TORUS MANIFOLDS WITH NON-ABELIAN SYMMETRIES

MICHAEL WIEMELE

Abstract. Let $G$ be a connected compact non-abelian Lie group and $T$ be a maximal torus of $G$. A torus manifold with $G$-action is defined to be a smooth connected closed oriented manifold of dimension $2\dim T$ with an almost effective action of $G$ such that $M^G \neq \emptyset$. We show that if there is a torus manifold $M$ with $G$-action, then the action of a finite covering group of $G$ factors through $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod SO(2l_i) \times T^0$. The action of $\tilde{G}$ on $M$ restricts to an action of $\tilde{G}' = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod U(l_i) \times T^0$ which has the same orbits as the $\tilde{G}$-action.

We define invariants of torus manifolds with $G$-action which determine their $\tilde{G}'$-equivariant diffeomorphism type. We call these invariants admissible 5-tuples. A simply connected torus manifold with $G$-action is determined by its admissible 5-tuple up to a $\tilde{G}'$-equivariant diffeomorphism. Furthermore, we prove that all admissible 5-tuples may be realised by torus manifolds with $\tilde{G}''$-action, where $\tilde{G}''$ is a finite covering group of $\tilde{G}'$.

1. Introduction

A $2n$-dimensional smooth connected closed oriented manifold $M$ with an almost effective action of an $n$-dimensional torus $T$ is called a torus manifold if $M^T \neq \emptyset$. If each point of $M$ has an invariant open neighborhood which is weakly equivariantly diffeomorphic to an open subset of the standard action of $T$ on $\mathbb{C}^n$, then the orbit space $M/T$ is an $n$-dimensional manifold with corners [13] pp. 720-721. In this case $M$ is said to be quasitoric if $M/T$ is face preserving homeomorphic to a simple polytope $P$. In that case there are strong relations between the topology of $M$ and the combinatorics of $P$ [6] [5].

In this article we study torus manifolds for which the $T$-action may be extended by an action of a connected compact non-abelian Lie group $G$. To state our results, we introduce a bit more notation which we use to describe the structure of torus manifolds. A closed, connected submanifold $M_i$ of codimension two of a torus manifold $M$, which is pointwise fixed by a one-dimensional subtorus $\lambda(M_i)$ of $T$ and which contains a $T$-fixed point, is called a characteristic submanifold of $M$.

All characteristic submanifolds $M_i$ are orientable, and an orientation of $M_i$ determines a complex structure on the normal bundle $N(M_i, M)$ of $M_i$.

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We denote the set of unoriented characteristic submanifolds of $M$ by $\mathfrak{F}$. If $M$ is quasitoric, the characteristic submanifolds of $M$ are given by the preimages of the facets of $P$. In this case we identify $\mathfrak{F}$ with the set of facets of $P$.

Let $G$ be a connected compact non-abelian Lie group. We call a smooth connected closed oriented $G$-manifold $M$ a torus manifold with $G$-action if $G$ acts almost effectively on $M$, $\dim M = 2\, \text{rank} \ G$ and $MT \neq \emptyset$ for a maximal torus $T$ of $G$. That means that $M$ with the action of $T$ is a torus manifold. Because all maximal tori of $G$ are conjugated, $M$ together with the action of any other maximal torus $T'$ is also a torus manifold. Moreover, for all choices of a maximal torus of $G$, we get up to weakly equivariant diffeomorphism the same torus manifold. The $G$-action on $M$ induces an action of the Weyl group $W(G)$ of $G$ on $\mathfrak{F}$ and the $T$-equivariant cohomology of $M$. Results of Masuda [14] and Davis-Januszkiewicz [6] make a comparison of these actions possible. From this comparison we get a description of the action on $\mathfrak{F}$ and the isomorphism type of $W(G)$. Namely, there is a partition of $\mathfrak{F} = \mathfrak{F}_0 \sqcup \ldots \sqcup \mathfrak{F}_t$ and a finite covering group $\tilde{G} = \prod_{j=1}^t G_j \times T^0$ of $G$ such that each $G_{j_0}$ is non-abelian and $W(G_{j_0})$ acts transitively on $\mathfrak{F}_{j_0}$ and trivially on $\mathfrak{F}_j$, $j \neq j_0$, and the orientation of each $M_i \in \mathfrak{F}_j$, $j \neq j_0$, is preserved by $W(G_{j_0})$ (see section 2).

We call such $G_i$ the elementary factors of $\tilde{G}$.

By looking at the orbits of the $T$-fixed points, we find that we may assume without loss of generality that all elementary factors are isomorphic to $SU(l_i + 1)$, $SO(2l_i)$ or $SO(2l_i + 1)$ (see section 3). If $M$ is quasitoric, then all elementary factors are isomorphic to $SU(l_i + 1)$.

Now assume $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1)$ elementary. Then the restriction of the action of $G_1$ to $U(l_1)$ has the same orbits as the $G_1$-action (see section 6). The following theorem shows that the classification of simply connected torus manifolds with $\tilde{G}$-action reduces to the classification of torus manifolds with $U(l_1) \times G_2$-action.

**Theorem 1.1 (Theorem 6.3).** Let $M, M'$ be two simply connected torus manifolds with $\tilde{G}$-action, $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1)$ elementary. Then $M$ and $M'$ are $\tilde{G}$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.

By applying a blow up construction along the fixed points of an elementary factor of $\tilde{G}$ isomorphic to $SU(l_1 + 1)$ or $SO(2l_1 + 1)$, we get a fiber bundle over a complex or real projective space with some torus manifold as the fiber.

This construction may be reversed, and we call the inverse construction a blow down. With this notation we get:

**Theorem 1.2 (Corollaries 5.6, 5.14, 7.2, Theorem 7.3).** Let $\tilde{G} = G_1 \times G_2$ and let $M$ be a torus manifold with $G$-action such that $G_1$ is elementary and $l_2 = \text{rank} \ G_2$.

Then:

- If $G_1 = SU(l_1 + 1)$ and $\#\mathfrak{F}_1 = 2$ in the case $l_1 = 1$, then $M$ is the blow down of a fiber bundle $\tilde{M}$ over $\mathbb{C}P^{l_1}$ with the fiber being some $2l_2$-dimensional torus manifold with $G_2$-action along an invariant submanifold of codimension two. Here the $G_1$-action on $\tilde{M}$ covers the standard action of $SU(l_1 + 1)$ on $\mathbb{C}P^{l_1}$.

- If $G_1 = SO(2l_1 + 1)$ and $\#\mathfrak{F}_1 = 1$ in the case $l_1 = 1$, then $M$ is a blow down of a fiber bundle $\tilde{M}$ over $\mathbb{R}P^{2l_1}$ with the fiber being some $2l_2$-dimensional.
torus manifold with $G_2$-action along an invariant submanifold of codimension one or a Cartesian product of a $2l_1$-dimensional sphere and a $2l_2$-dimensional torus manifold with $G_2$-action. In the first case the $G_1$-action on $\tilde{M}$ covers the standard action of $SO(2l_1 + 1)$ on $\mathbb{R}P^{2l_1}$. In the second case $G_1$ acts in the usual way on $S^{2l_1}$.

If all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_1 + 1)$ or $SU(l_1 + 1)$, then we may iterate this construction. By this iteration we get a complete classification of torus manifolds with $\tilde{G}$-action up to a $\tilde{G}$-equivariant diffeomorphism in terms of admissible 5-tuples (Theorem 8.5). For general $G$ we have $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times SO(2l_1) \times T^{l_0}$. We may restrict the action of $\tilde{G}$ to $\prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod U(l_i) \times T^{l_0}$. Therefore we get invariants for torus manifolds with $G$-action from the above classification. With Theorem 1.1 we see that these invariants determine the $G$-equivariant diffeomorphism type of simply connected torus manifolds with $G$-action.

At the end we apply our classification to get more explicit results in special cases. These are:

For the special case $G_2 = \{1\}$ we get:

**Corollary 1.3** (Corollary 3.6). Assume that $G$ is elementary and $M$ is a torus manifold with $G$-action. Then $M$ is equivariantly diffeomorphic to $S^{2l}$ or $\mathbb{C}P^l$ if $G = SO(2l + 1), SO(2l)$ or $G = SU(l + 1)$, respectively.

We recover certain results of Kuroki [13, 11, 12] who gave a classification of torus manifolds with $G$-action and $\dim M/G \leq 1$ (see Corollaries 8.10 and 8.11).

For quasitoric manifolds we have the following result.

**Theorem 1.4** (Corollary 8.9). If $G$ is semi-simple and $M$ is a quasitoric manifold with $G$-action, then

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i + 1)$$

and $M$ is equivariantly diffeomorphic to a product of complex projective spaces.

Furthermore, we give an explicit classification of simply connected torus manifolds with $G$-action such that $\tilde{G}$ is semi-simple and has two simple factors.

**Theorem 1.5** (Corollaries 3.6, 8.12, 8.14). Let $\tilde{G} = G_1 \times G_2$ with $G_i$ simple and $M$ be a simply connected torus manifold with $G$-action. Then $M$ is one of the following:

$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2},$ \hspace{1cm} $\mathbb{C}P^{l_1} \times S^{2l_2},$ \hspace{1cm} $\#_i(S^{2l_1} \times S^{2l_2}),$ \hspace{1cm} $S^{2l_1 + 2l_2}.$

The $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism.

The paper is organized as follows. In section 2 we investigate the action of the Weyl group of $G$ on $\mathfrak{g}$ and $H^*_T(M)$. In section 3 we determine the orbit types of the $T$-fixed points in $M$ and the isomorphism types of the elementary factors of $G$. In section 4 the basic properties of the blow up construction are established. In section 5 we give an argument which reduces the classification problem for actions with an elementary factor $G_1 = SU(l_1 + 1)$ are studied. In section 6 we give an argument which reduces the classification problem for actions with an elementary factor $G_1 = SO(2l_1)$ to that with an elementary factor $SU(l_1)$. In section 7 we classify torus manifolds with $G$-action with elementary factor $G_1 = SO(2l_1 + 1)$. In section 8 we iterate the classification results of the previous sections.
sections and illustrate them with some applications. There are two appendices with preliminary facts on Lie groups and torus manifolds.

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2. The action of the Weyl group on \( \mathfrak{g} \)

Let \( G \) be a compact connected Lie group of rank \( n \) and \( T \) be a maximal torus of \( G \). Moreover, let \( M \) be a torus manifold with \( G \)-action. That means that \( G \) acts almost effectively on the \( 2n \)-dimensional smooth closed connected oriented manifold \( M \) such that \( MT \neq \emptyset \). We call a closed connected submanifold \( M_i \) of codimension two of \( M \), which is pointwise fixed by a one-dimensional subtorus \( \lambda(M_i) \) of \( T \) and which contains a \( T \)-fixed point, a characteristic submanifold of \( M \). If \( g \) is an element of the normalizer \( N_GT \) of \( T \) in \( G \), then, for every characteristic submanifold \( M_i \), \( gM_i \) is also a characteristic submanifold. Therefore there are actions of \( N_GT \) and the Weyl group of \( G \) on \( \mathfrak{g} \).

In this section we describe this action of the Weyl group of \( G \) on \( \mathfrak{g} \). At first we recall the definition of the equivariant cohomology of a \( G \)-space \( X \). Let \( EG \to BG \) be a universal principal \( G \)-bundle. Then \( EG \) is a contractible free right \( G \)-space. If \( T \) is a maximal torus of \( G \), then we may identify \( ET = EG \) and \( BT = EG/T \). The Borel construction \( X_G \) of \( X \) is the orbit space of the right action \( ((e,x),g) \mapsto (eg, g^{-1}x) \) on \( EG \times X \). The equivariant cohomology \( H^*_G(X) \) of \( X \) is defined as the cohomology of \( X_G \).

In this section we take all cohomology groups with coefficients in \( \mathbb{Q} \). The \( G \)-action on \( EG \times X \) induces a right action of the normalizer of \( T \) on \( X_T \). Therefore it induces a left action of the Weyl group of \( G \) on the \( T \)-equivariant cohomology of \( X \).

Now let \( X = M \) be a torus manifold with \( G \)-action. Denote the characteristic submanifolds of \( M \) by \( M_i \), \( i = 1, \ldots, m \). Then, for any \( g \in N_GT \), \( M_{g(i)} = gM_i \) is also a characteristic submanifold which depends only on the class \( w = [g] \in W(G) = N_GT/T \). Therefore we get an action of the Weyl group of \( G \) on \( \mathfrak{g} \). Notice that \( M_i \in \mathfrak{g} \) is a fixed point of the \( W(G) \)-action on \( \mathfrak{g} \) if and only if it is invariant under the action of \( N_GT \) on \( M \).

A choice of an orientation for each characteristic submanifold of \( M \) together with an orientation for \( M \) is called an omniorientation of \( M \). If we fix an omniorientation for \( M \), then the \( T \)-equivariant Poincaré dual \( \tau_i \) of \( M_i \) is well defined.

It is the image of the Thom class of \( N(M_i,M)_T \) under the natural map

\[
\psi : H^2(N(M_i,M)_T,N(M_i,M)_T - (M_i)_T) \to H^2(M_T,M_T - (M_i)_T) \to H^2_T(M).
\]

Because of the uniqueness of the Thom class [17, p. 110] and because \( \psi \) commutes with the action of \( W(G) \), we have

\begin{equation}
\tau_{g(i)} = \pm g^* \tau_i.
\end{equation}

Here the minus sign occurs if and only if \( g_{|M_i} : M_i \to M_{g(i)} \) is orientation reversing. We say that the class \( [g] \in W(G) \) acts orientation preserving at \( M_i \) if this map is orientation preserving. If \( [g] \) acts orientation preserving at all characteristic submanifolds, then we say that \( [g] \) preserves the omniorientation of \( M \).
Let \( S = H^{>0}(BT) \) and \( \hat{H}^2_+(M) = H^2_+(M)/S \)-torsion. Because \( M^T \neq \emptyset \), there is an injection \( H^2(BT) \hookrightarrow H^2_+(M) \) and

\[
H^2(BT) \cap S \text{-torsion} = \{0\}.
\]

By [14, pp. 240-241], the \( \tau_i \) are linearly independent in \( \hat{H}^*(T) \). By Lemma 3.2 of [14, p. 246], they form a basis of \( \hat{H}^2(T) \). The Lie algebra \( LG \) of \( G \) may be endowed with a Euclidean inner product which is invariant for the adjoint representation. This allows us to identify the Weyl group \( W(G) \) of \( G \) with a group of orthogonal transformations on the Lie algebra \( LT \). It is generated by reflections in the walls of the Weyl chambers of \( G \) [4, pp. 192-193]. In the following we say that an element of \( W(G) \) is a reflection if and only if it acts as a reflection on \( H^2(BT) \).

Here we say that \( A \in \text{Gl}(L) \) acts as a reflection on the \( \mathbb{Q} \)-vector space \( L \) if there is a decomposition \( L = L_+ \oplus L_- \) with \( \dim_{\mathbb{Q}} L_- = 1 \) and \( A|_{L_\pm} = \pm \text{Id} \). Notice that \( A \in \text{Gl}(L) \) acts as a reflection on \( L \) if and only if \( \text{ord} A = 2 \) and \( \text{trace}(A, L) = \dim_{\mathbb{Q}} L - 2 \).

\textbf{Lemma 2.1.} Let \( w \in W(G) \) be a reflection. Then there are the following possibilities for the action of \( w \) on \( \mathfrak{g} \):

1. \( w \) fixes all except exactly two elements of \( \mathfrak{g} \). It acts orientation preserving at all characteristic submanifolds.
2. \( w \) fixes all except exactly two elements of \( \mathfrak{g} \). Denote the elements of \( \mathfrak{g} \) which are not fixed by \( w \) by \( M_1, M_2 \). The action of \( w \) is orientation preserving at all characteristic submanifolds of \( M \) except \( M_1, M_2 \). It is orientation reversing at \( M_1, M_2 \).
3. \( w \) fixes all elements of \( \mathfrak{g} \). It acts orientation reversing at exactly one characteristic submanifold of \( M \).

\textbf{Proof.} Using the arguments given before Lemma 2.1, we have the following commutative diagram of \( W(G) \)-representations with exact rows and columns:

\[
\begin{array}{c}
S \text{-torsion in } H^2_+(M) \\
\downarrow \\
0 \longrightarrow H^2(BT) \longrightarrow H^2_+(M) \longrightarrow \hat{H}^2_+(M) \longrightarrow H^2(M) \\
\downarrow \\
\hat{H}^2_+(M) \\
\downarrow \\
0.
\end{array}
\]

Here \( \phi \) denotes the natural map \( H^2_+(M) \rightarrow H^2(M) \).

Because \( G \) is connected, the \( W(G) \)-action on \( H^2(M) \) is trivial. By (2.2) the \( S \)-torsion in \( H^2_+(M) \) injects into \( H^2(M) \). Therefore \( W(G) \) acts trivially on the \( S \)-torsion in \( H^2_+(M) \).

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Because \( w \) is a reflection, we have trace\((w, H^2(BT))\) = \( \text{dim}_Q H^2(BT) - 2 \). From the exact row in the diagram we get
\[
\text{trace}(w, \hat{H}^2_1(M)) = \text{trace}(w, H^2(BT)) + \text{trace}(w, \text{im } \phi)
= \text{dim}_Q H^2(BT) - 2 + \text{dim}_Q \text{im } \phi
= \text{dim}_Q \hat{H}^2_1(M) - 2.
\]

Similarly we get
\[
\text{trace}(w, \hat{H}^2_2(M)) = \text{trace}(w, H^2(BT)) - \text{trace}(w, S\text{-torsion in } H^2_2(M))
= \text{dim}_Q \hat{H}^2_2(M) - 2.
\]

Now the statement follows from (2.1) because the \( \tau_i \) form a basis of \( \hat{H}^2_2(M) \). \( \square \)

**Lemma 2.2.** An element \( w \in W(G) \) acts as a reflection on \( \hat{H}^2_2(M) \) if and only if it is a reflection.

**Proof.** Because, by (2.2), \( H^2(BT) \) injects into \( \hat{H}^2_2(M) \), \( W(G) \) acts effectively on \( \hat{H}^2_2(M) \). Therefore we may identify \( W(G) \) with a subgroup of \( \text{Gl}(\hat{H}^2_2(M)) \).

If \( w \in W(G) \), then, as in the proof of Lemma 2.1, we see that
\[
\text{dim}_Q H^2(BT) - \text{trace}(w, H^2(BT)) = \text{dim}_Q \hat{H}^2_2(M) - \text{trace}(w, \hat{H}^2_2(M)).
\]

Therefore, by the remark before Lemma 2.1 an element of \( W(G) \) of order two is a reflection if and only if it acts as a reflection on \( \hat{H}^2_2(M) \). \( \square \)

Let \( \mathfrak{g}_0 \) be the set of characteristic submanifolds which are fixed by the \( W(G) \)-action on \( \mathfrak{z} \) and at which \( W(G) \) acts orientation preserving. Furthermore let \( \mathfrak{g}_i, i = 1, \ldots, k \), be the other orbits of the \( W(G) \)-action on \( \mathfrak{z} \) and \( V_i \) the subspace of \( \hat{H}^2_2(M) \) spanned by the \( \tau_i \) with \( M_j \in \mathfrak{g}_i \). Then \( W(G) \) acts trivially on \( V_0 \). For \( i > 0 \), let \( W_i \) be the subgroup of \( W(G) \) which is generated by the reflections which act non-trivially on \( V_i \). Then, by Lemma 2.1 \( W_i \) acts trivially on \( V_j, j \neq i \).

By (2.2), \( H^2(BT) \) injects into \( \hat{H}^2_2(M) \). Therefore \( W(G) \) acts effectively on \( \hat{H}^2_2(M) \). This fact implies that the subgroups \( W_i, i = 1, \ldots, k \), of \( W(G) \) pairwise commute and \( (W_1, \ldots, W_i) \cap W_{i+1} = \{1\} \) for all \( i = 1, \ldots, k-1 \). Here \( (W_1, \ldots, W_i) \) denotes the subgroup of \( W(G) \) which is generated by \( W_1, \ldots, W_i \). Hence, we have an injective group homomorphism \( \prod W_i \to W(G), (w_1, \ldots, w_k) \mapsto w_1 \cdots w_k \).

**Lemma 2.3.** The group homomorphism \( \prod W_i \to W(G), (w_1, \ldots, w_k) \mapsto w_1 \cdots w_k \) is an isomorphism.

**Proof.** Because \( W(G) \) is generated by reflections and each reflection is contained in a \( W_i \), the above homomorphism is surjective. As noted before, it is injective. Therefore it is an isomorphism. \( \square \)

**Lemma 2.4.** For each pair \( M_{j_1}, M_{j_2} \in \mathfrak{g}_i, i > 0 \), with \( M_{j_1} \neq M_{j_2} \), there is a reflection \( w \in W_i \) with \( w(M_{j_1}) = M_{j_2} \).

**Proof.** Because \( \mathfrak{g}_i \) is an orbit of the \( W(G) \)-action on \( \mathfrak{z} \) and \( W(G) \) is generated by reflections, there is an \( M'_{j_1} \in \mathfrak{g}_i \) with \( M'_{j_1} \neq M_{j_2} \) and a reflection \( w \in W_i \) with \( w(M'_{j_1}) = M_{j_2} \).
Because \( W_i \) is generated by reflections and acts transitively on \( \mathfrak{S}_i \), the natural map \( W_i \to S(\mathfrak{S}_i) \) to the permutation group \( S(\mathfrak{S}_i) \) of \( \mathfrak{S}_i \) is a surjection by Lemma 2.1 and Lemma 3.10 of [H] p. 51]. Therefore there is a \( w' \in W_i \) with

\[
w'(M_{j_1}) = M'_{j_1}, \quad w'(M'_{j_1}) = M_{j_1}, \quad w'(M_{j_2}) = M_{j_2}.
\]

Now \( w'^{-1}ww' \in W_i \) is a reflection with the required properties. \( \square \)

It follows from Lemma 2.1 that for each pair \( M_{j_1}, M_{j_2} \in \mathfrak{S}_i, i > 0 \), with \( M_{j_1} \neq M_{j_2} \), there are at most two reflections which map \( M_{j_1} \) to \( M_{j_2} \).

If \( M_{j_1}', M_{j_2}' \in \mathfrak{S}_i \) is another pair with \( M_{j_1}' \neq M_{j_2}' \), then one sees as in the proof of Lemma 2.4 that there is a \( w' \in W_i \) with

\[
w'(M_{j_1}') = M_{j_1}, \quad w'(M_{j_2}') = M_{j_2}.
\]

Therefore there is a bijection

\[
\{ w \in W_i; w \text{ reflection}, w(M_{j_1}) = M_{j_2} \} \to \{ w \in W_i; w \text{ reflection}, w(M_{j_2}) = M_{j_1} \},
\]

\[
w \mapsto w'^{-1}ww'.
\]

In particular, the number of reflections which map \( M_{j_1} \) to \( M_{j_2} \) does not depend on the choice of \( M_{j_1}, M_{j_2} \in \mathfrak{S}_i \).

**Lemma 2.5.** Assume \( \#\mathfrak{S}_i > 1 \) and \( i > 0 \). If for each pair \( M_{j_1}, M_{j_2} \in \mathfrak{S}_i \) with \( M_{j_1} \neq M_{j_2} \) there is exactly one reflection in \( W_i \), which maps \( M_{j_1} \) to \( M_{j_2} \), then \( W_i \) is isomorphic to \( S(\mathfrak{S}_i) \cong W(SU(l_i + 1)) \), with \( l_i + 1 = \#\mathfrak{S}_i \).

**Proof.** First we show that there is no reflection of the third type as described in Lemma 2.4 in \( W_i \). Assume that \( w' \in W_i \) is a reflection of the third type. Then let \( M_1 \in \mathfrak{S}_i \) be the characteristic submanifold at which \( w' \) acts orientation reversing. Furthermore, let \( M_1 \neq M_2 \in \mathfrak{S}_i \).

Then by Lemma 2.4 there is a reflection \( w \in W_i \) such that \( wM_1 = M_2 \). Hence, \( w'ww' \) is a reflection with \( w'ww'M_1 = M_2 \). Because \( w \) and \( w'ww' \) have a different orientation behaviour at \( M_1 \), we have \( w \neq w'ww' \), contradicting our assumption.

To prove the lemma, it is sufficient to show that the kernel of the natural map \( W_i \to S(\mathfrak{S}_i) \) is trivial. Let \( w \) be an element of this kernel. Then for each \( \tau_j \in V_i \), we have

\[
w\tau_j = \pm \tau_j.
\]

If we have \( w\tau_j = \tau_j \) for all \( \tau_j \in V_i \), then \( w = \text{Id} \).

Now assume that \( w\tau_{j_0} = -\tau_{j_0} \) for a \( \tau_{j_0} \in V_i \). Then there are reflections \( w_1, \ldots, w_n \in W_i \), \( n \geq 2 \), with \( -\tau_{j_0} = w\tau_{j_0} = w_1 \ldots w_n\tau_{j_0} \). After removing some of the \( w_i \), we may assume that

\[
w_i \ldots w_n\tau_{j_0} \neq \pm \tau_{j_0} \quad \text{for all } i = 2, \ldots, n,
\]

\[
w_{i+1} \ldots w_n\tau_{j_0} \neq \pm w_i \ldots w_n\tau_{j_0} \quad \text{for all } i = 2, \ldots, n.
\]

Therefore, by Lemma 2.1 we have \( w_i\tau_{j_0} = \tau_{j_0} \) for \( 2 \leq i < n \). This equation together with \( w\tau_{j_0} = -\tau_{j_0} \) implies

\[
w_n \ldots w_{i+1}w_iw_2 \ldots w_n\tau_{j_0} = -w_n\tau_{j_0}.
\]

Therefore \( w_n \ldots w_{i+1}w_iw_2 \ldots w_nM_{j_0} = w_nM_{j_0} \).

But \( w_n \ldots w_{i+1}w_iw_2 \ldots w_n \) is a reflection. Therefore, by assumption, we have

\[
w_n \ldots w_{i}w_2 \ldots w_n = w_n.
and

\[ w_n \tau_j = w_n w_{n-1} \ldots w_2 w_1 w_2 \ldots w_n \tau_j = -w_n \tau_j. \]

Because \( w_n \tau_j \neq 0 \), this is impossible. Hence, our assumption that \( w \tau_j = -\tau_j \) is false.

Therefore the kernel is trivial. \( \square \)

To get the isomorphism type of \( W_i \) in the case where there is a pair \( M_{j_1}, M_{j_2} \in \mathfrak{g}_i, \ i > 0 \), with \( M_{j_1} \neq M_{j_2} \) and exactly two reflections in \( W_i \) which map \( M_{j_1} \) to \( M_{j_2} \), we first give a description of the Weyl groups of some Lie groups.

Let \( L \) be an \( l \)-dimensional \( \mathbb{Q} \)-vector space with basis \( e_1, \ldots, e_l \). For \( 1 \leq i < j \leq l \) let \( f_{ij \pm}, g_i \in \text{GL}(L) \) such that

\[
  f_{ij + e_k} = \begin{cases} e_i & \text{if } k = j, \\ e_j & \text{if } k = i, \\ e_k & \text{else}, \end{cases} \]

\[
  f_{ij - e_k} = \begin{cases} -e_i & \text{if } k = j, \\ -e_j & \text{if } k = i, \\ e_k & \text{else}, \end{cases} \]

\[
  g_i e_k = \begin{cases} -e_i & \text{if } k = i, \\ e_i & \text{else}. \end{cases} \]

Then we have the following isomorphisms of groups \([4\text{ pp. 171-172}]):

\[
  W(SU(l-1)) \cong S(l) \cong \langle f_{ij + 1}, 1 \leq i < j \leq l \rangle, \\
  W(SO(2l)) \cong \langle f_{ij \pm}, 1 \leq i < j \leq l \rangle, \\
  W(SO(2l + 1)) \cong W(Sp(l)) \cong \langle f_{ij \pm}, g_i, 1 \leq i < j \leq l \rangle.
\]

From this description and Lemma 2.1, we get:

**Lemma 2.6.** If for each pair \( M_{j_1}, M_{j_2} \in \mathfrak{g}_i, \ i > 0 \), with \( M_{j_1} \neq M_{j_2} \) there are exactly two reflections in \( W_i \) which map \( M_{j_1} \) to \( M_{j_2} \), then with \( l_i = \# \mathfrak{g}_i \) we have:

1. \( W_i \cong W(SO(2l_i)) \) if there is no reflection of the third type as described in Lemma 2.1 in \( W_i \).
2. \( W_i \cong W(SO(2l_i + 1)) \cong W(Sp(l_i)) \) if there is a reflection of the third type in \( W_i \).

By \([4\text{ p. 233})], G has a finite covering group \( \tilde{G} \) such that \( \tilde{G} = \prod_i G_i \times T^{l_0} \), where the \( G_i \) are simple and simply connected compact Lie groups. The Weyl group of \( G \) is given by \( W(G) = \prod_i W(G_i) \).

We call two reflections \( w, w' \in W(G) \) equivalent if there are reflections \( w_1, \ldots, w_k \in W(G) \) such that

\[
  w = w_1, \quad w' = w_k, \quad [w_i, w_{i+1}] \neq 1.
\]

Here \( [w_i, w_{i+1}] \) denotes the commutator of \( w_i \) and \( w_{i+1} \). Because the Dynkin diagram of a simple Lie group is connected, each \( W(G_i) \) is generated by equivalent reflections. Therefore each \( W(G_i) \) is contained in a \( W_j \). Therefore we get \( W_i = \prod_{j \in \mathcal{J}_i} W(G_j) \). Using Lemmas 2.4 and 2.6 we deduce:

\[
  W_i = \begin{cases} W(G_j) & \text{for some } j \text{ if } W_i \cong W(SO(4)), \\
  W(G_{j_1}) \times W(G_{j_2}) & \text{with } G_{j_1} \cong G_{j_2} \cong SU(2) \text{ if } W_i \cong W(SO(4)). \end{cases}
\]
Therefore we may write \( \tilde{G} = \prod_i G_i \times T_i^0 \) with \( W_i = W(G_i) \) and \( G_i \) simple and simply connected or \( G_i = \text{Spin}(4) \). In the following we will call these \( G_i \) the elementary factors of \( \tilde{G} \).

We summarize the above discussion in the following lemma.

**Lemma 2.7.** Let \( M \) be a torus manifold with \( G \)-action and \( \tilde{G} \) as above. Then all \( G_i \) are non-exceptional, i.e. \( G_i = \text{SU}(l_i + 1), \text{Spin}(2l_i), \text{Spin}(2l_i + 1), \text{Sp}(l_i) \).

The Weyl group of an elementary factor \( G_i \) of \( \tilde{G} \) acts transitively on \( \#h_i \) and trivially on \( \#h_j, j \neq i \).

For a given isomorphism type of \( G_i \), there are at most two possible values of \( \#h_i \). The possible values of \( \#h_i \) are listed in the following table:

<table>
<thead>
<tr>
<th>( G_i )</th>
<th>( #h_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SU}(2) = \text{Spin}(3) = \text{Sp}(1) )</td>
<td>1, 2</td>
</tr>
<tr>
<td>( \text{Spin}(4) )</td>
<td>2</td>
</tr>
<tr>
<td>( \text{Spin}(5) = \text{Sp}(2) )</td>
<td>2</td>
</tr>
<tr>
<td>( \text{SU}(4) = \text{Spin}(6) )</td>
<td>3, 4</td>
</tr>
<tr>
<td>( \text{SU}(l_i + 1), l_i \neq 1, 3 )</td>
<td>( l_i + 1 )</td>
</tr>
<tr>
<td>( \text{Spin}(2l_i + 1), l_i &gt; 2 )</td>
<td>( l_i )</td>
</tr>
<tr>
<td>( \text{Spin}(2l_i), l_i &gt; 3 )</td>
<td>( l_i )</td>
</tr>
<tr>
<td>( \text{Sp}(l_i), l_i &gt; 2 )</td>
<td>( l_i )</td>
</tr>
</tbody>
</table>

If we restrict our attention to quasitoric manifolds with \( G \)-action, then we get a much shorter list of possible isomorphism types of the elementary factors. In fact, if \( M \) is a quasitoric manifold with \( G \)-action, then, as shown in the next lemma, all elementary factors of \( G \) are isomorphic to \( \text{SU}(l_i + 1) \) for some \( l_i \geq 1 \).

**Lemma 2.8.** Let \( M \) be a quasitoric manifold with \( G \)-action. Then there is a covering group \( \tilde{G} \) of \( G \) with \( \tilde{G} = \prod_{i=1}^{k_1} \text{SU}(l_i + 1) \times T_i^0 \).

**Proof.** First we show for \( i > 0 \):

\[
W_i \cong S(\#h_i). \tag{2.3}
\]

To do so, it is sufficient to prove that there is an omniorientation on \( M \) which is preserved by the action of \( W(G) \). This is true if for every characteristic submanifold \( M_i \) and \( g \in N_GT \) such that \( gM_i = M_i, g \) preserves the orientation of \( M_i \). Since \( G \) is connected, \( g \) preserves the orientation of \( M \) and acts trivially on \( H^2(M) \).

Because each vertex of the orbit polytope \( P \) of \( M \) is the intersection of exactly \( n \) facets of \( P \), every fixed point of the \( T \)-action on \( M \) is the transverse intersection of exactly \( n \) characteristic submanifolds. Thus, the Poincaré dual \( PD(M_i) \in H^2(M) \) of \( M_i \) is non-zero because \( M_i \cap M^T \neq \emptyset \). Therefore \( g \) preserves the orientation of \( M_i \) since otherwise

\[
PD(M_i) = \frac{1}{2} (PD(M_i) + PD(M_i))
\]

\[
= \frac{1}{2} (PD(M_i) + g^*PD(M_i)) \quad (g \text{ acts trivially on } H^2(M))
\]

\[
= \frac{1}{2} (PD(M_i) - PD(M_i)) \quad (g \text{ reverses the orientation of } M_i)
\]

\[
= 0.
\]
This establishes \([2,3]\). Recall that all simple compact simply connected Lie groups having a Weyl group isomorphic to some symmetric group are isomorphic to some \(SU(l_i+1)\). Therefore all elementary factors of \(\tilde{G}\) are isomorphic to \(SU(l_i+1)\). From this the statement follows. □

Remark 2.9. In [15] Masuda and Panov show that the cohomology with coefficients in \(Z\) of a torus manifold \(M\) is generated by its degree-two part if and only if the torus action on \(M\) is locally standard and the orbit space \(M/T\) is a homology polytope. That means that all faces of \(M/T\) are acyclic and all intersections of facets of \(M/T\) are connected. In particular, each \(T\)-fixed point is the transverse intersection of \(n\) characteristic submanifolds. Therefore the above lemma also holds in this case.

For a characteristic submanifold \(M_i\) of \(M\), let \(\lambda(M_i)\) denote the one-dimensional subtorus of \(T\) which fixes \(M_i\) pointwise. The normalizer \(N_{G}T\) of \(T\) in \(G\) acts by conjugation on the set of one-dimensional subtori of \(T\). The following lemma shows that

\[\lambda : \mathfrak{g} \to \{\text{one-dimensional subtori of } T\}\]

is \(N_{G}T\)-equivariant.

Lemma 2.10. Let \(M\) be a torus manifold with \(G\)-action, \(g \in N_{G}T\) and \(M_i \subset M\) be a characteristic submanifold. Then we have:

1. \(\lambda(gM_i) = g\lambda(M_i)g^{-1}\).
2. If \(gM_i = M_i\), then \(g\) acts orientation preserving on \(M_i\) if and only if \(\lambda(M_i) \to \lambda(M_i)\ t \to gtg^{-1}\) is orientation preserving.

Proof. First we prove (1). Let \(x \in M_i\) be a generic point. Then the identity component \(T_{x}^{0}\) of the stabilizer of \(x\) in \(T\) is given by \(T_{x}^{0} = \lambda(M_i)\). Therefore we have

\[\lambda(gM_i) = T_{gx}^{0} = gtg^{-1} = g\lambda(M_i)g^{-1}.

Now we shall prove (2). An orientation of \(M_i\) induces a complex structure on \(N(M_i, M)\). We fix an isomorphism \(\rho : \lambda(M_i) \to S^1\) such that the action of \(t \in \lambda(M_i)\) on \(N(M_i, M)\) is given by multiplication with \(\rho(t)^m\), \(m > 0\). The differential \(Dg : N(M_i, M) \to N(M_i, M)\) is orientation preserving if and only if it is complex linear. Otherwise it is complex anti-linear. Therefore for \(v \in N(M_i, M)\) we have

\[\rho(gtg^{-1})^m v = (Dt)(Dg)^{-1}v = (Dg)\rho(t)^m(Dg)^{-1}v = \rho(t^{\pm m})v = \rho(t^{\pm 1})^m v.

This equation implies that \(\rho(gtg^{-1}t^{\pm 1}) \in \mathbb{Z}/m\mathbb{Z}\). Because \(\lambda(M_i)\) is connected and \(\mathbb{Z}/m\mathbb{Z}\) is discrete, \(gtg^{-1} = t^{\pm 1}\) follows, where the plus sign arises if and only if \(g\) acts orientation preserving on \(M_i\). □

3. \(G\)-action on \(M\)

In this section we consider torus manifolds with \(G\)-action such that \(\tilde{G}\) has only one elementary factor \(G_1\), i.e. \(\tilde{G} = G_1 \times T^{l_0}\). There are two cases:

1. There is a \(T\)-fixed point which is not fixed by \(G_1\).
2. There is a \(G\)-fixed point.

We first discuss the case where there is a \(T\)-fixed point which is not fixed by \(G_1\).
Lemma 3.1. Let $\tilde{G} = G_1 \times T^o$ with $G_1$ elementary, rank $G_1 = l_1$ and $M$ a torus manifold with $G$-action of dimension $2n = 2(l_0 + l_1)$. If there is an $x \in M^T$ which is not fixed by the action of $G_1$, then
\begin{enumerate}
  \item $G_1 = SU(l_1 + 1)$ or $G_1 = Spin(2l_1 + 1)$, and the stabilizer of $x$ in $G_1$ is conjugated to $S(U(l_1) \times U(1))$ or $Spin(2l_1)$, respectively.
  \item The $G_1$-orbit of $x$ equals the component of $M^{T^o}$ which contains $x$.
\end{enumerate}
Moreover, if $G_1 = SU(4)$, one has $\#\mathcal{F}_1 = 4$.

Proof. The $G_1$-orbit of $x$ is contained in the component $N$ of $M^{T^o}$ containing $x$. Therefore we have
\[ \text{codim} G_{1x} = \dim G_1 / G_{1x} = \dim G_1 x \leq \dim N \leq 2l_1. \]
Furthermore the stabilizer $G_{1x}$ of $x$ has maximal rank $l_1$. In particular, its identity component $G_1^0$ is a closed connected maximal rank subgroup.

Next we use the theory of Lie groups to determine the isomorphism types of $G_1$ and $G_{1x}$. At first we consider the case $G_1 \neq Spin(4)$. From the classification of closed connected maximal rank subgroups of a compact Lie group given in [2, p. 219] we get the following connected maximal rank subgroups $H$ of maximal dimension:

\[
\begin{array}{ccc}
G_1 & H & \text{codim } H \\
SU(2) = Spin(3) = Sp(1) & S(U(1) \times U(1)) & 2 \\
Spin(5) = Sp(2) & Spin(4) & 4 \\
SU(4) = Spin(6) & S(U(3) \times U(1)) & 6 \\
SU(l_1 + 1), l_1 \neq 1, 3 & S(U(l_1) \times U(1)) & 2l_1 \\
Spin(2l_1 + 1), l_1 > 2 & Spin(2l_1) & 2l_1 \\
Spin(2l_1), l_1 > 3 & Spin(2l_1 - 2) \times Spin(2) & 4l_1 - 4 \\
Sp(l_1), l_1 > 2 & Sp(l_1 - 1) \times Sp(1) & 4l_1 - 4 \\
\end{array}
\]

Because $H$ is unique up to conjugation and
\[ \text{codim } H \leq \text{codim } G_{1x}^0 = \text{codim } G_{1x} \leq 2l_1, \]
we see $G_1 = SU(l_1 + 1)$ or $G_1 = Spin(2l_1 + 1)$. Moreover, $G_{1x}$ is conjugated to a subgroup of $G_1$ which contains $S(U(l_1) \times U(1))$ or $Spin(2l_1)$, respectively.

If $l_1 > 1$, then $S(U(l_1) \times U(1))$ is a maximal subgroup of $SU(l_1 + 1)$ by Lemma A.1. Therefore, if $G_1 = SU(l_1 + 1)$ and $l_1 > 1$, then $G_{1x}$ is conjugated to $S(U(l_1) \times U(1))$. Because $\dim S(U(l_1) \times U(1)) = 2l_1 \geq \dim N \geq \text{codim } G_{1x}$, we have $G_{1x} = N$ in this case.

If $G_1 = Spin(2l_1 + 1)$, $l_1 \geq 1$, then by Lemma A.3 there are two proper subgroups of $G_1$ which contain $Spin(2l_1)$, $Spin(2l_1)$ and its normalizer $H_0$. Because of dimension reasons we have $N = G_{1x}$. Because $Spin(2l_1 + 1)/H_0$ is not orientable and $M^{T^o}$ is orientable, $G_{1x} = Spin(2l_1)$ follows. The case $G_1 = SU(2)$ is included in the discussion in this paragraph because $SU(2) = Spin(3)$.

Now we prove the last statement of the lemma. If $G_1 = SU(4)$, then $G_{1x}$ is $G_1$-equivariantly diffeomorphic to $\mathbb{C}P^3$ by the above discussion. Because $\mathbb{C}P^3$ has four characteristic submanifolds with pairwise non-trivial intersections, by Lemmas [3, 2] and [3, 3] there are four characteristic submanifolds $M_1, \ldots, M_4$ which intersect
transversely with $G_1 x = N$. Because $G_1 x$ is a component of $M^{T_0}$ we have by Lemma B.1 that $\lambda(M_i) \not\in T_0$. Therefore $\lambda(M_i)$ is not fixed pointwise by the action of $W(G_1)$ on $T$. Here $W(G_1)$ acts on $T$ by conjugation. Now it follows with Lemma B.10 that $M_1, \ldots, M_t$ belong to $\mathfrak{g}_1$.

Now we turn to the case $G_1 = \text{Spin}(4) = SU(2) \times SU(2)$. Then there are the following proper closed connected maximal rank subgroups $H$ of $G_1$ of codimension at most 4:

$SU(2) \times S(U(1) \times U(1)), \ S(U(1) \times U(1)) \times SU(2), \ S(U(1) \times U(1)) \times S(U(1) \times U(1))$.

The last has codimension four in $G_1$. The others have codimension two in $G_1$. At first assume that $G_1 x$ has dimension four. Then we have

$$G_{1x}^0 = S(U(1) \times U(1)) \times S(U(1) \times U(1)).$$

There are five proper subgroups of Spin(4) which contain $S(U(1) \times U(1)) \times S(U(1) \times U(1))$ as a maximal connected subgroup, namely:

$$H'_1 = S(U(1) \times U(1)) \times S(U(1) \times U(1)),$$
$$H'_2 = N_{SU(2)} S(U(1) \times U(1)) \times S(U(1) \times U(1)),$$
$$H'_3 = S(U(1) \times U(1)) \times N_{SU(2)} S(U(1) \times U(1)),$$
$$H'_4 = N_{SU(2)} S(U(1) \times U(1)) \times N_{SU(2)} S(U(1) \times U(1)),$$
$$H'_5 = \{(g_1, g_2) \in N_{SU(2)} S(U(1) \times U(1)) \times N_{SU(2)} S(U(1) \times U(1)); g_1 \in S(U(1) \times U(1)) \Leftrightarrow g_2 \in S(U(1) \times U(1))\}.$$

Therefore $G_1 x$ is $G_1$-equivariantly diffeomorphic to one of the following spaces:

$$\text{Spin}(4)/H'_1 = S^2 \times S^2,$$
$$\text{Spin}(4)/H'_2 = S^2 \times \mathbb{Z}_2 \times S^2 = \text{orientable double cover of } \mathbb{R}P^2 \times \mathbb{R}P^2,$$
$$\text{Spin}(4)/H'_3 = \mathbb{R}P^2 \times S^2,$$
$$\text{Spin}(4)/H'_4 = S^2 \times \mathbb{R}P^2,$$
$$\text{Spin}(4)/H'_5 = \mathbb{R}P^2 \times \mathbb{R}P^2.$$

Since $G_1 x = M^{T_0}$ is orientable, the latter three do not occur.

For $N = G_1 x = S^2 \times S^2, S^2 \times \mathbb{Z}_2 \times S^2$, let $N^{(1)}$ be the union of the $T$-orbits in $N$ of dimension less than or equal to one. Then $W(G_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the orbit space $N^{(1)}/T$. This space is given by one of the following graphs:

\[ (S^2 \times S^2)^{(1)}/T, \quad (S^2 \times \mathbb{Z}_2 \times S^2)^{(1)}/T \]

Here the edges correspond to orbits of dimension one and the vertices to the fixed points. The arrows indicate the action of the generators $w_1, w_2 \in W(G_1)$ on this space. Let $M_1, M_2$ be the two characteristic submanifolds of $M$ which intersect
Remark 3.2. If, in the situation of Lemma 3.1, \( T \cap G_1 \) is the standard maximal torus of \( G_1 \), then it follows by Proposition 2 of [8, p. 325] that \( G_1 x \) is conjugated to the groups given in Lemma 3.1 (1) by an element of the normalizer of the maximal torus.

**Lemma 3.3.** In the situation of the previous lemma \( x \) is contained in the intersection of exactly \( l_1 \) characteristic submanifolds belonging to \( \mathfrak{g}_1 \).

**Proof.** Because \( N = G_1 x \) has dimension \( 2l_1 \), \( x \) is contained in exactly \( l_1 \) characteristic submanifolds of \( N \). By Lemmas [B.2] and [B.3] we know that they are components of intersections of characteristic submanifolds \( M_{i_1}, \ldots, M_{i_l} \) of \( M \) with \( N \).

Because \( G_1 x \) is a component of \( M^{T^0} \), \( \lambda(M_i) \) is not a subgroup of \( T^0 \) for \( i = 1, \ldots, l_1 \) by Lemmas [B.1] and [B.3]. Therefore \( \lambda(M_i) \) is not fixed pointwise by \( W(G_1) \). By Lemma [2.10], this implies that \( M_i \) belongs to \( \mathfrak{g}_1 \).

By Lemmas [B.3] and [B.1], \( G_1 x \) is the intersection of \( l_0 \) characteristic submanifolds \( M_{i_1+1}, \ldots, M_{i_n} \) of \( M \). We show that these manifolds do not belong to \( \mathfrak{g}_1 \). Assume that there is an \( i \geq l_1 + 1 \) such that \( M_i \) belongs to \( \mathfrak{g}_1 \). Because \( W(G_1) \) acts transitively on \( \mathfrak{g}_1 \), there is a \( w \in W(G_1) \) with \( w(M_i) = M_j, j \leq l_1 \). But this is impossible because \( M_i \supset G_1 x \not\subset M_j \). \( \square \)

Now we turn to the case where there is a \( T \)-fixed point which is fixed by \( G_1 \).

**Lemma 3.4.** Let \( \tilde{G} = G_1 \times T^0 \) with \( G_1 \) elementary, rank \( G_1 = l_1 \) and \( M \) a torus manifold with \( G \)-action of dimension \( 2n = 2(l_0 + l_1) \). If there is a \( T \)-fixed point \( x \in M^T \) which is fixed by \( G_1 \), then \( G_1 = SU(l_1 + 1) \) or \( G_1 = Spin(2l_1) \).

Moreover, if \( G_1 \not= Spin(8) \) one has:

- (3.1) \( T_xM = V_1 \oplus V_2 \otimes_{C} W_1 \) if \( G_1 = SU(l_1 + 1) \) and \( \# \mathfrak{g}_1 = 4 \) in the case \( l_1 = 3 \),
- (3.2) \( T_xM = V_3 \oplus W_2 \) if \( G_1 = Spin(2l_1) \) and \( \# \mathfrak{g}_1 = 3 \) in the case \( l_1 = 3 \),

where \( W_1 \) is the standard complex representation of \( SU(l_1 + 1) \) or its dual, \( W_2 \) is the standard real representation of \( SO(2l_1) \) and the \( V_i \) are complex \( T^0 \)-representations.
In the case $G_1 = \text{Spin}(8)$, one may change the action of $G_1$ on $M$ by an automorphism of $G_1$, which is independent of $x$, to reach the situation described in (3.2).

Furthermore, we have $x \in \bigcap_{M_i \in \mathcal{S}} M_i$. If $l_1 = 1$, then we have $\#\mathcal{S}_1 = 2$.

Proof. Let $M_1, \ldots, M_n$ be the characteristic submanifolds of $M$ which intersect in $x$. Then the weight spaces of the $G$-representation $T_x M$ are given by

$$N_x(M_1, M), \ldots, N_x(M_n, M).$$

For $g \in N_G T$ we have $M_i = gM_j$ if and only if $N_x(M_i, M) = gN_x(M_j, M)$. Because $G_1$ acts non-trivially on $T_x M$, there is at least one $M_i, i \in \{1, \ldots, n\}$, such that $M_i \in \mathcal{S}_1$.

In the following a weight space of $T_x M$ together with a choice of an orientation for this weight space is called an oriented weight space of $T_x M$. The action of $G_1$ on $T_x M$ induces an action of $W(G_1)$ on the set of oriented weight spaces of $T_x M$.

Because $W(G_1)$ acts transitively on $\mathcal{S}_1$ and $x$ is a $G$-fixed point, we have

$$\frac{1}{2} \# \{\text{oriented weight spaces of } T_x M \text{ which are not fixed by } W(G_1)\} = \#\mathcal{S}_1$$

and $x \in \bigcap_{M_i \in \mathcal{S}} M_i$.

For the $G$-representation $T_x M$ we have

$$T_x M = N_x(M^{T_{l_0}}, M) \oplus T_x M^{T_{l_0}}.$$

If $l_0 = 0$, then we have $N_x(M^{T_{l_0}}, M) = \{0\}$. Otherwise the action of $T^{l_0}$ induces a complex structure on $N_x(M^{T_{l_0}}, M)$. By [4, p. 68] and [4, p. 82], we have

$$N_x(M^{T_{l_0}}, M) = \bigoplus_i V_i \otimes_C W_i,$$

where the $V_i$ are one-dimensional complex $T^{l_0}$-representations and the $W_i$ are irreducible complex $G_1$-representations. Since $T^{l_0}$ acts almost effectively on $M$, there are at least $n - l_1$ summands in this decomposition. Therefore we get

$$\dim_C W_i = \dim_C N_x(M^{T_{l_0}}, M) - \sum_{j \neq i} \dim_C V_j \otimes_C W_j \leq n - (n - l_1 - 1) = l_1 + 1.$$

Furthermore,

$$\dim_R T_x M^{T_{l_0}} \leq 2(n - l_0) = 2l_1.$$

If there is a $W_{i_0}$ with $\dim_C W_{i_0} = l_1 + 1$, then from equation (3.5) we get, for all other $W_i$,

$$\dim_C W_i = \dim_C N_x(M^{T_{l_0}}, M) - \dim_C V_{i_0} \otimes_C W_{i_0} - \sum_{j \neq i, i_0} \dim_C V_j \otimes_C W_j \leq 1.$$

So they are one-dimensional. Therefore they are trivial. Furthermore, we have

$$\dim_C N_x(M^{T_{l_0}}, M) = \sum_i \dim_C V_i \otimes_C W_i \geq n$$

because there are at least $n - l_1$ summands in the decomposition (3.5). Therefore $T_x M^{T_{l_0}}$ is zero-dimensional in this case.
If $\dim_R T_xM^{T^0} = 2l_1$, then we have
$$\dim_C W_i = \dim_C N_x(M^{T^0}, M) - \sum_{j \neq i} \dim_C V_j \otimes_C W_j \leq 1.$$ 
Therefore all $W_i$ are one-dimensional, so they are trivial in this case.

There are the following lower bounds $d_R$, $d_C$ for the dimension of real and complex non-trivial irreducible representations of $G_1$ [19 pp. 53-54]:

<table>
<thead>
<tr>
<th>$G_1$</th>
<th>$d_R$</th>
<th>$d_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(2) = Spin(3) = Sp(1)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$Spin(4)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$Spin(5) = Sp(2)$</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$SU(4) = Spin(6)$</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$SU(l_1 + 1), l_1 \neq 1, 3$</td>
<td>$2l_1 + 2$</td>
<td>$l_1 + 1$</td>
</tr>
<tr>
<td>$Spin(2l_1 + 1), l_1 &gt; 2$</td>
<td>$2l_1 + 1$</td>
<td>$2l_1 + 1$</td>
</tr>
<tr>
<td>$Sp(l_1), l_1 &gt; 2$</td>
<td>$2l_1 + 1$</td>
<td>$2l_1$</td>
</tr>
</tbody>
</table>

In [19 pp. 53-54] the dominant weights of the $G_1$-representations realising these bounds are also given. They are important in the discussion below.

Because $G_1$ acts non-trivially on $T_xM$, one of the $W_i$’s or $T_xM^{T^0}$ is a non-trivial $G_1$-representation. Therefore we have $d_R \leq 2l_1$ or $d_C \leq l_1 + 1$ by (3.6) and (3.7). Therefore $G_1 \neq Sp(l_1), l_1 > 1$, and $G_1 \neq Spin(2l_1 + 1), l_1 > 1$.

If $G_1 = Spin(2l_1), l_1 > 3$, then all $W_i$ are trivial because
$$\dim_C W_i \leq l_1 + 1 < 2l_1 = d_C.$$ 
Moreover, $T_xM^{T^0}$ has dimension $2l_1$. Therefore it is the standard real $SO(2l_1)$-representation if $l_1 > 4$. If $l_1 = 4$, then there are three eight-dimensional real representations of Spin(8), namely the standard real $SO(8)$-representation and the two half spinor representations. They have three different kernels. Notice that the kernel of the $G_1$-representation $T_xM^{T^0}$ is equal to the kernel of the $G_1$-action on $M$. Therefore, if one of them is isomorphic to $T_xM^{T^0}$, then it is isomorphic to $T_yM^{T^0}$ for all $y \in M^T$. So we may -- after changing the action of Spin(8) on $M$ by an automorphism -- assume that $T_xM^{T^0}$ is the standard real $SO(8)$-representation.

If $G_1 = SU(l_1 + 1), l_1 \neq 1, 3$, then only one $W_i$ is non-trivial and $T_xM^{T^0}$ has dimension zero. The non-trivial $W_i$ is the standard representation of $SU(l_1 + 1)$ or its dual depending on the complex structure of $N_x(M^{T^0}, M)$.

If $G_1 = SU(4)$, then there is one real representation of dimension 6 and two complex representations of dimension 4. If the first representation occurs in the decomposition of $T_xM$, then, by (3.3), we have $\# 3_1 = 3$. If one of the others occurs, then $\# 3_1 = 4$.

If $G_1 = SU(2)$, then there is one non-trivial $W_i$ of dimension 2. Therefore, by (3.3), one has $\# 3_1 = 2$.

If $G_1 = Spin(4)$, then $T_xM$ is almost faithful representation. Because all almost faithful complex representations of Spin(4) have at least dimension four, there is no $W_i$ of dimension three.

If there is one $W_i$ of dimension two, then we see as in (3.3) that all other $W_i$ and $T_xM^{T^0}$ have dimension less than or equal to two. Because there is no non-trivial two-dimensional real Spin(4)-representation, there is another $W_i$ of dimension two.
Therefore there are eight oriented weight spaces of $T_x M$ which are not fixed by the action on $W(G_1)$. But this contradicts (3.3) because $\#\tilde{S}_1 = 2$.

Therefore all $W_i$ are one-dimensional. Hence, they are trivial. $T_z M^{T_0}$ has to be the standard four-dimensional real representation of Spin(4).

With the Lemmas 3.1 and 3.4, we see that there is no elementary factor of $\tilde{G}$, which is isomorphic to $Sp(l_1)$ for $l_1 > 2$.

Now let $G_1 = \text{Spin}(2l)$. If $l = 3$, we assume $\#\tilde{S}_1 = 3$. Then, by looking at the $G_1$-representation $T_x M$, one sees with Lemma 3.4 that the $G_1$-action factors through $SO(2l)$.

Now let $G_1 = \text{Spin}(2l+1)$, $l > 1$. Then, by Lemma 3.1, we have $G_1x \cong \text{Spin}(2l)$. Because the $G_1x$-action on $N_x(G_1x, M)$ is trivial by Lemma 3.4, the $G_1$-action factors through $SO(2l+1)$.

In the case $G_1 = \text{Spin}(3)$ and $\#\tilde{S}_1 = 1$ we have $G_1x = S^2$. The characteristic submanifold $M_1 \in \tilde{S}_1$ intersects $G_1x$ transversely in $x$. Because $\#\tilde{S}_1 = 1$, $\lambda(M_1)$ is invariant under the action of $W(G_1)$ on the maximal torus of $G$. Because, by Lemma 3.10 the non-trivial element of $W(G_1)$ reverses the orientation of $\lambda(M_1)$, it is a maximal torus of $G_1$. Therefore the center of $G_1$ acts trivially on $M$. Hence, the $G_1$-action on $M$ factors through $SO(3)$.

If in the case $G_1 = \text{Spin}(3)$ and $\#\tilde{S}_1 = 2$ the principal orbit type of the $G_1$-action is given by $\text{Spin}(3)/\text{Spin}(2)$, then the $G_1$-action factors through $SO(3)$.

Therefore in the following we may replace an elementary factor $G_i$ of $\tilde{G}$ isomorphic to Spin$(l)$, which satisfies the above conditions, by $SO(l)$.

**Convention 3.5.** If we say that an elementary factor $G_i$ is isomorphic to $SU(2)$ or $SU(4)$, then we mean that $\#\tilde{S}_i = 2$ or $\#\tilde{S}_i = 4$, respectively. Conversely, if we say that $G_i$ is isomorphic to $SO(3)$ we mean that $\#\tilde{S}_i = 1$ or $\#\tilde{S}_i = 2$ and the $SO(3)$-action has principal orbit type $SO(3)/SO(2)$. If we say $G_i = SO(6)$, then we mean $\#\tilde{S}_i = 3$.

**Corollary 3.6.** Assume that $G$ is elementary. Then $M$ is equivariantly diffeomorphic to $\mathbb{C}P^{d_1}$ or $M = S^{2d_1}$ if $\tilde{G} = SU(l_1 + 1)$ or $\tilde{G} = SO(2l_1 + 1), SO(2l_1)$, respectively.

**Proof.** If $G$ is elementary, we may assume that $G = \tilde{G} = SO(2l_1), SO(2l_1 + 1), SU(l_1 + 1)$ and dim $M = 2l_1$.

If $G = SO(2l_1)$, then, by Lemmas 3.1 and 3.4, the principal orbit type of the $SO(2l_1)$-action is given by $SO(2l_1)/SO(2l_1 - 1)$, which has codimension one in $M$.

The group $SO(2l_1 - 1) \times O(1))$ is the only proper subgroup of $SO(2l_1)$ which contains $SO(2l_1 - 1)$ properly. Because $SO(2l_1)/SO(2l_1 - 1) \times O(1)) = \mathbb{R}P^{2l_1 - 1}$ is orientable, all orbits of the $SO(2l_1)$-action are of types $SO(2l_1)/SO(2l_1 - 1) = SO(2l_1)/SO(2l_1 - 1)$.

By 3 pp. 206-207, we have

$$M = D^{2l_1}_1 \cup_\phi D^{2l_1}_2,$$

where $SO(2l_1)$ acts on the disks $D^{2l_1}_i$ in the usual way and

$$\phi : S^{2l_1 - 1} \to SO(2l_1)/SO(2l_1 - 1) \to S^{2l_1 - 1} = SO(2l_1)/SO(2l_1 - 1)$$

is given by $gSO(2l_1 - 1) \mapsto gnSO(2l_1 - 1)$, where $n \in N_{SO(2l_1)}SO(2l_1 - 1) = S(O(2l_1 - 1) \times O(1))$.

Therefore $\phi = \pm \text{Id}_{S^{2l_1 - 1}}$ and $M = S^{2l_1}$. 


If $G = SO(2l_1 + 1)$, then
\[ M = SO(2l_1 + 1)/SO(2l_1) = S^{2l_1}. \]
follows directly from Lemmas 3.1 and 3.4.

Now assume $G = SU(l_1 + 1)$. Because $\dim M = 2l_1$, the intersection of $l_1 + 1$ pairwise distinct characteristic submanifolds of $M$ is empty. By Lemma 3.4 no $T$-fixed point is fixed by $G$. Therefore from Lemma 3.1 we get
\[ M = SU(l_1 + 1)/S(U(l_1) \times U(1)) = \mathbb{C}P^1. \]

\[ \square \]

**Remark 3.7.** Another proof of this statement follows from the classification given in section 8

### 4. Blowing up

In this section we describe blow ups of torus manifolds with $G$-action. They are used in the following sections to construct from a torus manifold $M$ with $G$-action another torus manifold $\hat{M}$ with $G$-action, such that an elementary factor of the covering group $\hat{G}$ of $G$ has no fixed point in $\hat{M}$.

References for this construction are [17, pp. 602-611] and [18, pp. 269-270].

As before we write $\hat{G} = \prod_{i=1}^k G_i \times T^\alpha$ with $G_i$ elementary and $T^\alpha$ a torus.

We will see in sections 5 and 7 that there are the following two cases:

1. A component $N$ of $M^{G_1}$ has odd codimension in $M$.
2. A component $N$ of $M^{G_1}$ has even codimension in $M$, and there is a $g \in Z(\hat{G})$ such that $g$ acts trivially on $N$ and $g^2$ acts as $-\text{Id}$ on $N(N,M)$.

In the second case the action of $g$ on $N(N,M)$ induces a $G$-invariant complex structure. We equip $N(N,M)$ with this structure. Let $E = N(N,M) \oplus K$, where $K = \mathbb{R}$ in the first case and $K = \mathbb{C}$ in the second case.

In the following we call case (1) the real case and case (2) the complex case.

**Lemma 4.1.** The projectivication $P_K(E)$ is orientable.

**Proof.** Because $M$ is orientable the total space of the normal bundle of $N$ in $M$ is orientable. Therefore
\[ E = N(N,M) \oplus K = N(N,M) \times \mathbb{K} \]
and the associated sphere bundle $S(E)$ are orientable.

Let $Z_K = \mathbb{Z}/2\mathbb{Z}$ if $K = \mathbb{R}$ and $Z_K = S^1$ if $K = \mathbb{C}$. Then $Z_K$ acts on $E$ and $S(E)$ by multiplication on the fibers. Now $P_K(E)$ is given by $S(E)/Z_K$. If $K = \mathbb{C}$, then $Z_K$ is connected. Therefore it acts orientation preserving on $S(E)$.

If $K = \mathbb{R}$, then $\dim E$ is even. Therefore the restriction of the $Z_K$-action to a fiber of $E$ is orientation preserving. Hence, it preserves the orientation of $S(E)$.

Because the action of $Z_K$ is orientation preserving on $S(E)$, $P_K(E)$ is orientable.

\[ \square \]

Choose a $G$-invariant Riemannian metric on $N(N,M)$ and a $G$-equivariant closed tubular neighborhood $B$ around $N$. Then one may identify
\[ B = \{ z_0 \in N(N,M); |z_0| \leq 1 \} = \{ (z_0 : 1) \in P_K(E); |z_0| \leq 1 \}. \]

By gluing the complements of the interior of $B$ in $M$ and $P_K(E)$ along the boundary of $B$, we get a new torus manifold with $G$-action $\hat{M}$, the blow up of
$M$ along $N$. It is easy to see, using isotopies of tubular neighborhoods, that the $G$-equivariant diffeomorphism type of $\tilde{M}$ does not depend on the choices of the Riemannian metric and the tubular neighborhood.

$\tilde{M}$ is oriented in such a way that the induced orientation on $M - \tilde{B}$ coincides with the orientation induced from $M$. This forces the inclusion of $P_K(E) - \tilde{B}$ to be orientation reversing. Because $G_1$ is elementary there is no one-dimensional $G_1$-invariant subbundle of $N(N, M)$. Therefore we have $\#\pi_0(M^{G_1}) = \#\pi_0(M^{G_1}) - 1$.

So by iterating this process over all components of $M^{G_1}$ one ends up at a torus manifold $\tilde{M}'$ with $G$-action without $G_1$-fixed points. In the following we will call $\tilde{M}'$ the blow up of $M$ along $M^{G_1}$.

**Lemma 4.2.** There is a $G$-equivariant map $F : \tilde{M} \to M$ which maps the exceptional submanifold $M_0 = P_K(N(N, M) \oplus \{0\})$ to $N$ and is the identity on $M - \tilde{B}$. Moreover, $F$ restricts to a diffeomorphism $\tilde{M} - M_0 \to M - N$. Its restriction to $M_0$ is the bundle projection $P_K(N(N, M) \oplus \{0\}) \to N$.

**Proof.** The $G$-equivariant map

$$f : P_K(E) - \tilde{B} \to B \ (z_0 : z_1) \mapsto (z_0 \tilde{z}_1 : |z_0|^2) \ (z_0 \in N(N, M), z_1 \in \mathbb{K})$$

is the identity on $\partial B$. Therefore it may be extended to a continuous map $h : \tilde{M} \to M$ which is the identity outside of $P_K(E) - \tilde{B}$.

Because $f|_{P_K(E) - \tilde{B} - M_0} : P_K(E) - \tilde{B} - M_0 \to B - N$ is a diffeomorphism there is a $G$-equivariant diffeomorphism $F' : \tilde{M} - M_0 \to M - N$ which is the identity outside $P_K(E) - \tilde{B} - M_0$ and coincides with $f$ near $M_0$ by [10, pp. 24-25]. Therefore $F'$ extends to a differentiable map $F : \tilde{M} \to M$ such that $F|_{M_0} = f|_{M_0}$ is the bundle projection. \hfill $\square$

**Lemma 4.3.** Let $H$ be a closed subgroup of $G$. Then there is a bijection

$$\{ \text{components of } M^H \not\subset N \} \to \{ \text{components of } \tilde{M}^H \not\subset M_0 \}$$

such that

$$N' \leftrightarrow \tilde{N}' = \left( P_K(N(N \cap N', N') \oplus \mathbb{K}) - \tilde{B} \right) \cup_{\partial B \cap N'} \left( N' - \tilde{B} \right)$$

and its inverse is given by

$$F(N'') \leftrightarrow N''$$,

where $N'$ is a component of $M^H$ and $N''$ is one of $\tilde{M}^H$. Here $F(N'')$ is the image of $N''$ under the map $F$ defined in Lemma 4.2. For a component $N'$ of $M^H$, we call $\tilde{N}'$ the proper transform of $N'$.

**Proof.** At first we calculate the fixed point set of the $H$-action on $M$:

$$\tilde{M}^H = \left( \left( P_K(E) - \tilde{B} \right) \cup_{\partial B} \left( M - \tilde{B} \right) \right)^H = \left( P_K(E) - \tilde{B} \right)^H \cup_{\partial B^H} \left( M - \tilde{B} \right)^H.$$ 

Because $H$ is compact, there are pairwise distinct $i$-dimensional non-trivial irreducible $H$-representations $V_{ij}$ and $H$-vector bundles $E_{ij}$ over $N^H$ such that

$$N(N, M)|_{N^H} = N(N, M)|_{N^H}^H \oplus \bigoplus_i \bigoplus_j E_{ij},$$

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and the $H$-representation on each fiber of $E_{ij}$ is isomorphic to $\mathbb{K}^{d_{ij}} \otimes V_{ij}$, where $\mathbb{K}^{d_{ij}}$ denotes the trivial $H$-representation of dimension $d_{ij}$.

Now the $H$-fixed points in $P_{K}(E)$ are given by

$$P_{K}(E)^{H} = P_{K}(N(N, M) \oplus \mathbb{K})^{H}_{N, H} = P_{K}(N(N, M)_{N}^{H} \oplus \mathbb{K}) \prod_{j} P_{K}(E_{ij} \oplus \{0\}).$$

Because $N(N, M)_{N}^{H} = N(N^{H}, M^{H})$ we get

$$\tilde{M}^{H} = \left(\left(\prod_{j} P_{K}(E_{ij} \oplus \{0\})\right) \cup_{\partial B^{H}} \left(M - B^{H}\right)\right)^{H}_{M^{H}} \prod_{j} P_{K}(E_{ij} \oplus \{0\}),$$

where $N'$ runs through the connected components of $M^{H}$ which are not contained in $N$. Thus the statement follows. $\square$

By replacing $H$ in Lemma 4.3 by a one-dimensional subtorus of $T$, we get:

**Corollary 4.4.** There is a bijection between the characteristic submanifolds of $M$ and the characteristic submanifolds of $\tilde{M}$, which are not contained in $M_{0}$.

**Proof.** The only thing that is to be proved here is that for a characteristic submanifold $M_{i}$ of $\tilde{M}$, $\tilde{M}_{i}^{T}$ is non-empty. If $(M_{i} - N)^{T} \neq \emptyset$, then this is clear.

If $p \in (M_{i} \cap N)^{T}$, then $\left.P_{K}(N(M_{i} \cap N, M_{i}) \oplus \{0\})\right|_{p}$ is a $T$-invariant submanifold of $M_{i}$ which is diffeomorphic to $\mathbb{C}P^{k}$ or $\mathbb{R}P^{2k}$. Therefore it contains a $T$-fixed point. $\square$

This bijection is compatible with the action of the Weyl group of $G$ on the sets of characteristic submanifolds of $\tilde{M}$ and $M$.

In the real case the exceptional submanifold $M_{0}$ has codimension one in $\tilde{M}$ and is $G$-invariant. Because there is no $S^{1}$-representation of real dimension one, $M_{0}$ does not contain a characteristic submanifold of $\tilde{M}$ in this case.

In the complex case $M_{0}$ is $G$-invariant and may be a characteristic submanifold of $\tilde{M}$.

Therefore there is a bijection between the non-trivial orbits of the $W(G)$-actions on the sets of characteristic submanifolds of $M$ and $\tilde{M}$. Hence we get the same elementary factors for the $G$-actions on $\tilde{M}$ and $M$.

**Corollary 4.5.** Let $H$ be a closed subgroup of $G$ and $N'$ be a component of $\tilde{M}^{H}$ such that $N \cap N'$ has codimension one—in the real case— or two—in the complex case—in $N'$. Then $F$ induces an $(N_{G}H)^{0}$-equivariant diffeomorphism of $N'$ and $N'$.

**Proof.** Because of the dimension assumption the $(N_{G}H)^{0}$-equivariant map

$$f_{|P_{K}(N(N \cap N', N') \oplus \mathbb{K}) - \tilde{B} \cap N'} : P_{K}(N(N \cap N', N') \oplus \mathbb{K}) - \tilde{B} \cap N' \to B \cap N'$$
Lemma 4.7. If in the complex case is that $\tilde{M} - M_0$ is a diffeomorphism. Because the restriction of $F$ to $\tilde{M} - M_0$ is a $G$-equivariant diffeomorphism, the restriction $F|_{\tilde{N}'} : \tilde{N}' - M_0 \to N' - N$ is an $(N_G H)^0$-equivariant diffeomorphism. Therefore $F|_{\tilde{N}'} : \tilde{N}' \to N'$ is a diffeomorphism. □

Lemma 4.6. In the complex case let $\tilde{E} = N(N, M)^* \oplus \mathbb{C}$, where $N(N, M)^*$ is the normal bundle of $N$ in $M$ equipped with the dual complex structure. Then there is a $G$-equivariant diffeomorphism

$$\tilde{M} \to P_{\bar{C}}(\tilde{E}) - \tilde{B} \cup_{\partial B} M - \tilde{B}.$$ 

This means that the diffeomorphism type of $\tilde{M}$ does not change if we replace the complex structure on $N(N, M)$ by its dual.

Proof. We have $P_{\bar{C}}(E) = E/\sim$ and $P_{\bar{C}}(\tilde{E}) = E'/\sim'$, where

$$(z_0, z_1) \sim (z'_0, z'_1) \iff \exists t \in \mathbb{C}^* \ (tz_0, tz_1) = (z'_0, z'_1),$$

$$(z_0, z_1) \sim' (z'_0, z'_1) \iff \exists \bar{t} \in \mathbb{C}^* \ (\bar{t}z_0, \bar{t}z_1) = (z'_0, z'_1).$$

Therefore

$$E \to E \quad (z_0, z_1) \mapsto (z_0, \bar{z}_1)$$

induces a $G$-equivariant diffeomorphism $P_{\bar{C}}(E) - \tilde{B} \to P_{\bar{C}}(\tilde{E}) - \tilde{B}$ which is the identity on $\partial B$. By [10, pp. 24-25] the result follows. □

Lemma 4.7. If in the complex case $G_1 = SU(l_1 + 1)$ and codim $N = 2l_1 + 2$ or in the real case $G_1 = SO(2l_1 + 1)$ and codim $N = 2l_1 + 1$, then $F : \tilde{M} \to M$ induces a homeomorphism $\bar{F} : M/G_1 \to M/G_1$.

Proof. Because $F|_{\tilde{M} - M_0} : \tilde{M} - M_0 \to M - N$ is an equivariant diffeomorphism and $\tilde{M}/G_1, M/G_1$ are compact Hausdorff spaces, the only thing that has to be checked is that

$$F|_{P_k(N(N, M))} : P_k(N(N, M)) \to N$$

induces a homeomorphism of the orbit spaces. But this map is just the bundle map $P_k(N(N, M)) \to N$.

If $G_1 = SU(l_1 + 1)$, then, because of dimension reasons [19, pp. 53-54], the $G_1$-representation on the fibers of $N(N, M)$ is the standard representation of $G_1$ or its dual. If $G_1 = SO(2l_1 + 1)$, then, by [19, pp. 53-54], the $G_1$-representation on the fibers of $N(N, M)$ is the standard representation of $G_1$.

Thus, in both cases the $G_1$-action on the fibers of $P_k(N(N, M)) \to N$ is transitive. Therefore the statement follows. □

Remark 4.8. All statements proved above also hold for non-connected groups of the form $G \times K$, where $K$ is a finite group and $G$ is connected if we replace $N$ by a $K$-invariant union of components of $M^{G_1}$.

Now we want to reverse the construction of a blow up. Let $A$ be a closed $G$-manifold and $E \to A$ be a $G$-vector bundle such that $G_1$ acts trivially on $A$. If $E$ is even-dimensional, we assume that there is a $g \in Z(G)$ such that $g$ acts trivially on $A$ and $g^2$ acts on $E$ as $-\text{Id}$. In this case we equip $E$ with the complex structure induced by the action of $g$. 

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Assume that \( \tilde{M} \) is a \( G \)-manifold and there is a \( G \)-equivariant embedding of \( P_K(E) \hookrightarrow M \) such that the normal bundle of \( P_K(E) \) is isomorphic to the tautological bundle over \( P_K(E) \).

Then one may identify a closed \( G \)-equivariant tubular neighborhood \( B^c \) of \( P_K(E) \) in \( \tilde{M} \) with

\[
B^c = \{ (z_0 : 1) \in P_K(E \oplus K) : |z_0| \geq 1 \} \cup \{ (z_0 : 0) \in P_K(E \oplus K) \}.
\]

By gluing the complements of the interior of \( B^c \) in \( \tilde{M} \) and \( P_K(E \oplus K) \), we get a \( G \)-manifold \( M \) such that \( A \) is \( G \)-equivariantly diffeomorphic to a union of components of \( M^{G_1} \).

We call \( M \) the blow down of \( \tilde{M} \) along \( P_K(E) \).

It is easy to see that the \( G \)-equivariant diffeomorphism type of \( M \) does not depend on the choices of a metric on \( E \) and the tubular neighborhood of \( P_K(E) \) in \( M \) if \( G_1 \) acts transitively on the fibers of \( P_K(E) \to A \).

It is also easy to see that the blow up and blow down constructions are inverse to each other.

5. The case \( G_1 = SU(l_1 + 1) \)

In this section we discuss actions of groups which have a covering group of the form \( G_1 \times G_2 \), where \( G_1 = SU(l_1 + 1) \) is elementary and \( G_2 \) acts effectively on \( M \). It turns out that the blow up of \( M \) along \( M^{G_1} \) is a fiber bundle over \( \mathbb{CP}^{l_1} \). This fact leads to our first classification result.

The assumption on \( G_2 \) is no restriction on \( G \), because one may replace any covering group \( \tilde{G} \) by the quotient \( \tilde{G}/H \) where \( H \) is a finite subgroup of \( G_2 \) acting trivially on \( M \). Following Convention 3.3 we also assume \( \# \mathfrak{F}_1 = 2 \) or \( \# \mathfrak{F}_1 = 4 \) in the cases \( G_1 = SU(2) \) or \( G_1 = SU(4) \), respectively. Furthermore, we assume after conjugating \( T \) with some element of \( G_1 \) that \( T_1 = T \cap G_1 \) is the standard maximal torus of \( G_1 \).

5.1. The \( G_1 \)-action on \( M \). We have the following lemma:

**Lemma 5.1.** Let \( M \) be a torus manifold with \( G \)-action. Suppose \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SU(l_1 + 1) \) elementary. Then the \( W(S(U(l_1) \times U(1))) \)-action on \( \mathfrak{F}_1 \) has an orbit \( \mathfrak{F}_1' \) with \( l_1 \) elements and there is a component \( N_1 \) of \( \bigcap_{i \in \mathfrak{F}_1} M_i \) which contains a \( T \)-fixed point.

**Proof.** We know that \( W(S(U(l_1 + 1)) = S_{l_1 + 1} = S(\mathfrak{F}_1) \) and \( W(S(U(l_1) \times U(1))) = S_{l_1} \subset S_{l_1 + 1} \). Therefore the first statement follows. Let \( x \in M^T \). Then, by Lemmas 3.3 and 3.4 \( x \) is contained in the intersection of \( l_1 \) characteristic submanifolds of \( M \) belonging to \( \mathfrak{F}_1 \). Because \( W(G_1) = S(\mathfrak{F}_1) \) there is a \( g \in N_G, T_1 \) such that \( gx \in \bigcap_{i \in \mathfrak{F}_1} M_i \). Therefore the second statement follows. \( \square \)

**Remark 5.2.** We will see in Lemma 5.10 that \( \bigcap_{i \in \mathfrak{F}_1} M_i \) is connected.

**Lemma 5.3.** Let \( M \) be a torus manifold with \( G \)-action. Suppose \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SU(l_1 + 1) \) elementary. Furthermore, let \( N_1 \) be as in Lemma 5.1. Then there is a group homomorphism \( \psi : S(U(l_1) \times U(1)) \to Z(G_2) \) such that, with

\[
\begin{align*}
H_0 &= SU(l_1 + 1) \times \text{im } \psi_1, \\
H_1 &= S(U(l_1) \times U(1)) \times \text{im } \psi_1, \\
H_2 &= \{ (g, \psi_1(g)) \in H_1 : g \in S(U(l_1) \times U(1)) \},
\end{align*}
\]
Denote by $H_3$ the component of the identity of the isotropy subgroup of the torus $T$ that acts effectively on $M$ and trivially on $G_1x$. Therefore, in both cases, there is a homomorphism $\psi_1 : S(U(l_1) \times U(1)) \to S_1 \to T_2$ such that, for all $g \in S(U(l_1) \times U(1))$, $(g, \psi_1(g))$ acts trivially on $T_2N_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes \omega_i$.

Hence the component of the identity of the isotropy subgroup of the torus $T$ for generic points in $N_1$ is given by

$$H_3 = \{(t, \psi_1(t)) \in T_1 \times T_2\}.$$  

With Lemma \ref{lem:3.1}, we see that

$$H_3 = \langle (\lambda(M_t); M_t \in \mathfrak{g}_1, M_t \supset N_1) \rangle.$$  

Because the Weyl group of $G_2$ acts trivially and orientation preserving on $\mathfrak{g}_1$, $\lambda(M_t), M_t \in \mathfrak{g}_1$, is pointwise fixed by the action of $W(G_2)$ on $T$ by Lemma \ref{lem:2.10}

It follows from \ref{lem:3.2} that $H_3$ is pointwise fixed by the action of $W(G_2)$ on $T$. Here $W(G_2)$ acts on $T$ by conjugation. Therefore the image of $\psi_1$ is contained in the center of $G_2$. Furthermore $\im \psi_1$ is the projection of $\lambda(M_t), M_t \in \mathfrak{g}_1$, to $T_2$.

Because $H_3$ commutes with $G_2$ it follows that $N_1$ is $G_2$-invariant. So we have proved the first and the third statement.
Now we turn to the second and fourth parts.

Because $T_xN_1 = (T_xM)^{H_2} = (T_xM)^{H_2}$, $N_1$ is a component of $M^{H_2}$. Because, by Lemma A.2, $H_1$ is the only proper closed connected subgroup of $H_0$ which contains $H_2$ properly, for $y \in N_1$ there are the following possibilities:

- $H^0_{0y} = H_0$,
- $H^0_{0y} = H_1$ and $\dim H_0y = 2l_1$,
- $H^0_{0y} = H_2$ and $\dim H_0y = 2l_1 + 1$,

where $H^0_{0y}$ is the identity component of the stabilizer of $y$ in $H_0$. If $g \in H_0$ such that $gy \in N_1$, then we have $H^0_{0gy} = gH^0_{0y}g^{-1} \in \{H_0, H_1, H_2\}$. Therefore

$$
g \in N_{H_0}H^0_{0y} = \begin{cases} H_0 & \text{if } y \in M^{H_0}, \\
H_1 & \text{if } y \notin M^{H_0} \text{ and } l_1 > 1,
N_{G_1}T_1 \times \im \psi_1 & \text{if } H^0_{0y} = H_1 \text{ and } l_1 = 1,
T_1 \times \im \psi_1 & \text{if } H^0_{0y} = H_2, l_1 = 1 \text{ and } \im \psi_1 \neq \{1\}.
\end{cases}
$$

Now let $y \in N_1$ such that $H^0_{0y} \neq H_0$. Because $N_1$ is a component of $M^{H_2}$ and $H_0y$ is $H_2$ invariant, $N_1 \cap H_0y$ is a union of some components of $(H_0y)^{H_2}$. Therefore $N_1 \cap H_0y$ is a submanifold of $M$. Moreover,

$$
T_yN_1 \cap T_yH_0y = (T_yM)^{H_2} \cap T_yH_0y = (T_yH_0y)^{H_2} = T_y(N_1 \cap H_0y).
$$

Hence,

$$
\dim T_yN_1 \cap T_yH_0y = \dim N_1 \cap H_0y \leq \dim H_1y
= \dim H_1/H^0_{0y} = \begin{cases} 0 & \text{if } H^0_{0y} = H_1, \\
1 & \text{if } H^0_{0y} = H_2 \text{ and } \im \psi_1 \neq \{1\}
\end{cases}
$$

follows. Therefore $N_1$ intersects $H_0y$ transversely in $y$. It follows, by Lemma A.5, that $GN_1 - N_1^{H_0} = H_0N_1 - N_1^{H_0}$ is an open subset of $M$.

Because $M$ is connected and $\codim M^{H_0} \geq 4$, $M - M^{H_0}$ is connected. Since $(M - M^{H_0}) \cap H_0N_1 = H_0N_1 - N_1^{H_0}$ is closed in $M - M^{H_0}$, we have $M - M^{H_0} = H_0N_1 - N_1^{H_0}$. Hence

$$
M = (M - M^{H_0}) \amalg M^{H_0} = 
(H_0N_1 - N_1^{H_0}) \amalg M^{H_0}
= (H_0N_1 - N_1^{H_0}) \amalg (M^{H_0} \cap N_1) \amalg (M^{H_0} - N_1^{H_0})
= H_0N_1 \amalg (M^{H_0} - N_1^{H_0}).
$$

Because $N_1$ is a component of $M^{H_2}$, $N_1^{H_0}$ is a union of components of $M^{H_0}$. Therefore $M^{H_0} - N_1^{H_0}$ is closed in $M$. Because $H_0N_1$ is closed in $M$ it follows that $M = GN_1 = H_0N_1 = G_1N_1$. \hfill \Box

The following lemma guarantees together with Lemma A.3 that if $l_1 > 1$, then the homomorphism $\psi_1$ is independent of all choices made in its construction, namely the choice of $N_1$ and of $x \in N_1^T$.

**Lemma 5.4.** In the situation of Lemma 5.3 let $T' = T_2$ or $T' = \im \psi_1$. Then the principal orbit type of the $G_1 \times T'$-action on $M$ is given by $(G_1 \times T')/H_2$. 

Proof. Let $H \subset G_1 \times T'$ be a principal isotropy subgroup. Then, by Lemma 5.3 we may assume $H \supset H_2$. Consider the projection 
\[ \pi_1 : G_1 \times T' \to G_1 \]
on the first factor.

At first we show that the restriction of $\pi_1$ to $H$ is injective. Because $(G_1 \times T')_x \cap T' = T'_x$ for all $x \in M$ and the $T'$-action on $M$ is effective, there is an $x \in M$ such that 
\[ (G_1 \times T')_x \cap T' = \{1\}. \]

Furthermore, there is a $g \in G_1 \times T'$ such that $(G_1 \times T')_x \cap gHg^{-1} = T'_x$.

Because $T'$ is contained in the center of $G_1 \times T'$, we get 
\[ gHg^{-1} \cap T' = \{1\}, \]
\[ H \cap g^{-1}T'g = \{1\}, \]
\[ H \cap T' = \{1\}. \]

Therefore the restriction of $\pi_1$ to $H$ is injective.

Furthermore, $\pi_1(H) \supset \pi_1(H_2) = S(U(l_1) \times U(1))$. Therefore, by Lemma A.1 we have 
\[ \pi_1(H) = \begin{cases} SU(l_1 + 1), S(U(l_1) \times U(1)) & \text{if } l_1 > 1, \\ SU(l_1 + 1), S(U(l_1) \times U(1)), N_{G_1}T_1 & \text{if } l_1 = 1. \end{cases} \]

There is a left inverse $\phi : \pi_1(H) \to H \hookrightarrow G_1 \times T'$ to $\pi_1|_H$. Therefore there is a group homomorphism $\psi' : \pi_1(H) \to T'$ such that 
\[ H = \phi(\pi_1(H)) = \{(g, \psi'(g)) \in G_1 \times T'; g \in \pi_1(H)\}. \]

Because $H_2$ is a subgroup of $H$, we see that $\psi'|_{S(U(l_1) \times U(1))} = \psi_1$.

At first we discuss the cases $\pi_1(H) = SU(l_1 + 1)$ and $\pi_1(H) = S(U(l_1) \times U(1))$.

Because $T'$ is abelian we have in these cases 
\[ H = \phi(\pi_1(H)) = \begin{cases} G_1 & \text{if } \pi_1(H) = SU(l_1 + 1), \\ H_2 & \text{if } \pi_1(H) = S(U(l_1) \times U(1)). \end{cases} \]

The first case does not occur because $G_1$ acts non-trivially on $M$.

Now we discuss the case $l_1 = 1$ and $\pi_1(H) = N_{G_1}T_1$. Because for $t \in T_1$ and $g \in N_{G_1}T_1$ we have 
\[ \psi'(t)^{-1} = \psi'(gtg^{-1}) = \psi'(g)\psi'(t)\psi'(g)^{-1} = \psi'(t), \]

it follows that $\psi_1$ is trivial in this case.

Let $x \in M^T$. Then it follows by the definition of $\psi_1$ in the proof of Lemma 5.3 that $x$ is not a fixed point of $G_1$. By Lemma 3.1 we know that 
\[ G_{1x} = S(U(l_1) \times U(1)) = T_1. \]

Therefore $(G_1 \times T')_x = T_1 \times T'$ is abelian. But $H$ is non-abelian if $\pi_1(H) = N_{G_1}T_1$. This is a contradiction because $H$ is conjugated to a subgroup of $(G_1 \times T')_x$. □

If $l_1 = 1$, we have $\#\mathcal{F}_1 = 2$ and $W(S(U(l_1) \times U(1))) = \{1\}$. Therefore there are two choices for $N_1$. Denote them by $M_1$ and $M_2$.

Lemma 5.5. In the situation described above let $\psi_i$ be the homomorphism constructed for $M_i$, $i = 1, 2$. Then we have $\psi_1 = \psi_2^{-1}$. 
Proof. By (5.1) and (5.2), we have
\[ \lambda(M_1) = \{(t, \psi(t)) \in H_1; \ t \in S(U(1) \times U(1))\}. \]
Now, with Lemma 2.10 we see
\[ \lambda(M_1) = g\lambda(M_2)g^{-1} = \{(t^{-1}, \psi_2(t)) \in H_1; \ t \in S(U(1) \times U(1))\} \]
\[ = \{(t, \psi_2(t^{-1})) \in H_1; \ t \in S(U(1) \times U(1))\}, \]
where \( g \in N_G, T_1 = T_1. \) Therefore the result follows. \( \square \)

**Corollary 5.6.** If in the situation of Lemma 5.3 the \( G_1 \)-action on \( M \) has no fixed point, then \( M \) is the total space of a \( G \)-equivariant fiber bundle over \( \mathbb{C}P^1 \) with fiber some torus manifold. More precisely \( M = H_0 \times H, N_1. \)

Proof. \( H_0 \times H, N_1 \) is defined to be the space \( H_0 \times N_1/ \sim_1, \) where
\[ (g_1, y_1) \sim_1 (g_2, y_2) \]
\[ \iff \exists h \in H_1 \ g_1 h^{-1} = g_2 \text{ and } hy_1 = y_2. \]
By Lemma 5.3 we have that \( M = H_0 N_1 = (H_0 \times N_1)/ \sim_2, \) where
\[ (g_1, y_1) \sim_2 (g_2, y_2) \]
\[ \iff g_1 y_1 = g_2 y_2. \]
We show that the two equivalence relations \( \sim_1, \sim_2 \) are equal.
For \( (g_1, y_1), (g_2, y_2) \in H_0 \times N_1 \) we have
\[ g_1 y_1 = g_2 y_2 \]
\[ \iff \exists h \in N_{H_0}, H_0 y_1 \ g_1 h^{-1} = g_2 \text{ and } hy_1 = y_2 \]
\[ \iff \exists h \in H_1 \ g_1 h^{-1} = g_2 \text{ and } hy_1 = y_2. \]
For the last equivalence we have to show the implication from the second to the third line. If \( l_1 > 1, \) \( N_{H_0}, H_0 y_1 \) is equal to \( H_1 \) because \( y_1 \) is not an \( H_0 \)-fixed point. So we have \( h \in H_1. \)
If \( l_1 = 1, \) then \( N_1 \) is a characteristic submanifold of \( M \) belonging to \( \mathfrak{g}_1. \) If \( H_0 y_1 \) is \( H_2 \) we are done because \( N_{H_1}, H_0 y_1 = H_1. \)
Now assume that \( H_0 y_1 = H_1 \) and there is an \( h \in N_G, T_1 \times \text{im} \psi_1 - T_1 \times \text{im} \psi_1 \) such that \( y_2 = hy_1 \in N_1. \) Then \( y_2 \in N_1 \cap N_2 \subset M_{T_1 \times \text{im} \psi_1} \), where \( N_2 \) is the other characteristic submanifold of \( M \) belonging to \( \mathfrak{g}_1. \)
As shown in the proof of Lemma 5.3 \( N_1 \) intersects \( H_0 y_2 \) transversely in \( y_2. \)
Therefore one has
\[ T_{y_2} N_1 \oplus T_{y_2} H_0 y_2 = T_{y_2} M = T_{y_2} N_2 \oplus T_{y_2} H_0 y_2 \]
as \( T_1 \times \text{im} \psi_1 \)-representations. This implies
\[ T_{y_2} N_1 = T_{y_2} N_2 \]
as \( T_1 \times \text{im} \psi_1 \)-representations. Therefore \( T_1 \times \text{im} \psi_1 \) acts trivially on both \( N_1 \) and \( N_2. \) Therefore we have \( \text{im} \psi_1 = \{1\} \) and \( \lambda(N_1) = \lambda(N_2) = T_1. \) Hence, we get a contradiction because the intersection of \( N_1 \) and \( N_2 \) is non-empty. \( \square \)

**Corollary 5.7.** In the situation of Lemma 5.3 we have \( M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{g}_1} M_i. \)
Proof: At first let \( l_1 > 1 \). By Lemma 5.3 we know \( M^{H_0} \subset M^{G_1} \subset N_1 \). Therefore \( M^{G_1} \subset \bigcap_{g \in N_{G_1}, T_1} gN_1 = \bigcap_{M_i \in \mathfrak{g}_1} M_i \). There is a \( g \in N_{G_1}, T_1 - T_1 \) with \( gH_2g^{-1} \not\subset H_1 \). Thus, the subgroup \( \langle H, gH_2g^{-1} \rangle \) of \( H_0 \), which is generated by \( H_2 \) and \( gH_2g^{-1} \), contains \( H_2 \) as a proper subgroup. Therefore \( \langle H_2, gH_2g^{-1} \rangle = H_0 \) follows by Lemma A.2. Because \( H_2 \) acts trivially on \( N_1 \), this equation implies

\[
M^{H_0} \supset \bigcap_{g \in N_{G_1}, T_1} gN_1 = \bigcap_{M_i \in \mathfrak{g}_1} M_i.
\]

Now let \( l_1 = 1 \). Then \( \mathfrak{g}_1 \) contains two characteristic submanifolds \( M_1 \) and \( M_2 \).

As in the first case one can show that \( M^{H_0} \subset M^{G_1} \subset M_1 \cap M_2 \).

So \( M^{H_0} \supset M_1 \cap M_2 \) remains to be shown. Assume that there is a \( y \in M_1 \cap M_2 - M^{H_0} \). Then we also have \( y \in M^{H_1} \). Now the above assumption leads to a contradiction as in the proof of Corollary 5.6. \( \square \)

Corollary 5.8. If in the situation of Lemma 5.3 \( \psi_1 \) is trivial, then \( M^{G_1} \) is empty. Otherwise the normal bundle of \( M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{g}_1} M_i \) possesses a \( G \)-invariant complex structure. It is induced by the action of some element \( g \in \text{im} \psi_1 \). Furthermore, it is unique up to conjugation.

Proof. If \( \psi_1 \) is trivial, then \( \langle \lambda(M_i); M_i \in \mathfrak{g}_1 \rangle \) is contained in the \( l_1 \)-dimensional maximal torus of \( G \) by Lemma 5.3. By Corollary 5.7 and Lemma B.1 it follows that \( M^{H_0} \) is empty.

If \( \psi_1 \) is non-trivial, then for \( y \in M^{H_0} \) we have

\[
N_y(M^{H_0}, M) = V_{\mathbb{C}} \oplus V_{\mathbb{R}},
\]

where \( \text{im} \psi_1 \) acts non-trivially on the \( H_0 \)-representation \( V_{\mathbb{C}} \) and trivially on the \( H_0 \)-representation \( V_{\mathbb{R}} \). Clearly \( V_{\mathbb{C}} \) has at least real dimension two, and the action of \( \text{im} \psi_1 \) induces an \( H_0 \)-invariant complex structure on \( V_{\mathbb{C}} \). Because \( M^{H_0} \) has codimension \( 2l_1 + 2 \) by Corollary 5.7 and Lemma B.1 the dimension of \( V_{\mathbb{R}} \) is at most \( 2l_1 \). So it follows from [19 pp. 53-54] that \( V_{\mathbb{R}} \) is trivial if \( l_1 \neq 3 \).

If \( l_1 = 3 \), we have \( SU(4) = \text{Spin}(6) \), and there are two possibilities:

1. \( V_{\mathbb{R}} \) is trivial.
2. \( V_{\mathbb{R}} \) is the standard representation of \( SO(6) \) and \( V_{\mathbb{C}} \) a one-dimensional complex representation of \( \text{im} \psi_1 \).

Because the principal orbits are dense in \( M \), it follows with the slice theorem that the principal orbit types of the \( H_0 \)-actions on \( N_y(M^{H_0}, M) \) and \( M \) are equal.

Therefore in the second case the principal orbit type of the \( H_0 \)-action on \( M \) is given by \( \text{Spin}(6) \times S^1/\text{Spin}(5) \times \{1\} \). Therefore we see with Lemma 5.3 that the second case does not occur.

Because of dimension reasons we get

\[
N_y(M^{H_0}, M) = V_{\mathbb{C}} = W \otimes_{\mathbb{C}} V,
\]

where \( W \) is the standard complex representation of \( SU(l_1 + 1) \) or its dual and \( V \) is a complex one-dimensional \( \text{im} \psi_1 \)-representation. Because \( \text{im} \psi_1 \subset Z(G) \), we see that \( N(M^{H_0}, M) \) has a \( G \)-invariant complex structure, which is induced by the action of some \( g \in \text{im} \psi_1 \).
Next we prove the uniqueness of this complex structure. Assume that there is another \( g' \in Z(G) \cap G_y \) whose action induces a complex structure on \( N_y(M^{H_0}, M) \). Then \( g' \) induces a \( - \)-with respect to the complex structure induced by \( g \)-complex linear \( H_0 \)-equivariant map

\[
J : N_y(M^{H_0}, M) \to N_y(M^{H_0}, M)
\]

with \( J^2 + \text{Id} = 0 \). Because \( N_y(M^{H_0}, M) \) is an irreducible \( H_0 \)-representation, it follows by Schur’s Lemma that \( J \) is multiplication with \( \pm i \). Therefore \( g' \) induces up to conjugation the same complex structure as \( g \). \( \square \)

**Corollary 5.9.** If in the situation of Lemma 5.3 \( M^{G_1} = M^{H_0} \neq \emptyset \), then \( \ker \psi_1 = SU(l_1) \).

**Proof.** Let \( y \in M^{H_0} \). Then by the proof of Corollary 5.8 we have

\[
N_y(M^{H_0}, M) = W \otimes_{\mathbb{C}} V,
\]

where \( W \) is the standard complex \( SU(l_1 + 1) \)-representation or its dual and \( V \) is a one-dimensional complex \( \psi_1 \)-representation. Furthermore, \( \text{im} \psi_1 \) acts effectively on \( M \).

Because the principal orbits are dense in \( M \), it follows by the slice theorem that the principal orbit types of the \( H_0 \)-actions on \( N_y(M^{H_0}, M) \) and \( M \) are equal. Therefore a principal isotropy subgroup of the \( H_0 \)-action on \( M \) is given by

\[
H = \left\{ (g, g_{l+1}^{\pm 1}) \in H_1; g = \begin{pmatrix} A & 0 \\ 0 & \text{g}^{l+1} \end{pmatrix} \in S(U(l_1) \times U(1)) \text{ with } A \in U(l_1) \right\}.
\]

Now the statement follows by the uniqueness of the principal orbit type and Lemmas 5.4 and A.3. \( \square \)

**Lemma 5.10.** In the situation of Lemma 5.3, the intersection \( \bigcap_{M_i \in \mathfrak{H}_1} M_i = N_1 \) is connected.

**Proof.** Let \( \tilde{M} \) be the blow up of \( M \) along \( M^{G_1} \) and \( \tilde{N}_1 \) be the proper transform of \( N_1 \) in \( M \). By Corollary 5.6 we have \( M = H_0 \times H_1, \tilde{N}_1 \), which is a fiber bundle over \( \mathbb{C}P^1 \). The characteristic submanifolds of \( M \), which are permuted by \( W(G_1) \), are given by the preimages of the characteristic submanifolds of \( \mathbb{C}P^1 \) under the bundle map. By Corollary 5.4 and the discussion following this corollary, they are also given by the proper transforms \( M_i \) of the characteristic submanifolds \( M_i \in \mathfrak{H}_1 \) of \( M \). Because \( l_1 \) characteristic submanifolds of \( \mathbb{C}P^1 \) intersect in a single point, we see that \( \bigcap_{M_i \in \mathfrak{H}_1} M_i = \tilde{N}_1 \). Therefore this intersection is connected. Because \( \bigcap_{M_i \in \mathfrak{H}_1} M_i \) is mapped by \( F \) to \( \bigcap_{M_i \in \mathfrak{H}_1} M_i \), we see that \( \bigcap_{M_i \in \mathfrak{H}_1} M_i = \tilde{N}_1 \) is connected. \( \square \)

5.2. **Blowing up along \( M^{G_1} \).** By blowing up a torus manifold \( M \) with \( G \)-action along \( M^{G_1} \), one gets a torus manifold \( \tilde{M} \) without \( G_1 \)-fixed points.

Denote by \( \tilde{N}_1 \) the proper transform of \( N_1 \) as defined in Lemma 5.1. Then by Corollary 5.5 there is an \( (H_1, G_2) \)-equivariant diffeomorphism \( F : \tilde{N}_1 \to N_1 \).

As in section 4, we denote by \( M_0 = P_C(N(M^{G_1}, M) \oplus \{0\}) \) the exceptional submanifold of \( M \). Because \( M_0 \cap \tilde{N}_1 \) is mapped by this diffeomorphism to \( M^{G_1} = M^{H_0} = N_1^{H_0}, H_1 \) acts trivially on \( M_0 \cap \tilde{N}_1 \). By Corollary 5.6 we know that \( \tilde{M} \) is diffeomorphic to \( H_0 \times H_1, \tilde{N}_1 = H_0 \times H_1 N_1 \).
A natural question arising here is: When is a torus manifold of this form a blow up of another torus manifold with $G$-action? We claim that this is the case if and only if $N_1$ has a codimension two submanifold, which is fixed by the $H_1$-action and $\ker \psi_1 = SU(l_1)$.

**Lemma 5.11.** Let $N_1$ be a torus manifold with $G_2$-action, $A$ be a closed codimension two submanifold of $N_1$, $\psi_1 \in \text{Hom}(SU(l_1) \times U(1), Z(G_2))$ such that $\text{im} \psi_1$ acts trivially on $A$ and $\ker \psi_1 = SU(l_1)$. Also let

$$
\begin{align*}
H_0 &= SU(l_1 + 1) \times \text{im} \psi_1, \\
H_1 &= SU\langle l_1 \times U(1) \rangle \times \text{im} \psi_1, \\
H_2 &= \{ (g, \psi_1(g)) ; g \in SU\langle l_1 \times U(1) \rangle \}.
\end{align*}
$$

1. Then $H_1$ acts on $N_1$ by $(g, t) x = \psi_1(g)^{-1} tx$, where $x \in N_1$ and $(g, t) \in H_1$.
2. Assume that $Z(G_2)$ acts effectively on $N_1$ and let $y \in A$ and $V$ be the one-dimensional complex $H_1$-representation $N_y(A, N_1)$. Then $V$ extends to an $l_1 + 1$-dimensional complex representation of $H_0$. Therefore there is an $l_1 + 1$-dimensional complex $G$-vector bundle $E'$ over $A$ which contains $N(A, N_1)$ as a subbundle.
3. Then the normal bundle of $H_0 / H_1 \times A$ in $H_0 \times H_1, N_1$ is isomorphic to the tautological bundle over $P_\mathbb{C}(E' \oplus \{0\})$.

The lemma guarantees together with the discussion at the end of section 4 that one can remove $H_0 / H_1 \times A$ from $H_0 \times H_1, N_1$ and replace it by $A$ to get a torus manifold with $G$-action $M$ such that $M / H_0 = A$. The blow up of $M$ along $A$ is $H_0 \times H_1, N_1$.

**Proof.** (1) is trivial.

(2) For $i = 1, \ldots, l_1 + 1$ let

$$
\lambda_i : T_1 \to S^1, \begin{pmatrix} g_1 \\ \vdots \\ g_{l_1 + 1} \end{pmatrix} \mapsto g_i
$$

and $\mu : \text{im} \psi_1 \to S^1$ be the character of the $\text{im} \psi_1$ representation $N_y(A, N_1)$. Then $\mu$ is an isomorphism.

Also, by [4, p. 176] the character ring of the maximal torus $T_1 \times \text{im} \psi_1$ of $H_1 = SU\langle l_1 \times U(1) \rangle \times \text{im} \psi_1$ is given by

$$
R(T_1 \times \text{im} \psi_1) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{l_1 + 1}, \mu, \mu^{-1}] / (\lambda_1 \cdots \lambda_{l_1 + 1} - 1).
$$

With this notation, the character of $V$ is given by $\mu \lambda_{l_1 + 1}^{\pm 1}$. Therefore the $H_0$-representation $W$ with the character $\mu \sum_{i=1}^{l_1 + 1} \lambda_i^{\pm 1}$ is $l_1 + 1$-dimensional and $V \subset W$.

Let $G_2 = G'_2 \times \text{im} \psi_1$ and $E'' = N(A, N_1)$ be equipped with the action of $G'_2$ but without the action of $H_1$. Then $E' = E'' \otimes \mathbb{C}$ $W$ is a $G$-vector bundle with the required features.
Now we turn to (3). The normal bundle of \( H_0/H_1 \times A \) in \( H_0 \times H_1, N_1 \) is given by \( H_0 \times H_1, N(A, N_1) \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
H_0 \times H_1, N(A, N_1) & \xrightarrow{f} & P_\mathbb{C}(E' \oplus \{0\}) \times E' \\
\pi_1 \downarrow & & \pi_2 \downarrow \\
H_0/H_1 \times A & \xrightarrow{g} & P_\mathbb{C}(E' \oplus \{0\}),
\end{array}
\]

where the vertical maps are the natural projections and \( f, g \) are given by

\[
f([h_1, h_2] : m) = ([m \otimes h_2 h_1 e_1], m \otimes h_2 h_1 e_1)
\]

and

\[
g([h_1, h_2], q) = [m_q \otimes h_2 h_1 e_1],
\]

where \( e_1 \in W - \{0\} \) is fixed such that for all \( g' \in S(U(l_1) \times U(1)), \psi_1(g')g' e_1 = e_1 \)

and \( m_q \neq 0 \) is some element of the fiber of \( N(A, N_1) \) over \( q \in A \).

The map \( f \) induces an isomorphism of the normal bundle of \( H_0/H_1 \times A \) in \( H_0 \times H_1, N_1 \) and the tautological bundle over \( P_\mathbb{C}(E' \oplus \{0\}) \). \( \square \)

### 5.3. Admissible triples

Now we are in the position to state our first classification theorem. To do so, we need the following definition.

**Definition 5.12.** Let \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SU(l_1 + 1) \). Then a triple \((\psi, N, A)\) with

- \( \psi \in \text{Hom}(S(U(l_1) \times U(1)), Z(G_2)) \),
- \( N \) a torus manifold with \( G_2 \)-action,
- \( A \) the empty set or a closed codimension two submanifold of \( N \), such that \( \text{im} \psi \) acts trivially on \( A \) and \( \ker \psi = SU(l_1) \) if \( A \neq \emptyset \)

is called *admissible* for \((\tilde{G}, G_1)\). We say that two admissible triples \((\psi, N, A), (\psi', N', A')\) for \((\tilde{G}, G_1)\) are equivalent if there is a \( G_2 \)-equivariant diffeomorphism \( \phi : N \to N' \) such that \( \phi(A) = A' \) and

\[
\psi = \begin{cases} 
\psi' & \text{if } l_1 > 1, \\
\psi'^{+1} & \text{if } l_1 = 1.
\end{cases}
\]

**Theorem 5.13.** Let \( \tilde{G} = G_1 \times G_2 \) with \( G_1 = SU(l_1 + 1) \). There is a one-to-one-correspondence between the \( \tilde{G} \)-equivariant diffeomorphism classes of torus manifolds with \( \tilde{G} \)-action such that \( G_1 \) is elementary and the equivalence classes of admissible triples for \((\tilde{G}, G_1)\).

**Proof.** Let \( M \) be a torus manifold with \( \tilde{G} \)-action such that \( G_1 \) is elementary. Then, by Corollaries 5.7 and 5.9 \((\psi_1, N_1, M^{H_0})\) is an admissible triple, where \( \psi_1 \) is defined as in Lemma 5.3 and \( N_1 \) is defined as in Lemma 5.1.

Let \((\psi, N, A)\) be an admissible triple for \((\tilde{G}, G_1)\). If \( A \neq \emptyset \), then, by Lemma 5.11 the blow down of \( H_0 \times_{H_1} N \) along \( H_0/H_1 \times A \) is a torus manifold with \( \tilde{G} \)-action. If \( A = \emptyset \), then we have the torus manifold \( H_0 \times_{H_1} N \).
We show that these two operations are inverse to each other. Let $M$ be a torus manifold with $G$-action. If $M_{H_0} = \emptyset$, then, by Corollary 5.6, we have $M = H_0 \times H_1 N_1$. If $M_{H_0} \neq \emptyset$, then by the discussion before Lemma 5.11 $M$ is the blow down of $H_0 \times H_1 N_1$ along $H_0/H_1 \times M_{H_0}$.

Now assume $l_1 > 1$. Let $(\psi, N, A)$ be an admissible triple with $A \neq \emptyset$ and $M$ be the blow down of $H_0 \times H_1 N$ along $H_0/H_1 \times A$. Then, by the remark after Lemma 5.11 we have $A = M_{H_0}$. By Lemma 5.10 and Corollary 4.5 we have $N = N_1$. With Lemmas 5.3 and 4.3 one sees that $\psi \equiv \psi_1$, where $\psi_1$ is the homomorphism defined in Lemma 5.3 for $M$.

Now let $(\psi, N, 0)$ be an admissible triple and $M = H_0 \times H_1 N$. Then we have $M_{H_0} = \emptyset$. By Lemma 5.10 we have $N = N_1$. As in the first case one sees $\psi \equiv \psi_1$.

Now assume $l_1 = 1$. Let $(\psi, N, A)$ be an admissible triple with $A \neq \emptyset$ and $M$ be the blow down of $H_0 \times H_1 N$ along $H_0/H_1 \times A$. Then, by the remark after Lemma 5.11 $A = M_{H_0}$. By Lemma 5.9 we have two choices for $N_1$ and $\psi \equiv \psi_1^{\pm 1}$. Because the two choices for $N_1$ lead to equivalent admissible triples we recover the equivalence class of $(\psi, N, A)$. In the case $A = \emptyset$ a similar argument completes the proof of the theorem.

**Corollary 5.14.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$. Then the torus manifolds with $G$-action such that $G_1$ is elementary and $M^{G_1} \neq \emptyset$ are given by blow downs of fiber bundles over $\mathbb{C}P^{l_1}$ with fiber some torus manifold with $G_2$-action along a submanifold of codimension two.

Now we specialise our classification result to special classes of torus manifolds.

**Theorem 5.15.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$, $M$ be a torus manifold with $G$-action and $(\psi, N, A)$ be the admissible triple for $(\tilde{G}, G_1)$ corresponding to $M$. Then $H^*(M; \mathbb{Z})$ is generated by its degree two part if and only if $H^*(N; \mathbb{Z})$ is generated by its degree two part and $A$ is connected.

**Proof.** To make the notation simpler we omit the coefficients of the cohomology in the proof. If $H^*(M)$ is generated by its degree two part, then $H^*(N)$ is generated by its degree two part by [15, p. 716]. Moreover, $A$ is connected by [13, p. 738] and Corollary 5.7.

Now assume that $H^*(N)$ is generated by its degree two part and $A = \emptyset$. Then by Poincaré duality $H_{odd}(N) = 0$. Therefore by a universal coefficient theorem $H^*(N) = Hom(H_*(N), \mathbb{Z})$ is torsion free. By Corollary 4.10 $M$ is a fiber bundle over $\mathbb{C}P^{l_1}$ with fiber $N$. Because the Serre spectral sequence of this fibration degenerates, we have

$$H^*(M) \cong H^*(\mathbb{C}P^{l_1}) \otimes H^*(N)$$

as a $H^*(\mathbb{C}P^{l_1})$-modul. Because $H^*(N)$ is generated by its degree two part, it follows that the cohomology of $M$ is generated by its degree two part.

Now we turn to the general case $A \neq \emptyset$. Then, by [13, p. 716], $H^*(A)$ is generated by its degree two part. Moreover, $H^*(N) \to H^*(A)$ is surjective. Let $\tilde{M}$ be the blow up of $M$ along $A$ and $F : \tilde{M} \to M$ be the map defined in section 4.

Because, by Lemma 4.2 $F$ is the identity outside some open tubular neighborhood of $A \times \mathbb{C}P^{l_1}$, the induced homomorphism $F^* : H^*(M, A) \to H^*(\tilde{M}, A \times \mathbb{C}P^{l_1})$ is an isomorphism by excision. Furthermore, the push forward $F_* : H^*(M) \to H^*(\tilde{M})$ is a section of $F^* : H^*(M) \to H^*(\tilde{M})$. Therefore $F^* : H^*(M) \to H^*(\tilde{M})$ is injective and $H^{odd}(M)$ vanishes.
Because $A$ is connected, we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^2(M, A \times \mathbb{C}P^l_1) & \longrightarrow & H^2(N, A) \\
\downarrow & & \downarrow \\
H^2(\mathbb{C}P^l_1) & \longrightarrow & H^2(M) \longrightarrow H^2(N) \longrightarrow 0 \\
\downarrow & & \downarrow \\
H^2(\mathbb{C}P^l_1) & \longrightarrow & H^2(A \times \mathbb{C}P^l_1) \longrightarrow H^2(A) \longrightarrow 0 \\
\downarrow & & \downarrow \\
H^3(M, A \times \mathbb{C}P^l_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

Now from the snake lemma it follows that

\[H^2(M, A) \cong F^* \cdot H^2(M, A \times \mathbb{C}P^l_1) \cong H^2(N, A)\]

and

\[H^3(M, A) \cong F^* \cdot H^3(M, A \times \mathbb{C}P^l_1) \cong 0.\]

Because $\iota_{NM} = F \circ \iota_{N\tilde{M}}$, where $\iota_{NM}, \iota_{N\tilde{M}}$ are the inclusions of $N$ in $M$ and $\tilde{M}$, the left arrow in the following diagram is an isomorphism:

\[
\begin{array}{ccc}
0 & \longrightarrow & H^2(M, A) \longrightarrow H^2(M) \longrightarrow H^2(A) \longrightarrow 0 \\
\iota_{NM} \downarrow & & \iota_{N\tilde{M}} \downarrow & & \text{Id} \downarrow \\
0 & \longrightarrow & H^2(N, A) \longrightarrow H^2(N) \longrightarrow H^2(A) \longrightarrow 0.
\end{array}
\]

Therefore it follows from the five lemma that

\[H^2(M) \cong H^2(N)\]

and

\[H^2(\tilde{M}) \cong H^2(\mathbb{C}P^l_1) \oplus H^2(N) \cong H^2(\mathbb{C}P^l_1) \oplus H^2(M).\]

Let $t \in H^2(\mathbb{C}P^l_1)$ be a generator of $H^*(\mathbb{C}P^l_1)$ and $x \in H^*(M)$. Then, because $H^*(\tilde{M})$ is generated by its degree two part, there are sums of products $x_i \in H^*(M)$ of elements of $H^2(M)$ such that

\[x = F_i F^*(x) = F_i \left( \sum F^*(x_i) t^i \right) = \sum x_i F_i(t^i).\]

Therefore it remains to show that $F_i(t^i)$ is a product of elements of $H^2(M)$.

The $l_1 + 1$ characteristic submanifolds $\tilde{M}_1, \ldots, \tilde{M}_{l_1+1}$ of $\tilde{M}$ which are permuted by $W(G_1)$ are the preimages of the characteristic submanifolds of $\mathbb{C}P^l_1$ under the projection $\tilde{M} \to \mathbb{C}P^l_1$. Therefore they can be oriented in such a way that $t$ is the Poincaré dual of each of them.
Because $F$ restricts to a diffeomorphism $M - A \times \mathbb{CP}^l \to M - A$ and $F(M_i) = M_i$, $F_i(t^i), i \leq l_1$, is the Poincaré dual $PD\left(\bigcap_{1 \leq k \leq i} M_k\right)$ of the intersection $\bigcap_{1 \leq k \leq i} M_k$ of characteristic submanifolds of $M$, which belong to $\mathcal{F}_i$. Therefore for $i \leq l_1$ we have

$$F_i(t^i) = PD\left(\bigcap_{1 \leq k \leq i} M_k\right) = F_i(t^i).$$

Because $t^i = 0$ for $i > l_1$, the statement follows. \hfill $\square$

**Theorem 5.16.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$, $M$ be a torus manifold with $\tilde{G}$-action and $(\psi, N, A)$ be the admissible triple for $(\tilde{G}, G_1)$ corresponding to $M$. Then $M$ is quasitoric if and only if $N$ is quasitoric and $A$ is connected.

**Proof.** At first assume that $M$ is quasitoric. Then $N$ is quasitoric and $A$ is connected because all intersections of characteristic submanifolds of $M$ are quasitoric and connected.

Now assume that $N$ is quasitoric and $A \subset N$ is connected. Then, by Theorem 5.15 and \cite[p. 738]{15}, the $T$-action on $M$ is locally standard and $M/T$ is a homology polytope. We have to show that $M/T$ is face preserving homeomorphic to a simple polytope.

Let $T_2 = T \cap G_2$. Then the orbit space $N/T_2$ is face preserving homeomorphic to a simple polytope $P$. Because $A$ is connected, $A/T_2$ is a facet $F_1$ of $P$.

With the notation from Lemma 5.11 let

$$B = \{(z_0 : 1) \in P_{\mathbb{C}}(E' \oplus \mathbb{C}); z_0 \in E', |z_0| \leq 1\}.$$  

Then the orbit space of the $T$-action on $B$ is given by $F_1 \times \Delta^{l_1 + 1}$.

Let $B'$ be a closed $\tilde{G}$-invariant tubular neighborhood of $H_0/H_1 \times A$ in $H_0 \times H_1$. Then the bundle projection $\partial B' \to H_0/H_1 \times A$ extends to an equivariant map

$$H_0 \times H_1 \times N - \hat{B'} \to H_0 \times H_1 \times N,$$

which induces a face preserving homeomorphism

$$\left(H_0 \times H_1 \times N - \hat{B'}\right)/T \cong P \times \Delta^{l_1}.$$

Now $M$ is given by gluing $B$ and $H_0 \times H_1 \times N - \hat{B'}$ along the boundaries $\partial B, \partial B'$. The corresponding gluing of the orbit spaces is illustrated in Figure 1 for the case

![Figure 1. The orbit space of a blow down](Image of Figure 1)
dim $N = 2$ and $l_1 = 1$. Because the gluing map $f : \partial B \to \partial B'$ is $\tilde{G}$-equivariant and $G_1$ acts transitively on the fibers of $\partial B \to A$ and $\partial B' \to A$, it induces a map

$$\hat{f} : F_1 \times \Delta^{l_1} \to \partial B/T \to \partial B'/T = F_1 \times \Delta^{l_1}, \quad (x, y) \mapsto (\hat{f}_1(x), \hat{f}_2(x, y)),$$

where $\hat{f}_1 : F_1 \to F_1$ is a face preserving homeomorphism and $\hat{f}_2 : F_1 \times \Delta^{l_1} \to \Delta^{l_1}$ such that, for all $x \in F_1$, $\hat{f}_2(x, \cdot)$ is a face preserving homeomorphism of $\Delta^{l_1}$.

Now fix embeddings

$$\Delta^{l_1+1} \hookrightarrow \mathbb{R}^{l_1+1} \text{ and } P \hookrightarrow \mathbb{R}^{n-l_1-1} \times [0, 1]$$

such that $\Delta^{l_1} \subseteq \mathbb{R}^{l_1} \times \{1\}$, $\Delta^{l_1+1} = \text{conv}(0, \Delta^{l_1})$ and $P \cap \mathbb{R}^{n-l_1-1} \times \{0\} = F_1$.

Denote by $p_1 : \mathbb{R}^{l_1+1} \to \mathbb{R}$ and $p_2 : \mathbb{R}^{n-l_1} \to \mathbb{R}$ the projections on the last coordinate. For $\epsilon > 0$ small enough, $P$ and $P \cap \{p_2 \geq \epsilon\}$ are combinatorially equivalent. Therefore there is a face preserving homeomorphism

$$g_1 : P \to P \cap \{p_2 \geq \epsilon\}$$

such that $g_1(F_1) = P \cap \{p_2 = \epsilon\}$ and $g_1(F_1) = F_i \cap \{p_2 \geq \epsilon\}$ for the other facets of $P$. The map

$$g_2 : F_1 \times [0, 1] \to P \cap \{p_2 \leq \epsilon\}, \quad (x, y) \mapsto x(1 - y) + yg_1(x)$$

is a face preserving homeomorphism with $p_2 \circ g_2(x, y) = \epsilon y$ for all $(x, y) \in F_1 \times [0, 1]$.

Now let

$$\hat{P} = P \times \Delta^{l_1+1} \cap \{p_1 = p_2\} \subseteq \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1},$$

$$\hat{P}_1 = P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \geq \epsilon\} \subseteq \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1},$$

$$\hat{P}_2 = P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \leq \epsilon\} \subseteq \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}.$$

Then there are face preserving homeomorphisms

$$h_1 : P \times \Delta^{l_1} \to \hat{P}_1 \quad (x, y) \mapsto (g_1(x), p_2(g_1(x))y)$$

and

$$h_2 : F_1 \times \Delta^{l_1+1} \to \hat{P}_2 \quad (x, y) \mapsto (g_2(x, p_1(y)), \epsilon y).$$

We claim that $\hat{P}$ and $M/T$ are face preserving homeomorphic. This is the case if

$$\hat{f}^{-1} \circ h_1^{-1} \circ h_2 : F_1 \times \Delta^{l_1} \to F_1 \times \Delta^{l_1}$$

extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. Now for $(x, y) \in F_1 \times \Delta^{l_1}$ we have

$$\hat{f}^{-1} \circ h_1^{-1} \circ h_2(x, y) = \hat{f}^{-1} \circ h_1^{-1}(g_2(x, p_1(y)), \epsilon y)$$

$$= \hat{f}^{-1} \circ h_1^{-1}(g_2(x, 1), \epsilon y)$$

$$= \hat{f}^{-1}(g_1^{-1} \circ g_2(x, 1), y)$$

$$= (\hat{f}_1^{-1}(x), (\hat{f}_2(x, \cdot))^{-1}(y)).$$

Because $\Delta^{l_1+1}$ is the cone over $\Delta^{l_1}$, this map extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. \hfill \Box

**Lemma 5.17.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$, $M$ be a torus manifold with $G$-action and $(\psi, N, A)$ be the admissible triple for $(\tilde{G}, G_1)$ corresponding to $M$. Then there is an isomorphism $\pi_1(N) \to \pi_1(M)$. 

Proof. Let \( \tilde{M} \) be the blow up of \( M \) along \( A \). Then, by [10, p. 270], there is a isomorphism \( \pi_1(\tilde{M}) \to \pi_1(M) \).

Now, by Corollary 5.6, \( M \) is the total space of a fiber bundle over \( \mathbb{C}P^{l_1} \) with fiber \( N \). Therefore there is an exact sequence

\[
\pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1}) \to \pi_1(N) \to \pi_1(\tilde{M}) \to 0.
\]

Because the torus action on \( N \) has fixed points, there is a section in this bundle. Hence, \( \pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1}) \) is surjective.

6. The case \( G_1 = SO(2l_1) \)

In this section we study torus manifolds with \( G \)-action, where \( \tilde{G} = G_1 \times G_2 \) and \( G_1 = SO(2l_1) \) is elementary. It turns out that the restriction of the action of \( G_1 \) to \( U(l_1) \) on such a manifold has the same orbits as the action of \( SO(2l_1) \). Therefore the results of the previous section may be applied to construct invariants for such manifolds. For simply connected torus manifolds with \( G \)-action these invariants determine their \( \tilde{G} \)-equivariant diffeomorphism type.

Let \( \tilde{G} = G_1 \times G_2 \), where \( G_1 = SO(2l_1) \) is elementary, and let \( M \) be a torus manifold with \( G \)-action. Then, by Lemmas 3.1 and 3.4, one sees that the principal orbit type of the \( G_1 \)-action is given by \( SO(2l_1)/SO(2l_1-1) \). Therefore the \( G_1 \)-action has only three orbit types, \( SO(2l_1)/SO(2l_1-1) \), \( SO(2l_1)/SO(2l_1-1) \times O(1) \) and \( SO(2l_1)/SO(2l_1) \). The induced action of \( U(l_1) \) has the same orbits, which are of types \( U(l_1)/U(l_1-1) \), \( U(l_1)/U(l_1-1) \times \mathbb{Z}_2 \) and \( U(l_1)/U(l_1) \), respectively. Here \( \langle U(l_1-1), \mathbb{Z}_2 \rangle \) denotes the subgroup of \( U(l_1) \), which is generated by \( U(l_1-1) \) and the diagonal matrix with all entries equal to \(-1\).

Let \( S = S^1 \). Then there is a finite covering

\[
SU(l_1) \times S \to U(l_1) \quad (A, s) \mapsto sA.
\]

So we may replace the factor \( G_1 \) of \( \tilde{G} \) by \( SU(l_1) \) and \( G_2 \) by \( S \times G_2 \) to reach the situation of the previous section.

Let \( x \in M^T \) and \( T_2 = T \cap G_2 \). Then we may assume by Lemma 3.4 that the \( G_1 \times T_2 \)-representation \( T_xM \) is given by

\[
T_xM = V \oplus W;
\]

where \( V \) is a complex representation of \( T_2 \) and \( W \) is the standard real representation of \( G_1 \). Therefore

\[
T_2M = V \oplus V_0 \otimes_{\mathbb{C}} W_0
\]

as a \( SU(l_1) \times S \times T_2 \)-representation, where \( V_0 \) is the standard complex one-dimensional representation of \( S \) and \( W_0 \) is the standard complex representation of \( SU(l_1) \).

Therefore the group homomorphism \( \psi_1 \) and the groups \( H_0, H_1, H_2 \) introduced in Lemma 5.3 have the following form:

\[
\text{im} \psi_1 = S
\]

and

\[
H_0 = SU(l_1) \times S,
H_1 = S(U(l_1-1) \times U(1)) \times S,
H_2 = \left\{ (g, g^{-1}_{l_1+1}) \in H_1 \mid g = \begin{pmatrix} A & 0 \\ 0 & g_{l_1+1} \end{pmatrix} \text{ with } A \in U(l_1-1) \right\}.
\]
Lemma 6.1. Let $M$ be a torus manifold with $U(l_1) \times G_2$-action such that all $U(l_1)$-orbits are of type $U(l_1)/U(l_1 - 1)$ or $U(l_1)/U(l_1)$. Then the action of $U(l_1) \times G_2$ on $M$ extends to an action of $SO(2l_1) \times G_2$ if and only if there is a $G_2$-equivariant $\mathbb{Z}_2$-principal bundle $P_M^\prime$ such that

$$P_M = S^1 \times_{\mathbb{Z}_2} P_M^\prime,$$

where the action of $G_2$ on $S^1$ is trivial.

Proof. If the action extends to an $SO(2l_1) \times G_2$-action, then $SO(2l_1) \times G_2$ acts on $M \times M^{U(l_1)}$. Therefore $P_M^\prime = \left( M \times M^{U(l_1)} \right)^{SO(2l_1-1)} \to X$ is such a $G_2$-equivariant $\mathbb{Z}_2$-principal bundle.

If there is such a $G_2$-equivariant $\mathbb{Z}_2$-bundle $P_M^\prime$, then by a $G_2$-equivariant version of Jänich’s Klassifikationssatz, there is a torus manifold $M'$ with $SO(2l_1) \times G_2$-action with $M'/U(l_1) = X$ and $P_M = S^1 \times_{\mathbb{Z}_2} P_M^\prime = P_M^\prime$. Therefore $M'$ and $M$ are $U(l_1) \times G_2$-equivariantly diffeomorphic.

Lemma 6.2. Let $M, M'$ be torus manifolds with $SO(2l_1) \times G_2$-action such that there are no exceptional $SO(2l_1)$-orbits and $H_1(M; \mathbb{Z})$ and $H_1(M'; \mathbb{Z})$ are torsion.
If there is a $U(l_1) \times G_2$-equivariant diffeomorphism $f : M \to M'$, then there is an $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M \to M'$. Moreover, $g$ and $f$ induce the same map on $M/U(l_1) - B$, where $B$ is a collar of $\partial(M/U(l_1))$.

**Proof.** The map $f$ induces a $G_2$-equivariant diffeomorphism $\hat{f} : X = M/\text{SO}(2l_1) \to M'/\text{SO}(2l_1)$. We use this map to identify these spaces. It follows from [3, p. 91] and the equality $H^i(X; \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ that $H^1(X; \mathbb{Z})$ is torsion. Hence, $H^1(X; \mathbb{Z}) = 0$.

Recall that for the universal principal $\mathbb{Z}_2$-bundle $P \to \mathbb{R}P^\infty$, the first Chern class of the principal $S^1$-bundle $S^1 \times_{\mathbb{Z}_2} P \to \mathbb{R}P^\infty$ is given by $\delta w_1(P)$, where $\delta : H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^2(\mathbb{R}P^\infty; \mathbb{Z})$ is the Bockstein homomorphism and $w_1(P)$ is the first Stiefel-Whitney class of $P$. By naturality, this relation also holds for any principal $\mathbb{Z}_2$-bundle over $X$. Because $H^1(X; \mathbb{Z}) = 0$, the Bockstein homomorphism $\delta : H^1(X; \mathbb{Z}_2) \to H^2(X; \mathbb{Z})$ is injective.

Hence, the principal $S^1$-bundle $P_M \to X$ has up to isomorphism at most one restriction of structure group to $\mathbb{Z}_2$. Therefore the two restrictions of the structure group induced by the $SO(2l_1)$-actions on $M, M'$ are the same up to a $G_2$-equivariant isomorphism.

Therefore, by the proof of Jänich’s Klassifikationssatz, there is an $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M \to M'$, which induces the same map as $f$ outside a neighborhood of $\partial X$. □

Now we turn to the case where there are exceptional $SO(2l_1)$-orbits. Then we have:

**Theorem 6.3.** Let $M, M'$ be two simply connected torus manifolds with $SO(2l_1) \times G_2$-action. Then $M$ and $M'$ are $SO(2l_1) \times G_2$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.

**Proof.** In this proof we take all cohomology groups with coefficients in $\mathbb{Z}$. Let $f : M \to M'$ be a $U(l_1) \times G_2$-equivariant diffeomorphism. Moreover, let $A, A'$ be the union of the exceptional $U(l_1)$-orbits in $M, M'$, respectively. Because the $U(l_1)$-representation $N_x(M^{U(l_1)}, M)$ is the standard representation for all $x \in M^{U(l_1)}$, there are invariant neighborhoods of $M^{U(l_1)}$ and $M^{U(l_1)}$ which do not contain any exceptional orbit. Hence, $A, A'$ are closed submanifolds of $M, M'$.

Denote by $D, D'$ the unit disc bundle in $N(A, M)$ and $N(A', M')$, respectively. Let $h : D \to B \subset M$ and $h' : D' \to B' \subset M'$ be $SO(2l_1) \times G_2$-equivariant tubular neighborhoods of $A$ and $A'$.

Then, by Theorems 4.6 and 8.3 of [10, pp. 10, 19], we may assume that $f(B) = B'$ and that $h'^{-1} \circ f \circ h$ is a linear map.

It is sufficient to show the following:

1. There is an $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M - \bar{B} \to M' - \bar{B}'$ such that $g$ and $f$ induce the same maps on $(\partial B)/U(l_1)$.

2. The map $g$ extends to an $SO(2l_1) \times G_2$-equivariant diffeomorphism $M \to M'$.

If $H_1(M - \bar{B})$ is torsion, we may apply the arguments from the proof of Lemma 6.2 to show (1). Therefore we show that $H_1(M - \bar{B})$ is torsion.

Let $A_1, \ldots, A_k$ be the orientable components of $A$ of codimension two in $M$. We fix orientations for each of these components and for $M$. Let $\tau_1, \ldots, \tau_k \in H^2(M)$ be the Poincaré duals for $A_1, \ldots, A_k$. Because $H_1(M) = 0$, it follows from a universal
coefficient theorem and Poincaré duality that
\[ H^2(M) \cong \text{Hom}(H_2(M), \mathbb{Z}) \cong \text{Hom}(H^{2n-2}(M), \mathbb{Z}), \]
where an isomorphism is given by
\[ \alpha \mapsto (\beta \mapsto \langle \beta \alpha, [M] \rangle). \]
Here we have \( \dim M = 2n. \) In particular, \( H^2(M) \) is torsion free.

We claim that the \( \tau_1, \ldots, \tau_k \) are linear independent. Let \( a_1, \ldots, a_k \in \mathbb{Z} \) such that
\[ 0 = \sum_{i=1}^{k} a_i \tau_i. \]
Then we have \( 0 = a_i \iota_A^* \tau_i, \) where \( \iota_A : A_i \to M \) is the inclusion. By restricting to an orbit \( O \) contained in \( A_i, \) we get
\[ 0 = a_i \iota_{O, A_i}^* \tau_i \in H^2(SO(2l_1)/SO(2l_1 - 1) \times O(1))) = \mathbb{Z}_2. \]
Because \( N(A_i, M)|_O = SO(2l_1)/SO(2l_1 - 1) \times \mathbb{Z}_2 \mathbb{R}^2 \) with \( \mathbb{Z}_2 \) acting on \( \mathbb{R}^2 \) by multiplication with \(-1, \) it follows that \( \iota_{O, A_i}^* \tau_i \neq 0. \) Therefore \( a_i \) is divisible by two.

Hence, we may replace \( a_i \mapsto \frac{1}{2} a_i \) in (6.1). Since the above arguments then hold for the new \( a_i, \) we see that the original \( a_i \) are divisible by arbitrary high powers of two. Therefore they must vanish.

There is an exact sequence
\[ H^{2n-2}(M) \to H^{2n-2}(A) \to H^{2n-1}(M, A) \to 0. \]
Because, by [3, p. 185], there are no components of \( A \) which have codimension one in \( M, \) there is an isomorphism
\[ H^{2n-2}(A) \cong \mathbb{Z}^k \oplus (\mathbb{Z}_2)^{k_1}, \]
where \( k_1 \) is the number of non-orientable components of codimension two of \( A. \) Let
\[ \phi : H^{2n-2}(A) \to \mathbb{Z}^k, \]
\[ \alpha \mapsto (\langle \alpha, [A_1] \rangle, \ldots, \langle \alpha, [A_k] \rangle). \]
Because the \( \tau_1, \ldots, \tau_k \) are linear independent, it follows that \( \phi \circ \iota_A^* : H^{2n-2}(M) \to \mathbb{Z}^k \) has rank \( k. \)

Therefore, from the exactness of the above sequence, it follows that \( H^{2n-1}(M, A) \) is torsion. By Poincaré duality and excision, it follows that \( H_1(M - B) \) is torsion. Hence we have proven (1).

Now we prove (2). By Theorem 9.4 of [10, p. 24], it is sufficient to show that
\[ k = h^{l-1} \circ g \circ h : \partial D \to \partial D' \]
extends to an \( SO(2l_1) \times G_2 \)-equivariant diffeomorphism \( D \to D'. \)

Let \( O \) be an \( SO(2l_1) \)-orbit in \( A \) and \( S \to O \) be the restriction of the sphere bundle \( \partial D \to A \) to \( O. \) Because \( f \) and \( g \) induce the same maps on the orbit space \( (\partial B)/U(l_1) \) and \( S \) is \( SO(2l_1) \)-invariant, we have \( k(S) = h^{l-1} \circ f \circ h(S) = S'. \) Because \( h^{l-1} \circ f \circ h : D \to D' \) is a linear map, we see that \( S' \) is the restriction of the sphere bundle \( \partial D' \to A' \) to an \( SO(2l_1) \)-orbit \( O'. \)

We may choose \( SO(2l_1) \)-equivariant bundle isomorphisms
\[ k_1 : SO(2l_1)/SO(2l_1 - 1) \times \mathbb{Z}_2 S^m \to S \]
and
\[ k'_1 : SO(2l_1)/SO(2l_1 - 1) \times \mathbb{Z}_2 S^m \to S'. \]
Because $f$ and $g$ induce the same maps on the orbit space $S/\SO(2l_1) = S^m/\mathbb{Z}_2 = \mathbb{R}^m$ and $h^{-1} \circ f \circ h$ is a linear map, it follows that $k_l^{-1} \circ k \circ k_1$ is of the form

$$[g \SO(2l_1 - 1), x] \mapsto [g z \SO(2l_1 - 1), \pm Ax] = [g \SO(2l_1 - 1), \pm Ax],$$

where $z \in S(O(2l_1 - 1) \times O(1))/\SO(2l_1 - 1) = \mathbb{Z}_2$ and $A \in O(m + 1)$. Therefore $k$ is linear on each fiber. Hence, it extends to an $\SO(2l_1) \times G_2$-equivariant diffeomorphism $D \to D'$.

Let $M$ be a simply connected torus manifold with $\SO(2l_1) \times G_2$-action. By Theorem 6.13 there is an admissible triple $(\psi, N, A)$ corresponding to $M$ equipped with the action of $SU(l_1) \times S \times G_2$ as above. The admissible triple $(\psi, N, A)$ determines the $SU(l_1) \times S \times G_2$-equivariant diffeomorphism type of $M$. With Theorem 6.3 we see that the $\SO(2l_1) \times G_2$-equivariant diffeomorphism type of $M$ is determined by $(\psi, N, A)$.

**Lemma 6.4.** Let $M$ be a torus manifold with $G_1 \times G_2$-action, where $G_1 = \SO(2l_1)$ is elementary and $G_2$ is a not necessarily connected Lie group. If $M^{\SO(2l_1)}$ is connected, then $G_2$ acts orientation preserving on $N(M^{\SO(2l_1)}, M)$. Therefore $G_2$ acts orientation preserving on $M$ if and only if it acts orientation preserving on $M^{\SO(2l_1)}$.

**Proof.** Let $g \in G_2$, $x \in M^{\SO(2l_1)}$ and $y = gx \in M^{\SO(2l_1)}$. Because $M^{\SO(2l_1)}$ is connected there is an orientation preserving $\SO(2l_1)$-invariant isomorphism

$$N_x(M^{\SO(2l_1)}, M) \cong N_y(M^{\SO(2l_1)}, M).$$

Therefore $g : N_x(M^{\SO(2l_1)}, M) \to N_y(M^{\SO(2l_1)}, M)$ induces an automorphism $\phi$ of the $\SO(2l_1)$-representation $N_x(M^{\SO(2l_1)}, M)$ which is orientation preserving if and only if $g$ is orientation preserving.

Because, by Lemma 6.3, $N_x(M^{\SO(2l_1)}, M)$ is just the standard real representation of $\SO(2l_1)$, its complexification $N_x(M^{\SO(2l_1)}, M) \otimes \mathbb{C}$ is an irreducible complex representation. Therefore, by Schur’s Lemma, there is a $\lambda \in \mathbb{C} - \{0\}$ such that for all $a \in N_x(M^{\SO(2l_1)}, M)$,

$$\phi(a) \otimes 1 = \phi c(a \otimes 1) = a \otimes \lambda.$$

This equation implies that $\lambda \in \mathbb{R} - \{0\}$ and $\phi(a) = \lambda a$. Therefore $\phi$ is orientation preserving.

7. The case $G_1 = \SO(2l_1 + 1)$

In this section we discuss actions of groups, which have a covering group, whose action on $M$ factors through $\tilde{G} = G_1 \times G_2$ with $G_1 = \SO(2l_1 + 1)$ elementary. In the case $G_1 = \SO(3)$ we also assume $\# \mathbb{F}_1 = 1$ or that the principal orbit type of the $\SO(3)$-action on $M$ is given by $\SO(3)/\SO(2)$.

It is shown that a torus manifold with $\tilde{G}$-action is a product of a sphere and a torus manifold with $G_2$-action or the blow up along the fixed points of $G_1$ is a fiber bundle over a real projective space.

We assume that $T_1 = T \cap G_1$ is the standard maximal torus of $G_1$. 


determines the SU
\( \text{Lemma 6.4.} \)
7.1. The $G_1$-action on $M$.

**Lemma 7.1.** Let $\hat{G} = G_1 \times G_2$ with $G_1 = SO(2l_1+1)$ and let $M$ be a torus manifold with $G$-action such that $G_1$ is elementary. If $l_1 > 1$ there is, by Lemma 3.3 a component $N_1$ of $\bigcap_{i \in S_1} M_i$ with $N_1^T \neq \emptyset$. If $l_1 = 1$ let $N_1$ be a characteristic submanifold belonging to $S_1$. Then:

1. $N_1$ is a component of $M^{SO(2l_1)}$.
2. $M = G_1 N_1$.

**Proof.** Let $x \in N_1^T$. Then, by Lemmas 3.1, 3.4 and Remark 3.2, $G_1 x = SO(2l_1)$.

Let $T_2$ be the maximal torus $T \cap G_2$ of $G_2$. On the tangent space of $M$ in $x$ we have the $SO(2l_1) \times T_2$-representation

$$T_x M = N_x (G_1 x, M) \oplus T_x G_1 x.$$  

By Lemma 3.1, $T_2$ acts trivially on $G_1 x$. Moreover, $T_2$ acts almost effectively on $N_x (G_1 x, M)$. Therefore it follows by dimensional reasons that $N_x (G_1 x, M)$ splits as a sum of complex one-dimensional $SO(2l_1) \times T_2$-representations. If $l_1 > 1$, $SO(2l_1)$ has no non-trivial one-dimensional complex representations. Therefore we have

$$T_x M = \bigoplus V_i \oplus W,$$

where the $V_i$ are one-dimensional complex representations of $T_2$ and $W$ is the standard real representation of $SO(2l_1)$.

If $l_1 = 1$ and $\# S_1 = 2$, then $SO(2l_1)$ acts trivially on $N_x (G_1 x, M)$ because $SO(3)/SO(2)$ is the principal orbit type of the $SO(3)$-action on $M$ [3, p. 181].

If $l_1 = 1$ and $\# S_1 = 1$, then, by the discussion leading to Convention 3.5, $SO(2)$ acts trivially on $N_x (G_1 x, M)$. Therefore in these cases $T_x M$ splits as in (7.1).

Because $N_x (G_1 x, M)$ is the tangent space of $N_1$ in $x$ the maximal torus $T_1$ of $G_1$ acts trivially on $N_1$. Therefore $N_1$ is the component of $M^{T_1}$ which contains $x$. Because $T_x N_1 = (T_x M)^{T_1} = (T_x M)^{SO(2l_1)}$, $N_1$ is a component of $M^{SO(2l_1)}$.

Now we prove (2). Let $y \in N_1$. Then there are the following possibilities:

- $G_{1y} = G_1$.
- $G_{1y} = S(O(2l_1) \times O(1))$ and $\dim G_{1y} = 2l_1$.
- $G_{1y} = SO(2l_1)$ and $\dim G_{1y} = 2l_1$.

If $g \in G_1$ such that $gy \in N_1$, then

$$gG_{1y}g^{-1} = G_{1yy} \in \{ S(O(2l_1) \times O(1)), SO(2l_1), G_1 \}$$

and

$$y \in N_{G_1} G_{1y} = \begin{cases} G_1 & \text{if } y \in M^{G_1}, \\ S(O(2l_1) \times O(1)) & \text{if } y \notin M^{G_1}. \end{cases}$$

Therefore $G_{1y} \cap N_1 \subset S(O(2l_1) \times O(1))y$ contains at most two elements. If $y$ is not fixed by $G_1$, then one sees as in the proof of Lemma 3.3 that $G_{1y}$ and $N_1$ intersect transversely in $y$.

Therefore $G_1 (N_1 - N_1^{G_1})$ is open in $M - M^{G_1}$ by Lemma A.3. Because $M^{G_1}$ has codimension at least three, $M - M^{G_1}$ is connected. But

$$G_1 \left( N_1 - N_1^{G_1} \right) = G_1 N_1 \cap (M - M^{G_1})$$

is also closed in $M - M^{G_1}$. Hence,

$$M - M^{G_1} = G_1 \left( N_1 - N_1^{G_1} \right) = G_1 N_1 - N_1^{G_1}.$$
Therefore one sees as in the proof of Lemma 5.3 that
\[ M = G_1 N_1 \Pi \left( M^{G_1} - N_1^{G_1} \right). \]

Because \( G_1 N_1 \) and \( M^{G_1} - N_1^{G_1} \) are closed in \( M \), the statement follows. \( \square \)

**Corollary 7.2.** If in the situation of Lemma 7.1 the \( G_1 \)-action on \( M \) has no fixed point in \( M \), then \( M = SO(2l_1 + 1)/SO(2l_1) \times N_1 \) or \( M = SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 \)

\[ \text{where } \mathbb{Z}_2 = S(O(2l_1) \times O(1))/SO(2l_1). \]

In the second case the \( \mathbb{Z}_2 \)-action on \( N_1 \) is orientation reversing.

If \( l_1 = 1 \) and \( \# \mathcal{F}_1 = 1 \), then we have \( M = SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 N_1 \). If \( l_1 = 1 \) and \( \# \mathcal{F}_1 = 2 \), then we have \( M = SO(2l_1 + 1)/SO(2l_1) \times N_1 \).

**Proof.** Let \( g \in S(O(2l_1) \times O(1)) = N_{G_1} SO(2l_1) \). Then \( gN_1 \) is a component of \( M^{SO(2l_1)} \). Because \( N_1 \subset M^{SO(2l_1)} \), \( gN_1 \) only depends on the class
\[ gSO(2l_1) \in S(O(2l_1) \times O(1))/SO(2l_1) = \mathbb{Z}_2. \]

Therefore there are two cases:

1. There is a \( g \in S(O(2l_1) \times O(1)) \) such that \( gN_1 \neq N_1 \).
2. The submanifold \( N_1 \) is \( SO(2l_1) \times O(1) \)-invariant, i.e. \( gN_1 = N_1 \) for all \( g \in S(O(2l_1) \times O(1)) \).

If \( l_1 = 1 \) and \( \# \mathcal{F}_1 = 1 \), then \( N_1 \) is the only characteristic submanifold of \( M \) belonging to \( \mathcal{F}_1 \). Therefore only the second case occurs.

If \( l_1 = 1 \) and \( \# \mathcal{F}_1 = 2 \), then there is a \( g_1 \in N_{G_1} T_1 \) such that \( N_1 \neq g_1 N_1 \). Therefore we are in the first case.

In general we have \( M = G_1 \times N_1 / \sim \) with
\[
\begin{align*}
(g_1, y_1) & \sim (g_2, y_2) \\
g_1 y_1 & = g_2 y_2 \\
g_2^{-1} g_1 y_1 & = y_2 \\
g_2^{-1} g_1 & \in S(O(2l_1) \times O(1)) \text{ and } g_2^{-1} g_1 y_1 = y_2.
\end{align*}
\]

In case \( \mathcal{F}_1 \) the last statement is equivalent to
\[
g_2^{-1} g_1 \in SO(2l_1) \text{ and } g_2^{-1} g_1 y_1 = y_2.
\]

Therefore we get \( M = SO(2l_1 + 1)/SO(2l_1) \times N_1 \).

In case \( \mathcal{F}_1 \) we have as in the proof of Corollary 5.6
\[
M = SO(2l_1 + 1) \times S(O(2l_1) \times O(1)) N_1 = SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 N_1.
\]

This equation implies that \( M \) is the orbit space of a diagonal \( \mathbb{Z}_2 \)-action on
\[
SO(2l_1 + 1)/SO(2l_1) \times N_1.
\]

Because \( M \) is orientable this action has to be orientation preserving. But the \( \mathbb{Z}_2 \)-action on \( SO(2l_1 + 1)/SO(2l_1) \) is orientation reversing. Therefore the \( \mathbb{Z}_2 \)-action on \( N_1 \) is also orientation reversing. \( \square \)

**Corollary 7.3.** In the situation of Lemma 7.1 \( M^{G_1} \subset N_1 \) is empty or has codimension one in \( N_1 \).
Proof. By Lemma 7.1 it is clear that $M^{G_1} \subset N_1$. For $y \in M^{G_1}$ consider the $G_1$-representation $T_y M$. Because $N_1$ is a component of $M^{SO(2l_1)}$, the restriction of $T_y M$ to $SO(2l_1)$ equals the $SO(2l_1)$-representation $T_x M$, where $x \in N_1^T$.

Because, by Lemma 7.2 $T_x M$ is a direct sum of a trivial representation and the standard real representation of $SO(2l_1)$ and $T_1 \subset SO(2l_1)$, $T_y M$ is a sum of a trivial and the standard real representation of $SO(2l_1 + 1)$ by [2] p. 167. Therefore $M^{G_1} \subset N_1$ has codimension one. □

7.2. Blowing up along $M^{G_1}$. As in section 5 we discuss the question as to when a manifold of the form given in Corollary 7.2 is a blow up.

If $M$ is the blow up of $M$ along $M^{G_1}$, then there is an equivariant embedding of $P(E(N(M^{G_1}, M)) \rightarrow M$. Therefore the $G_1$-action on $M$ has an orbit of type $SO(2l_1 + 1)/SO(2l_1) \times O(1))$. This fact shows that $M$ is of the form

$$SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1,$$

where $N_1$ is the proper transform of $N_1$. By Corollary 7.3 $N_1$ and $N_1$ are $G_2$-equivariantly diffeomorphic. Because $M^{G_1}$ has codimension one in $N_1$, the $Z_2$-action on $N_1$ has a fixed point component of codimension one.

The following lemma shows that these two conditions are sufficient.

**Lemma 7.4.** Let $N_1$ be a torus manifold with $G_2$-action. Assume that there are a non-trivial orientation reversing action of $Z_2 = S((O(2l_1) \times O(1))/SO(2l_1))$ on $N_1$, which commutes with the action of $G_2$, and a closed codimension one submanifold $A$ of $N_1$, on which $Z_2$ acts trivially.

Let $E' = N(A, N_1)$ be equipped with the action of $G_2$ induced from the action on $N_1$ and the trivial action of $Z_2$. Denote by $W$ the standard real representation of $SO(2l_1 + 1)$. Then:

1. $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$ is orientable.
2. The normal bundle of $SO(2l_1 + 1)/SO(2l_1) \times O(1)) \times A$ in $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$ is isomorphic to the tautological bundle over $P_k(E' \otimes W \oplus \{0\})$.

The lemma guarantees, together with the discussion at the end of section 4, that one may remove $SO(2l_1 + 1)/SO(2l_1) \times O(1)) \times A$ from $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$ and replace it by $A$ to get a torus manifold with $G$-action $M$ such that $M^{SO(2l_1 + 1)} = A$. The blow up of $M$ along $A$ is $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$.

**Proof.** The diagonal $Z_2$-action on $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$ is orientation preserving. Therefore $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$ is orientable.

The normal bundle of $SO(2l_1 + 1)/SO(2l_1) \times O(1)) \times A$ in $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N_1$ is given by $SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N(A, N_1)$. Consider the following commutative diagram:

$$\begin{array}{cc}
SO(2l_1 + 1)/SO(2l_1) \times_{Z_2} N(A, N_1) & \xrightarrow{f} & P_k(E' \otimes W) \times E' \otimes W \\
\pi_1 \downarrow & & \pi_1 \downarrow \\
SO(2l_1 + 1)/SO(2l_1) \times O(1)) \times A & \xrightarrow{g} & P_k(E' \otimes W),
\end{array}$$

where the vertical maps are the natural projections and $f, g$ are given by

$$f([hSO(2l_1) : m]) = ([m \otimes he_1], m \otimes he_1)$$

and replace it by
and

\[ g(hS(O(2l_1) \times O(1)), q) = [m_q \otimes h e_1], \]

where \( e_1 \in W - \{0\} \) is fixed such that for all \( g' \in SO(2l_1) \), \( g' e_1 = e_1 \) and \( m_q \neq 0 \) is some element of the fiber of \( E' \) over \( q \).

The map \( f \) induces an isomorphism of the normal bundle of \( SO(2l_1)/SO(2l_1) \times (O(1) \times A) \) in \( SO(2l_1)/SO(2l_1) \times_{Z_2} N_1 \) and the tautological bundle over \( P_z(E' \otimes W \oplus \{0\}) \). \( \square \)

**Lemma 7.5.** If \( l_1 > 1 \), in the situation of Lemma 7.1 then \( \bigcap_{M_i \in \mathfrak{F}_1} M_i = M^{SO(2l_1)} \) has at most two components. It has two components if and only if \( M = S^{2l_1} \times N_1 \).

**Proof.** If \( M = S^{2l_1} \times N_1 \), then \( \bigcap_{M_i \in \mathfrak{F}_1} M_i = \{N, S\} \times N_1 \), where \( N, S \) are the north and the south poles of the sphere, respectively. Otherwise the blow up of \( M \) along \( M^{SO(2l_1+1)} \) is given by \( S^{2l_1} \times_{Z_2} N_1 \), which is a fiber bundle over \( \mathbb{R}P^{2l_1} \). The characteristic submanifolds of \( S^{2l_1} \times_{Z_2} N_1 \), which are permuted by \( W(G_1) \), are given by the preimages of the following submanifolds of \( \mathbb{R}P^{2l_1} \):

\[ \mathbb{R}P^{2l_1-2} = \{(x_1 : x_2 : \cdots : x_{2l_1-2} : 0 : 0 : x_{2l_1+1} : \cdots : x_{2l_1+1}) \in \mathbb{R}P^{2l_1} : i = 1, \ldots, l_1 \}. \]

These characteristic submanifolds are also given by the proper transforms \( \tilde{M}_i \) of the characteristic submanifolds \( M_i \in \mathfrak{F}_1 \) of \( M \). Because

\[ \bigcap_{i=1}^{l_1} \mathbb{R}P^{2l_1-2} = \{(0 : 0 : \cdots : 0 : 1)\}, \]

it follows that

\[ \bigcap_{M_i \in \mathfrak{F}_1} \tilde{M}_i = \tilde{N}_1 = \tilde{M}^{SO(2l_1)}. \]

Therefore, with Lemma 4.3 and Corollary 7.3

\[ \bigcap_{M_i \in \mathfrak{F}_1} M_i = N_1 = M^{SO(2l_1)} \]

follows. In particular, \( \bigcap_{M_i \in \mathfrak{F}_1} M_i \) is connected. \( \square \)

**Lemma 7.6.** If \( l_1 = 1 \), in the situation of Lemma 7.1 then the following statements are equivalent:

- \( M^{SO(2)} \) has two components.
- \( \# \mathfrak{F}_1 = 2 \).
- \( M = S^2 \times N_1 \).

If \( l_1 = 1 \) and \( \# \mathfrak{F}_1 = 1 \), then \( M^{SO(2)} \) is connected.

**Proof.** At first we prove that all components of \( M^{SO(2)} \) are characteristic submanifolds of \( M \) belonging to \( \mathfrak{F}_1 \). By Lemma 7.1 \( N_1 \) is a characteristic submanifold of \( M \) and a component of \( M^{SO(2)} \) such that \( G_1 N_1 = M \). Therefore, if \( x \in M^{SO(2)} \), then there is a \( g \in N_{G_1} SO(2) \) such that \( g^{-1} x \in N_1 \). This implies \( x \in g N_1 \). Because \( gN_1 \) is a characteristic submanifold belonging to \( \mathfrak{F}_1 \) and a component of \( M^{SO(2)} \), it follows that \( M^{SO(2)} \) is a union of characteristic submanifolds of \( M \) belonging to \( \mathfrak{F}_1 \).

Now assume that \( \# \mathfrak{F}_1 = 1 \). Then we have \( M^{SO(2)} = N_1 \). Therefore \( M^{SO(2)} \) is connected.
Now assume that $M = SO(3)/SO(2) \times N_1$. Then it is clear that $M^{SO(2)}$ has two components.

Now assume that $M^{SO(2)}$ has two components. Because these components are characteristic submanifolds belonging to $\mathcal{F}_1$, it follows that $\# \mathcal{F}_1 = 2$.

Now assume that $\# \mathcal{F}_1 = 2$. If there is no $G_1$-fixed point, then it follows from Corollary 7.2 that

$$M = SO(3)/SO(2) \times N_1.$$ 

Assume that there is a $G_1$-fixed point in $M$. Then the blow up of $M$ along $M^{G_1}$ contains an orbit of type $SO(3)/S(O(2) \times O(1))$. Now Corollary 7.2 implies $\# \mathcal{F}_1 = 1$. Therefore there is no $G_1$-fixed point if $\# \mathcal{F}_1 = 2$. \hfill \Box

### 7.3. Admissible pairs

We are now in the position to state another classification theorem. To do so, we use the following definition.

**Definition 7.7.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$. Then a pair $(N, A)$ with

- $N$ a torus manifold with $G_2 \times \mathbb{Z}_2$-action such that the $\mathbb{Z}_2$-action is orientation reversing or trivial,
- $A \subset N$ the empty set or a closed $G_2 \times \mathbb{Z}_2$-invariant submanifold of codimension one, on which $\mathbb{Z}_2$ acts trivially, such that if $A \neq \emptyset$, then $\mathbb{Z}_2$ acts non-trivially on $N$,

is called admissible for $(\tilde{G}, G_1)$.

We say that two admissible pairs $(N, A), (N', A')$ are equivalent if there is a $G_2 \times \mathbb{Z}_2$-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A) = A'$.

**Theorem 7.8.** Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$. There is a one-to-one correspondence between the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-actions such that $G_1$ is elementary and equivalence classes of admissible pairs for $(\tilde{G}, G_1)$.

**Proof.** Let $M$ be a torus manifold with $\tilde{G}$-action. If $\bigcap_{M_i \in \mathcal{F}_1} M_i$ has two components and $l_1 > 1$ or $\# \mathcal{F}_1 = 2$ and $l_1 = 1$, then we assign to $M$ the admissible pair $\Phi(M) = (N_1, \emptyset)$, where $N_1$ is a component of $\bigcap_{M_i \in \mathcal{F}_1} M_i$ or a characteristic submanifold belonging to $\mathcal{F}_1$ in the case $l_1 = 1$. The action of $\mathbb{Z}_2$ is trivial in this case.

If $\bigcap_{M_i \in \mathcal{F}_1} M_i$ is connected and $l_1 > 1$ or $\# \mathcal{F}_1 = 1$ and $l_1 = 1$, then we assign to $M$ the pair

$$\Phi(M) = \left( \bigcap_{M_i \in \mathcal{F}_1} M_i, M^{SO(2l_1+1)} \right).$$

Because $\bigcap_{M_i \in \mathcal{F}_1} M_i = M^{SO(2l_1)}$, there is a non-trivial action of $\mathbb{Z}_2 = S(O(2l_1) \times O(1))/SO(2l_1)$ on $\bigcap_{M_i \in \mathcal{F}_1} M_i$.

Now let $(N, A)$ be an admissible pair for $(\tilde{G}, G_1)$. If the $\mathbb{Z}_2$-action on $N$ is trivial, we have $A = \emptyset$ and assign to $(N, \emptyset)$ the torus manifold with $\tilde{G}$-action $\Psi((N, \emptyset)) = S^{2l_1} \times N$.

If the $\mathbb{Z}_2$-action on $N$ is non-trivial, we assign to $(N, A)$ the blow down $\Psi((N, A))$ of $SO(2l_1 + 1)/SO(2l_1) \times \mathbb{Z}_2 N$ along $SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \times A$.

By Lemma 7.8 it is clear that this construction gives a one-to-one correspondence between torus manifolds with $\tilde{G}$-action such that $\bigcap_{M_i \in \mathcal{F}_1} M_i$ has two components.
and \( l_1 > 1 \) and admissible pairs with trivial \( \mathbb{Z}_2 \)-action. With Lemma 7.6 we see that an analogous statement holds for \( l_1 = 1 \) and \( \# \mathbb{S}_1 = 2 \).

Now let \((N, A)\) be an admissible pair such that \( \mathbb{Z}_2 \) acts non-trivially on \( N \). Then the discussion after Lemma 7.4 shows that \( \Phi(\Psi((N, A))) \) is equivalent to \((N, A)\).

If \( M \) is a torus manifold with \( G_1 \times G_2 \)-action such that \( G_1 \) is elementary and \( N_1 = \bigcap_{M_i \in \mathbb{S}_1} M_i \) is connected, the blow up of \( M \) along \( M^{SO(2l_1+1)} \) is given by

\[ SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1. \]

Therefore we find that \( \Psi(\Phi(M)) \) is equivariantly diffeomorphic to \( M \). \( \square \)

8. Classification

Here we use the results of the previous sections to state a classification of torus manifolds with \( G \)-action. We do not consider actions of groups, which have \( SO(2l_1) \) as an elementary factor, because as explained in section 6 these factors may be replaced by \( SU(l_1) \times S^1 \). We get the classification by iterating the constructions given in Theorem 5.13 and Theorem 7.8.

We illustrate this iteration in the case that all elementary factors of \( G \) are isomorphic to \( SU(l_1 + 1) \). Let \( G = \prod_{i=1}^{k} G_i \times T^{d_0} \) and \( M \) be a torus manifold with \( G \)-action such that all \( G_i \) are elementary and isomorphic to \( SU(l_1 + 1) \).

In Theorem 5.13 we constructed a triple \((\psi_1, N_1, A_1)\), which determines the \( \tilde{G} \)-equivariant diffeomorphism type of \( M \). Here \( N_1 \) is a torus manifold with \( \prod_{i=2}^{k} G_i \times T^{d_0} \)-action. Therefore there is a triple \((\psi_2, N_2, A_2)\) which determines the \( \prod_{i=2}^{k} G_i \times T^{d_0} \)-equivariant diffeomorphism type of \( N_1 \). Because \( N_2 \subset N_1 \) such that \( G_2 N_2 = N_1 \) and \( A_1 \) is \( G_2 \)-invariant, we have \( G_2 (A_1 \cap N_2) = A_1 \). Therefore the \( G \)-equivariant diffeomorphism type of \( M \) is determined by

\[ (\psi_1 \times \psi_2, N_2, A_1 \cap N_2, A_2). \]

Continuing in this manner leads to a triple

\[ (\psi, N, (A_1, \ldots, A_k)), \]

where \( \psi \in \text{Hom} \left( \prod_{i=1}^{k} S(U(l_i) \times U(1)), T^{d_0} \right) \), \( N \) is a \( 2l_0 \)-dimensional torus manifold and the \( A_i \) are codimension two submanifolds of \( N \) or empty.

The iteration becomes more complicated if there are more than one elementary factors of \( \tilde{G} \) isomorphic to \( SO(2l_1 + 1) \). To illustrate what happens here, we discuss the case \( \tilde{G} = G_1 \times G_2 \times T^{d_0} \) where the \( G_i \) are elementary and isomorphic to \( SO(2l_1 + 1) \).

Then, by Theorem 7.8 there is an admissible pair \((N_1, B_1)\) for \((\tilde{G}, G_1)\) corresponding to \( M \), where \( N_1 \) is a torus manifold with \( G_2 \times T^{d_0} \times (\mathbb{Z}_2)_1 \)-action. By Lemmas 7.3 and 7.6 we have two cases:

1. \( N_1^{SO(2l_2)} \) has two components.
2. \( N_1^{SO(2l_2)} \) is connected.

In the first case we have

\[ N_1 = SO(2l_2 + 1)/SO(2l_2) \times N_2, \]

where \( N_2 \) is a \( 2l_0 \)-dimensional torus manifold. The action of \((\mathbb{Z}_2)_1 \) on \( N_1 \) commutes with the action of \( G_2 \times T^{d_0} \). Therefore the action of \((\mathbb{Z}_2)_1 \) on \( N_1 \) splits as a product of an action on \( SO(2l_2 + 1)/SO(2l_2) \) and an action on \( N_2 \). Because there is only one non-trivial action of \( \mathbb{Z}_2 \) on \( SO(2l_2 + 1)/SO(2l_2) \) which commutes with the
action of $SO(2l_2 + 1)$, the $G_2 \times T^{l_0} \times (Z_2)_1$-equivariant diffeomorphism type of $N_1$ is completely determined by a pair $(N_2, a_{12})$, where $N_2$ is equipped with the action of $T^{l_0} \times (Z_2)_1$ and $a_{12} \in \{0, 1\}$ is non-zero if and only if the $(Z_2)_1$-action on $SO(2l_2 + 1)/SO(2l_2)$ is non-trivial.

In the second case the $G_2 \times T^{l_0}$-equivariant diffeomorphism type of $N_1$ is determined by a pair $(N_2, B_2)$, where $N_2 = N_1^{SO(2l_2)}$. Because $N_2$ is $(Z_2)_1$-invariant in this case, $N_2$ is a torus manifold with $T^{l_0} \times (Z_2)_1 \times (Z_2)_2$-action, where $(Z_2)_2 = S(O(2l_2) \times O(1))/SO(2l_2)$. We put $a_{12} = 0$ in this case.

As in the case where there are only elementary factors isomorphic to $SU(l_i + 1)$, one sees that the $G_1 \times G_2 \times T^{l_0}$-equivariant diffeomorphism type of $M$ is determined by

$$(N_2, (N_2 \cap B_1, B_2), a_{12}).$$

There are some relations between $a_{12}$ and $B_1$. For example, if $a_{12} = 1$, then there are no $(Z_2)_1$-fixed points in $N_1$. Therefore $B_1$ has to be empty.

If there are more than two elementary factors of $G$ isomorphic to $SO(2l_1 + 1)$, we have to introduce more numbers $a_{ij}$. There are some relations between the $a_{ij}$ coming from the fact that $M$ is required to be orientable. This will be explained in the proof of Lemma 8.3.

8.1. Admissible 5-tuples. We use the following definition to make the above constructions more formal.

**Definition 8.1.** Let $\hat{G} = \prod_{i=1}^{k} G_i \times G'$ with

$$G_i = \begin{cases} SU(l_i + 1) & \text{if } i \leq k_0, \\ SO(2l_i + 1) & \text{if } i > k_0 \end{cases}$$

and $k_0 \in \{0, \ldots, k\}$. Then a 5-tuple

$$(\psi, N, (A_i)_{i=1,\ldots,k_0}, (B_i)_{i=k_0+1,\ldots,k}, (a_{ij})_{k_0+1 \leq i < j \leq k})$$

with

1. $\psi \in \text{Hom}(\prod_{i=1}^{k_0} S(U(l_i) \times U(1)), Z(G'))$ and $\psi_i = \psi|_{S(U(l_i) \times U(1))}$,

2. $N$ a torus manifold with $G' \times \prod_{i=k_0+1}^{k} (Z_2)_1$-action,

3. $A_i \subset N$ the empty set or a $G' \times \prod_{i=k_0+1}^{k} (Z_2)_1$-invariant closed submanifold of codimension two, on which im $\psi_i$ acts trivially, such that if $A_i \neq \emptyset$, then ker $\psi_i = SU(l_i)$,

4. $B_i \subset N$ the empty set or a $G' \times \prod_{i=k_0+1}^{k} (Z_2)_1$-invariant closed submanifold of codimension one, on which $(Z_2)_1$ acts trivially, such that if $B_i \neq \emptyset$, then the action of $(Z_2)_1$ on $N$ is non-trivial,

5. $a_{ij} \in \{0, 1\}$ such that

(a) if $a_{ij} = 1$, then:

(i) the action of $(Z_2)_j$ on $N$ is trivial,

(ii) $a_{ik} = 0$ for $k > j$,

(iii) $B_i = \emptyset$,

(b) if the action of $(Z_2)_i$ on $N$ is non-trivial, then it is orientation preserving if and only if $\sum_{j>i} a_{ij}$ is odd,
Remark 8.2. By Lemma B.3, two submanifolds $A_1, A_2$ of $N$ satisfying condition (3) intersect transversely if and only if no component of $A_1$ is a component of $A_2$.

By Lemma B.4, two submanifolds $A_1, B_1$ of $N$ satisfying conditions (3) and (4), respectively, always intersect transversely.

By Lemma B.5, two submanifolds $B_1, B_2$ of $N$ satisfying condition (4) intersect transversely if and only if no component of $B_1$ is a component of $B_2$.

Lemma 8.3. Let $\tilde{G}$ be as above. Then there is a one-to-one correspondence between the equivalence classes of admissible 5-tuples

$$(\psi, N, (A_i)_{i=1}, \ldots, k_0), (B_i)_{i=k_0+1}, \ldots, k, (a_{ij})_{k_0+1 \leq i < j \leq k})$$

for $(\tilde{G}, \prod_{i=1}^k G_i)$ and the equivalence classes of admissible 5-tuples

$$(\psi', N', (A'_i)_{i=1}, \ldots, k_0), (B'_i)_{i=k_0+1}, \ldots, k-1, (a'_{ij})_{k_0+1 \leq i < j \leq k-1})$$

for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is elementary for the $G_k \times G'$-action on $N'$.

Proof. At first assume that $G_k = SU(l_k + 1)$. Let $(\psi, N, (A_i)_{i=1}, \ldots, k-1, \emptyset, \emptyset)$ be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is elementary for the $G_k \times G'$-action on $N$.

Let $(\psi_k, N_k, A_k)$ be the admissible triple for $(G_k \times G', G_k)$ which corresponds to $N$ under the correspondence given in Theorem 5.13. Then $N_k$ is a submanifold of $N$. By Lemma B.1, $A_i, i = 1, \ldots, k-1$, intersects $N_k$ transversely. Therefore $N_k \cap A_i$ has codimension 2 in $N_k$. Because $A_i = G_k(N_k \cap A_i)$, $N_k \cap A_i$ has no component which is contained in $A_k$ or $N_k \cap A_j, j \neq i$. Therefore by

$$(\psi \times \psi_k, N_k, (A_1 \cap N_k, \ldots, A_{k-1} \cap N_k, A_k), \emptyset, \emptyset)$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$ is given.

Now let

$$(\psi \times \psi_k, N_k, (A_1, \ldots, A_k), \emptyset, \emptyset)$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$. Let $H_0 = G_k \times \text{im} \psi_k$ and $H_1 = S(U(l_k) \times U(1)) \times \text{im} \psi_k$. 

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Then, by Lemma 5.11 the blow down of $N = N_0 \times H_i, N_k$ along $H_0/H_1 \times A_k$ is a torus manifold with $G_k \times G'$-action. By Lemma 4.3 $F(H_0 \times H_1, A_i) = G_k(F(A_i)), i < k$, are submanifolds of $N$ satisfying condition (3) of Definition 8.1. Because $F(A_i)$ and $F(A_j), i < j < k$, have no components in common, $G_k F(A_i)$ and $G_k F(A_j)$ intersect transversely. Therefore by

$$(\psi, N, (G_k F(A_1), \ldots, G_k F(A_{k-1})), \emptyset, \emptyset)$$

an admissible triple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ is given.

As in the proof of Theorem 7.8 one sees that this construction leads to a one-to-one correspondence.

Now assume that $G_k = SO(2l_k + 1)$. Let

$$(8.1) \quad (\psi, N, (A_i)_{i=1, \ldots, k}, (B_i)_{i=k_0+1, \ldots, k-1}, (a_{ij})_{k_0+1 \leq i < j \leq k-1})$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is elementary for the $G_k \times G'$-action on $N$.

At first assume that, for the $G_k$-action on $N$, $N^{SO(2l_k)}$ is connected. Let $(N_k, B_k)$ be the admissible pair for $(G_k \times G', G_k)$ which corresponds to $N$ under the correspondence given in Theorem 7.8 Then $N_k$ is a submanifold of $N$ which is invariant under the action of $G' \times \prod_{i=k_0+1}^{k-1} (\mathbb{Z}_2)_{i}$, where $(\mathbb{Z}_2)_{k} = S(O(2l_k) \times O(1))/SO(2l_k)$. For $i < k$, let $a_{ik} = 0$.

We claim that by

$$(8.2) \quad (\psi, N, (A_1 \cap N_k, \ldots, A_k \cap N_k), (B_{k_0+1} \cap N_k, \ldots, B_{k-1} \cap N_k, B_k), (a_{ij}))$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ is given.

At first note that, for $i = 1, \ldots, k-1$, the $A_i$ and $B_i$ intersect $N_k$ transversely by Lemmas 8.1 and 8.3. Therefore $A_i \cap N_k$ and $B_i \cap N_k$ has codimension two or one, respectively, in $N_k$.

One sees as in the case $G_k = SU(l_k + 1)$ that the $N_k \cap A_i$ and $N_k \cap B_i$ intersect pairwise transversely.

Now we verify condition (3) of Definition 8.1 for the 5-tuple $8.2$. By Lemma 6.4 $(\mathbb{Z}_2)_{i}, i < k$, acts orientation preserving on $N$ if and only if it acts orientation preserving on $N_k$. This proves $8.1$ because $8.1$ is an admissible 5-tuple and $a_{ik} = 0$.

Because, by Lemma 7.1 $G_k N_k = N$, $(\mathbb{Z}_2)_{i}, i < k$, acts trivially on $N_k$ if and only if it acts trivially on $N$. This proves $(5c)$ and $(5a)$ because $(5c)$ and $(5a)$ hold for the admissible 5-tuple $8.1$ and $a_{ik} = 0$.

Because $a_{ik} = 0$, $(5(a)ii)$ and $(5(a)iii)$ are clear.

Now assume that $N^{SO(2l_k)}$ is non-connected. Then, by Lemmas 6.2 and 7.6 we have

$N = SO(2l_k+1)/SO(2l_k) \times N_k$.

In this case the $(\mathbb{Z}_2)_i$-action, $i < k$, on $N$ commutes with the action of $SO(2l_k+1)$. Therefore it splits in a product of an action on $SO(2l_k+1)/SO(2l_k)$ and an action on $N_k$. We put $a_{ik} = 1$ if the $(\mathbb{Z}_2)_i$-action on $SO(2l_k+1)/SO(2l_k)$ is non-trivial and $a_{ik} = 0$ otherwise. Because there is only one non-trivial action of $\mathbb{Z}_2$ on $SO(2l_k+1)/SO(2l_k)$ which commutes with the action of $SO(2l_k+1)$, we may recover the action of $(\mathbb{Z}_2)_i$ on $N$ from the action on $N_k$ and $a_{ik}$.
We identify $SO(2l_k)/SO(2l_k) \times N_k$ with $N_k$ and equip it with the trivial action of $(\mathbb{Z}_2)_k = S(\mathbb{O}(2l_k) \times O(1))/SO(2l_k)$. We claim that by
\begin{equation}
(\psi, N_k, (A_1 \cap N_k, \ldots, A_{k_0} \cap N_k), (B_{k_0+1} \cap N_k, \ldots, B_{k-1} \cap N_k, \emptyset), (a_{ij}))
\end{equation}
an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$ is given.

Conditions $\roman{9}$ and $\roman{10}$ of Definition $8.1$ and the transversality condition are verified as in the previous cases.

Therefore we only have to verify condition $\roman{5}$. Because the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k + 1)/SO(2l_k)$ is orientation reversing, the $(\mathbb{Z}_2)_i$-action, $i < k$ on $N_k$ has the same orientation behavior as the action on $N$ if and only if the $(\mathbb{Z}_2)_i$-action on $SO(2l_k + 1)/SO(2l_k)$ is trivial. By the definition of $a_{ik}$, this is the case if and only if $a_{ik} = 0$. Therefore $\roman{5}$ follows because $\roman{9}$ is an admissible 5-tuple and $(\mathbb{Z}_2)_k$ acts trivially on $N_k$.

If the $(\mathbb{Z}_2)_i$-action on $N_k$ is trivial and non-trivial on $SO(2l_k + 1)/SO(2l_k)$, then the $(\mathbb{Z}_2)_i$-action on $N$ is orientation reversing. Therefore $\sum_{j>i} a_{ij}$ is odd.

The $(\mathbb{Z}_2)_i$-actions on $N_k$ and $SO(2l_k + 1)/SO(2l_k)$ are trivial if and only if the $(\mathbb{Z}_2)_i$-action on $N$ is trivial. Therefore $\sum_{j>i} a_{ij}$ is odd or trivial. This verifies $\roman{9}$. If there is a $j < i$ such that $a_{ji} = 1$, then $(\mathbb{Z}_2)_j$ acts trivially on $N$ because the admissible 5-tuple $\roman{9}$ satisfies $\roman{5}$. Therefore $a_{ik} = 0$. This proves $\roman{5}$. If the $(\mathbb{Z}_2)_i$-action on $SO(2l_k + 1)/SO(2l_k)$ is non-trivial, the action on $N$ has no fixed points. Therefore $B_i = \emptyset$. This proves $\roman{5}$. The property $\roman{5}$. is clear.

Now let

$$(\psi, N_k, (A_1, \ldots, A_{k_0}), (B_{k_0+1}, \ldots, B_k), (a_{ij}))$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$. At first assume that $(\mathbb{Z}_2)_k$ acts non-trivially on $N_k$. Then the blow down $N$ of $\tilde{N} = SO(2l_k + 1)/SO(2l_k) \times (\mathbb{Z}_2)_k N_k$ along $SO(2l_k + 1)/SO(2l_k) \times (\mathbb{Z}_2)_k B_k$ is a torus manifold with $G_k \times G' \times \prod_{i=k_0+1}^{k-1} (\mathbb{Z}_2)_i$-action. As in the case $G_k = SU(l_k + 1)$ one sees that

$$(\psi, N, (G_k F(A_1), \ldots, G_k F(A_{k_0})), (G_k F(B_{k_0+1}), \ldots, G_k F(B_{k-1})), (a_{ij}))$$

is an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$.

If $(\mathbb{Z}_2)_k$ acts trivially on $N_k$, then put

$$N = SO(2l_k + 1)/SO(2l_k) \times N_k.$$ 

Here $(\mathbb{Z}_2)_i$, $i < k$, acts by the product action of the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k + 1)/SO(2l_k)$ and the action on $N_k$ if $a_{ik} = 1$. Otherwise $(\mathbb{Z}_2)_i$, acts by the product action of the trivial action on $SO(2l_k + 1)/SO(2l_k)$ and the action on $N_k$. Now by

$$(\psi, N, (SO(2l_k + 1)/SO(2l_k) \times A_1, \ldots, SO(2l_k + 1)/SO(2l_k) \times A_{k_0}),$$

$$ (SO(2l_k + 1)/SO(2l_k) \times B_{k_0+1}, \ldots, SO(2l_k + 1)/SO(2l_k) \times B_{k-1}), (a_{ij}))$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ is given.

As in the proof of Theorem $\roman{28}$ one sees that this construction leads to a one-to-one correspondence.
Let $\tilde{G} = \prod_i G_i \times T^{l_0}$ and
\[ (\psi, M, (A_i), (B_i), (a_{ij})) \]
be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that $G_k$ is an elementary factor of $\prod_{i \geq k} G_i \times T^{l_0}$ for the action on $M$. Furthermore, let
\[ (\psi', N, (A'_i), (B'_i), (a'_{ij})) \]
be the admissible 5-tuple for $(\tilde{G}, \prod_i G_i)$ corresponding to $(\psi, M, (A_i), (B_i), (a_{ij}))$. Then the following lemma shows that $G_1$ acts non-trivially on $N$. Furthermore, assume that $G''_1$ is a product of elementary factors for the action on $M$.

Then $N$ is a torus manifold with $G_1 \times G' \times T^{l_0}$-action for some $l_0 \geq 0$ and $G_1$ is an elementary factor of $\tilde{G}$, with respect to the action on $M$, if and only if it is an elementary factor of $G_1 \times G' \times T^{l_0}$, with respect to the action on $N$.

**Lemma 8.4.** Let $\tilde{G} = G_1 \times G' \times G''$, $M$ be a torus manifold with $\tilde{G}$-action and $N$ be a component of an intersection of characteristic submanifolds of $M$ which is $G_1 \times G'$-invariant and contains a $T$-fixed point $x$ such that $G_1$ acts non-trivially on $N$. Furthermore, assume that $G''_1$ is a product of elementary factors for the action on $M$.

Then $N$ is a torus manifold with $G_1 \times G' \times T^{l_0}$-action for some $l_0 \geq 0$ and $G_1$ is an elementary factor of $\tilde{G}$, with respect to the action on $M$, if and only if it is an elementary factor of $G_1 \times G' \times T^{l_0}$, with respect to the action on $N$.

**Proof.** Assume that $G_1$ is an elementary factor for one of the two actions on $M$ and $N$. Then $G_1$ is isomorphic to a simple group or Spin(4). If $G_1$ is simple and not isomorphic to $SU(2)$, then the statement is clear.

Therefore there are two cases, $G_1 = SU(2), Spin(4)$.

If $x$ is not fixed by $G_1$, then $G_1 = SU(2)$ is elementary for both actions on $N$ and $M$ by Lemma 5.1. Therefore we may assume that $x \in N^{G_1} \subset M^{G_1}$. Then there is a bijection
\[ \mathfrak{F}_{xM} \rightarrow \mathfrak{F}_{xN} \sqcup \mathfrak{F}_{xN}^l, \]
where
\[ \mathfrak{F}_{xM} = \{ \text{characteristic submanifolds of } M \text{ containing } x \}, \]
\[ \mathfrak{F}_{xN} = \{ \text{characteristic submanifolds of } N \text{ containing } x \}, \]
\[ \mathfrak{F}_{xN}^l = \{ \text{characteristic submanifolds of } M \text{ containing } N \}. \]

This bijection is compatible with the actions of the Weyl group of $G_x$.

At first assume that $G_1 = SU(2)$ is elementary for the action on $M$ but not for the action on $N$. Then there is another simple factor $G_2 = SU(2)$ of $G_1 \times G' \times T_{l_0}$ such that $G_1 \times G_2$ is elementary for the action on $N$. At first assume that $G_2$ is elementary for the action on $M$.

Let $w_i \in W(G_i)$, $i = 1, 2$, be generators. Then there are two non-trivial $W(G_1 \times G_2)$-orbits $\mathfrak{F}_1, \mathfrak{F}_2$ in $\mathfrak{F}_{xM}$. We have:
\- $\#\mathfrak{F}_i = 2$, $i = 1, 2$,
\- $w_i$, $i = 1, 2$, acts non-trivially on $\mathfrak{F}_i$ and trivially on the other orbit.
But because \( G_1 \times G_2 \) is elementary for the action on \( N \), there is exactly one non-trivial \( W(G_1 \times G_2) \)-orbit \( \mathfrak{F}_1' \) in \( \mathfrak{F}_{xN} \). We have:

- \( \# \mathfrak{F}_1' = 2 \),
- \( w_i, i = 1, 2 \), acts non-trivially on \( \mathfrak{F}_1' \).

This is a contradiction.

If \( G_2 \) is not elementary, then \( G_2 \) is a simple factor of an elementary factor. In this case the action of \( W(G_1 \times G_2) \) on \( \mathfrak{F}_{xM} \) behaves as in the first case. Therefore we also get a contradiction in this case.

Under the assumption that \( G_1 = \text{Spin}(4) \) is elementary for the action on \( M \), a similar argument shows that \( G_1 \) is elementary for the action on \( N \).

Therefore \( G_1 \) is elementary for the action on \( N \) if it is elementary for the action on \( M \).

If \( G_1 \) is elementary for the action on \( N \) but not elementary for the action on \( M \), then it is a simple factor of an elementary factor \( G_1' \neq G_1 \) of \( \hat{G} \) or a product \( G_2' \times G_3' \) of elementary factors \( G_2' \) and \( G_3' \) of \( \hat{G} \). But because \( G'' \) is a product of elementary factors, it contains all elementary factors of \( \hat{G} \) which have non-trivial intersection with \( G'' \). Because \( G_1 \) is not contained in \( G'' \), it follows that \( G_1', G_2' \) and \( G_3' \) are subgroups of \( G_1 \times G' \). Therefore, by the above argument, \( G_1' \) or \( G_2' \) and \( G_3' \) are elementary for the action on \( N \). Because elementary factors cannot contain each other, we get a contradiction to the assumption that \( G_1 \) is elementary for the action on \( N \). \( \square \)

Recall from section 4 that if \( M \) is a torus manifold with \( G \)-action, then we may assume that all elementary factors of \( G \) are isomorphic to \( SU(l_i + 1) \), \( SO(2l_i + 1) \) or \( SO(2l_i) \). That means \( \hat{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod SO(2l_i) \times T^{k_0} \). Because, as described in section 6, we may replace elementary factors isomorphic to \( SO(2l_i) \) by \( SU(l_i) \times S^1 \), the following theorem may be used to construct invariants of torus manifolds with \( \hat{G} \)-action. By Theorem 5.3 these invariants determine the \( \hat{G} \)-equivariant diffeomorphism type of simply connected torus manifolds with \( \hat{G} \)-action.

**Theorem 8.5.** Let \( \hat{G} = \prod_{i=1}^k G_i \times T^{k_0} \) with

\[
G_i = \begin{cases} 
SU(l_i + 1) & \text{if } i \leq k_0, \\
SO(2l_i + 1) & \text{if } i > k_0
\end{cases}
\]

and \( k_0 \in \{0, \ldots, k\} \). Then there is a one-to-one correspondence between the equivalence classes of admissible 5-tuples for \( \hat{G} \) and the \( \hat{G} \)-equivariant diffeomorphism classes of torus manifolds with \( \hat{G} \)-action such that all \( G_i \) are elementary.

**Proof:** This follows from Lemma 5.3 and Lemma 8.4 by induction. \( \square \)

Using Lemma 2.3 and Theorem 5.5 we get the following result for quasitoric manifolds.

**Theorem 8.6.** Let \( \hat{G} = \prod_{i=1}^k G_i \times T^{k_0} \) with \( G_i = SU(l_i + 1) \). Then there is a one-to-one correspondence between the equivalence classes of admissible 5-tuples for \( \hat{G} \) of the form

\[(\psi, N, (A_i)_{1 \leq i \leq k}, \emptyset, \emptyset)\]

with \( N \) quasitoric, \( A_i, 1 \leq i \leq k \), connected and the \( \hat{G} \)-equivariant diffeomorphism classes of quasitoric manifolds with \( \hat{G} \)-action.
**Remark 8.7.** Remark [2.3] and Theorem [5.15] lead to a similar result for torus manifolds with $G$-actions whose cohomologies are generated by their degree two parts.

**Corollary 8.8.** Let $\tilde{G} = \prod_{i=1}^{l_0} G_i \times T^{l_0}$ with $G_i$ elementary and $M$ a torus manifold with $G$-action. Then $M/G$ has dimension $l_0 + \# \{ G_i; G_i = SO(2l_i) \}$.

**Proof.** At first we discuss the case where all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_i + 1)$ or $SU(l_i + 1)$, i.e. $\# \{ G_i; G_i = SO(2l_i) \} = 0$. By Lemma [4.7] replacing $M$ by the blow up $\tilde{M}$ of $M$ along the fixed points of $G_1$ does not change the orbit space. Therefore, by Corollaries [5.6] and [7.2] we have up to finite coverings

$$M/G = (M/G_1)/\left( \prod_{i \geq 2} G_i \times T^{l_0} \right) = (\tilde{M}/G_1)/\left( \prod_{i \geq 2} G_i \times T^{l_0} \right),$$

where $N_1$ is the $\prod_{i \geq 2} G_i \times T^{l_0}$-manifold from the admissible 5-tuple for $(\tilde{G}, G_1)$ corresponding to $M$. Here $H_0, H_1$ are defined as in Lemma [6.3] if $G_1 = SU(l_i + 1)$. If $G_1 = SO(2l_i + 1)$, we have $H_0 = SO(2l_i + 1)$ and $H_1 = SO(2l_1) \times O(1)$.

By iterating this argument we find that $M/G = N/T^{l_0}$ up to finite coverings, where $N$ is the $T^{l_0}$-manifold from the admissible 5-tuple for $\tilde{G}$ corresponding to $M$.

Now we study the case $l'_0 = \# \{ G_i; G_i = SO(2l_i) \} \neq 0$. As discussed in section [4] the orbits of the $G$-action on $M$ do not change if we replace an elementary factor isomorphic to $SO(2l_i)$ by $SU(l_i + 1) \times S^1$. Therefore this replacement does not change the dimension of the orbit space, but it increases $l_0$ by one and decreases $l'_0$ by one. Therefore the statement follows by induction on $l'_0$. □

**8.2. Applications.** Now we apply our classification results to special cases. We first discuss the case where $M$ is a torus manifold with $G$-action such that $G$ is semi-simple and $H^\ast(M; \mathbb{Z})$ is generated by its degree two part.

**Corollary 8.9.** If $G$ is semi-simple and $M$ is a torus manifold with $G$-action such that $H^\ast(M; \mathbb{Z})$ is generated by its degree two part, then

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i + 1)$$

and

$$M = \prod_{i=1}^{k} \mathbb{C}P^{l_i},$$

where each $SU(l_i + 1)$ acts in the usual way on $\mathbb{C}P^{l_i}$ and trivially on $\mathbb{C}P^{l_j}$, $j \neq i$.

**Proof.** By Lemma [2.3] and Remark [2.9] all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i + 1)$. Because $G$ is semi-simple, there is only one admissible 5-tuple for $\tilde{G}$, namely (const, pt, $\emptyset$, $\emptyset$, $\emptyset$). It corresponds to a product of complex projective spaces. □

Next we discuss torus manifolds $M$ with $G$-action such that $\dim M/G \leq 1$. With Theorem [8.5] we recover the following two results of S. Kuroki [15, 11]:

**Corollary 8.10.** Let $M$ be a simply connected torus manifold with $G$-action such that $M$ is a homogeneous $G$-manifold. Then $M$ is a product of even-dimensional spheres and complex projective spaces.
Proof. By Corollary 8.8, the center of $G$ is zero-dimensional. Moreover, all elementary factors of $G$ are isomorphic to $SU(l_i + 1)$ or $SO(2l_i + 1)$. Therefore the admissible 5-tuple corresponding to $M$ is given by

$$(\text{const}, \text{pt}, 0, 0, (a_{ij})),$$

where the $a_{ij} \in \{0, 1\}$ are unknown. In particular, no elementary factor of $G$ has a fixed point in $M$. Therefore, by Corollaries 5.6 and 7.2, $M$ splits into a direct product of complex projective spaces and even-dimensional spheres. \hfill \square

**Corollary 8.11.** If the $G$-action on the simply connected torus manifold $M$ has an orbit of codimension one, then $M$ is the projectivication of a complex vector bundle or a sphere bundle over a product of complex projective spaces and even-dimensional spheres.

Proof. By Corollary 8.8, we may assume that there is a covering group $\tilde{G} = S^1 \times \prod_i G_i$ of $G$ with $G_i$ elementary and $G_i = SU(l_i + 1)$ or $G_i = SO(2l_i + 1)$. We assume that the $G_i$ are sorted in such a way that

- $G_i = SO(2l_i + 1)$ and $G_i$ has no fixed point in $M$ if $i \leq k_0$,
- $G_i = SU(l_i + 1)$ and $G_i$ has no fixed point in $M$ if $k_0 + 1 \leq i \leq k_1$,
- $G_i = SU(l_i + 1), SO(2l_i + 1)$ and $G_i$ has fixed points in $M$ if $i \geq k_1 + 1$,

where $k_0 \leq k_1$ are some constants.

By Corollaries 5.6 and 7.2 we know that $M$ is of the form

$$M = \prod_{i=1}^{k_0} S^{2l_i} \times H_{0k_0+1} \times H_{1k_0+1} \left( H_{0k_0+2} \times H_{1k_0+2} \left( \cdots (H_{0k_1} \times H_{1k_1} M') \cdots ) \right) \right),$$

where

$$H_{0i} = SU(l_i + 1) \times \text{im} \psi_i,$$

$$H_{1i} = S(U(l_i + 1) \times U(1)) \times \text{im} \psi_i,$$

for $i = k_0 + 1, \ldots, k_1$, and $M'$ is a torus manifold with $\tilde{G}'$-action, where $\tilde{G}' = \prod_{i=k_1+1}^{k_2} G_i \times S^1$.

Because the action of $H_{1i}$ on $H_{0j}$, $j > i$, is trivial and the actions of the $H_{1i}$ on $M'$ commute, $M$ may be written as

$$M = \prod_{i=1}^{k_0} S^{2l_i} \times \left( \prod_{i=k_0+1}^{k_1} H_{0i} \times \prod_{i=1}^{k_1} H_{1i} M' \right).$$

Therefore $M$ is a fiber bundle over a product of even-dimensional spheres and complex projective spaces with fiber $M'$.

Let $(\psi, N', (A_i), (B_i), (a_{ij}))$ be the admissible 5-tuple for $\tilde{G}'$ corresponding to $M'$. Because $\dim N' = 2$ and all $G_i$, $i > k_1$, have fixed points in $M'$, we have

$$N' = S^2, \quad A_i \neq \emptyset, \quad B_i \neq \emptyset.$$

Because the $S^1$-action on $S^2$ has only two fixed points, $N$ and $S$, there are at most two elementary factors isomorphic to $SU(l_i + 1)$. The orientation reversing involutions of $S^2$ which commute with the $S^1$-action and have fixed points are given by “reflections” at $S^1$-orbits. Therefore there is at most one elementary
factor isomorphic to \(SO(2l_i + 1)\). If there is such a factor, then there is at most one \(G_i\) isomorphic to \(SU(l_i + 1)\) because \(N\) is mapped to \(S\) by such a reflection. Let

\[
\phi_i : S(U(l_i) \times U(1)) \to U(1) \quad \begin{pmatrix} A & 0 \\ 0 & g \end{pmatrix} \mapsto g \quad (A \in U(l_i), g \in U(1)).
\]

Then we have the following admissible 5-tuples:

<table>
<thead>
<tr>
<th>(\tilde{G}')</th>
<th>5-tuple</th>
<th>(M')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{S}^4)</td>
<td>((0, \mathbb{S}^2, \emptyset, \emptyset, \emptyset))</td>
<td>(\mathbb{S}^2)</td>
</tr>
<tr>
<td>(\mathbb{S}^4 \times SU(l_1 + 1))</td>
<td>((\phi_1^1, \mathbb{S}^2, {N}, \emptyset, \emptyset))</td>
<td>(\mathbb{C}P^{l_1+1})</td>
</tr>
<tr>
<td>(\mathbb{S}^4 \times SU(l_1 + 1))</td>
<td>((\phi_2^1, \mathbb{S}^2, {N}, \emptyset, \emptyset))</td>
<td>(S^{2l_1+2})</td>
</tr>
<tr>
<td>(\mathbb{S}^4 \times SO(2l_1 + 1))</td>
<td>((0, \mathbb{S}^2, \emptyset, \emptyset, \emptyset))</td>
<td>(S^{2l_1+2})</td>
</tr>
<tr>
<td>(\mathbb{S}^4 \times SU(l_1 + 1) \times SU(l_2 + 1))</td>
<td>((\phi_1^1 \phi_2^1, \mathbb{S}^2, {{N}, {S}}, \emptyset, \emptyset))</td>
<td>(\mathbb{C}P^{l_1+l_2+1})</td>
</tr>
<tr>
<td>(\mathbb{S}^4 \times SU(l_1 + 1) \times SO(2l_2 + 1))</td>
<td>((\phi_1^1, \mathbb{S}^2, {N}, \emptyset, \emptyset))</td>
<td>(S^{2l_1+2l_2+2})</td>
</tr>
</tbody>
</table>

Therefore the statement follows.

Now we turn to the case where \(M\) is a torus manifold with \(G\)-action such that \(G\) is semi-simple and has exactly two elementary factors \(G_1, G_2\). We start with a discussion of the case where \(G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)\).

**Corollary 8.12.** Let \(\tilde{G} = G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)\) with \(G_1\) and \(G_2\) elementary of rank \(l_1, l_2\), respectively, and let \(M\) be a torus manifold with \(G\)-action. Then \(M\) is one of the following:

\[
\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}, \mathbb{C}P^{l_1} \times \mathbb{S}^{2l_2}, \mathbb{S}^{2l_1} \times \mathbb{S}^{2l_2}, S^{2l_1} \times Z_2 \times S^{2l_2}, S^{2l_1} \times S^{2l_2}, \mathbb{S}^{2l_1} \times Z_2 \times S^{2l_2}.
\]

Here \(S^{l_1}\) denotes the \(l\)-sphere together with the \(Z_2\)-action generated by the anti-podal map and \(S^{2l_2}\) the \(l\)-sphere together with the \(Z_2\)-action generated by a reflection at a hyperplane.

Furthermore, the \(\tilde{G}\)-actions on these spaces is unique up to equivariant diffeomorphism.

**Proof.** First assume that \(G_1, G_2 \neq SO(2l)\). Then we have the following possibilities for the admissible 5-tuple of \(M\):

<table>
<thead>
<tr>
<th>(G_1)</th>
<th>(G_2)</th>
<th>5-tuple</th>
<th>(M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(SU(l_1 + 1))</td>
<td>(SU(l_2 + 1))</td>
<td>((\text{const, pt, } \emptyset, \emptyset, \emptyset))</td>
<td>(\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2})</td>
</tr>
<tr>
<td>(SU(l_1 + 1))</td>
<td>(SO(2l_2 + 1))</td>
<td>((\text{const, pt, } \emptyset, \emptyset, \emptyset))</td>
<td>(\mathbb{C}P^{l_1} \times \mathbb{S}^{2l_2})</td>
</tr>
<tr>
<td>(SO(2l_1 + 1))</td>
<td>(SO(2l_2 + 1))</td>
<td>((0, \text{pt, } \emptyset, \emptyset, a_{12} = 0))</td>
<td>(\mathbb{S}^{2l_1} \times \mathbb{S}^{2l_2})</td>
</tr>
<tr>
<td>(SO(2l_1 + 1))</td>
<td>(SO(2l_2 + 1))</td>
<td>((0, \text{pt, } \emptyset, \emptyset, a_{12} = 1))</td>
<td>(S^{2l_1} \times Z_2 \times S^{2l_2})</td>
</tr>
</tbody>
</table>

If \(G_1 = SU(l_1 + 1)\) and \(G_2 = SO(2l_2)\), then by Corollary 3.6, there is one admissible triple for \((G, G_1)\), namely \((\text{const, } S^{2l_2}, \emptyset)\). It corresponds to \(\mathbb{C}P^{l_1} \times \mathbb{S}^{2l_2}\).

Now assume that \(G_1 = SO(2l_1 + 1)\) and \(G_2 = SO(2l_2)\). Let \((N, B)\) be the admissible pair for \((G, G_1)\) corresponding to \(M\). Then, by Corollary 3.6, we have \(N = S^{2l_2}\). Up to equivariant diffeomorphism there are two orientation reversing involutions on \(S^{2l_2}\) which commute with the action of \(G_2\), the anti-podal map and a reflection at a hyperplane in \(\mathbb{R}^{2l_2+1}\). Therefore we have four possibilities for \(M\): \(\mathbb{S}^{2l_1} \times S^{2l_2}, S^{2l_1+2l_2}, S^{2l_1} \times Z_2 \times S^{2l_2}, S^{2l_1+2l_2}\).
For the discussion of the case $G_1 \times G_2 = SO(2l_1) \times SO(2l_2)$ we need the following lemma.

**Lemma 8.13.** Let $\tilde{G} = SO(2l_1) \times S^1$ and $M$ be a simply connected torus manifold with $G$-action such that $SO(2l_1)$ is an elementary factor of $\tilde{G}$, $S^1$ acts effectively on $M$ and $M^{S^1}$ has codimension two in $M$.

Then $M$ is equivariantly diffeomorphic to $\#_i(S^2 \times S^{2l_1})$ or $S^{2l_1+2}$.

Here the action of $\tilde{G}$ on $S^{2l_1+2}$ is given by the restriction of the usual $SO(2l_1+3)$-action to $\tilde{G}$. The action of $\tilde{G}$ on $S^2 \times S^{2l_1}$ is the product action of the usual action of $S^1$ and $SO(2l_1)$ on $S^2$ and $S^{2l_1}$, respectively. Moreover, the connected sums are equivariant.

**Proof.** As described in section 6, we may replace $\tilde{G}$ by $SU(l_1) \times S^1$. Let $(\psi, N, A)$ be the admissible triple corresponding to $\tilde{M}$. Then $\psi$ is completely determined by the discussion in section 6, and $A = N^{S^1} = M^{SU(l_1)}$. Furthermore $S$ and $S^1$ act effectively on $N$. All components of $N^{S^1}$ and $N^{S^1}$ have codimension two in $N$.

By Lemma 5.17, $N$ is simply connected.

Denote by $\tilde{M}$ the blow up of $M$ along $A$. Because all $T$-fixed points of $M$ are contained in $A$, we have $l_1 \#MT = #\tilde{MT}$. On the other hand, $M$ is a fiber bundle with fiber $N$ over $CP^{l_1-1}$. Therefore we have $l_1 \#N^{S \times S^1} = #\tilde{MT}$.

From this $#MT = #N^{S \times S^1}$ follows.

Because $S$ and $S^1$ both act effectively on $N$ such that their fixed point sets have codimension two, it follows from the classification of simply connected four-dimensional $T^2$-manifolds given in [20] pp. 547, 549 that the $T$-equivariant diffeomorphism type of $N$ is determined by $#MT$ and that $#MT$ is even.

Therefore the $S \times S^1 \times SU(l_1)$-equivariant diffeomorphism type of $M$ is uniquely determined by $#MT = \chi(M)$. It follows from Theorem 6.3 that the $SO(2l_1) \times S^1$-equivariant diffeomorphism type of $M$ is uniquely determined by $\chi(M)$. Because

$$M_k = \begin{cases} \#_{L=1}(S^2 \times S^{2l_1})_i & \text{if } k \geq 1, \\ S^{2l_1+2} & \text{if } k = 0 \end{cases}$$

possesses an action of $\tilde{G}$ and $\chi(M_k) = 2k + 2$, the statement follows. \qed

**Corollary 8.14.** Let $\tilde{G} = SO(2l_1) \times SO(2l_2)$ and $M$ be a simply connected torus manifold with $G$-action such that $SO(2l_1)$, $SO(2l_2)$ are elementary factors of $\tilde{G}$.

Then $M$ is equivariantly diffeomorphic to $\#_i(S^{2l_1} \times S^{2l_2})$, or $M = S^{2l_1+2l_2}$.

Here the action of $\tilde{G}$ on $S^{2l_1+2}$ is given by the restriction of the usual $SO(2l_1 + 2l_2 + 1)$-action to $\tilde{G}$. The action of $\tilde{G}$ on $S^{2l_1} \times S^{2l_2}$ is the product action of the usual action of $SO(2l_1)$ and $SO(2l_2)$ on $S^{2l_1}$ and $S^{2l_2}$, respectively. Moreover, the connected sums are equivariant.

**Proof.** As described in section 6, we may replace $\tilde{G}$ by $SU(l_1) \times S \times SO(2l_2)$. Let $(\psi, N, A)$ be the admissible triple for $(SU(l_1) \times S \times SO(2l_2), SU(l_1))$ corresponding to $M$. Then $\psi$ is completely determined by the discussion in section 6 and $A = N^{S^1}$. Furthermore, $S$ acts effectively on $N$ such that $N^{S^1}$ has codimension two.

By Lemma 5.17, $N$ is simply connected. Therefore, by Lemma 8.13, the equivariant diffeomorphism type of $N$ is uniquely determined by $\chi(N) \in 2Z$. Because all other parts of the triple $(\psi, N, A)$ are determined by the discussion in section 6 and the equivariant diffeomorphism type of $N$, it follows that the equivariant diffeomorphism type of $M$ is determined by $\chi(N)$. Let $T_2$ be the maximal torus $T \cap SO(2l_2)$.
of \( SO(2l_2) \). Then as in the proof of Lemma 8.13 one sees that
\[
\chi(M) = \#M^T = \#N^S \times T^2 = \chi(N).
\]

Therefore the equivariant diffeomorphism type of \( M \) is uniquely determined by
\[
\chi(M) \in 2\mathbb{Z}. \quad \text{Because}
\]
\[
M_k = \begin{cases} 
\#_{k-1}(S^{2l_1} \times S^{2l_2}) & \text{if } k \geq 1, \\
S^{2l_1+2l_2} & \text{if } k = 0
\end{cases}
\]
possesses an action of \( \tilde{G} \) and \( \chi(M_k) = 2k + 2 \), the statement follows. \( \square \)

At the end of this section we give a classification of four-dimensional torus manifolds with \( G \)-action.

**Corollary 8.15.** Let \( M \) be a four-dimensional torus manifold with \( G \)-action and \( G \) be a non-abelian Lie group of rank two. Then \( M \) is one of the following:

\[
\mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1, S^4, S_1^2 \times_{Z_2} S_1^2, S_1^2 \times_{Z_2} S_2^2
\]

or an \( S^2 \)-bundle over \( \mathbb{C}P^1 \). Here \( S_1^2 \) denotes the two-sphere together with the \( Z_2 \)-action generated by the anti-podal map and \( S_2^2 \) the two-sphere together with the \( Z_2 \)-action generated by a reflection at a hyperplane.

**Proof.** Let \( \tilde{G} \) be a covering group of \( G \). Then there are the following possibilities using Convention 3.5

\[
\tilde{G} = SU(3), SU(2) \times SU(2), SU(2) \times S^1, SU(2) \times SO(3), SO(3) \times SO(3), SO(3) \times S^1, Spin(4), SO(5).
\]

If \( \tilde{G} = \text{Spin}(4) \), we replace it by \( SU(2) \times S^1 \) as before.

Then we have the following admissible 5-tuples:

<table>
<thead>
<tr>
<th>( \tilde{G} )</th>
<th>5-tuple</th>
<th>( M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SU(3) )</td>
<td>(const, pt, ( 0 ), ( 0 ), ( 0 ))</td>
<td>( \mathbb{C}P^2 )</td>
</tr>
<tr>
<td>( SU(2) \times SU(2) )</td>
<td>(const, pt, ( 0 ), ( 0 ), ( 0 ))</td>
<td>( \mathbb{C}P^1 \times \mathbb{C}P^1 )</td>
</tr>
<tr>
<td>( SU(2) \times S^1 )</td>
<td>(( \psi ), ( S^2 ), ( 0 ), ( 0 )) ( (\psi, S^2, N, 0, 0) )</td>
<td>( S^2 )-bundle over ( \mathbb{C}P^1 )</td>
</tr>
<tr>
<td>( SU(2) \times SO(3) )</td>
<td>(const, pt, ( 0 ), ( 0 ), ( 0 ))</td>
<td>( \mathbb{C}P^1 \times S^2 )</td>
</tr>
<tr>
<td>( SO(3) \times SO(3) )</td>
<td>(( 0 ), ( pt, 0 ), ( a_{12} = 1 )) ( (0, pt, 0, a_{12} = 0) )</td>
<td>( S_1^2 \times_{Z_2} S_1^2 ) ( S_2^2 \times S^2 )</td>
</tr>
<tr>
<td>( SO(3) \times S^1 )</td>
<td>(( 0 ), ( S^2 ), ( 0 ), ( 0 )) ( (0, S_1^2, 0, 0, 0) ) ( (0, S_2^2, 0, 0, 0) )</td>
<td>( S_1^2 \times_{Z_2} S_1^2 ) ( S_1^2 \times_{Z_2} S_2^2 ) ( S_2^2 \times S^2 )</td>
</tr>
<tr>
<td>( SO(5) )</td>
<td>(( 0 ), ( pt, 0 ), ( 0 ), ( 0 ))</td>
<td>( S^4 )</td>
</tr>
</tbody>
</table>

Here \( \psi \) is a group homomorphism \( S(U(1) \times U(1)) \rightarrow S^1 \). \( \square \)
 Appendix A. Lie groups

Lemma A.1. Let \( l > 1 \). Then \( S(U(l) \times U(1)) \) is a maximal subgroup of \( SU(l+1) \).

Proof. Let \( H \) be a subgroup of \( SU(l+1) \) with \( S(U(l) \times U(1)) \subset H \subset SU(l+1) \). 

Because \( S(U(l) \times U(1)) \) is a maximal connected subgroup of \( SU(l+1) \), the identity component of \( H \) has to be \( S(U(l) \times U(1)) \). Therefore \( H \) is contained in the normalizer of \( S(U(l) \times U(1)) \). Because \( l > 1 \),

\[
N_{SU(l+1)} S(U(l) \times U(1))/S(U(l) \times U(1))
\]

\[
= (SU(l+1)/S(U(l) \times U(1)))^{SU(l+1)} = (\mathbb{C}P^l)^{SU(l+1)}
\]

is just one point. Therefore \( H = S(U(l) \times U(1)) \) follows. \( \square \)

Lemma A.2. Let \( \psi : S(U(l) \times U(1)) \to S^1 \) be a non-trivial group homomorphism and

\[
H_0 = SU(l+1) \times S^1,
H_1 = S(U(l) \times U(1)) \times S^1,
H_2 = \{(g, \psi(g)), g \in S(U(l) \times U(1))\}.
\]

Then \( H_1 \) is the only connected proper closed subgroup of \( H_0 \) which contains \( H_2 \) properly.

Proof. Let \( H_2 \subset H \subset H_0 \) be a closed connected subgroup. Then we have

\[
\text{rank } H_0 \geq \text{rank } H \geq \text{rank } H_2 = \text{rank } H_0 - 1.
\]

At first assume that \( \text{rank } H = \text{rank } H_0 \). Then we have by \([18\text{ p. 297}]\)

\[
H = H' \times S^1,
\]

where \( H' \) is a connected subgroup of maximal rank of \( SU(l+1) \). Let \( \pi_1 : H_0 \to SU(l+1) \) be the projection to the first factor. Because \( H' = \pi_1(H) \supset \pi_1(H_2) = S(U(l) \times U(1)) \) and \( S(U(l) \times U(1)) \) is a maximal connected subgroup of \( SU(l+1) \), we have by Lemma A.1 that \( H = H_1 \) or \( H = H_0 \).

Now assume that \( \text{rank } H = \text{rank } H_2 \). Then there is a non-trivial group homomorphism \( H \to S^1 \). Therefore locally \( H \) is a product \( H' \times S^1 \), where \( H' \) is a simple group which contains \( SU(l) \) as a maximal rank subgroup. By \([2\text{ p. 219}]\), we have

\[
H' = E_7, E_8, G_2, SU(l).
\]

If \( H' = SU(l) \), then we have \( H = H_2 \). Therefore we have to show that the other cases do not occur.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( \text{dim } H_0 )</th>
<th>( \text{dim } H' \times S^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>81</td>
<td>( \dim E_7 \times S^1 = 134 )</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>( \dim E_8 \times S^1 = 249 )</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>( \dim G_2 \times S^1 = 15 )</td>
</tr>
</tbody>
</table>

Therefore the first two cases do not occur. Because there is no \( G_2 \)-representation of dimension less than seven, the third case does not occur. \( \square \)

Lemma A.3. Let \( T \) be a torus and \( \psi_1, \psi_2 : S(U(l) \times U(1)) \to T \) be two group homomorphisms. Furthermore, let, for \( i = 1, 2 \),

\[
H_i = \{(g, \psi_i(g)), g \in S(U(l+1) \times T; g \in S(U(l) \times U(1))\}
\]
be the graph of $\psi_1$. Then:

1. If $l > 1$, then $H_1$ and $H_2$ are conjugated in $SU(l + 1) \times T$ if and only if $\psi_1 = \psi_2$.
2. If $l = 1$, then $H_1$ and $H_2$ are conjugated in $SU(l + 1) \times T$ if and only if $\psi_1 = \psi_2^{+1}$.

**Proof.** At first assume that $H_1$ and $H_2$ are conjugated in $SU(l + 1) \times T$. Let $g' \in SU(l + 1) \times T$ such that

$$H_1 = g'H_2g'^{-1}.$$ 

Because $T$ is contained in the center of $SU(l + 1) \times T$, we may assume that $g' = (g, 1) \in SU(l + 1) \times \{1\}$. Let $\pi_1 : SU(l + 1) \times T \to SU(l + 1)$ be the projection on the first factor. Then:

$$S(U(l) \times U(1)) = \pi_1(H_1) = g\pi_1(H_2)g^{-1} = gS(U(l) \times U(1))g^{-1}.$$ 

By Lemma A.3 it follows that

$$g \in N_{SU(l + 1)}S(U(l) \times U(1)) = \begin{cases} S(U(l) \times U(1)) & \text{if } l > 1, \\ N_{SU(2)}S(U(1) \times U(1)) & \text{if } l = 1. \end{cases}$$

Now for $h \in S(U(l) \times U(1))$ we have

$$(h, \psi_1(h)) = g'(g^{-1}hg, \psi_1(h))g'^{-1}.$$ 

Now $(g^{-1}hg, \psi_1(h))$ lies in $H_2$. Therefore we may write

$$g'(g^{-1}hg, \psi_1(h))g'^{-1} = g'(g^{-1}hg, \psi_2(g^{-1}hg))g'^{-1} = (h, \psi_2(g^{-1}hg)).$$

If $l > 1$ we have

$$\psi_2(g^{-1}hg) = \psi_2(g)^{-1}\psi_2(h)\psi_2(g) = \psi_2(h).$$

Otherwise we have

$$\psi_2(g^{-1}hg) = \psi_2(h^{\pm 1}) = \psi_2(h)^{\pm 1}.$$ 

The other implications are trivial. Therefore the statement follows. \hfill \Box

**Lemma A.4.** Let $l \geq 1$. $Spin(2l)$ is a maximal connected subgroup of $Spin(2l + 1)$. Its normalizer consists of two components.

**Proof.** By [2, p. 219], $Spin(2l)$ is a maximal connected subgroup of $Spin(2l + 1)$ and

$$N_{Spin(2l+1)}Spin(2l)/Spin(2l) = (Spin(2l + 1)/Spin(2l))^{Spin(2l)} = (S^{2l})^{Spin(2l)}$$

consists of two points. Therefore the second statement follows. \hfill \Box

**Lemma A.5.** Let $G$ be a Lie group which acts on the manifold $M$. Furthermore, let $N \subset M$ be a submanifold. If the intersection of $Gx$ and $N$ is transverse in $x$ for all $x \in N$, then $GN$ is open in $M$.

**Proof.** We will show that $f : G \times N \to M$, $(h, x) \mapsto hx$ is a submersion. Because a submersion is an open map, it follows that $GN = f(G \times N)$ is open in $M$. For
If \( g \in G \), let
\[
l_g : G \times N \to G \times N,
\]
\((h, x) \mapsto (gh, x)\)
and
\[
l'_g : M \to M,
\]
x \mapsto gx.

Then we have for all \( g \in G \)
\[
f = l'_g \circ f \circ l_{g^{-1}}.
\]
Now for \((g, x) \in G \times N\) we have
\[
D_{(g, x)} f = D_{l'_g} D_{(e, x)} f D_{(g, x)} l_{g^{-1}}.
\]
Because \( Gx \) and \( N \) intersect transversely in \( x \), the differential \( D_{(e, x)} f \) is surjective. Because \( l'_g, l_{g^{-1}} \) are diffeomorphisms, it follows that \( D_{(g, x)} f \) is surjective. Therefore \( f \) is a submersion. \( \square \)

**APPENDIX B. GENERALITIES ON TORUS MANIFOLDS**

**Lemma B.1.** Let \( M \) be a torus manifold and \( M_1, \ldots, M_k \) be pairwise distinct characteristic submanifolds of \( M \) with \( N = M_1 \cap \cdots \cap M_k \neq \emptyset \). Then each \( M_i \) intersects transversely with \( \bigcap_{j=1}^{k-1} M_j \). Therefore \( N \) is a submanifold of \( M \) with codim \( N = 2k \) and \( \dim(\lambda(M_1), \ldots, \lambda(M_k)) = k \). Furthermore, \( N \) is the union of some components of \( M^{(\lambda(M_1), \ldots, \lambda(M_k))} \).

**Proof.** We prove the lemma by induction on \( k \). Let \( k \geq 1 \) and \( x \in N \). Then we have
\[
T_x M = \bigcap_{i=1}^k T_x M_i \oplus \bigoplus_j V_j,
\]
where the \( V_j \) are one-dimensional complex \( \langle \lambda(M_1), \ldots, \lambda(M_k) \rangle \)-representations. Since the \( M_i \) have codimension two in \( M \), each \( \lambda(M_i) \) acts non-trivially on exactly one \( V_j \).

If codim \( \bigcap_{i=1}^k T_x M_i < 2k \), then there are \( i_1 \) and \( i_2 \) such that \( V_{j_{i_1}} = V_{j_{i_2}} \). Therefore
\[
T_x M_{i_1} = T_x M_{i_2} = T_x M^{(\lambda(M_{i_1}), \lambda(M_{i_2}))}
\]
has codimension two.

Since \( \langle \lambda(M_{i_1}), \lambda(M_{i_2}) \rangle \) has dimension two, it does not act almost effectively on \( M \). This is a contradiction. Therefore \( \bigcap_{i=1}^k T_x M_i \) has codimension \( 2k \). By induction hypothesis \( \bigcap_{i=1}^{k-1} M_i \) is a submanifold of codimension \( 2k - 2 \) and \( T_x \bigcap_{i=1}^{k-1} M_i = \bigcap_{i=1}^{k-1} T_x M_i \). Thus, \( M_k \) and \( \bigcap_{i=1}^{k-1} M_i \) intersect transversely. Therefore \( N \) is a submanifold of \( M \) of codimension \( 2k \).

If \( \langle \lambda(M_1), \ldots, \lambda(M_k) \rangle \) has dimension smaller than \( k \), then the weights of the \( V_j \) are linear dependent. Therefore there is \((a_1, \ldots, a_k) \in \mathbb{Z}^k - \{0\}\) such that
\[
\mathbb{C} = V_1^{a_1} \otimes \cdots \otimes V_k^{a_k},
\]
where \( \mathbb{C} \) denotes the trivial \( \langle \lambda(M_1), \ldots, \lambda(M_k) \rangle \)-representation. This gives a contradiction because each \( \lambda(M_i) \) acts non-trivially on exactly one \( V_j \).
By Lemma B.1, the intersection of the characteristic submanifolds is non-trivial.

\[ \langle \lambda(M_1), \ldots, \lambda(M_k) \rangle \]

has dimension \( k \), \( M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \) has dimension at most \( 2n - 2k \). But \( N \) is contained in \( M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \) and has dimension \( 2n - 2k \). Therefore it is the union of some components of \( M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \).

**Lemma B.2.** Let \( M \) be a torus manifold of dimension \( 2n \) and \( N \) be a component of the intersection of \( k (\leq n) \) characteristic submanifolds \( M_1, \ldots, M_k \) of \( M \) with \( N^T \neq \emptyset \). Then \( N \) is a torus manifold. Moreover, the characteristic submanifolds of \( N \) are given by the components of intersections of characteristic submanifolds \( M_i \neq M_1, \ldots, M_k \) of \( M \) with \( N \), which contain a \( T \)-fixed point.

**Proof.** Let \( M_i \neq M_1, \ldots, M_k \) be a characteristic submanifold of \( M \) with \( (M_i \cap N)^T \neq \emptyset \). Then, by Lemma [B.1] each component of \( M_i \cap N \) which contains a \( T \)-fixed point has codimension two in \( N \). This means that they are characteristic submanifolds of \( N \).

Now let \( N_1 \subset N \) be a characteristic submanifold and \( x \in N_1^T \). Then we have

\[ T_x M = T_x N_1 \oplus V_0 \oplus N_x(N, M) \]

as \( T \)-representations with \( V_0 \) a one-dimensional complex \( T \)-representation. Let \( M_i \) be the characteristic submanifold of \( M \) which corresponds to \( V_0 \). Then \( N_1 \) is the component of the intersection \( M_i \cap N \) which contains \( x \).

**Lemma B.3.** Let \( M \) be a \( 2n \)-dimensional torus manifold and \( T' \) be a subtorus of \( T \). If \( N \) is a component of \( M^{T'} \) which contains a \( T \)-fixed point \( x \), then \( N \) is a component of the intersection of some characteristic submanifolds of \( M \).

**Proof.** By Lemma [B.1] the intersection of the characteristic submanifolds \( M_1, \ldots, M_k \) is a union of some components of \( M^{\langle \lambda(M_1), \ldots, \lambda(M_k) \rangle} \).

Therefore we have to show that there are characteristic submanifolds \( M_1, \ldots, M_k \) of \( M \) such that

\[ T_x N = T_x (M_1 \cap \cdots \cap M_k) \]

There are \( n \) characteristic submanifolds \( M_1, \ldots, M_n \) which intersect transversely in \( x \). Therefore we have

\[ T_x M = N_x(M_1, M) \oplus \cdots \oplus N_x(M_n, M) \]

We may assume that there is a \( 1 \leq k \leq n \) such that \( T' \) acts trivially on \( N_x(M_i, M) \) for \( i > k \) and non-trivially on \( N_x(M_i, M) \) for \( i \leq k \). Then we have

\[ T_x N = (T_x M)^{T'} = N_x(M_{k+1}, M) \oplus \cdots \oplus N_x(M_n, M) = T_x (M_1 \cap \cdots \cap M_k) \]

**Lemma B.4.** Let \( M \) be a torus manifold with \( T^n \times \mathbb{Z}_2 \)-action such that \( \mathbb{Z}_2 \) acts non-trivially on \( M \). Furthermore, let \( B \subset M \) be a submanifold of codimension one on which \( \mathbb{Z}_2 \) acts trivially and let \( N \) be the intersection of characteristic submanifolds \( M_1, \ldots, M_k \) of \( M \). Then \( B \) and \( N \) intersect transversely.

**Proof.** Let \( x \in B \cap N \); then we have the \( \langle \lambda(M_1), \ldots, \lambda(M_k) \rangle \times \mathbb{Z}_2 \)-representation \( T_x M \). It decomposes as the sum of the eigenspaces of the non-trivial element of \( \mathbb{Z}_2 \). Because \( B \) has codimension one the eigenspace to the eigenvalue \(-1\) is...
one-dimensional. Because the irreducible non-trivial torus representations are two-dimensional, we have

\[ T_x N = (T_x M)^{(\lambda(M_1), \ldots, \lambda(M_k))} = T_x M^{(\lambda(M_1), \ldots, \lambda(M_k))} \times \mathbb{Z}^2 \oplus T_x (B, M)^{(\lambda(M_1), \ldots, \lambda(M_k))} \]

This means that the intersection is transverse. \(\square\)

**Lemma B.5.** Let \( M^{2n} \) be a \((\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2 \)-manifold such that \((\mathbb{Z}_2)^i \) acts non-trivially on \( M \). Furthermore, let \( B_i \subset M \), \( i = 1, 2 \), be closed connected submanifolds of codimension one such that \((\mathbb{Z}_2)^i \) acts trivially on \( B_i \). Then the following statements are equivalent:

1. \( B_1, B_2 \) intersect transversely,
2. \( B_1 \neq B_2 \),
3. \((\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2 \) acts effectively on \( M \) or \( B_1 \cap B_2 = \emptyset \).

**Proof.** Denote by \( V_i \) the non-trivial real irreducible representation of \((\mathbb{Z}_2)^i \). Let \( x \in B_1 \cap B_2 \). Then for the \((\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2 \)-representation \( T_x M \) there are two possibilities:

\[ T_x M = \begin{cases} \mathbb{R}^{2n-1} \oplus V_1 \oplus V_2, \\ \mathbb{R}^{2n-2} \oplus V_1 \oplus V_2. \end{cases} \]

In the first case \( B_i \), \( i = 1, 2 \), is the component of \( M^{(\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2} \) containing \( x \) and \((\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2 \) acts non-effectively on \( M \). In the second case \((\mathbb{Z}_2)^1 \times (\mathbb{Z}_2)^2 \) acts effectively on \( M \) and \( B_1, B_2 \) intersect transversely in \( x \).

All conditions given in the lemma imply that we are in the second case or \( B_1 \cap B_2 = \emptyset \). Therefore they are equivalent. \(\square\)

**Remark B.6.** Lemmas B.1, B.4 also hold if we do not require that a characteristic manifold contains a \( T \)-fixed point.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FRIBOURG, CHEMIN DU MUSÉE 23, CH-1700 FRIBOURG, SWITZERLAND

E-mail address: michwiem@web.de
Current address: MPI for Mathematics, Vivatsgasse 7, D-53111 Bonn, Germany
E-mail address: wiemeler@mpim-bonn.mpg.de

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