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THE COMPLEX CROWN FOR HOMOGENEOUS HARMONIC SPACES

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Abstract. The complex crown of a noncompact Riemannian symmetric space $X = G/K$ is generalized to the case of homogeneous harmonic spaces $S = NA$. We prove that every eigenfunction of the Laplace-Beltrami operator on S extends holomorphically to the crown, and that the crown is the maximal S-invariant domain in $S_{\mathbb{C}}$ with this property.

1. Introduction

Let X be a simply connected homogeneous harmonic Riemannian space. Then according to [7], Corollary 1.2, X is isometric (up to scaling of the metric) to one of the following spaces:

(i) $\mathbb{R}^n$.
(ii) S^n, $P^k(\mathbb{C})$, $P^l(\mathbb{H})$, or $P^2(\mathbb{O})$, i.e., a compact rank one symmetric space.
(iii) $H^n(\mathbb{R})$, $H^k(\mathbb{C})$, $H^l(\mathbb{H})$, or $H^2(\mathbb{O})$, i.e., a noncompact rank one symmetric space.
(iv) A solvable Lie group $S = A \ltimes N$, where N is of Heisenberg-type and $A \simeq \mathbb{R}^+$ acts on N by anisotropic dilations preserving the grading.

We denote by $\mathcal{L}$ the Laplace-Beltrami operator on X. In case (i), $\mathcal{L}$-eigenfunctions on $\mathbb{R}^n$ extend to holomorphic functions on $\mathbb{C}^n$. Likewise, in case (ii), $\mathcal{L}$-eigenfunctions on $X = U/K$ admit a holomorphic continuation to the whole affine complexification $X_{\mathbb{C}} = U_{\mathbb{C}}/K_{\mathbb{C}}$ of X. This is no longer true in case (iii). However, in this case, and more generally for a noncompact Riemannian symmetric space of any rank $X = G/K$, there exists a *unique* G-invariant domain $\mathrm{Cr}(X)$ of $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ containing X, the *complex crown*, with the following property ([13], [11]):

Every $\mathcal{L}$-eigenfunction on X admits a holomorphic extension to $\mathrm{Cr}(X)$*, and this domain is maximal for this property.*

The objective of this paper is to obtain an analogous theory for the spaces in (iv) above.

We note that all spaces in (iii), except for $H^n(\mathbb{R})$, fall into class (iv) by identifying the symmetric space $X = G/K$ with the NA-part in the Iwasawa decomposition $G = NAK$ of a noncompact simple Lie group G of real rank one, and by suitably scaling the metric.

Our investigations start with a new model of the crown domain for the rank one symmetric spaces X. We describe $\mathrm{Cr}(X)$ in terms of the Iwasawa coordinates A and N only. Henceforth we refer to this new model as the *mixed model* of the

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crown. In section 2 we provide the mixed model for the two basic cases, i.e., the symmetric spaces associated with the groups $G = \mathrm{Sl}(2,\mathbb{R})$ and $G = \mathrm{SU}(2,1)$.

Starting from the two basic cases, reduction of symmetry allows us to obtain a mixed model for all rank one symmetric spaces and motivates a definition of $\mathrm{Cr}(S)$ for the remaining spaces in (iv). This is worked out in section 3.

Finally, in section 4, we use recent results from [11] to prove the holomorphic extension of $\mathcal{L}$-eigenfunctions on S to the crown domain $\mathrm{Cr}(S)$ and establish maximality of $\mathrm{Cr}(S)$ with respect to this property.

2. Mixed model for the crown domain

The crown domain can be realized inside the complexification of an Iwasawa AN-group. The goal of this section is to make this explicit for the rank one groups $\mathrm{Sl}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$.

2.1. **Notation for rank one spaces.** Let G be a connected semi-simple Lie group of real rank one. We assume that $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is the universal complexification of G, and that $G_{\mathbb{C}}$ is simply connected. We fix an Iwasawa decomposition $G = NAK$ and form the Riemannian symmetric space

$$X = G/K.$$

The Cartan involution θ on G associated to K extends holomorphically to $G_{\mathbb{C}}$, say $\theta_{\mathbb{C}}$, and the $\theta_{\mathbb{C}}$-fixed points will be denoted by $K_{\mathbb{C}}$. We arrive at a totally real embedding

$$X \hookrightarrow X_{\mathbb{C}} := G_{\mathbb{C}}/K_{\mathbb{C}}, \quad gK \mapsto gK_{\mathbb{C}}.$$

Let $x_0 = K \in X$ be the base-point.

Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of G, K, A and N. Let $\Sigma^+ = \Sigma(\mathfrak{a}, \mathfrak{n})$ be the set of positive roots and put

$$\Omega := \{Y \in \mathfrak{a} \mid \forall \alpha \in \Sigma^+ \ |\alpha(Y)| < \pi/2\}.$$

Note that Ω is a symmetric interval in $\mathfrak{a} \simeq \mathbb{R}$. The crown domain of X is defined as

$$\mathrm{Cr}(X) := G \exp(i\Omega) \cdot x_0 \subset X_{\mathbb{C}}. \tag{2.1}$$

Let us point out that Ω is invariant under the Weyl group $\mathcal{W} = N_K(A)/Z_K(A) \simeq \mathbb{Z}_2$ and that $\exp(i\Omega)$ consists of elliptic elements in $G_{\mathbb{C}}$. We will refer to (2.1) as the *elliptic model* of $\mathrm{Cr}(X)$ (see [13] for the basic structure theory in these coordinates).

Let us define a domain Λ in $\mathfrak{n}$ by

$$\Lambda := \{Y \in \mathfrak{n} \mid \exp(iY) \cdot x_0 \in \mathrm{Cr}(X)\}_0,$$

where $\{\cdot\}_0$ refers to the connected component of $\{\cdot\}$ which contains 0.

The set Λ is explicitly determined in [10], Th. 8.11. Further by [10], Th. 8.3, we have

$$\mathrm{Cr}(X) = G \exp(i\Lambda) \cdot x_0. \tag{2.2}$$

We refer to (2.2) as the *unipotent model* of $\mathrm{Cr}(X)$.

Finally let us mention the fact that $\mathrm{Cr}(X) \subset N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0$, which brings us to the question of whether $\mathrm{Cr}(X)$ can be expressed in terms of A, N, Ω and Λ. This is indeed the case and will be considered in the following two subsections for the groups $\mathrm{Sl}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$.

2.2. **Mixed model for the upper half plane.** Let

$$G = \mathrm{Sl}(2,\mathbb{R}) \quad \text{and} \quad G_{\mathbb{C}} = \mathrm{Sl}(2,\mathbb{C}).$$

Our choices of A, N and K are as follows:

$$A = \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \mid t > 0 \right\},$$
$$A_{\mathbb{C}} = \left\{ a_z = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mid z \in \mathbb{C}^* \right\},$$
$$K = \mathrm{SO}(2,\mathbb{R}), \quad K_{\mathbb{C}} = \mathrm{SO}(2,\mathbb{C}),$$

and

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$
$$N_{\mathbb{C}} = \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

We will identify $X = G/K$ with the upper half plane $\mathbf{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ via the map

$$X \to \mathbf{H}, \quad gK \mapsto \frac{ai+b}{ci+d} \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right). \tag{2.3}$$

Note that $x_0 = i$ within our identification.

We view $X = \mathbf{H}$ inside of the complex projective space $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and note that $\mathbb{P}^1(\mathbb{C})$ is homogeneous for $G_{\mathbb{C}}$ with respect to the usual fractional linear action:

$$g(z) = \frac{az+b}{cz+d} \qquad \left(z \in \mathbb{P}^1(\mathbb{C}),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{C}} \right).$$

Let us remark that the map

$$X_{\mathbb{C}} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \backslash \mathrm{diag}, \quad gK_{\mathbb{C}} \mapsto (g(i), g(-i))$$

is a $G_{\mathbb{C}}$-equivariant holomorphic diffeomorphism. In the sequel we will identify $X_{\mathbb{C}}$ with $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \backslash \mathrm{diag}$, and emphasize that within this identification the embedding of (2.3) becomes

$$X \hookrightarrow X_{\mathbb{C}}, \quad z \mapsto (z, \overline{z}).$$

We will denote by $\overline{X}$ the lower half plane, and note that the *crown domain* for $\mathrm{Sl}(2,\mathbb{R})$ is given by

$$\mathrm{Cr}(X) = X \times \overline{X}.$$

We note that

$$\Omega = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in (-\pi/4, \pi/4) \right\}$$

and

$$\Lambda = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in (-1,1) \right\}.$$

With that we come to the mixed model for the crown which combines both parametrizations in an unexpected way.

We let $F := \{\pm \mathbf{1}\}$ be the center of G and note that

$$N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0 = \mathbb{C} \times \mathbb{C} \backslash \mathrm{diag},$$

and

$$N_{\mathbb{C}}A_{\mathbb{C}} \cdot x_0 \simeq N_{\mathbb{C}}A_{\mathbb{C}}/F.$$

Proposition 2.1. *Let $G = \mathrm{Sl}(2,\mathbb{R})$. Then the map*

$$NA \times \Omega \times \Lambda \to \mathrm{Cr}(X), \quad (na, H, Y) \mapsto na\exp(iH)\exp(iY) \cdot x_0$$

is an AN-equivariant diffeomorphism.

Proof. By the facts listed above we only have to show that the map is defined and onto. For that we first note that

$$\exp(i\Lambda) \cdot x_0 = \{((1+t)i, -(1-t)i) \mid t \in (-1,1)\},$$

and thus

$$A\exp(i\Lambda) = i\mathbb{R}^+ \times -i\mathbb{R}^+.$$

Consequently,

$$A\exp(i\Omega)\exp(i\Lambda) \cdot x_0 = \{(z,w) \in \mathrm{Cr}(X) \mid \arg(w) = -\pi + \arg(z)\},$$

and finally

$$NA\exp(i\Omega)\exp(i\Lambda) \cdot x_0 = \mathrm{Cr}(X),$$

as asserted. □

2.3. **Mixed model for** $\mathrm{SU}(2,1)$**.** Let $G = \mathrm{SU}(2,1)$ and $G_{\mathbb{C}} = \mathrm{Sl}(3,\mathbb{C})$. We let $G_{\mathbb{C}}$ act on $\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3 \setminus \{0\}/\sim)$ by projectivized linear transformations. We embed $\mathbb{C}^2$ into $\mathbb{P}^2(\mathbb{C})$ via $z \mapsto [z,1]$. Then G preserves the ball

$$X = \{z \in \mathbb{C}^2 \mid \|z\|_2 < 1\} \simeq G/K,$$

with the maximal compact subgroup $K = S(U(2) \times U(1))$ stabilizing the origin $x_0 = 0 \in X$. Note that an element

$$g = \begin{pmatrix} A & u \\ v^t & \alpha \end{pmatrix} \in G,$$

with $u, v \in \mathbb{C}^2$, acts on $z \in X$ by

$$g(z) = \frac{Az + u}{v^t \cdot z + \alpha}.$$

Let $\tau : G_{\mathbb{C}} \to G_{\mathbb{C}}$ be the holomorphic involution with fixed point set $\mathrm{SO}(2,1;\mathbb{C})$. In the sequel we will view $\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})$ as a $G_{\mathbb{C}}$-space with the twisted action

$$g \cdot (z,w) = (g(z), \tau(g)(w)) \qquad (z, w \in \mathbb{P}^2(\mathbb{C}); g \in G_{\mathbb{C}}).$$

Now, as X is Hermitian, the crown is given by the double (see [13], Th. 7.7)

$$\mathrm{Cr}(X) = X \times X,$$

but with X embedded in $\mathrm{Cr}(X)$ as $z \mapsto (z, \overline{z})$. Note that this embedding is G-equivariant. Let

$$A = \left\{ a_t := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

set $Y_t := \log a_t$, and note that

$$\Omega = \{Y_t \mid -\pi/4 < t < \pi/4\}.$$

According to [10], Th. 8.11, we have

$$\Lambda = \Big\{ Z_{a,b} := \begin{pmatrix} ib & a & -ib \\ -\overline{a} & 0 & \overline{a} \\ ib & a & -ib \end{pmatrix} \mid a \in \mathbb{C}, b \in \mathbb{R}; \\ |a|^2 + |b| < 1/2 \Big\}.$$

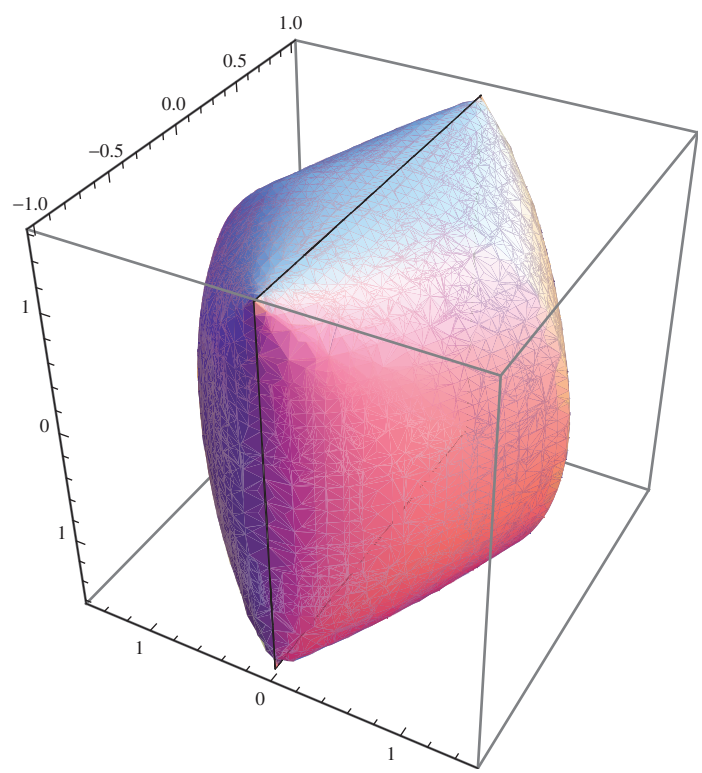
We now define a subset $\mathcal{D} \subset \mathfrak{a} \times \mathfrak{n}$ by

$$\mathcal{D} := \{(Y_t, Z_{a,b}) \in \mathfrak{a} \times \mathfrak{n} \mid (1 - 2|a|^2 - 2|b|)\cos(2t) > (1 - \cos(2t))|a|^2\}_0.$$

If $\pi_{\mathfrak{a}} : \mathfrak{a} \times \mathfrak{n} \to \mathfrak{a}$ denotes the first coordinate projection and $\pi_{\mathfrak{n}}$ resp. the second, we record:

- $\mathcal{D} \subset \Omega \times \Lambda$,
- $\pi_{\mathfrak{a}}(\mathcal{D}) = \Omega$,
- $\pi_{\mathfrak{n}}(\mathcal{D}) = \Lambda$.

In the coordinates $(2a, 2b, 2t)$ (with a real) we can visualize $\mathcal{D}$ as the interior of the following solid in $\mathbb{R}^3$:



Proposition 2.2. *Let $G = \mathrm{SU}(2,1)$. Then the map*

$$\Phi : NA \times \mathcal{D} \to \mathrm{Cr}(X), \quad (na, (Y, Z)) \mapsto na \exp(iY) \exp(iZ) \cdot x_0$$

is a diffeomorphism.

Proof. All that we have to show is that the map is defined and onto.

To begin with we show that Φ is defined, i.e., $\operatorname{Im}\Phi \subset \operatorname{Cr}(X)$. Set $n_{a,b} := \exp(iZ_{a,b})$. Let $M = Z_K(A)$. By M-invariance it is no loss of generality to assume that a is real. Then

$$n_{a,b} = \begin{pmatrix} 1-b+a^2/2 & ia & b-a^2/2 \\ -ia & 1 & ia \\ -b+a^2/2 & ia & 1+b-a^2/2 \end{pmatrix}.$$

We have to show that:

$$a_{i\phi} n_{a,b}(0) \in X \quad \text{and} \quad \tau(a_{i\phi} n_{a,b})(0) \ \in X,$$

for all parameters $(\phi, a, b) \in \mathcal{D}$. Note that $\tau(a_{i\phi}) = a_{i\phi}$ and $\tau(n_{a,b}) = n_{a,-b}$. Now

$$n_{a,b}(0) = \frac{1}{1+b-a^2/2}(b-a^2/2, ia).$$

Note that $n_{a,b}(0) \in X$ if and only if

$$(b-a^2/2)^2 + a^2 < (1+b-a^2/2)^2$$

or

$$-2b + 2a^2 < 1.$$

Combined with $n_{a,-b}(0) \in X$ we arrive at $2|b| + 2a^2 < 1$, which is the defining condition of Λ.

We move on and apply $a_{i\phi}$ to $n_{a,\pm b}(0)$, and obtain that

$$a_{i\phi} n_{a,\pm b}(0) = \frac{\left(\cos\phi \frac{\pm b - a^2/2}{1 \pm b - a^2/2} + i\sin\phi, \frac{ia}{1 \pm b - a^2/2}\right)}{\cos\phi + i\sin\phi \frac{\pm b - a^2/2}{1 \pm b - a^2/2}}.$$

Hence $a_{i\phi} n_{a,\pm b}(0) \in X$ if and only if

$$\cos^2\phi + \sin^2\phi \frac{(\pm b - a^2/2)^2}{(1 \pm b - a^2/2)^2} > \cos^2\phi \frac{(\pm b - a^2/2)^2}{(1 \pm b - a^2/2)^2} + \\ + \sin^2\phi + \frac{a^2}{(1 \pm b - a^2/2)^2}$$

or, equivalently, after clearing denominators,

$$(1 \pm b - a^2/2)^2 \cos^2\phi + (\pm b - a^2/2)^2 \sin^2\phi > (\pm b - a^2/2)^2 \cos^2\phi \\ + (1 \pm b - a^2/2)^2 \sin^2\phi + a^2.$$

Simplifying further we arrive at

$$\cos^2\phi + 2(\pm b - a^2/2)\cos^2\phi > \sin^2\phi + 2(\pm b - a^2/2)\sin^2\phi + a^2$$

or, equivalently,

$$(1 \pm 2b - 2a^2)\cos(2\phi) > (1 - \cos 2\phi)a^2.$$

However, this is the defining condition for $\mathcal{D}$.

To see that the map is onto we observe that $\operatorname{Cr}(X) \subset N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0$. Hence there exists a domain $\mathcal{D}' \subset \mathfrak{a} + \mathfrak{n}$ such that

$$\operatorname{Cr}(X) = NA \cdot \{\exp(iY)\exp(iZ) \mid (Y, Z) \in \mathcal{D}'\}.$$

If $\mathcal{D}'$ is strictly larger than $\mathcal{D}$, then $\mathcal{D}'$ would contain a boundary point of $\mathcal{D}$. Our computations show that this is not possible, and the proof of the proposition is concluded. $\square$

3. The crown for homogeneous harmonic spaces

Let S be a simply connected noncompact homogeneous harmonic space. According to [7], Corollary 1.2, there are the following possibilities for S:

(i) $S = \mathbb{R}^n$.
(ii) S is the AN-part of a noncompact simple Lie group G of real rank one.
(iii) $S = A \ltimes N$ with N a nilpotent group of Heisenberg-type and $A \simeq \mathbb{R}$ acting on N by graduation preserving scalings.

The case of $S = \mathbb{R}^n$ we will not consider; groups under (ii) are referred to as symmetric solvable harmonic groups. We mention that all spaces in (ii), except for those associated to $G = \mathrm{SO}_o(1,n)$, are of the type in (iii). Most issues of the Lorentz groups $G = \mathrm{SO}_o(1,n)$ readily reduce to $\mathrm{SO}_o(1,2) \simeq \mathrm{PSl}(2,\mathbb{R})$, where comprehensive treatments are available. In fact, for the real hyperbolic spaces $X = H^n(\mathbb{R}) = \mathrm{SO}_o(1,n)/\mathrm{SO}(n)$ we obtain the following result, analogous to Proposition 2.1.

Proposition 3.1. *Let $G = \mathrm{SO}_o(1,n)$ $(n \geq 2)$. Then the map*

$$NA \times \Omega \times \Lambda \to \mathrm{Cr}(X), \quad (na, H, Y) \mapsto na \exp(iH) \exp(iY) \cdot x_0$$

is a diffeomorphism.

We will now focus on type (iii). In the sequel we recall some basic facts about H-type groups and their solvable harmonic extensions. We refer to [14] for a more comprehensive treatment and references. After that we introduce the crown domain for such harmonic extensions.

3.1. **H-type Lie algebras and groups.** Let $\mathfrak{n}$ be a real nilpotent Lie algebra of step two (that is, $[\mathfrak{n},\mathfrak{n}] \neq \{0\}$ and $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = \{0\}$), equipped with an inner product $\langle \cdot, \cdot \rangle$ and associated norm $|\cdot|$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement in $\mathfrak{n}$. Then

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad [\mathfrak{v},\mathfrak{z}] = 0, \quad [\mathfrak{n},\mathfrak{n}] = [\mathfrak{v},\mathfrak{v}] \subset \mathfrak{z}.$$

For $Z \in \mathfrak{z}$ let $J_Z : \mathfrak{v} \to \mathfrak{v}$ be the linear map defined by

$$\langle J_Z V, V' \rangle = \langle Z, [V, V'] \rangle, \quad \forall V, V' \in \mathfrak{v}.$$

Then $\mathfrak{n}$ is called a *Heisenberg-type* algebra (or *H-type* algebra, for short) if

$$J_Z^2 = -|Z|^2 \,\mathrm{id}_{\mathfrak{v}} \qquad (Z \in \mathfrak{z}). \tag{3.1}$$

A connected and simply connected Lie group N is called an H-type group if its Lie algebra $\mathfrak{n} = \mathrm{Lie}(N)$ is an H-type algebra; see [9].

We let $p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z} (\geq 1)$. Condition (3.1) implies that p is even and $[\mathfrak{v},\mathfrak{v}] = \mathfrak{z}$.

Moreover, (3.1) implies that the map $Z \to J_Z$ extends to a representation of the real Clifford algebra $\mathrm{Cl}(\mathfrak{z}) \cong \mathrm{Cl}_q$ on $\mathfrak{v}$. This procedure can be reversed and yields a general method for constructing H-type algebras.

Since $\mathfrak{n}$ is nilpotent, the exponential map $\exp : \mathfrak{n} \mapsto N$ is a diffeomorphism. The Campbell-Hausdorff formula implies the following product law in N:

$$\exp X \cdot \exp X' = \exp\left(X + X' + \tfrac{1}{2}[X, X']\right), \quad \forall X, X' \in \mathfrak{n}.$$

This is sometimes written as

$$(V, Z) \cdot (V', Z') = \left(V + V', Z + Z' + \tfrac{1}{2}[V, V']\right),$$

using the exponential chart to parametrize the elements $n = \exp(V + Z)$ by the couples $(V, Z) \in \mathfrak{v} \oplus \mathfrak{z} = \mathfrak{n}$.

3.1.1. *Reduction theory.* We conclude this section with reduction theory for H-type Lie algebras to Heisenberg algebras.

Let $\mathfrak{z}_1 = \mathbb{R}Z_1$ be a one-dimensional subspace of $\mathfrak{z}$ and $\mathfrak{z}_1^\perp$ be its orthogonal complement in $\mathfrak{z}$. We assume that $|Z_1| = 1$ and set $J_1 := J_{Z_1}$. We form the quotient algebra

$$\mathfrak{n}_1 := \mathfrak{n}/\mathfrak{z}_1^\perp$$

and record that $\mathfrak{n}_1$ is two-step nilpotent. Let $p_1 : \mathfrak{z} \to \mathfrak{z}_1$ be the orthogonal projection. If we identify $\mathfrak{n}_1$ with the vector space $\mathfrak{v} \oplus \mathfrak{z}_1$ via the linear map

$$\mathfrak{n}_1 \to \mathfrak{v} \oplus \mathfrak{z}_1, \quad (V, Z) + \mathfrak{z}_1^\perp \mapsto (V, p_1(Z)),$$

then the bracket in $\mathfrak{n}_1$ becomes in the new coordinates

$$[(V, Z_1), (V', Z_1)] = (0, \langle Z_1, [V, V']\rangle Z_1),$$

where $V, V' \in \mathfrak{v}$. Since $\langle Z_1, [V, V']\rangle = \langle J_1 V, V'\rangle$, we see that J_1 determines a Lie algebra automorphism of $\mathfrak{n}_1$ and thus $\mathfrak{n}_1$ is isomorphic to the $p + 1$-dimensional Heisenberg algebra.

3.2. **Harmonic solvable extensions of H-type groups.** Let $\mathfrak{n}$ be an H-type algebra with associated H-type group N. Let $\mathfrak{a}$ be a one-dimensional Lie algebra with an inner product. Write $\mathfrak{a} = \mathbb{R}H$, where H is a unit vector in $\mathfrak{a}$. Let $A = \exp \mathfrak{a}$ be a one-dimensional Lie group with Lie algebra $\mathfrak{a}$ and isomorphic to $\mathbb{R}^+$ (the multiplicative group of positive real numbers). Let the elements $a_t = \exp(tH) \in A$ act on N by the dilations $(V, Z) \to (e^{t/2}V, e^t Z)$ for $t \in \mathbb{R}$, and let S be the associated semi-direct product of N and A:

$$S = NA = N \rtimes A.$$

The action of A on N becomes the inner automorphism

$$a_t \exp(V + Z) a_t^{-1} = \exp\left(e^{t/2}V + e^t Z\right),$$

and the product in S is given by

$$\exp(V + Z)a_t \exp(V' + Z')a_{t'} = \exp(V + Z)\exp(e^{t/2}V' + e^t Z')a_{t+t'}.$$

S is a connected and simply connected Lie group with Lie algebra

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$$

and Lie bracket defined by linearity and the requirement that

$$[H, V] = \frac{1}{2}V, \qquad [H, Z] = Z, \qquad \forall V \in \mathfrak{v},\ \forall Z \in \mathfrak{z}.$$

The map $(V, Z, tH) \to \exp(V + Z)\exp(tH)$ is a diffeomorphism of $\mathfrak{s}$ onto S. If we parametrize the elements $na = \exp(V + Z)\exp(tH) \in NA$ by the triples $(V, Z, t) \in \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}$, then the product law reads

$$(V, Z, t) \cdot (V', Z', t') = \left(V + e^{t/2}V', Z + e^t Z' + \tfrac{1}{2}e^{t/2}[V, V'], t + t'\right),$$

for all $V, V' \in \mathfrak{v}$, $Z, Z' \in \mathfrak{z}$, $t, t' \in \mathbb{R}$. For $n = (V, Z, 0) \in N$ and $a_t = (0, 0, t) \in A$ we consistently get $na_t = (V, Z, t)$.

We extend the inner products on $\mathfrak{n}$ and $\mathfrak{a}$ to an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{s}$ by linearity and the requirement that $\mathfrak{n}$ be orthogonal to $\mathfrak{a}$. The left-invariant Riemannian metric on S defined by this inner product turns S into a harmonic solvable group [5, 6].

3.3. **The complexification and the crown.** Let $N_{\mathbb{C}}$ be the simply connected Lie group with Lie algebra $\mathfrak{n}_{\mathbb{C}}$ and set $A_{\mathbb{C}} = \mathbb{C}^*$. Then $A_{\mathbb{C}}$ acts on $N_{\mathbb{C}}$ by $t\cdot(V,Z) := (tV, t^2Z)$ for $t \in \mathbb{C}^*$ and $(V,Z) \in N_{\mathbb{C}}$.

We define a complexification $S_{\mathbb{C}}$ of S by

$$S_{\mathbb{C}} = N_{\mathbb{C}} \rtimes A_{\mathbb{C}}.$$

We identify the Lie algebra $\mathfrak{a}$ with $\mathbb{R}$ by sending $H \mapsto 1/2$. For $z \in \mathbb{C}$ we often set $a_z := \exp(zH)$ and note that a_z corresponds to $e^{z/2} \in \mathbb{C}^*$. In particular,

$$\{z \in \mathbb{C} : \ \exp(zH) = e\} = 4\pi i\mathbb{Z}.$$

It follows that the exponential map $\exp : \mathfrak{a}_{\mathbb{C}} \to A_{\mathbb{C}}$ is certainly injective if restricted to $\mathfrak{a} \oplus iH(-2\pi, 2\pi]$.

Motivated by our discussion of $\mathrm{SU}(2,1)$ and the discussed $\mathrm{SU}(n,1)$-reduction, we define the following sets for a more general harmonic AN-group:

$$\Omega = \{tH \in \mathfrak{a} : \ |t| < \tfrac{\pi}{2}\},$$
$$\Lambda = \{(V,Z) \in \mathfrak{n} : \ \tfrac{1}{2}|V|^2 + |Z| < 1\},$$
$$\mathcal{D} = \left\{(V,Z,t) \in \mathfrak{s} : \ \cos t(1 - \tfrac{1}{2}|V|^2 - |Z|) > \tfrac{1}{4}(1-\cos t)|V|^2\right\}_0,$$
$$D = \{\exp(itH)\exp(iV + iZ) : \ (V,Z,t) \in \mathcal{D}\} \subset S_{\mathbb{C}}.$$

Here as usual $\{\cdot\}_0$ denotes the connected component of $\{\cdot\}$ containing 0, and we write (V,Z,t) for the element $V + Z + tH$ of $\mathfrak{s}$.

Definition 3.2. The complex crown of S will be defined as the following subset of $S_{\mathbb{C}}$:

$$Cr(S) = NAD \subset NA\exp(i\Omega)\exp(i\Lambda) \subset S_{\mathbb{C}}.$$

It is easy to see that $\mathrm{Cr}(S)$ is open and simply connected in $S_{\mathbb{C}}$.

We conclude with the proof of the mixed model of the crown for rank one symmetric spaces. In this case we let $S = AN$ be the solvable part of the rank one group $G = NAK$. Note that $F := A_{\mathbb{C}} \cap K_{\mathbb{C}}$ is isomorphic to $\mathbb{Z}_2$ and that there is an $S_{\mathbb{C}}$-equivariant embedding $S_{\mathbb{C}}/F \to X_{\mathbb{C}}$. As $\exp(2i\Omega) \cap F = \{e\}$, we thus arrive at an S-equivariant embedding $\mathrm{Cr}(S) \hookrightarrow X_{\mathbb{C}}$.

Theorem 3.3. *Let $X = G/K$ be a noncompact Riemannian symmetric space of rank one, $X \neq H^n(\mathbb{R})$. Let $S = NA$, and let $\mathrm{Cr}(S)$ be embedded in $X_{\mathbb{C}}$ by $z \mapsto z\cdot x_0$. Then*

$$\mathrm{Cr}(X) = \mathrm{Cr}(S).$$

Proof. We first show that $\mathrm{Cr}(S) \subset \mathrm{Cr}(X)$. We have to show that $\exp(itH)\exp(iY)\cdot x_0 \in \mathrm{Cr}(X)$ for all $(Y,t) \in \mathcal{D}$. Let $M = Z_K(A)$ and let $S_{\mathfrak{v}}$, $S_{\mathfrak{z}}$ denote the unit spheres in $\mathfrak{v}$ and $\mathfrak{z}$. Since the group $\mathrm{Ad}(M)$ acts transitively on $S_{\mathfrak{v}} \times S_{\mathfrak{z}}$ (see, e.g., [3], Theorem 6.2, for a simple proof within the framework of H-type groups), the assertion is reduced to $G = \mathrm{SU}(2,1)$, where it was shown in Proposition 2.2 above.

For the converse inclusion let $X_1 = G_1/K_1 \subset X = G/K$ be a subdomain with $K_1 \subset K$ and $\Omega_1 = \Omega$. We claim that $\partial\,\mathrm{Cr}(X_1) \subset \partial\,\mathrm{Cr}(X)$. In fact, if $z \in \partial\,\mathrm{Cr}(X_1)$, then Lemma 2.3 (ii) in [13] shows that z is the limit of a sequence

$x_n := g_n \exp(iY_n) \cdot x_0$ with $g_n \in G_1$ and $\Omega_1 \ni Y_n \to Y \in \partial\Omega_1$. On the other hand, assuming $z \in \mathrm{Cr}(X)$ we would also have $z = g\exp(iZ) \cdot x_0$ with $g \in G$ and $Z \in \Omega$. By following the proof of the cited lemma one sees that $g^{-1}g_n$ must be bounded, from which a contradiction is easily reached. Hence $z \in \partial\,\mathrm{Cr}(X)$.

In order to conclude the proof of the theorem, it is enough to show that the elements which are in the boundary of D do not lie in $\mathrm{Cr}(X)$, i.e.,

$$\{\exp(itH)\exp(iV + iZ) : \quad (V, Z, t) \in \partial\mathcal{D}\} \cap \mathrm{Cr}(X) = \emptyset.$$

This again reduces to $G = \mathrm{SU}(2,1)$ and thus follows from $\partial\,\mathrm{Cr}(X_1) \subset \partial\,\mathrm{Cr}(X)$. $\square$

4. Geometric analysis

Let S be a simply connected noncompact homogeneous harmonic space, $S \neq \mathbb{R}^n$. In this section we will show that the eigenfunctions of the Laplace-Beltrami operator on S extend holomorphically to $\mathrm{Cr}(S)$ and that $\mathrm{Cr}(S)$ is maximal with respect to this property.

4.1. **Holomorphic extension of eigenfunctions.** For $z \in D$ we consider the following totally real embedding of S into $\mathrm{Cr}(S)$:

$$S \hookrightarrow \mathrm{Cr}(S), \quad s \mapsto sz.$$

Now let $\mathcal{L}$ be the Laplace-Beltrami operator on S, explicitly given by (see [14])

$$\mathcal{L} := \sum_{j=1}^{p} V_j^2 + \sum_{i=1}^{q} Z_i^2 + H^2 - 2\rho H,$$

where $2\rho = (p/2) + q$ and the $(V_j)_j$ and $(Z_i)_i$ form an orthonormal basis of $\mathfrak{v}$ and $\mathfrak{z}$, respectively. Here we consider the elements $X \in \mathfrak{s}$ as left-invariant vector fields on S. Hence it is clear that $\mathcal{L}$ extends to a left $S_{\mathbb{C}}$-invariant holomorphic differential operator on $S_{\mathbb{C}}$, which we denote by $\mathcal{L}_{\mathbb{C}}$. Now if $M \subset S_{\mathbb{C}}$ is a totally real analytic submanifold, then we can restrict $\mathcal{L}_{\mathbb{C}}$ to M, in symbols $\mathcal{L}_M$, in the following way: if f is a real analytic function near $m \in M$ and $f_{\mathbb{C}}$ is a holomorphic extension of f in a complex neighborhood of m in $S_{\mathbb{C}}$, then set

$$(\mathcal{L}_M f)(m) := (\mathcal{L}_{\mathbb{C}} f_{\mathbb{C}})(m).$$

Then:

Proposition 4.1. *For all $z \in D$ the restriction $\mathcal{L}_{Sz}$ is elliptic.*

Proof. To illustrate what is going on we first give a proof for those S related to $G = \mathrm{Sl}(2,\mathbb{R})$. We consider this as the degenerate case where $\mathfrak{v} = \{0\}$ so that $\mathrm{ad}(H)$ acts on $\mathfrak{n} = \mathfrak{z} = \mathbb{R}Z$ as the identity map (see [3], section 3(d)). Here

$$H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The set $\mathcal{D}$ degenerates to the product

$$\mathcal{D} = \Omega \times \Lambda = \{tH : \quad |t| < \tfrac{\pi}{2}\} \times \{xZ : \quad |x| < 1\},$$

and $D = \exp(i\Omega)\exp(i\Lambda)$, in agreement with Proposition 2.1. The Laplace-Beltrami operator on S becomes $\mathcal{L} = Z^2 + H^2 - H$. Let $z = \exp(itH)\exp(ixZ) \in D$. Using

$[H, Z] = Z$ we get

$$\mathrm{Ad}(z)H = H - ie^{it}xZ,$$
$$\mathrm{Ad}(z)Z = e^{it}Z.$$

It follows that the leading symbol, or principal part, of $\mathcal{L}_{Sz}$ is given by

$$[\mathcal{L}_{Sz}]_{\mathrm{prin}} = (H - ie^{it}xZ)^2 + e^{2it}Z^2.$$

Let us verify that $\mathcal{L}_{Sz}$ is elliptic. The associated quadratic form is given by the matrix

$$L(z) := \begin{pmatrix} 1 & -ixe^{it} \\ -ixe^{it} & e^{2it}(1-x^2) \end{pmatrix}.$$

We have to show that $\langle L(z)\xi, \xi\rangle = 0$ has no solution for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2\backslash\{0\}$. We look at

$$\xi_1^2 - 2ixe^{it}\xi_1\xi_2 + e^{2it}(1-x^2)\xi_2^2 = 0.$$

Now $\xi_2 = 0$ is readily excluded, and we remain with the quadric

$$\xi^2 - 2ixe^{it}\xi + e^{2it}(1-x^2) = 0,$$

whose solutions are

$$\begin{aligned}\lambda_{1,2} &= ixe^{it} \pm \sqrt{-x^2e^{2it} - (1-x^2)e^{2it}} \\ &= ie^{it}(x \pm 1).\end{aligned}$$

These are never real in the domain $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $-1 < x < 1$. The same proof works for those S related to $\mathrm{SO}_o(1, n)$, $n \geq 2$.

To put the computation above in a more abstract framework: it is to show here that for $z \in D$ the operator $L(z) : \mathfrak{s}_{\mathbb{C}} \to \mathfrak{s}_{\mathbb{C}}$ defined by

$$L(z) := \mathrm{Ad}(z)^t\,\mathrm{Ad}(z) \tag{4.1}$$

is elliptic in the sense that $\langle L(z)\xi, \xi\rangle = 0$ for $\xi \in \mathfrak{s}$ implies $\xi = 0$.

Thus, for the sequel we assume that $\mathfrak{n}$ is H-type, i.e., $\mathfrak{v}$ and $\mathfrak{z}$ are $\neq \{0\}$. We will reduce to the case where N is a Heisenberg group, i.e., S is related to $\mathrm{SU}(1, n)$. We use the already introduced technique of reduction to Heisenberg groups. Therefore, let $Z_1 \in \mathfrak{z}$ be a normalized element and $\mathfrak{n}_1 = \mathfrak{n}/\mathfrak{z}_1^{\perp}$ be as before. We let S_1 be the harmonic group associated to N_1 and note that there is a natural group homomorphism $S \to S_1$ which extends to a holomorphic map $S_{\mathbb{C}} \to (S_1)_{\mathbb{C}}$ which maps $\mathrm{Cr}(S)$ onto $\mathrm{Cr}(S_1)$. Now the assertion is true for $\mathrm{Cr}(S_1)$ in view of [11] (proof of Th. 3.2) and Theorem 3.3. Let M be the group of automorphisms of S preserving the inner product on $\mathfrak{s}$. By [3], Proposition 6.1 and Remark 6.3, the group $\mathrm{Ad}(M)$ acts transitively on the unit sphere in $\mathfrak{z}$. Thus the ellipticity of (4.1) is true for S if it is true for all choices of S_1, and the assertion follows. □

As a consequence of this fact, we obtain as in [11] that:

Theorem 4.2. *Every $\mathcal{L}$-eigenfunction on S extends to a holomorphic function on* $\mathrm{Cr}(S)$.

Proof (analogous to the proof of Th. 3.2 of [11]). As $\mathcal{L}$ is elliptic, the regularity theorem for elliptic differential operators implies that all eigenfunctions f are analytic and extend holomorphically on a neighborhood of $\mathbf{1}$ in $S_{\mathbb{C}}$. This neighborhood can be chosen independently of f.

Let $0 \leq t \leq 1$ and define $\mathcal{D}_t := t\mathcal{D}$ and correspondingly D_t. It follows that all eigenfunctions f extend holomorphically to a common domain of the form $UD_t \subset S_{\mathbb{C}}$ for some $0 < t \leq 1$ and some open neighborhood U of $\mathbf{1}$ in S. As $\mathcal{L}$ is S-invariant we can replace f by any left S-translate and conclude that f extends to $SD_t \subset S_{\mathbb{C}}$.

If $t = 1$, we are finished. Otherwise, let t be such that all eigenfunctions f extend to SD_t, but for each $t < t' \leq 1$ some eigenfunction does not extend to $SD_{t'}$. Let $(Y, r) \in \partial\mathcal{D}_t$ and let $z = \exp(irH)\exp(iY)$. By our previous proposition, $\mathcal{L}_{Sz}$ is elliptic. Now it comes down to choosing appropriate local coordinates to see that f extends holomorphically on a complex cone based at z. The condition (9.4.16) in [8], Cor. 9.4.9, is verified so that this corollary applies. We conclude that there exists an open neighborhood of z on which all eigenfunctions f are holomorphic. By compactness of ∂D_t we then find $t' > t$ and an open neighborhood U of $\mathbf{1}$ in S, both independent of f, such that f extends holomorphically to $UD_{t'}$. As before we obtain that f extends to $SD_{t'}$. As this is valid for all f a contradiction is reached and the theorem follows. □

4.2. **Maximality of** $\mathrm{Cr}(S)$**.** We begin with a collection of some facts about Poisson kernels on S. Let us denote by $\theta : S \to S$ the geodesic symmetry, centered at the identity. Every $z \in S_{\mathbb{C}}$ can be uniquely written as $z = n(z)a(z)$ with $n \in N_{\mathbb{C}}$ and $a \in A_{\mathbb{C}}$. As $\mathrm{Cr}(S)$ is simply connected, we obtain for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ a holomorphic map

$$a^\lambda : \mathrm{Cr}(S) \to \mathbb{C},\ z \mapsto e^{\lambda \log a(z)}.$$

The function $P_\lambda := a^{\rho - i\lambda} \circ \theta$ on S is referred to as *Poisson kernel on S with parameter* λ. We note that both a^λ and P_λ are $\mathcal{L}$-eigenfunctions [4, 1].

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we define the *spherical function* with parameter λ by

$$\phi_\lambda(s) := b \int_N P_\lambda(ns) P_{-\lambda}(n)\, dn \qquad (s \in S),$$

where $b = 2^{q-1}\Gamma(\frac{p+q+1}{2})/\pi^{\frac{p+q+1}{2}}$ is a constant such that $\phi_\lambda(e) = 1$.

Let us denote by $\mathcal{X} \subset \mathfrak{a}_{\mathbb{C}}^*$ the parameter set for nontrivial positive definite spherical functions on S and note that $\mathfrak{a}^* \subset \mathcal{X}$.

Theorem 4.3. *The crown domain* $\mathrm{Cr}(S)$ *is the unique largest S-invariant domain in $S_{\mathbb{C}}$ containing S with the property that all $\mathcal{L}$-eigenfunctions extend holomorphically to* $\mathrm{Cr}(S)$.

Proof. Let us first discuss the case of symmetric S, i.e. $S \simeq X = G/K$ and $\mathrm{Cr}(S) = \mathrm{Cr}(X)$. Fix $\lambda \in \mathcal{X}$. In this case it was shown in Section 5 of [12] (with corrigendum in [10], Remark 4.8) that $\mathrm{Cr}(X)$ is the unique largest G-invariant domain in $X_{\mathbb{C}}$ containing X to which ϕ_λ extends holomorphically. We will now adapt this method to the S-invariant situation.

Let $z \in \partial\,\mathrm{Cr}(S)$. Recall the elliptic boundary of $\mathrm{Cr}(X)$ in $X_{\mathbb{C}}$, $\partial_{\mathrm{ell}}\,\mathrm{Cr}(X) = G\exp(i\partial\Omega) \cdot x_0 \subset \partial\,\mathrm{Cr}(X)$. Let us first assume that $z \in \partial\,\mathrm{Cr}(S) \cap \partial_{\mathrm{ell}}\,\mathrm{Cr}(X)$. Assume that ϕ_λ extends holomorphically to an open neighborhood $U \subset S_{\mathbb{C}}$ of z. We find $r > 1$ and $z_0 \in \exp(ir\partial\Omega)$ and $g \in G$ such that $gz_0 \in U$.

Set $\psi_\lambda := \phi_\lambda \circ g^{-1}$. Then ψ_λ is an $\mathcal{L}$-eigenfunction. Now it was shown in [12] (Lemma 5.3 and Th. 5.4 with proof) that there exists an $s \in S$ and a sequence $(s_n)_{n\in\mathbb{N}} \subset S$ with $\lim_{n\to\infty} s_n = s \in S$ such that

$$0 \leq \phi_\lambda(s_n z_0) \nearrow \infty.$$

Now Lemma 5.3 in [12] readily modifies and allows us to replace the sequence $(s_n)_{n\in N}$ by a sequence $(g^{-1}u_n g)_{n\in\mathbb{N}}$ with $u_n \in S$ converging to $u \in S$. Hence we get that

$$0 \leq \psi_\lambda(u_n z) \nearrow \infty\,, \tag{4.2}$$

a contradiction. If z is not in the elliptic boundary, then we argue as in [10], Rem. 4.8, and reach the same conclusion.

For nonsymmetric S we apply $\mathrm{SU}(2,1)$-reduction (see [2]). If $z \in \mathrm{Cr}(S)$, then we can put $z \in \mathrm{Cr}(S_1)$ with $\mathrm{Cr}(S_1)$ an $\mathrm{SU}(2,1)$-crown contained in $\mathrm{Cr}(S)$. Let $\mathcal{X}_1$ be the set of equivalence classes of nontrivial positive definite spherical functions on S_1 and note that $\mathcal{X}_1 = \mathfrak{a}^* \amalg i(-c\rho_1, c\rho_1)$ for some $c > 0$. Disintegration theory of unitary representations yields a positive Radon measure μ on $\mathcal{X}_1$ such that

$$\phi_\lambda|_{S_1} = \int_{\mathcal{X}_1 \cup \{\mathbf{1}\}} \phi^1_\sigma \; d\mu(\sigma).$$

Here ϕ^1_σ stands for the spherical function on S_1 with parameter σ. If we can exclude the case that $\operatorname{supp}\mu = \{\mathbf{1}\}$, then the assertion follows with (4.2) as before.

Finally $\operatorname{supp}\mu = \{\mathbf{1}\}$ cannot happen, as $\phi_\lambda|_A$ is not constant and $A \subset S_1$. □

Corollary 4.4. *The geodesic symmetry extends to a holomorphic involutive map* $\theta : \mathrm{Cr}(S) \to \mathrm{Cr}(S)$.

Proof. For all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the Poisson kernel $P_\lambda = a^{\rho - i\lambda} \circ \theta$ extends to a holomorphic function on $\mathrm{Cr}(S)$. Thus the map θ must extend to a holomorphic involutive map

$$\theta : \mathrm{Cr}(S) \to S_{\mathbb{C}}.$$

To see that $\theta(\mathrm{Cr}(S)) = \mathrm{Cr}(S)$, we observe that $\theta(\mathrm{Cr}(S))$ is an open connected, S-invariant neighborhood of S in $S_{\mathbb{C}}$ (θ being holomorphic and involutive). Suppose this neighborhood is not contained in $\mathrm{Cr}(S)$. Then there exists a boundary point $z \in \partial\,\mathrm{Cr}(S)$ and a $q \in \mathrm{Cr}(S)$ such that $z = \theta(q)$. However, then

$$P_\lambda(z) = a^{\rho - i\lambda}(q).$$

Now the right-hand side in this formula is well defined $\forall \lambda \in \mathfrak{a}^*_{\mathbb{C}}$, whereas the left-hand side blows up for some $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (if it did not, then ϕ_λ for some $\lambda \in \mathcal{X}$ would extend holomorphically to an S-invariant domain strictly containing $\mathrm{Cr}(S)$). It follows that $\theta(\mathrm{Cr}(S)) \subseteq \mathrm{Cr}(S)$. The opposite inclusion is clear from the involutivity of θ. □

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References

[1] J.-P. Anker, E. Damek, and C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Scuola Norm. Sup. Pisa **23** (1996) 643-679. MR1469569 (99a:22014)

[2] M.G. Cowling, A.H. Dooley, A. Koranyi, and F. Ricci, *H-type groups and Iwasawa decomposition*, Adv. Math. **87** (1991) 1-41. MR1102963 (92e:22017)

[3] M.G. Cowling, A.H. Dooley, A. Koranyi, and F. Ricci, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal. **8** (1998) 199-237. MR1705176 (2000m:53071)

[4] E. Damek, *A Poisson kernel on Heisenberg type nilpotent groups*, Colloq. Math. **53** (1987) 239-247. MR924068 (89d:22006)

[5] E. Damek and F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. **27** (1992) 139-142. MR1142682 (93b:53043)

[6] E. Damek and F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. **2** (1992) 213-248. MR1164603 (93d:43006)

[7] J. Heber, *On harmonic and asymptotically harmonic spaces*, Geom. Funct. Anal. **16** (2006) 869-890. MR2255384 (2007e:53060)

[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer Grundlehren 256, 1983. MR717035 (85g:35002a)

[9] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. **258** (1980) 147-153. MR554324 (81c:58059)

[10] B. Krötz and E. M. Opdam, *Analysis on the crown domain*, Geom. Funct. Anal. **18** (2008) 1326-1421. MR2465692 (2010a:22011)

[11] B. Krötz and H. Schlichtkrull, *Holomorphic extension of eigenfunctions*, Math. Ann. **345** (2009) 835-841. MR2545868 (2010k:22016)

[12] B. Krötz, G. Ólafsson and R.J. Stanton, *The Image of the Heat Kernel Transform on Riemannian Symmetric Spaces of the Noncompact Type*, Int. Math. Res. Not. 2005, no. 22, 1307-1329. MR2152539 (2006e:58036)

[13] B. Krötz and R.J. Stanton, *Holomorphic extensions of representations. II. Geometry and Harmonic Analysis*, Geom. Funct. Anal. **15** (2005) 190-245. MR2140631 (2006d:43010)

[14] F. Rouviere, *Espaces de Damek-Ricci, geometrie et analyse*, Semin. Congr. 7, Soc. Math. France (2003) 45-100. MR2038648 (2004m:22015)

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