooth generic fiber $f_1 : Y_1 \rightarrow X_1$, and admissible special fiber $f_0 : Y_0 \rightarrow X_0$. If we choose local monodromy generators for $\pi_1^{\text{tame}}(Y_1 \setminus \{Q_1, Q_2, Q_3, Q_4\})$ which are compatible with the degeneration to X_0 , we then find that if we have a branched cover of X_1 corresponding to a Hurwitz factorization (g_1, g_2, g_3, g_4) , the induced admissible cover of X_0 will have monodromy given by (g_1, g_2, ρ) over X_0^1 and (ρ^{-1}, g_3, g_4) over X_0^2 , where $\rho = g_3g_4$. The multiplicity of the admissible cover arises because it may be possible to apply different simultaneous conjugations to (g_1, g_2, ρ) and to (ρ^{-1}, g_3, g_4) while maintaining the relationship between ρ and ρ^{-1} . It is well known that when ρ is a pure-cycle of order m , the admissible cover has multiplicity m , although we recover this fact independently in our situation as part of the Hurwitz number calculation of [15].

To calculate more generally the multiplicity of an admissible cover of the above type, we define the action of the braid operator B_3 on the set of Hurwitz factorizations as

$$B_3 \cdot (g_1, g_2, g_3, g_4) = (g_1g_2g_1g_2^{-1}g_1^{-1}, g_1g_2g_1^{-1}, g_3, g_4).$$

One easily checks that B_3 preserves Hurwitz factorizations of a given ramification type. The multiplicity of a given admissible cover is the length of the orbit of B_3 acting on the corresponding Hurwitz factorization ([24]).

In this context, we can give the following sharper statement of Theorem 2.1 (b), phrased in somewhat different language in [15].

Theorem 2.3. *Given a genus-0 ramification type $(d; e_1, e_2, e_3, e_4)$, with $e_1 \leq e_2 \leq e_3 \leq e_4$ the only possibilities for an admissible cover of type $(d; e_1, e_2, *, e_3, e_4)$ are type $(d; e_1, e_2, *m, e_3, e_4)$ or type $(d; e_1, e_2, *_{e_1-e_2}, e_3, e_4)$.*

- (a) *Fix $m \geq 1$. There is at most one admissible cover of type $(d; e_1, e_2, *m, e_3, e_4)$, and if such a cover exists, it has multiplicity m . Such a cover must satisfy $m \equiv e_2 - e_1 + 1 \pmod{2}$, and under this restriction, the cover exists if and only if*

$$e_2 - e_1 + 1 \leq m \leq 2d + 1 - e_3 - e_4$$

in the case $d + 1 \leq e_2 + e_3$, and if and only if

$$e_4 - e_3 + 1 \leq m \leq 2d + 1 - e_3 - e_4$$

in the case that $d + 1 \geq e_2 + e_3$.

(b) The number of admissible covers of type $(d; e_1, e_2, *_{e_1-e_2}, e_3, e_4)$ is

$$\begin{cases} e_1(d+1-e_1-e_2) & \text{if } d+1 \leq e_2+e_3, \\ (e_3+e_4-d-1)(d+1-e_4) & \text{if } d+1 \geq e_2+e_3. \end{cases}$$

Such admissible covers have multiplicity 1, and the component cover of type $(d; e_1, e_2, e_1-e_2)$ is uniquely determined, so the admissible cover is determined by its second component cover and the gluing of the component covers to one another over the node. Moreover, the gluing over the node is unique when $e_1 \neq e_2$. When $e_1 = e_2$, there are always two possibilities for gluing except for a single admissible cover in the case that e_3, e_4 , and d are all even.

Proof. We briefly explain how this follows from Theorem 4.2 of [15]. As stated above, the possibilities for admissible covers are determined by pairs (g_1, g_2, ρ) , (ρ^{-1}, g_3, g_4) , where (g_1, g_2, g_3, g_4) is a Hurwitz factorization of type $(d; e_1, e_2, e_3, e_4)$. *Loc. cit.* immediately implies that ρ is always either a single cycle of length $m \geq 1$ or the product of two disjoint cycles.

For (a), we find from part (i) of *loc. cit.* that the ranges for m (which is e_3+e_4-2k in the notation of *loc. cit.*) are as asserted, and that for a given m , the number of possibilities with ρ an m -cycle is precisely m , when counted with multiplicity. On the other hand, in this case both component covers are three-point pure-cycle covers, and thus uniquely determined (see Theorem 2.1 (a)). Thus the admissible cover is unique in this case, with multiplicity m .

For (b), we see by inspection of the description of part (ii) of *loc. cit.* that g_1 is disjoint from g_2 . It immediately follows that the braid action is trivial, so the multiplicity is always 1, and the asserted count of covers follows immediately from the proof of Proposition 4.10 of *loc. cit.* Moreover, the component cover of type $(d; e_1, e_2, e_1-e_2)$ is a disjoint union of covers of type $(e_1; e_1, e_1)$ and $(e_2; e_2, e_2)$ (as well as $d - e_1 - e_2$ copies of the trivial cover), so it is uniquely determined, as asserted. Furthermore, we see that the second component cover is always a single connected cover of degree d , and g_1, g_2 are recovered as the disjoint cycles of ρ^{-1} , so the gluing is unique when $e_1 \neq e_2$. When $e_1 = e_2$, it is possible to swap g_1 and g_2 , so we see that there are two possibilities for gluing. The argument of Lemma 2.2 shows that we do in fact obtain two distinct admissible covers in this way, except for a single cover occurring when e_3, e_4 and d are all even. \square

3. GROUP THEORY

In several contexts, we will have to calculate monodromy groups other than those treated by Theorem 2.1 (c). We will also have to pass between counting mere covers and counting G -Galois covers. In this section, we give basic group-theoretic results to address these topics.

Since we restrict our attention to covers of prime degree, the following proposition will be helpful.

Proposition 3.1. *Let p be a prime number and G a transitive group on p letters. Suppose that G contains a pure cycle of length $1 < e < p - 2$. Then G is either A_p or S_p .*

Moreover, if $e = p - 2$, and G is neither A_p nor S_p , then $p = 2^s + 1$ for some s , and G contains a unique minimal normal subgroup isomorphic to $\text{PSL}_2(2^s)$, and

is contained in $\mathrm{PTL}_2(2^s) \simeq \mathrm{PSL}_2(2^s) \rtimes \mathbb{Z}/s\mathbb{Z}$. If $e = p - 1$, and G is not S_p , then $G \simeq \mathbb{F}_p \rtimes \mathbb{F}_p^*$.

Note that this does not contradict the exceptional case $d = 6$ and $G = S_5$ in Theorem 2.1 (c), since we assume that the degree d is prime.

Proof. Since p is prime, G is necessarily primitive, and a theorem usually attributed to Marggraff ([14]) then states that G is at least $(p - e + 1)$ -transitive. The 2-transitive permutation groups have been classified by Cameron (Section 5 of [8]; see also Feit, Corollary 4.2 of [9] for an earlier statement in our situation of prime degree). Specifically, G has a unique minimal normal subgroup which is either elementary abelian or one of several possible simple groups.

When $e \leq p - 2$, we have that $p - e + 1 \geq 3$. Since G is at least 3-transitive, one easily checks that the elementary abelian case is not possible: indeed, one checks directly that if a subgroup of a 3-transitive group inside S_p contains an element of prime order ℓ , then it is not possible for all its conjugates to commute with one another. Similarly, most possibilities in the simple case cannot be 3-transitive. If G is not S_p or A_p , then G must have a unique minimal normal subgroup N which is isomorphic to a Mathieu group M_{11}, M_{23} , or to $N = \mathrm{PSL}_2(2^s)$. We then have that G is a subgroup of $\mathrm{Aut}(N)$. For $N = M_{11}, M_{23}$, we have $N = G = \mathrm{Aut}(N)$, and it is easy to check that the Mathieu groups M_{11} and M_{23} do not contain any single cycles of order less than p , for example with the computer algebra package GAP. Therefore these cases do not occur. The group $\mathrm{PSL}_2(2^s)$ occurs in degree $2^s + 1$ (by its action on $\mathbb{P}_{\mathbb{F}_{2^s}}^1$), so can only occur if $p = 2^s + 1$. In this case, we have that G is a subgroup of $\mathrm{Aut}(\mathrm{PSL}_2(2^s)) = \mathrm{PTL}_2(2^s)$ and G is at most 3-transitive, so we have $e = p - 2$, as desired.

Finally, if $e = p - 1$, one can give an elementary argument to rule out possibilities other than S_p or $\mathbb{F}_p \rtimes \mathbb{F}_p^*$; see also Müller, Theorem 6.2 of [16]. \square

We illustrate the utility of the proposition with:

Corollary 3.2. *Fix e_1, e_2, e_3, e_4 with $2 \leq e_i \leq p$ for each i , with $e_1 + e_2 \leq p$, and satisfying $2p + 2 = \sum_i e_i$. For $p > 5$, any genus-0 cover of type $(p; e_1 - e_2, e_3, e_4)$ has monodromy group S_p or A_p , with the latter case occurring precisely when e_3 and e_4 are odd, and $e_1 + e_2$ is even. For $p = 5$, the only exceptional case is type $(5; 2-2, 4, 4)$, where the monodromy group is $\mathbb{F}_5 \rtimes \mathbb{F}_5^*$.*

Proof. Without loss of generality, we assume $e_1 \leq e_2$ and $e_3 \leq e_4$. Applying Proposition 3.1, it is clear that the only possible exception occurs for types with $e_3, e_4 \geq p - 2$. We thus have to treat types $(p; 3-3, p - 2, p - 2)$, $(p; 2-4, p - 2, p - 2)$, $(p; 2-2, p - 2, p)$, $(p; 2-3, p - 2, p - 1)$, and $(p; 2-2, p - 1, p - 1)$. The fourth case cannot be exceptional since G contains both a $(p - 2)$ -cycle and a $(p - 1)$ -cycle, and the last case is also ruled out for $p > 5$ because $\mathbb{F}_p \rtimes \mathbb{F}_p^*$ does not contain a 2-2-cycle.

For the first three cases, we must have that $p = 2^s + 1$ for some s and that G is a subgroup of $\Gamma := \mathrm{PTL}_2(2^s)$. Since $p = 2^s + 1$ is a Fermat prime number, we have that s is a power of 2. Moreover, since $\mathrm{PSL}_2(4) = A_5$ as permutation groups in S_5 , we may assume $s \geq 4$. Since s is even, any element of order 3 in $\Gamma \simeq \mathrm{PSL}_2(2^s) \rtimes \mathbb{Z}/s\mathbb{Z}$ must lie inside $\mathrm{PSL}_2(2^s)$, and a nontrivial element of this group can fix at most 2 letters. Thus, in order to contain a 3-3-cycle, we would have to have $6 \leq p = 2^s + 1 \leq 8$, which contradicts the hypothesis $s \geq 4$. This rules out the first case. In the second case, if we square the 2-4-cycle we obtain a 2-2-cycle.

To complete the argument for both the second and third cases it is thus enough to check directly that if $s > 4$, an element of order 2 cannot fix precisely $p - 4$ letters, ruling out a 2-2-cycle in this case. It remains only to check directly that $\text{PTL}_2(16)$ does not contain a 2-2-cycle, which one can do directly with GAP. \square

Because the theory of stable reduction is developed in the G -Galois context, it is convenient to be able to pass back and forth between the context of mere covers and of G -Galois covers. The following easy result relates the number of mere covers to the number of G -Galois covers in the case we are interested in.

Lemma 3.3. *Let $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a (mere) cover of degree d with monodromy group $G = A_d$ (respectively, S_d). Then the number of G -Galois structures on the Galois closure of f is exactly 2 (respectively, 1).*

Proof. The case that $G = S_d$ is clear, since conjugacy by S_d is then the same as conjugacy by G . Suppose $G = A_d$, and let $X = \{(g_1, \dots, g_r) \mid \prod_i g_i = 1, \langle g_i \rangle = A_d\}$. Since the centralizer $C_{S_d}(A_d)$ of A_d in S_d is trivial, it follows that S_d acts freely on X , so the number of elements in $X_f \subseteq X$ corresponding to f as a mere cover is $|S_d|$. Since the center of $G = A_d$ is trivial, G also acts freely on X , and X_f breaks into two equivalence classes of G -Galois covers, each of size $|A_d|$. \square

4. STABLE REDUCTION

In this section, we recall some generalities on stable reduction of Galois covers of curves, and prove a few simple lemmas as a prelude to our main calculations. The main references for this section are [26] and [3]. Since these sources only consider the case of G -Galois covers, we restrict to this situation here as well. Lemma 3.3 implies that we may translate results on good or bad reduction of G -Galois covers to results on the stable reduction of the mere covers, so this is no restriction.

Let R be a discrete valuation ring with fraction field K of characteristic zero and residue field an algebraically closed field k of characteristic $p > 0$. Let $f : V = \mathbb{P}_K^1 \rightarrow X = \mathbb{P}_K^1$ be a degree- p cover branched at r points $Q_1 = 0, Q_2 = 1, \dots, Q_r = \infty$ over K with ramification type $(p; C_1, \dots, C_r)$. For the moment, we do not assume that the C_i are the conjugacy classes of a single cycle. We denote the Galois closure of f by $g : Y \rightarrow \mathbb{P}^1$ and let G be its Galois group. Note that G is a transitive subgroup of S_p and thus has order divisible by p . Write $H := \text{Gal}(Y, V)$, a subgroup of index p .

We suppose that $Q_i \not\equiv Q_j \pmod{p}$, for $i \neq j$, in other words, that $(X; \{Q_i\})$ has good reduction as a marked curve. We assume moreover that g has bad reduction to characteristic p and denote by $\bar{g} : \bar{Y} \rightarrow \bar{X}$ its *stable reduction*. The stable reduction \bar{g} is defined as follows. After replacing K by a finite extension, there exists a unique stable model \mathcal{Y} of Y as defined in [26]. We define $\mathcal{X} = \mathcal{Y}/G$. The stable reduction $\bar{g} : \bar{Y} := \mathcal{Y} \otimes_R k \rightarrow \bar{X} := \mathcal{X} \otimes_R k$ is a finite map between semistable curves in characteristic p ; we call such maps *stable G -maps*. We refer to [26], Definition 2.14, for a precise definition.

Roughly speaking, the theory of stable reduction proceeds in two steps: first, one understands the possibilities for stable G -maps, and then one counts the number of characteristic-0 covers reducing to each stable G -map.

We begin by describing what the stable reduction must look like. Since $(X; \{Q_i\})$ has good reduction to characteristic p , there exists a model $\mathcal{X}_0 \rightarrow \text{Spec}(R)$ such that the Q_i extend to disjoint sections. There is a unique irreducible component

\bar{X}_0 of \bar{X} , called the *original component*, on which the natural contraction map $\bar{X} \rightarrow \mathcal{X}_0 \otimes_R k$ is an isomorphism. The restriction of \bar{g} to \bar{X}_0 is inseparable.

Let $\mathbb{B} \subseteq \{1, 2, \dots, r\}$ consist of those indices i such that C_i is not the conjugacy class of a p -cycle. For $i \in \mathbb{B}$, we have that Q_i specializes to an irreducible component $\bar{X}_i \neq \bar{X}_0$ of \bar{X} . The restriction of \bar{g} to \bar{X}_i is separable, and \bar{X}_i intersects the rest of \bar{X} in a single point ξ_i . Let \bar{Y}_i be an irreducible component of \bar{Y} above \bar{X}_i , and write $\bar{g}_i : \bar{Y}_i \rightarrow \bar{X}_i$ for the restriction of \bar{g} to \bar{Y}_i . We denote by G_i the decomposition group of \bar{Y}_i . We call the components \bar{X}_i (resp. the covers \bar{g}_i) for $i \in \mathbb{B}$ the *primitive tails* (resp. the *primitive tail covers*) associated with the stable reduction. The following definition gives a characterization of those covers that can arise as primitive tail covers (compare to [26], Section 2.2).

Definition 4.1. Let k be an algebraically closed field of characteristic $p > 0$. Let C be a conjugacy class of S_p which is not the class of a p -cycle. A *primitive tail cover* of ramification type C is a G -Galois cover $\varphi_C : T_C \rightarrow \mathbb{P}_k^1$ defined over k which is branched at exactly two points $0, \infty$, satisfying the following conditions.

- (a) The Galois group G_C of φ_C is a subgroup of S_p and contains a subgroup H_C of index p such that $\bar{T}_C := T_C/H_C$ has genus 0.
- (b) The induced map $\bar{\varphi}_C : \bar{T}_C \rightarrow \mathbb{P}^1$ is tamely branched at $x = 0$, with conjugacy class C , and wildly branched at $x = \infty$.

If φ is a tail cover, we let $h = h(\varphi)$ be the conductor and $pm = pm(\varphi)$ the ramification index of a wild ramification point. Alternatively, the invariant h may be defined as the jump in the filtration of the higher ramification groups in the lower numbering. Note that there are several conventions for this numbering. Our numbering differs by one from that of [23].

We say that two primitive tail covers $\varphi_i : T_i \rightarrow \mathbb{P}_k^1$ are *isomorphic* if there exists a G -equivariant isomorphism $\iota : T_1 \rightarrow T_2$. Note that we do not require an isomorphism to send \bar{T}_1 to \bar{T}_2 .

Note that an isomorphism ι of primitive tail covers may be completed into a commuting square

$$\begin{array}{ccc} T_1 & \xrightarrow{\iota} & T_2 \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

In our situation, we have that $m \mid (p - 1)$. To see this, we remark that the inertia group of a wild ramification group is a subgroup of the normalizer $N_{S_p}(P)$ of a Sylow p -subgroup in S_p , which has order $p(p - 1)$. The invariants h, m describe the wild ramification of the tail cover φ_C , and only depend on the conjugacy class C . In Section 5, we will show this if C is the class of a single cycle or the product of 2 disjoint cycles, but this holds more generally.

In the more general setup of [26], Definition 2.9, it is required that $\sigma := h/m < 1$ as part of the definition of primitive tail cover. Lemma 5.1 implies that in our situation this follows from (a). Moreover, that lemma implies that $\gcd(h, m) = 1$. Summarizing, we find that (h, m) satisfy:

$$(4.1) \quad m \mid (p - 1), \quad 1 \leq h < m, \quad \gcd(h, m) = 1.$$

From this it follows that the number of primitive tail covers of fixed ramification type is finite.

In the more general setup of [26], there also exist so-called new tail covers, which are only branched at one point $x = \infty$, but otherwise satisfy the same properties as primitive tail covers. The following lemma states that in our situation no new tails occur. We explain the idea of the proof. Let $\varphi : T \rightarrow \mathbb{P}^1$ be a new tail cover. Its Galois group contains a subgroup of index p . We denote the corresponding degree- p cover by $\bar{\varphi} : \bar{T} \rightarrow \mathbb{P}_k^1$. We define h, m as above. In [20], Lemme 3.1.2, it is shown that the ramification invariant satisfies $\sigma = h/m > 1$. Applying the Riemann–Hurwitz formula to both φ and $T \rightarrow \bar{T}$, one finds that $g(\bar{T}) = (\sigma - 1)(p - 1)/2$. This contradicts the hypothesis that \bar{T} has genus zero.

Lemma 4.2. *The curve \bar{X} consists of at most $r + 1$ irreducible components: the original component \bar{X}_0 and primitive tails \bar{X}_i for all $i \in \mathbb{B}$.*

Proof. In the case that $r = 3$ this is proved in [26], Section 4.4, using that the cover is the Galois closure of a genus-0 cover of degree p . The general case is a straightforward generalization. \square

Due to the nonexistence of new tails in our case, we sometimes refer to primitive tail covers as tail covers in the sequel, if no confusion arises.

It remains to discuss the restriction of \bar{g} to the original component \bar{X}_0 . As mentioned above, this restriction is inseparable, and it is described by a so-called deformation datum ([26], Section 1.3).

In order to describe deformation data, we set some notation. Let \bar{Q}_i be the limit on \bar{X}_0 of the Q_i for $i \notin \mathbb{B}$, and set $\bar{Q}_i = \xi_i$ for $i \in \mathbb{B}$.

Definition 4.3. Let k be an algebraically closed field of characteristic p . A *deformation datum* is a pair (Z, ω) , where Z is a smooth projective curve over k together with a finite Galois cover $g : Z \rightarrow X = \mathbb{P}_k^1$, and ω is a meromorphic differential form on Z such that the following conditions hold.

- (a) Let H be the Galois group of $Z \rightarrow X$. Then

$$\beta^* \omega = \chi(\beta) \cdot \omega, \quad \text{for all } \beta \in H.$$

Here $\chi : H \rightarrow \mathbb{F}_p^\times$ is an injective character.

- (b) The differential form ω is either logarithmic, i.e. of the form $\omega = df/f$, or exact, i.e. of the form $d f$, for some meromorphic function f on Z .

Two deformation data (Z, ω_i) on a fixed Galois cover $Z \rightarrow X = \mathbb{P}_k^1$ are *isomorphic* if and only if they differ by a constant $\epsilon \in \mathbb{F}_p^\times$.

The motivation for the notion of isomorphism of deformation data comes from the correspondence between logarithmic (resp. exact) differential forms and μ_p -torsors (resp. α_p -torsors); see for example [25], Section 2.2. Note that the cover $Z \rightarrow X$ is necessarily cyclic, of degree dividing $p - 1$. In particular, the degree of this cover is prime to p .

To a G -Galois cover $g : Y \rightarrow \mathbb{P}^1$ with bad reduction, we may associate a deformation datum, as follows. Choose an irreducible component \bar{Y}_0 of \bar{Y} above the original component \bar{X}_0 . Since the restriction $\bar{g}_0 : \bar{Y}_0 \rightarrow \bar{X}_0$ is inseparable and $G \subset S_p$, it follows that the inertia group I of \bar{Y}_0 is cyclic of order p , i.e. a Sylow p -subgroup of G . Since the inertia group is normal in the decomposition group, the decomposition group G_0 of \bar{Y}_0 is a subgroup of $N_{S_p}(I) \simeq \mathbb{F}_p \rtimes_\chi \mathbb{F}_p^*$, where $\chi : \mathbb{F}_p^* \rightarrow \mathbb{F}_p^*$ is an injective character. It follows that the map \bar{g}_0 factors as $\bar{g}_0 : \bar{Y}_0 \rightarrow \bar{Z}_0 \rightarrow \bar{X}_0$, where $\bar{Y}_0 \rightarrow \bar{Z}_0$ is inseparable of degree p and $\bar{Z}_0 \rightarrow \bar{X}_0$ is separable. We conclude

that the Galois group $H_0 := \text{Gal}(\bar{Z}_0, \bar{X}_0)$ is a subgroup of $\mathbb{F}_p^* \simeq \mathbb{Z}/(p-1)\mathbb{Z}$. In particular, it follows that

$$(4.2) \quad G_0 \simeq I \rtimes_{\chi} H_0.$$

The inseparable map $\bar{Y}_0 \rightarrow \bar{Z}_0$ is characterized by a differential form ω on \bar{Z}_0 satisfying the properties of Definition 4.3; see [26], Section 1.3.2.

The differential form ω is logarithmic if $\bar{Y}_0 \rightarrow \bar{Z}_0$ is a μ_p -torsor and exact if this map is an α_p -torsor. A differential form is logarithmic if and only if it is fixed by the Cartier operator \mathcal{C} and exact if and only if it is killed by \mathcal{C} . (See for example [10], exercise 9.6, for the definition of the Cartier operator and an outline of these properties.) Wewers ([25], Lemma 2.8) shows that in the case of covers branched at $r = 3$ points the differential form is always logarithmic.

The deformation datum (Z, ω) associated with g satisfies the following compatibilities with the tail covers. See the discussion in Section 1.3 of [26], in particular Propositions 1.7 and 1.8 and Equation (2), for proofs of these statements. For $i \in \mathbb{B}$, we let h_i (resp. pm_i) be the conductor (resp. ramification index) of a wild ramification point of the corresponding tail cover of type C_i , as defined above. We put $\sigma_i = h_i/m_i$. We also use the convention $\sigma_i = 0$ for $i \notin \mathbb{B}$.

- (a) If C_i is the conjugacy class of a p -cycle, then \bar{Q}_i is unbranched in $\bar{Z}_0 \rightarrow \bar{X}_0$ and ω has a simple pole at all points of \bar{Z}_0 above \bar{Q}_i .
- (b) Otherwise, $\bar{Z}_0 \rightarrow \bar{X}_0$ is branched of order m_i at \bar{Q}_i , and ω has a zero of order $h_i - 1$ at the points of \bar{Z}_0 above \bar{Q}_i .
- (c) The map $\bar{Z}_0 \rightarrow \bar{X}_0$ is unbranched outside $\{\bar{Q}_i\}$. All poles and zeros of ω are above the \bar{Q}_i .
- (d) The invariants σ_i satisfy $\sum_{i \in \mathbb{B}} \sigma_i = r - 2$.

The set (σ_i) is called the *signature* of the deformation datum (Z, ω) .

Proposition 4.4. *Suppose that $r = 3, 4$. We fix rational numbers $(\sigma_1, \dots, \sigma_r)$ with $\sigma_i \in \frac{1}{p-1}\mathbb{Z}$ and $0 \leq \sigma_i < 1$, and $\sum_{i=1}^r \sigma_i = r - 2$. We fix points $\bar{Q}_1 = 0, \bar{Q}_2 = 1, \dots, \bar{Q}_r = \infty$ on $\bar{X}_0 \simeq \mathbb{P}_k^1$. Then there exists a deformation datum of signature (σ_i) , unique up to scaling. If further the \bar{Q}_i are general, the deformation datum is logarithmic and unique up to isomorphism.*

Note that the condition that the \bar{Q}_i are general is vacuously satisfied in the case that $r = 3$, since the automorphisms of \mathbb{P}_k^1 act 3-transitively.

Proof. In the case that $r = 3$, the result is proved in [26]. (The proof in that case is similar to the proof in the case that $r = 4$, which we give here.) Suppose that $r = 4$. We write $\lambda = \bar{Q}_3 \in \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ and $\sigma_i = a_i/(p-1)$. (If ω is the deformation datum associated with \bar{g} , then $a_i = h_i(p-1)/m_i$.) Without loss of generality, we may assume that $a_1 \leq a_2 \leq a_3 \leq a_4$.

It is shown in [3], Chapter 3, that a deformation datum of signature (σ_i) consists of a differential form ω on the curve \bar{Z}_0 defined as follows. Let \bar{Z}_0 be a connected component of the (normalization of the) projective curve with Kummer equation

$$(4.3) \quad z^{p-1} = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3}.$$

Projection onto the x -coordinate defines a cover $\bar{Z}_0 \rightarrow \bar{X}_0$ of degree

$$m := \frac{p-1}{\text{gcd}(p-1, a_1, a_2, a_3, a_4)}.$$

The differential form ω may be written as

$$(4.4) \quad \omega = \epsilon \frac{z \, dx}{x(x-1)(x-\lambda)} = \epsilon \frac{x^{p-a_1}(x-1)^{p-1-a_2}(x-\lambda)^{p-1-a_3} z^p \, dx}{x^p(x-1)^p(x-\lambda)^p} \frac{dx}{x},$$

where $\epsilon \in k^\times$ is a unit.

To show the existence of the deformation datum, it suffices to show that one may choose ϵ such that ω is either logarithmic or exact, or, equivalently, such that ω is either fixed or killed by the Cartier operator \mathcal{C} .

Write

$$F := x^{p-a_1}(x-1)^{p-1-a_2}(x-\lambda)^{p-1-a_3} = \sum_i c_i x^i$$

and

$$\tilde{F} = \sum_{i \equiv 0 \pmod{p}} c_i^{1/p} x^{i/p}.$$

It follows from standard properties of the Cartier operator ([22]), (4.4), and the assumption that $a_1 + a_2 + a_3 + a_4 = 2(p-1)$ that

$$\mathcal{C}\omega = \epsilon^{1/p} \frac{\tilde{F} \cdot z}{x(x-1)(x-\lambda)} \frac{dx}{x}.$$

Since $\deg_x F = p + a_4 < 2p$ and $c_0 = 0$, it follows that $\tilde{F} = c_p^{1/p} x$. We conclude that $\mathcal{C}\omega = c_p^{1/p} \epsilon^{(1-p)/p} \omega$. One computes that

$$(4.5) \quad \begin{aligned} c_p &= \sum_{i+j=a_4} \binom{p-1-a_2}{i} \binom{p-1-a_3}{j} \lambda^j \\ &= (-1)^{a_4} \sum_{j=\max(0, a_2+a_4-p+1)}^{\min(a_4, p-1-a_3)} \binom{p-1-a_2}{a_4-j} \binom{p-1-a_3}{j} \lambda^j. \end{aligned}$$

The assumptions that $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq p-1$ and $a_1 + a_2 + a_3 + a_4 = 2(p-1)$ imply that c_p is nonzero as a polynomial in λ . It follows that ω defines an exact differential form if and only if λ is a zero of the polynomial c_p . Therefore ω is exact for finitely many λ .

We assume that $c_p(\lambda) \neq 0$. Since k is algebraically closed, we may choose $\epsilon \in k^\times$ such that $\epsilon^{p-1} = c_p$. Then $\mathcal{C}\omega = \omega$, and ω defines a logarithmic deformation datum. It is easy to see that ω is unique, up to multiplication by an element of \mathbb{F}_p^\times . This implies that the deformation datum is unique up to isomorphism. \square

5. THE TAIL COVERS

In Section 4, we have seen that associated to a Galois cover with bad reduction is a set of (primitive) tail covers. In this section, we analyze the possible tail covers for conjugacy classes of cycle type $e \neq p$ and cycle type $e_1 e_2$ of S_p . Recall from Theorem 2.3 that these are exactly the conjugacy classes which occur in the 3-point covers obtained as degeneration of the pure-cycle 4-point covers.

The following lemma shows the existence of primitive tail covers for the cases of interest. Note that in this lemma, we discuss uniqueness only as mere covers, and not as tail covers. The question of multiplicity when counted as tail covers in the single-cycle case will be addressed in Lemma 5.3.

- Lemma 5.1.** (a) *Let $2 \leq e < p - 1$ be an integer. There exists a primitive tail cover $\varphi_e : T_e \rightarrow \mathbb{P}_k^1$ of ramification type e . Its Galois group is A_p if e is odd and S_p if e is even. The wild branch point of φ_e has inertia group of order $p(p-1)/\gcd(p-1, e-1) =: pm_e$. The conductor is $h_e := (p-e)/\gcd(p-1, e-1)$.*
- (b) *In the case that $e = p - 1$, there exists a primitive tail cover φ_e of ramification type e , with Galois group $\mathbb{F}_p \rtimes \mathbb{F}_p^*$. The cover is totally branched above the wild branch point and has conductor $h_{p-1} = 1$.*
- (c) *Let $2 \leq e_1 \leq e_2 \leq p - 1$ be integers with $e_1 + e_2 \leq p$. There is a primitive tail cover $\varphi_{e_1, e_2} : T_{e_1, e_2} \rightarrow \mathbb{P}_k^1$ of ramification type $e_1 - e_2$. The wild branch point of φ_{e_1, e_2} has inertia group of order $p(p-1)/\gcd(p-1, e_1 + e_2 - 2) =: pm_{e_1, e_2}$. The conductor is $h_{e_1, e_2} := (p + 1 - e_1 - e_2)/\gcd(p-1, e_1 + e_2 - 2)$.*

In all three cases, the tail cover is unique with the given ramification when considered as a mere cover.

Proof. Let $2 \leq e \leq p - 1$ be an integer. We define the primitive tail cover φ_e as the Galois closure of the degree- p cover $\bar{\varphi}_e : \bar{T}_e := \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by

$$(5.1) \quad y^p + y^e = x, \quad (x, y) \mapsto x.$$

One checks that this is the unique degree- p cover (considered as a mere cover) between projective lines with one wild branch point and the required tame ramification.

The decomposition group G_e of φ_e is contained in S_p . We note that the normalizer in S_p of a Sylow p -subgroup has trivial center. Therefore the inertia group I of a wild ramification point of φ_e is contained in $\mathbb{F}_p \rtimes_{\chi} \mathbb{F}_p^*$, where $\chi : \mathbb{F}_p^* \rightarrow \mathbb{F}_p^*$ is an injective character. Therefore it follows from [7], Proposition 2.2.(i), that $\gcd(h_e, m_e) = 1$. The statement on the wild ramification now follows directly from the Riemann–Hurwitz formula applied to φ_e (as in [26], Lemma 4.10). Suppose that e is odd. Then $m_e = (p-1)/\gcd(p-1, e-1)$ divides $(p-1)/2$. Therefore in this case both the tame and the wild ramification groups are contained in A_p . This implies that the Galois group G_e of φ_e is a subgroup of A_p .

To prove (a), we suppose that $e \neq p - 1$. We show that the Galois group G_e of φ_e is A_p or S_p . Suppose that this is not the case. Proposition 3.1 implies that $e = p - 2 = 2^s - 1$. Moreover, G_e is a subgroup of $\mathrm{P}\Gamma\mathrm{L}_2(2^s) \simeq \mathrm{P}\mathrm{S}\mathrm{L}_2(2^s) \rtimes \mathbb{Z}/s\mathbb{Z}$. Since p is an odd prime strictly dividing the order of $\mathrm{P}\mathrm{G}\mathrm{L}_2(2^s)$, the normalizer of a Sylow p -subgroup I is dihedral of order $2p$ ([13], Satz II.7.3). Since s is prime to p , we conclude that the normalizer of I in $\mathrm{P}\Gamma\mathrm{L}_2(2^s)$ has order $2ps$. On the other hand, the computation of the wild ramification shows that the inertia group $I(\eta)$ of the wild ramification point η has order $pm_e = p(p-1)/2$, and $I(\eta)$ is also contained in $N_{S_p}(I)$, so could only be contained in $\mathrm{P}\Gamma\mathrm{L}_2(2^s)$ if $(p-1)/2 \leq 2s$. Thus the only possibility is $p = 17 = 2^4 + 1$. Hence if $G_e \not\subseteq S_p, A_p$, then $e = 15$ and $p = 17$. However, in this last case one may check using Magma that the specialization $x = 1$ of the polynomial has Galois group A_{17} . As before, we conclude that $G_e \simeq A_{17}$.

Now suppose that $e = p - 1$. One sees that the Galois closure of $\bar{\varphi}_{p-1}$ is in this case the cover $\varphi_{p-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ obtained by dividing out $\mathbb{F}_p \rtimes \mathbb{F}_p^* \subset \mathrm{P}\mathrm{G}\mathrm{L}_2(p) = \mathrm{Aut}(\mathbb{P}^1)$. This proves (b).

Let e_1, e_2 be as in the statement of (c). As in the proof of (a), we define φ_{e_1, e_2} as the Galois closure of a non-Galois cover $\bar{\varphi}_{e_1, e_2} : \bar{T}_{e_1, e_2} \rightarrow \mathbb{P}^1$ of degree p . The

cover $\bar{\varphi}_{e_1, e_2}$, if it exists, is given by an equation

$$(5.2) \quad F(y) := y^{e_1}(y-1)^{e_2}\tilde{F}(y) = x, \quad (x, y) \mapsto x,$$

where $\tilde{F}(y) = \sum_{i=0}^{p-e_1-e_2} c_i y^i$ has degree $p-e_1-e_2$. We may assume that $c_{p-e_1-e_2} = 1$.

Note that $\bar{\varphi}_{e_1, e_2}$ is totally ramified at $y = \infty$. The ramification points $y \neq \infty$ correspond to the zeros of $F'(y)$. The condition that $\bar{\varphi}_{e_1, e_2}$ has exactly three ramification points $y = 0, 1, \infty$ therefore yields the condition

$$F'(y) = \gamma y^{e_1-1}(y-1)^{e_2-1}$$

for some nonzero constant γ . Write $F'(y) = y^{e_1-1}(y-1)^{e_2-1}G(y)$.

Explicitly taking the derivative in (5.2), one computes that the coefficient of y^i in G equals

$$c_{i-1}(e_1 + e_2 + i - 1) - c_i(e_1 + i), \quad i \neq 0.$$

We conclude that the coefficients c_i of \tilde{F} satisfy the recursion

$$(5.3) \quad c_i = c_{i-1} \frac{e_1 + e_2 + i - 1}{e_1 + i}, \quad i = 1, \dots, p - e_1 - e_2.$$

This implies that the c_i are uniquely determined by $c_{p-e_1-e_2} = 1$. Conversely, it follows that the polynomial F defined by these c_i has the required tame ramification. The statement on the wild ramification follows from the Riemann–Hurwitz formula, as in the proof of (a). \square

It remains to analyze the number of tail covers, up to isomorphism, and their automorphism groups. Due to the nature of our argument, we will only need to carry out this analysis for the tails of ramification type e . From Lemma 5.1, it already follows that the map $\varphi_C : T_C \rightarrow \mathbb{P}^1$ is unique, when we do not rigidify in relation to the monodromy group. However, part of the datum of a tail cover is an isomorphism $\alpha : \text{Gal}(T_C, \mathbb{P}^1) \xrightarrow{\sim} G_C$. For every $\tau \in \text{Aut}(G_C)$, the tuple $(\varphi_C, \tau \circ \alpha)$ also defines a tail cover, which is potentially nonequivalent under the notion of isomorphism of tail covers given in Definition 4.1. Modification by τ leaves the cover unchanged as a G_C -Galois cover if and only if $\tau \in \text{Inn}(G_C)$. However, the weaker notion of equivalence for tail covers implies that τ leaves the cover unchanged as a tail cover if and only if τ can be described as conjugation by an element of $N_{\text{Aut}(T_C)}(G_C)$. Thus, the number of distinct tail covers corresponding to a given mere cover is the order of the cokernel of the map

$$N_{\text{Aut}(T_C)}(G_C) \rightarrow \text{Aut}(G_C)$$

given by conjugation. This motivates the first part of the following definition. The other group of automorphisms of the tail T_C we define below is used in Section 6.

Definition 5.2. Denote by $\text{Aut}_{G_C}(\varphi_C)$ the kernel of the map $N_{\text{Aut}(T_C)}(G_C) \rightarrow \text{Aut}(G_C)$. Equivalently, $\text{Aut}_{G_C}(\varphi_C)$ is the set of G_C -equivariant automorphisms of T_C . Denote by $\text{Aut}_{G_C}^0(\varphi_C) \subset \text{Aut}_{G_C}(\varphi_C)$ the subset of those automorphisms that fix a chosen ramification point η .

We conclude that the number of tail covers corresponding to φ_C is

$$(5.4) \quad \frac{|\text{Aut}(G_C)| |\text{Aut}_{G_C}(\varphi_C)|}{|N_{\text{Aut}(T_C)}(G_C)|}.$$

We now simultaneously compute these automorphism groups and show that in the single-cycle case, we have a unique tail cover.

Lemma 5.3. *Let $2 \leq e \leq p - 1$ be an integer.*

- (a) *The group $\text{Aut}_{G_e}(\varphi_e)$ is cyclic of order $(p - e)/2$ if e is odd and $p - e$ if e is even.*
- (b) *The group $\text{Aut}_{G_e}^0(\varphi_e)$ is cyclic of order h_e .*
- (c) *There is a unique primitive tail cover of type e .*

Proof. First note that the definition of $\text{Aut}_{G_e}(\varphi_e)$ implies that any element induces an automorphism of any intermediate cover of φ_e , and in particular induces automorphisms of \bar{T}_e and \mathbb{P}^1 . Choose a primitive $(p - e)$ th root of unity $\zeta \in \bar{\mathbb{F}}_p$. Then $\mu(x, y) = (\zeta^e x, \zeta y)$ is an automorphism of \bar{T}_e . One checks that μ generates the group of automorphisms of \bar{T}_e which induce automorphisms of \mathbb{P}^1 under φ_e , and that furthermore T_e is Galois over $\mathbb{P}^1/\langle\mu\rangle$, so in particular every element of $\langle\mu\rangle$ lifts to an automorphism of T_e . Taking the quotient by the action of μ , we obtain a diagram

$$(5.5) \quad \begin{array}{ccc} \bar{T}_e & \longrightarrow & \bar{T}'_e = \bar{T}_e/\langle\mu\rangle \\ \bar{\varphi}_e \downarrow & & \downarrow \bar{\psi}_e \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1/\langle\mu\rangle. \end{array}$$

Since we know the ramification of the other three maps, one computes that the tame ramification of $\bar{\psi}_e$ has cycle type $e-(p - e)$. Let $\psi_e : T'_e \rightarrow \mathbb{P}^1$ be the Galois closure of $\bar{\psi}_e$.

We now specialize to the case that e is odd. Since $G_e = A_p$ does not contain an element of cycle type $e-(p - e)$, it follows that the Galois group G' of ψ_e is S_p . Therefore it follows by degree considerations that the cover $T_e \rightarrow T'_e$ is cyclic of degree $(p - e)/2$. Denote by Q the Galois group of the cover $T_e \rightarrow \mathbb{P}^1/\langle\mu\rangle$. This is a group of order $p!(p - e)/2$, which contains normal subgroups isomorphic to A_p and $\mathbb{Z}/\frac{p-e}{2}\mathbb{Z}$, respectively. It follows that $Q = \mathbb{Z}/\frac{p-e}{2}\mathbb{Z} \rtimes S_p$. Note that $\text{Aut}_{G_e}(\varphi_e)$ is necessarily a subgroup of Q . In fact, it is precisely the subgroup of Q which commutes with every element of $A_p \subseteq Q$. One checks that the semidirect product cannot be split, and that $\text{Aut}_{G_e}(\varphi_e)$ is precisely the normal subgroup $\mathbb{Z}/\frac{p-e}{2}\mathbb{Z}$, that is the Galois group of T_e over T'_e .

To compute $\text{Aut}_{G_e}^0(\varphi_e)$ we need to compute the order of the inertia group of a wild ramification point of T_e in the map $T_e \rightarrow T'_e$. Write h'_e (resp. m'_e) for the conductor (resp. the order) of the tame ramification of a wild ramification point of ψ_e . A generalization of Abhyankar’s Lemma ([17], Lemma 4.1) implies that $h'_e/m'_e = |\mu|h_e/m_e$, where $|\mu| = (p - e)$ is the order of μ acting on \mathbb{P}^1_k . We conclude that $h'_e/m'_e = 1/(p - 1)$. Since m'_e divides $p - 1$, it follows that $h'_e = 1$ and $m'_e = p - 1$. We conclude that a wild ramification point of ψ_e has inertia group of order $pm'_e = p(p - 1) = pm_e \gcd(p - 1, e - 1)$.

Since we know the orders of the inertia groups of three of the four maps, we conclude that $\text{Aut}_{G_e}^0(\varphi_e)$ has order $h_e = (p - e)/\gcd(p - 1, e - 1)$. This proves (a) and (b) in the case that e is odd.

For (c), we simply observe that since $Q \subset N_{\text{Aut}(T_e)}(G_e)$, so since $\text{Aut}(A_p) = S_p$, we have

$$\frac{|\text{Aut}(G_e)||\text{Aut}_{G_e}(\varphi_e)|}{|N_{\text{Aut}(T_e)}(G_e)|} \leq \frac{p!\frac{p-e}{2}}{|Q|} = 1,$$

so the tail cover is unique, as desired.

We now treat the case that e is even. For (a), if $e < p - 1$, the Galois group of $\bar{\psi}_e$ is equal to the Galois group of $\bar{\varphi}_e$, which is isomorphic to S_p . We conclude that the degree of $T_e \rightarrow T'_e$ is $p - e$ in this case, and the group Q defined as above is a direct product $\mathbb{Z}/(p - e)\mathbb{Z} \times S_p$. Similar to the case that e is odd, we conclude that $\text{Aut}_{G_e}(\varphi_e)$ (resp. $\text{Aut}_{G_e}^0(\varphi_e)$) is cyclic of order $p - e$ (resp. h_e) in this case, as desired. On the other hand, if $e = p - 1$, we have that $p - e = 1$; hence μ is trivial, and we again conclude that (a) and (b) hold.

Finally, (c) is trivial for e even: if $e < p - 1$, the Galois group of φ_e is S_p and $\text{Aut}(S_p) = S_p$. Therefore there is a unique tail cover in this case. The same conclusion holds in the case that $e = p - 1$, since $G_{p-1} \simeq \mathbb{F}_p \rtimes_{\chi} \mathbb{F}_p^*$ and $\text{Aut}(G_{p-1}) = G_{p-1}$. The statement of the lemma follows. \square

Remark 5.4. In the case of e_1 - e_2 tail covers, there may be more than one structure on a given mere cover. However, we will not need to know this number for our argument.

6. REDUCTION OF 3-POINT COVERS

In this section, we (almost) compute the number of 3-point covers with bad reduction for ramification types $(p; e_1 - e_2, e_3, e_4)$. More precisely, we compute this number in the case that not both e_3 and $e_1 + e_2$ are even. In the remaining case, we only compute this number up to a factor 2, which is good enough for our purposes. The precise statement can be found in Corollary 6.9. Although we restrict to types of the above form, our strategy applies somewhat more generally.

The computation of the number of covers with bad reduction proceeds in two steps. First, we count the number of stable G -maps (Section 4) of a given type, i.e. the number of degenerate covers that may occur as the stable reduction of a cover with bad reduction. The second step consists in counting the number of covers which reduce to a given stable G -map. It turns out that in our situation this number only depends on the type and not on the stable G -map chosen.

We fix a type $\tau = (p; e_1 - e_2, e_3, e_4)$ satisfying the genus-0 condition $\sum_i e_i = 2p + 2$. We allow e_3 or e_4 to be p , although this is not the case that ultimately interests us; see below for an explanation. We do however assume throughout that we are not in the exceptional case $\tau = (5; 2-2, 4, 4)$.

In Lemma 2.2, we have counted non-Galois covers, but in this section, we deal with Galois covers. Let $G = G(\tau)$ be the Galois group of a cover of type τ . This group is well defined and either A_p or S_p , by Corollary 3.2. We write $\gamma(\tau)$ for the quotient of the number of Galois covers of type τ by the Hurwitz number $h(\tau)$ and call it the *Galois factor*. By Lemma 3.3, it follows that $\gamma(\tau)$ is 2 if G is A_p and 1 if it is S_p . The Galois factor $\gamma(\tau)$ will drop out from the formulas as soon as we pass back to the non-Galois situation in Section 7.

The following lemma counts the number of stable G -maps of type τ .

Lemma 6.1. *Let $\tau = (p; e_1 - e_2, e_3, e_4)$ be as above. The number of stable G -maps of type τ equals the number N_{e_1, e_2} of primitive tail covers of ramification type $e_1 - e_2$.*

Note that we count tail covers here up to the notion of isomorphism defined in Definition 4.1.

Proof. Recall that a stable G -map consists of a primitive tail cover for each conjugacy class C_i with $C_i \neq p$, together with a deformation datum, satisfying certain

compatibility conditions. Lemma 5.1 states that primitive tail covers φ_i of type C_i exist for i such that $C_i \neq p$. Moreover, by Proposition 4.4 we have a unique deformation datum. (Here we use that $r = 3$; therefore there is no choice for the \bar{Q}_i .) In particular, we know there exists at least one stable G -map φ of type τ . Lemma 5.3 states that there is a unique primitive tail cover for those $C_i \neq p$ which are single cycles. \square

As a second step, we need to count the number of covers in characteristic zero which reduce to a given stable G -map φ . We denote by $\tilde{L}(\varphi)$ the set of such lifts of φ and use a result of Wewers ([26]; see Lemma 6.4 below), which gives a formula for $|\tilde{L}(\varphi)|$. In particular, he shows that this number is always nonzero (Wewers ([26], Theorem 3). The main idea of Wewers' formula is that the number $|\tilde{L}(\varphi)|$ of lifts of the stable G -map φ may be expressed in terms of certain groups of automorphisms of φ . In particular, this number can be expressed in terms of purely characteristic- p information. In order to state the formula for $|\tilde{L}(\varphi)|$, we need to introduce another invariant.

Let $\varphi : \bar{Y} \rightarrow \bar{X}$ be a stable G -map of type τ . We denote by $\text{Aut}_G(\varphi)$ the set of G -equivariant automorphisms of \bar{Y} . Note that every $\gamma \in \text{Aut}_G(\varphi)$ acts on \bar{X} as well. This action stabilizes the original component \bar{X}_0 (Lemma 4.2).

Definition 6.2. Let $\text{Aut}_G^0(\varphi)$ be the group of G -equivariant automorphisms of \bar{Y} which induce the identity on the original component \bar{X}_0 .

The formula for $|\tilde{L}(\varphi)|$ given below is not stated in terms of the group $\text{Aut}_G^0(\varphi)$ itself, but in terms of certain restrictions of this group to the components of \bar{Y} . Though more difficult to state, these groups of automorphisms are more accessible in practice.

Let \bar{Y}_0 be an irreducible component of \bar{Y} above \bar{X}_0 whose inertia group is the fixed Sylow p -subgroup I of G . As in (4.2), we write $G_0 = I \rtimes_{\chi} H_0 \subset \mathbb{F}_p \rtimes_{\chi} \mathbb{F}_p^*$ for the decomposition group of \bar{Y}_0 . Wewers ([26], proof of Lemma 2.17) shows that the restriction $\gamma_0 := \gamma|_{\bar{Y}_0}$ of an element $\gamma \in \text{Aut}_G^0(\varphi)$ centralizes H_0 and normalizes I , i.e.

$$\gamma_0 \in C_{N_G(I)}(H_0).$$

Since $\bar{Y}|_{\bar{X}_0} = \text{Ind}_{G_0}^G \bar{Y}_0$ and γ is G -equivariant, it follows that the restriction of γ to $\varphi^{-1}(\bar{X}_0)$ is uniquely determined by $\gamma_0 \in C_{N_G(I)}(H_0)$. We denote the corresponding automorphism of $\varphi^{-1}(\bar{X}_0)$ also by γ_0 .

Definition 6.3. We denote by $n'(\varphi)$ the order of the subgroup consisting of those $\gamma_0 \in C_{N_G(I)}(H_0)$ such that there exists a $\gamma \in \text{Aut}_G^0(\varphi)$ with $\gamma|_{\bar{Y}_0} = \gamma_0$.

In the case that $G = A_p, S_p$, we remark that $N_G(I) \simeq \mathbb{F}_p \rtimes_{\chi} \mathbb{F}_p^*$. In particular, $C_{N_G(I)}(H_0) \subseteq \mathbb{F}_p^*$. We conclude that in this case the integer $n'(\varphi)$ divides $p - 1$.

We now state Wewers' formula for the number of lifts of the given stable G -map φ . This number is expressed in terms of $n'(\varphi)$, the conductor h_{C_i} of the tail φ_{C_i} (Lemma 5.1) and the groups $\text{Aut}_{G_{C_i}}^0(\varphi_{C_i})$ of tail automorphisms (Definition 5.2).

Lemma 6.4. *Let φ be a stable G -map of type $\tau = (p; e_1 - e_2, e_3, e_4)$, as above. We have that*

$$(6.1) \quad |\tilde{L}(\varphi)| = \frac{p-1}{n'(\varphi)} \left(\frac{h_{e_1, e_2}}{|\text{Aut}_{G_{e_1, e_2}}^0(\varphi_{e_1, e_2})|} \right).$$

Proof. This is a special case of [26], Corollary 4.8. The formula in *loc. cit.* has a contribution $h_{C_i}/|\text{Aut}_{G_{C_i}}^0(\varphi_{C_i})|$ for every primitive tail $i \in \mathbb{B}$. Lemma 5.3 (b) implies that the contribution of each single-cycle tail cancels out. \square

To compute the number of curves with bad reduction, we therefore need to compute the number $n'(\varphi)$ defined above. Wewers ([26, Lemma 2.17]) gives a concrete description of the group $\text{Aut}_G^0(\varphi)$, which could be used to calculate $n'(\varphi)$. However, there is a mistake in this part of the paper; therefore we do not use Wewers' description. Proposition 6.5 below is a replacement in our situation for the result of Wewers. For a more general version, we refer to the manuscript [21].

Recall that $\varphi_i : \bar{Y}_i \rightarrow \bar{X}_i$ for $i \in \mathbb{B}$ denotes a tail cover of φ and that ξ_i is the node connecting \bar{X}_0 to \bar{X}_i . For convenience, for $i \in \mathbb{B}$ we also introduce the notation $\widetilde{\text{Aut}}_{G_i}(\varphi_i)$ for the group of G -equivariant automorphisms of the induced tail cover $\text{Ind}_{G_i}^G(\varphi_i)$ (compare to Definition 5.2).

Suppose we are given $\gamma_0 \in C_{N_G(I)}(H_0)$, which we consider as acting on $\varphi^{-1}(\bar{X}_0)$. Then there exists an element $\widetilde{\gamma} \in \widetilde{\text{Aut}}_G^0(\varphi)$ which induces γ_0 if and only if for each $i \in \mathbb{B}$, there exists a $\gamma_i \in \widetilde{\text{Aut}}_{G_i}(\varphi_i)$ whose action on $\varphi_i^{-1}(\xi_i)$ is compatible with γ_0 . Since $n'(\varphi)$ is the cardinality of those γ_0 that extend, the following proposition gives a concrete way of analyzing this number. Moreover, the proposition shows that this number, and hence also $|\widetilde{L}(\varphi)|$, may be analyzed tail by tail.

Proposition 6.5. *Suppose $G = S_p$ or A_p , and we have a stable G -map $\varphi : \bar{Y} \rightarrow \bar{X}$. Then:*

- (a) *For $i \in \mathbb{B}$, the G -equivariant automorphisms of $\varphi^{-1}(\xi_i)$ form a cyclic group.*
- (b) *Given an element $\gamma_0 \in C_{N_G(I)}(H_0)$ and $i \in \mathbb{B}$, there exists $\gamma_i \in \widetilde{\text{Aut}}_{G_i}(\varphi_i)$ agreeing with the action of γ_0 on $\varphi^{-1}(\xi_i)$ if and only if there exists $\gamma'_i \in \widetilde{\text{Aut}}_{G_i}(\varphi_i)$ having the same orbit length on $\varphi^{-1}(\xi_i)$ as γ_0 has.*

Proof. For (a), we choose a point $\tilde{\xi}_i$ above ξ_i lying on the chosen component \bar{Y}_0 . One checks that the action of a G -equivariant automorphism γ on $\varphi^{-1}(\xi_i)$ is determined by where it sends $\tilde{\xi}_i$.

We note that the natural G -action on \bar{Y} identifies G with a subgroup of $\text{Aut}_G^0(\varphi)$. Using this identification, we may choose an element $g \in G$ such that $g(\tilde{\xi}_i) = \gamma(\tilde{\xi}_i)$. Note that $\gamma \neq g$ as an automorphism of \bar{Y} ; in fact, if γ, γ' are determined by g, g' , the composition law is that $\gamma \circ \gamma'$ corresponds to $g'g$.

We claim that the group of those $g \in G$ corresponding to a $\gamma \in \text{Aut}_G(\bar{Y})$ is cyclic. Part (a) follows from this claim.

A given $g \in G$ yields a choice of γ if and only if we have equality of stabilizers $G_{\tilde{\xi}_i} = G_{g(\tilde{\xi}_i)}$. Now, $G_{\tilde{\xi}_i}$ is necessarily contained in G_0 , since $\tilde{\xi}_i$ lies on \bar{Y}_0 . Since $I \subseteq G_{\tilde{\xi}_i}$, the condition $G_{\tilde{\xi}_i} = G_{g(\tilde{\xi}_i)}$ implies that $gIg^{-1} \subseteq G_0$. Since $I \triangleleft G_0$, we conclude that $g \in N_G(I)$. Since I fixes $\tilde{\xi}_i$, the choices of G may be taken modulo I , so we conclude that they lie in $N_G(I)/I$. Finally, since we assume that $G = S_p$ or A_p , the group $N_G(I)/I$ is cyclic: it is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$ if $G = S_p$ and to $\mathbb{Z}/(\frac{p-1}{2})\mathbb{Z}$ if $G = A_p$. This proves (a).

To prove (b), we recall that $\gamma_0 \in C_{N_G(I)}(H_0)$, and $C_{N_G(I)}(H_0)$ is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$ if $G = S_p$ (resp. $\mathbb{Z}/\frac{p-1}{2}\mathbb{Z}$ if $G = A_p$). Therefore the actions of both γ_0 and γ'_i on $\varphi^{-1}(\xi_i)$ lie in the same cyclic group. Choosing γ_i as an appropriate power of γ'_i satisfies the requirements of (b). \square

Corollary 6.6. *The cardinality of $\tilde{L}(\varphi)$ only depends on the ramification type τ and not on the choice of the stable G -map φ .*

Proof. We recall that $\varphi_{e_1-e_2}$ is unique as a mere cover (Lemma 5.1). Since the definition of $\text{Aut}_{G_{e_1-e_2}}^0(\varphi_{e_1-e_2})$ is independent of the G -structure, this group only depends on the conjugacy class e_1-e_2 . Therefore to prove the corollary, it suffices to show that $n'(\varphi)$ only depends on τ .

We recall that the stable G -map is almost determined by τ : Proposition 4.4 implies that the deformation datum is unique, and Lemma 5.1 implies that the tail covers are unique as mere covers. The definition of the integer $n'(\varphi)$ (Definition 6.3) implies that $n'(\varphi)$ is the cardinality of the set of $\gamma_0 \in C_{N_G(I)}(H_0)$ such that there exist γ_i for $i \in \mathbb{B}$ satisfying the condition of Proposition 6.5 (b). Obviously, this condition does not depend on the Galois structure of the tail covers. The corollary follows. \square

In what follows we write

$$(6.2) \quad \tilde{L}(\tau) = |\tilde{L}(\varphi)|$$

for the cardinality of the set of lifts of any stable G -map of type τ .

The difficulty we face in using Wewers' formula (6.1) directly is that we do not know the Galois group $G_{e_1-e_2}$ of the e_1-e_2 tail cover. This prevents us from directly computing $\tilde{L}(\tau)$, since we cannot verify the condition of Proposition 6.5.

We avoid this problem by using the following trick. We first consider covers of a modified type $\tau^* = (p; e_1-e_2, \varepsilon, p)$, with $\varepsilon = p + 2 - e_1 - e_2$. The covers of type τ^* all have bad reduction; therefore Wewers' formula (Lemma 6.4) yields a formula for $\tilde{L}(\tau^*)$. We then relate $\tilde{L}(\tau)$ and $\tilde{L}(\tau^*)$ (Proposition 6.8).

As above, we denote by $N_{e_1-e_2}$ the number of e_1-e_2 tails. Recall that $N_{e_1-e_2} \cdot \tilde{L}(\tau^*)$ is the number of Galois covers of type τ^* with bad reduction. (This follows from Lemma 6.1 and the definition of $\tilde{L}(\tau^*)$.) The following lemma expresses this number in terms of e_1, e_2 and the Galois factor $\gamma(\tau^*)$.

Lemma 6.7. *Let $\tau^* = (p; e_1-e_2, \varepsilon, p)$ be the modified type as above. Then*

$$N_{e_1-e_2} \cdot \tilde{L}(\tau^*) = \gamma(\tau^*)h(\tau^*) = \begin{cases} \gamma(\tau^*)(p + 1 - e_1 - e_2) & \text{if } e_1 \neq e_2, \\ \gamma(\tau^*)(p + 1 - e_1 - e_2)/2 & \text{otherwise.} \end{cases}$$

Proof. Lemma 2.2 implies that the Hurwitz number $h(\tau^*)$ equals $(p + 1 - e_1 - e_2)/2$ if $e_1 = e_2$ and $(p + 1 - e_1 - e_2)$ otherwise. All covers of type τ^* have bad reduction ([5]). (The reason is that p divides one of the ramification indices. The statement follows from our definition of the stable model; we assume that branch points do not coalesce modulo p .) Therefore the number $h(\tau^*)\gamma(\tau^*)$ of Galois covers equals the product of the number N_{e_1, e_2} of stable G -maps (Lemma 6.1) and the number $\tilde{L}(\tau^*)$ of lifts of a given stable G -map. \square

We now relate $\tilde{L}(\tau)$ and $\tilde{L}(\tau^*)$. A problem is that the Galois groups of covers with types τ and τ^* are not equal. By Corollary 3.2, the Galois group of a cover of type τ^* satisfies

$$G(\tau^*) = \begin{cases} A_p & \text{if } e_1 + e_2 \text{ is even, and} \\ S_p & \text{otherwise.} \end{cases}$$

In particular, we see that $G(\tau) = G(\tau^*)$ unless $e_1 + e_2$ and e_3 are both even. In this case we have that $G(\tau) = S_p$ and $G(\tau^*) = A_p$.

Proposition 6.8. *Let $\tau = (p; e_1-e_2, e_3, e_4)$ be a type satisfying the genus-0 condition, and let $\tau^* = (p; e_1-e_2, \varepsilon, p)$ be the corresponding modified type. If $G(\tau) = G(\tau^*)$ we have $\tilde{L}(\tau) = \tilde{L}(\tau^*)$. Otherwise, $\tilde{L}(\tau) \in \{\tilde{L}(\tau^*), \tilde{L}(\tau^*)/2\}$.*

Proof. The proof of Corollary 6.6 also implies that $n'(\varphi)$ only depends on the type. Therefore the integers $n'(\tau)$ and $n'(\tau^*)$ are well defined. This implies by Lemma 6.4 that

$$\tilde{L}(\tau) = \tilde{L}(\tau^*) \frac{n'(\tau^*)}{n'(\tau)}.$$

Therefore, to prove the proposition, it suffices to relate the numbers $n'(\tau)$ and $n'(\tau^*)$.

Let γ_0 be a generator of $C_{N_G(I)}(H_0)$. We ask which powers of γ_0 extend to an element of $\text{Aut}_G^0(\varphi)$ and we analyze this question tail by tail. Fix a tail \bar{X}_i , and suppose that it is a single-cycle tail of ramification type $e := e_i$. We claim that γ_0 extends to the single-cycle tail \bar{X}_i , i.e. that there exists an element γ_i satisfying the condition of Proposition 6.5.

First suppose that $e < p - 1$ is even. Thus $G = G_i = S_p$ (Lemma 5.1 (a)), and $\widetilde{\text{Aut}}_{G_i}(\varphi_i) = \text{Aut}_{G_i}(\varphi_i)$. Now, γ_0 acts on the fiber $\varphi^{-1}(\xi_i)$ with orbit length $(p - 1)/m_e = \text{gcd}(p - 1, p - e)$. On the other hand, by Lemma 5.1 we have that $h_e = (p - e)/\text{gcd}(p - 1, e - 1)$. Lemma 5.3 implies that if $\gamma_i \in \text{Aut}_{G_i}(\varphi_i)$ is a generator, then the order of γ_i is $p - e$, and also that $\text{Aut}_{G_i}^0(\varphi_i)$ has order h_e . We conclude that an orbit of γ_i has length $\text{gcd}(p - 1, e - 1) = \text{gcd}(p - 1, p - e)$, and thus by Proposition 6.5 that γ_0 extends to \bar{X}_i , as claimed.

The next case is that e is odd, and $G_i = A_p$. This proceeds exactly as before, except that both orbits in question have length $\text{gcd}(p - 1, p - e)/2$.

Now, suppose that e is odd, but $G_i = S_p$. Then the orbit length of γ_0 is $\text{gcd}(p - 1, p - e)$. In this case, $\widetilde{\text{Aut}}_{G_i}(\varphi_i)$ is equal to the G_i -equivariant automorphisms of $\text{Ind}_{A_p}^{S_p}(\varphi_i)$. This group contains induced copies of the G_i -equivariant automorphisms of φ_i , so in particular we know that we have elements of orbit length $\text{gcd}(p - 1, e - 1)/2$. The group $\widetilde{\text{Aut}}_{G_i}(\varphi_i)$ also contains a G_i -equivariant automorphism exchanging the two copies of φ_i , and whose square is the generator of the group $\text{Aut}_{G_i}(\varphi_i)$ of A_p -equivariant automorphisms of φ_i . (One may think of this automorphism as coming from the automorphism constructed in the proof of Lemma 5.3 inducing the isomorphism between the two different A_p -structures on the tail cover. The fact that there are two A_p -structures also follows from Lemma 3.3.) We thus have an element of $\widetilde{\text{Aut}}_{G_i}(\varphi_i)$ of orbit length $\text{gcd}(p - 1, e - 1)$, and γ_0 extends to the tail in this case as well.

Finally, if $e = p - 1$, then $m_i = p - 1$ and thus γ_0 acts as the identity on the fiber of ξ_i . The claim is trivially satisfied in this case.

It follows that extending γ_0 to the single-cycle tails imposes no condition on $n'(\tau)$ when $e < p$, and of course we do not have tails in the case that $e = p$. Therefore the only nontrivial condition imposed in extending γ_0 is the extension to the e_1-e_2 -tail.

In the case that $G(\tau) = G(\tau^*)$ we conclude the desired statement from Proposition 6.5, since the orbit lengths in question are clearly the same in both cases.

Suppose that $G(\tau) \neq G(\tau^*)$. This happens if and only if both $e_1 + e_2$ and e_3 are even. In this case we have that $G(\tau) = S_p$ and $G(\tau^*) = A_p$. It follows from

the above that the only conditions imposed on either $n'(\tau)$ or $n'(\tau^*)$ come from the e_1 - e_2 -tail. Since the orbit of γ_0 is twice as long in the case of τ , the answers can differ by at most a factor of 2 in this case, as desired. \square

The following corollary translates Proposition 6.8 into an estimate for the number of Galois covers of type $\tau = (p; e_1-e_2, e_3, e_4)$ with bad reduction. Theorem 1.2 is a special case.

Corollary 6.9. *Let $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p$ be integers with $\sum_i e_i = 2p + 2$ and $e_1 + e_2 \leq p$, and let $\tau = (p; e_1-e_2, e_3, e_4)$ be a ramification type with $\tau \neq (5; 2-2, 4, 4)$. The number of mere covers of type τ with bad reduction to characteristic p is equal to*

$$\begin{cases} \delta(\tau)(p + 1 - e_1 - e_2) & \text{if } e_1 \neq e_2, \\ \delta(\tau)(p + 1 - e_1 - e_2)/2 & \text{if } e_1 = e_2, \end{cases}$$

where $\delta(\tau) \in \{1, 2\}$, and $\delta = 1$ unless $e_1 + e_2$ and e_3 are both even.

Proof. We recall that the number of Galois covers of type τ with bad reduction is equal to

$$N_{e_1-e_2} \cdot \tilde{L}(\tau).$$

This follows from Lemma 6.1 and the definition of $\tilde{L}(\tau)$. Lemma 6.7 implies that

$$N_{e_1-e_2} \cdot \tilde{L}(\tau^*) = \begin{cases} \gamma(\tau^*)(p + 1 - e_1 - e_2) & \text{if } e_1 \neq e_2, \\ \gamma(\tau^*)(p + 1 - e_1 - e_2)/2 & \text{if } e_1 = e_2. \end{cases}$$

The definition of the Galois factor $\gamma(\tau)$, together with Proposition 6.8, implies that the number of mere covers of type τ with bad reduction is

$$\begin{cases} \frac{\gamma(\tau^*)}{\gamma(\tau)} \frac{\tilde{L}(\tau)}{\tilde{L}(\tau^*)} (p + 1 - e_1 - e_2) & \text{if } e_1 \neq e_2, \\ \frac{\gamma(\tau^*)}{\gamma(\tau)} \frac{\tilde{L}(\tau)}{\tilde{L}(\tau^*)} (p + 1 - e_1 - e_2)/2 & \text{if } e_1 = e_2. \end{cases}$$

Proposition 6.8 implies that $\tilde{L}(\tau)/\tilde{L}(\tau^*) \in \{1, 1/2\}$ and is equal to 1 unless $e_1 + e_2, e_3, e_4$ are all even. Moreover, if $\tilde{L}(\tau) \neq \tilde{L}(\tau^*)$, then $\gamma(\tau^*)/\gamma(\tau) = 2$. The statement of the corollary follows from this. \square

Remark 6.10. Similar to the proof of Corollary 6.9, one may show that every genus-0 three-point cover of type $(p; e_1, e_2, e_3)$ has bad reduction. We do not include this proof here, as a proof of this result using linear series already occurs in [18], Theorem 4.2.

7. REDUCTION OF ADMISSIBLE COVERS

In this section, we return to the case of non-Galois covers and use the results of Section 6 to compute the number of “admissible covers with good reduction”. We start by defining what we mean by this. As always, we fix a type $(p; e_1, e_2, e_3, e_4)$ with $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$ satisfying the genus-0 condition $\sum_i e_i = 2p + 2$. As in Section 2, we consider admissible degenerations of type $(p; e_1, e_2, *, e_3, e_4)$, which means that $Q_3 = \lambda \equiv Q_4 = \infty \pmod{p}$. Recall from Section 2 that in positive characteristic not every smooth cover degenerates to an admissible cover, as a degeneration might become inseparable. The number of admissible covers

(even counted with multiplicity) is still bounded by the number of smooth covers, but equality need not hold.

Definition 7.1. We define $h_p^{\text{adm}}(p; e_1, e_2, *, e_3, e_4)$ as the number of admissible covers (see the discussion preceding Theorem 2.3) of type $(p; e_1, e_2, *, e_3, e_4)$, counted with multiplicity, over an algebraically closed field of characteristic p .

The following proposition is the main result of this section.

Proposition 7.2. *The assumptions on the type $\tau = (p; e_1, e_2, e_3, e_4)$ are as above. Then*

$$h_p^{\text{adm}}(p; e_1, e_2, *, e_3, e_4) > h(p; e_1, e_2, e_3, e_4) - 2p$$

and

$$h_p^{\text{adm}}(p; e_1, e_2, *, e_3, e_4) = h(p; e_1, e_2, e_3, e_4) - p$$

unless $e_1 + e_2$ and e_3 are both even.

Proof. We begin by noting that in the case $\tau = (5; 2, 2, 4, 4)$ corresponding to the exceptional case of Corollary 3.2, the assertion of the proposition is automatic since by Theorem 2.1 (b) we have $h(5; 2, 2, 4, 4) = 8 < 10$. We may therefore assume that $\tau \neq (5; 2, 2, 4, 4)$.

We use the description of the admissible covers in characteristic zero (Theorem 2.3) and the results of Section 6 to estimate the number of admissible covers with good reduction to characteristic p , i.e. that remain separable.

We first consider the pure-cycle case, i.e. the case of Theorem 2.3 (a). Let m be an integer satisfying the conditions of *loc. cit.* We write $f_0 : V_0 \rightarrow X_0$ for the corresponding admissible cover. Recall from Section 2 that X_0 consists of two projective lines X_0^1, X_0^2 intersecting in one point. The admissible cover has type $(p; e_1, e_2, *, e_3, e_4)$, which means the component covers are of types $(p; e_1, e_2, m)$ and $(p; m, e_3, e_4)$. Since the product of three cycles cannot be trivial unless they all have overlapping support, we conclude that all the ramification of the component cover over X_0^i occurs in a single irreducible component, which we label Y_0^i . The remaining components are disjoint unions of \mathbb{P}^1 which map isomorphically to X_0^i . If $f_0^i : Y_0^i \rightarrow X_0^i$ are the restricted covers, we then have that f_0^1 has type $(d_1; e_1, e_2, m)$ where $d_1 = \frac{e_1 + e_2 + m - 1}{2}$, and f_0^2 has type $(d_2; m, e_3, e_4)$, where $d_2 = \frac{m + e_3 + e_4 - 1}{2}$. The admissible cover f_0 has good reduction to characteristic p if and only if both three-point covers f_0^i have good reduction.

It is shown in [18], Theorem 4.2, that a genus-0 three-point cover of type $(d; a, b, c)$ with $a, b, c < p$ has good reduction to characteristic p if and only if its degree d is strictly less than p . Since the degree d_2 of the cover f_0^2 is always at least as large as the other degree d_1 , it is enough to calculate when $d_2 < p$. The Riemann–Hurwitz formula implies that $d_2 = (m + e_3 + e_4 - 1)/2$. Therefore the condition $d_2 < p$ is equivalent to the inequality

$$e_3 + e_4 + m \leq 2p - 1.$$

Since we assumed the existence of an admissible cover with ρ an m -cycle, it follows from Theorem 2.3 (a) that $m \leq 2d + 1 - e_3 - e_4 = 2p + 1 - e_3 - e_4$. We find that $d_2 < p$ unless $m = 2p + 1 - e_3 - e_4$. We also note that the lower bound for m is always less than or equal to the upper bound, which is $2p + 1 - e_3 - e_4$. We thus conclude that there is a single admissible cover of this type with bad reduction, which has multiplicity $m = 2p + 1 - e_3 - e_4$.

We now consider the case of an admissible cover with ρ an e_1 - e_2 -cycle (Theorem 2.3 (b)). Let $f_0 : V_0 \rightarrow X_0$ be such an admissible cover in characteristic 0, as above. In particular, the restriction f_0^1 (resp. f_0^2) has type $(d_1; e_1, e_2, e_1 - e_2)$ (resp. $(d_2; e_1 - e_2, e_3, e_4)$). We write \tilde{f}_0 for the Galois closure of f_0 , and \tilde{f}_0^i for the corresponding restrictions. Let G^i be the Galois group of \tilde{f}_0^i . The assumptions on the e_i imply that p does not divide the order of the Galois group of \tilde{f}_0^1 ; therefore \tilde{f}_0^1 has good reduction to characteristic p . Moreover, the cover \tilde{f}_0^1 is uniquely determined by the triple (ρ^{-1}, g_3, g_4) . If $e_1 \neq e_2$, the gluing is likewise uniquely determined, while if $e_1 = e_2$ there are exactly 2 possibilities for the tuple (g_1, g_2, g_3, g_4) for a given triple (ρ^{-1}, g_3, g_4) . Therefore to count the number of admissible covers with bad reduction in this case, it suffices to consider the reduction behavior of the cover $\tilde{f}_0^2 : Y_0^2 \rightarrow X_0^2$.

Corollary 6.9 implies that whether or not e_1 equals e_2 , the number of admissible covers with bad reduction in the 2-cycle case is equal to $(p + 1 - e_1 - e_2)$ unless $e_1 + e_2$ and e_3 are both even, and bounded from above by $2(p + 1 - e_1 - e_2)$ always. We conclude using Theorem 2.3 that the total number of admissible covers with bad reduction counted with multiplicity is less than or equal to

$$(2p + 1 - e_3 - e_4) + 2(p + 1 - e_1 - e_2) = p + (p + 1 - e_1 - e_2) < 2p,$$

and equal to

$$(2p + 1 - e_3 - e_4) + (p + 1 - e_1 - e_2) = p$$

unless $e_1 + e_2$ and e_3 are both even. The proposition follows. \square

Remark 7.3. Theorem 4.2 of [18] does not need the assumption $d = p$. Therefore the proof of Proposition 7.2 in the single-cycle case shows the following stronger result. Let $(d; e_1, e_2, e_3, e_4)$ be a genus-0 type with $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$. Then the number of admissible covers with a single ramified point over the node and bad reduction to characteristic p is

$$(d - p + 1)(d + p + 1 - e_3 - e_4)$$

when either $d + 1 \geq e_2 + e_3$ or $d + 1 - e_1 < p$. Otherwise, all admissible covers have bad reduction.

8. PROOF OF THE MAIN RESULT

In this section, we count the number of mere covers with ramification type $(p; e_1, e_2, e_3, e_4)$ and bad reduction in the case that the branch points are generic. Equivalently, we compute the p -Hurwitz number $h_p(p; e_1, e_2, e_3, e_4)$.

Suppose that $r = 4$ and fix a genus-0 type $\tau = (p; e_1, e_2, e_3, e_4)$ with $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p$. We let $g : Y \rightarrow X = \mathbb{P}_K^1$ be a Galois cover of type τ defined over a local field K as in Section 4, such that $(X; \{Q_i\})$ is the generic r -marked curve of genus 0. It is no restriction to suppose that $Q_1 = 0, Q_2 = 1, Q_3 = \lambda, Q_4 = \infty$, where λ is transcendental over \mathbb{Q}_p . We suppose that g has bad reduction to characteristic p , and denote by $\bar{g} : \bar{Y} \rightarrow \bar{X}$ the stable reduction. We have seen in Section 4 that we may associate with \bar{g} a set of primitive tail covers (\bar{g}_i) and a deformation datum (\bar{Z}_0, ω) . The primitive tail covers \bar{g}_i for $i \in \mathbb{B} = \{1, 2, 3, 4\}$ are uniquely determined by the e_i (Lemma 5.1).

The following proposition shows that the number of covers with bad reduction is divisible by p in the case that the branch points are generic.

Proposition 8.1. *Suppose that $(X = \mathbb{P}_K^1; \{Q_i\})$ is the generic $r = 4$ -marked curve of genus zero. Then the number of mere covers of X of ramification type $(p; e_1, e_2, e_3, e_4)$ with bad reduction is nonzero and divisible by p .*

Proof. Since the number of Galois covers and the number of mere covers differ by a prime-to- p factor, it suffices to prove the proposition for Galois covers. The existence portion of the proposition is proved in [3], Proposition 2.4.1, and the divisibility by p in Lemma 3.4.1 of *loc. cit.* (in a more general setting). We briefly sketch the proof, which is easier in our case due to the simple structure of the stable reduction (Lemma 4.2). The idea of the proof is inspired by a result of [25], Section 3.

We begin by observing that away from the wild branch point ξ_i , the primitive tail cover \bar{g}_i is tamely ramified. Therefore we can lift this cover of affine curves to characteristic zero.

Let $\mathcal{X}_0 = \mathbb{P}_R^1$ be equipped with 4 sections $Q_1 = 0, Q_2 = 1, Q_3 = \lambda, Q_4 = \infty$, where $\lambda \in R$ is transcendental over \mathbb{Z}_p . Then (4.3) defines an m -cyclic cover $\mathcal{Z}_0 \rightarrow \mathcal{X}_0$. We write $Z \rightarrow X$ for its generic fiber. Proposition 4.4 implies the existence of a deformation datum (\bar{Z}_0, ω) . Associated with the deformation datum is a character $\chi: \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{F}_p^\times$ defined by $\chi(\beta) = \beta^* z/z \pmod{z}$. The logarithmic differential form ω corresponds to a p -torsion point $P_0 \in J(\bar{Z}_0)[p]_\chi$ on the Jacobian of \bar{Z}_0 . See for example [22]. (Here we use that the conjugacy classes C_i are conjugacy classes of prime-to- p elements. This implies that the differential form ω is holomorphic.)

Since $\sum_{i=1}^4 h_i = 2m$ and the branch points are generic, we have that $J(\bar{Z}_0)[p]_\chi \simeq \mathbb{Z}/p\mathbb{Z} \times \mu_p$ ([4], Proposition 2.9). After enlarging the discretely-valued field K , if necessary, we may choose a p -torsion point $P \in J(\mathcal{Z}_0 \otimes_R K)[p]_\chi$ lifting P_0 . It corresponds to an étale p -cyclic cover $W \rightarrow Z$. The cover $\psi: W \rightarrow X$ is Galois, with Galois group $N := \mathbb{Z}/p\mathbb{Z} \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$. It is easy to see that ψ has bad reduction, and that its deformation datum is (\bar{Z}_0, ω) .

By using formal patching ([20] or [26]), one now checks that there exists a map $g_R: \mathcal{Y} \rightarrow \mathcal{X}$ of stable curves over $\text{Spec}(R)$ whose generic fiber is a G -Galois cover of smooth curves, and whose special fiber defines the given tail covers and the deformation datum. Over a neighborhood of the original component g_R is the induced cover $\text{Ind}_N^G \mathcal{Z}_0 \rightarrow \mathcal{X}_0$. Over the tails, the cover g_R is induced by the lift of the tail covers. The fact that we can patch the tail covers with the cover over \mathcal{X}_0 follows from the observation that $h_i < m_i$ (Lemma 5.1), since locally there is a unique cover with this ramification ([26], Lemma 2.12). This proves the existence statement.

The divisibility by p now follows from the observation that the set of lifts P of the p -torsion point $P_0 \in J(\bar{Z}_0)[p]_\chi$ corresponding to the deformation datum is a μ_p -torsor. \square

We are now ready to prove our Theorem 1.1, as well as a slightly sharper version of Theorem 1.3.

Theorem 8.2. *Let p be an odd prime and k an algebraically closed field of characteristic p . Suppose we are given integers $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p$. There exists a dense open subset $U \subset \mathbb{P}_k^1$ such that for $\lambda \in U$ the number of degree- p covers with ramification type (e_1, e_2, e_3, e_4) over the branch points $(0, 1, \lambda, \infty)$ is given by the formula*

$$h_p(e_1, \dots, e_4) = \min_i (e_i(p+1-e_i)) - p.$$

Furthermore, unless both $e_1 + e_2$ and e_3 are even, every such cover has good degeneration under a degeneration of the base sending λ to ∞ .

Proof. Proposition 8.1 implies that the number of covers with ramification type $(p; e_1, e_2, e_3, e_4)$ and bad reduction is at least p , and divisible by p . Together with Theorem 2.1, this implies that the generic Hurwitz number $h_p(e_1, \dots, e_4)$ is at most $\min_i(e_i(p+1-e_i)) - p$. Proposition 7.2 implies that the number of admissible covers in characteristic p is strictly larger than $\min_i(e_i(p+1-e_i)) - 2p$. Since the number of separable covers can only decrease under specialization, we conclude that the generic Hurwitz number equals $\min_i(e_i(p+1-e_i)) - p$. This proves the first statement, and the second follows immediately from Proposition 7.2 in the situation that $e_1 + e_2$ and e_3 are not both even. \square

Remark 8.3. By using the results of [3] one can prove a stronger result than Theorem 8.2. We say that a $\lambda \in \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ is *supersingular* if it is a zero of the polynomial (4.5) and *ordinary* otherwise. Then the number of covers in characteristic p of type $(p; e_1, e_2, e_3, e_4)$ branched at $(0, 1, \lambda, \infty)$ is $h_p(p; e_1, e_2, e_3, e_4)$ if λ is ordinary and $h_p(p; e_1, e_2, e_3, e_4) - 1$ if λ is supersingular. To prove this result, one needs to study the stable reduction of the cover $\pi : \mathcal{H} \rightarrow \mathbb{P}_\lambda^1$ of the Hurwitz curve to the configuration space. We do not prove this result here, as it would require too many technical details.

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