ON RESTRICTION OF MAXIMAL MULTIPLIERS
IN WEIGHTED SETTINGS

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Abstract. We obtain restriction results of K. De Leeuw’s type for maximal operators defined through Fourier multipliers of either strong or weak type for weighted $L^p$ spaces with $1 \leq p < \infty$. Applications to the case of Hörmander-Mihlin multipliers, singular integral operators and Bochner-Riesz sums are given.

1. Introduction

In 1965 K. De Leeuw proved that if $m$ is a continuous function on $\mathbb{R}$ such that $m$ is a Fourier multiplier on $L^p(\mathbb{R})$, its restriction to the integers $m|\mathbb{Z}$ is a Fourier multiplier on $L^p(\mathbb{T})$. Moreover, its norm does not exceed the norm of $m$ as a multiplier on $L^p(\mathbb{R})$ (see [8, Proposition 3.3] and Jodeit’s article [12]).

In 1980 C. Kenig and P. Tomas extended De Leeuw’s result to maximal operators associated to a family of multipliers given by the dilations of a given one. More precisely, they proved that if $m$ is a continuous function and if $T_r$ denotes the multiplier operator associated to $m_r(\xi) = m(\xi/r)$, whenever $T^\# f(x) = \sup_{r > 0} |T_r f(x)|$ is a bounded operator on $L^p(\mathbb{R}^d)$ the same holds for the maximal operator on $L^p(\mathbb{T}^d)$ associated to the multipliers $m_r|\mathbb{Z}$. Furthermore, its norm does not exceed a constant times the norm of $T^\#$. They also obtained similar results for operators of weak type for $p > 1$ (see [13]).

In 2003, E. Berkson and T.A. Gillespie extended De Leeuw’s restriction result for multipliers on $L^p(\mathbb{R}, w)$ with $w$ a $1$-periodic weight belonging to $A_p(\mathbb{R})$ and $1 < p < \infty$. Such weights are said to be in the class $A_p(\mathbb{T})$. Their result is the following.

Theorem 1.1 ([4, Theorem 1.2]). Let $1 < p < \infty$ and let $w \in A_p(\mathbb{T})$. If $m$ is a continuous function on $\mathbb{R}$ such that it is a Fourier multiplier on $L^p(\mathbb{R}, w)$, then $m|\mathbb{Z}$ is a Fourier multiplier on $L^p(\mathbb{T}, w)$. Moreover, there is a constant $c_{p,w}$ depending only on $p$ and the $A_p$-constant of $w$, such that the norm of $m|\mathbb{Z}$ as a multiplier on $L^p(\mathbb{T}, w)$ does not exceed $c_{p,w}$ times the norm of $m$ as a multiplier on $L^p(\mathbb{R}, w)$.

This theorem has been recently improved by K. Andersen and P. Mohanty as follows.

Theorem 1.2 ([1, Theorem 1.1]). Let $1 < p < \infty$ and let $w \in L^1(\mathbb{T}^d)$. If $m$ is a continuous function on $\mathbb{R}^d$ such that it is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$,
Theorem 1.4. Let $m_\xi$ be a Fourier multiplier on $L^p(\mathbb{T}^d, w)$. Moreover, the norm of $m_\xi$ as a multiplier on $L^p(\mathbb{T}^d, w)$ does not exceed the norm of $m$ as a multiplier on $L^p(\mathbb{R}^d, w)$.

The purpose of this paper is twofold:

i) To give restriction results from $\mathbb{R}^d$ to $\mathbb{T}^d$ for Fourier multipliers and for associated maximal operators of weak type (and strong type) in any dimension and for $1 \leq p < \infty$. In particular, we shall prove the following.

**Theorem 1.3.** Let $1 \leq p < \infty$ and let $w$ be a periodic weight on $\mathbb{R}^d$ satisfying $w \in L^1(\mathbb{T}^d)$. Suppose that $\{m_j\}_j$ is a family of multipliers that are continuous functions satisfying that the associated maximal operator (see Definition 2.2 below) is bounded from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ (or to $L^p(\mathbb{R}^d, w)$). Then the maximal operator associated to their restriction to the integers $\{m_j|_{\mathbb{Z}^d}\}_j$ (see Definition 3.1 below) is bounded from $L^p(\mathbb{T}^d, w)$ to $L^{p,\infty}(\mathbb{T}^d, w)$ (resp. to $L^p(\mathbb{T}^d, w)$) and its operator norm does not exceed $c_p$ times the norm of the maximal operator associated to $\{m_j\}_j$, where $c_p$ is a constant that depends only on $p$.

ii) K. De Leeuw in [8] and J. Jodeit in [12] also proved some restriction results for strong Fourier multipliers on $L^p(\mathbb{R}^d, w)$ with $w$ a suitable weight in $A_p(\mathbb{R}^d)$ was given. Namely, [7, Corollary 4.13] states that if $m$ is a continuous and bounded function in $\mathbb{R}^d$ that is a Fourier multiplier on $L^p(\mathbb{R}^d, w)$ where $w = u \otimes v$ with $u \in A_p(\mathbb{R}^d)$, $v \in A_p(\mathbb{R}^d)$, then, for any $x \in \mathbb{R}^d$, the function $m(\xi, \cdot)$ is a Fourier multiplier on $L^p(\mathbb{R}^d, v)$. In this setting, we shall prove the following.

**Theorem 1.4.** Let $d = d_1 + d_2$, $1 \leq p < \infty$, $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$ and define $w(x, y) = u(x)v(y)$. Suppose that $\{m_j\}_j$ is a family of multipliers that are continuous functions satisfying that the associated maximal operator is bounded from $L^p(\mathbb{R}^d, w)$ to $L^{p,\infty}(\mathbb{R}^d, w)$ (or to $L^p(\mathbb{R}^d, w)$). Then, fixed $x \in \mathbb{R}^d$, the maximal operator associated to the family $\{m_j(\xi, \cdot)\}_j$ is bounded from $L^p(\mathbb{R}^{d_2}, v)$ to $L^{p,\infty}(\mathbb{R}^{d_2}, w)$ (resp. to $L^p(\mathbb{R}^{d_2}, v)$) and its operator norm does not exceed $c_{p,w}$ times the norm of the maximal operator associated to $\{m_j\}_j$, where $c_{p,w}$ is a constant that depends only on $p$, $d$ and the $A_p$-constant of $w$.

We want to emphasize that the techniques developed in this paper are different from those in [14, 7] where duality properties of Lebesgue spaces are strongly used. Our approach allows us to also consider the case of maximal multipliers of weak type $(1, 1)$, and deal with the difficulties derived from the fact that $L^{1,\infty}$ is not a Banach space. The endpoint case $p = 1$ is the weighted analogue of the results in [2, 15].

2. Definitions and notation

In this section we present some basic definitions needed for our consideration. Let $0 < p < \infty$ and let $(\mathcal{M}, \mu)$ be a $\sigma$-finite measure space. The space $L^{p,\infty}(\mu)$ is defined by the quasinorm $\|f\|_{L^{p,\infty}(\mu)} = \sup_{s > 0} (\mu f(s))^{1/p}$, where $\mu f(s) = \mu\{x : |f(x)| > s\}$. It is known (see [10, p. 485]) that, for every $q < p$,

$$\|f\|_{L^{p,\infty}(\mu)} \leq \sup \|f\chi_E\|_{L^q(\mu)} \mu(E)^{1/p-1/q} \leq c_{p,q} \|f\|_{L^{p,\infty}(\mu)},$$

where the supremum is taken on the family of sets of finite measure and $c_{p,q} = \frac{p}{p-q}$. The finiteness of the middle expression is called Kolmogorov’s condition.
If \( \nu \) is a positive measure absolutely continuous with respect to \( \mu \) and \( w \) denotes the Radon-Nykodym derivative of \( \nu \) with respect to \( \mu \), we shall write \( L^p(\nu) \) for \( L^p(\nu) \). If any confusion can arise, we shall write \( L^p(\mathcal{M}, \mu) \) and \( L^{p, \infty}(\mathcal{M}, \mu) \) to indicate the underlying measure space \( \mathcal{M} \).

Let \( C_c^\infty(\mathbb{R}^d) \) and \( S(\mathbb{R}^d) \) denote the class of infinitely differentiable functions with compact support and the Schwartz class of test functions, respectively. As usual, \( \mathfrak{B}(X, Y) \) indicates the set of bounded operators on \( X \) into \( Y \) and \( \mathcal{B}(X) = \mathfrak{B}(X, X) \).

A weight on \( \mathbb{R}^d \) is a locally integrable function \( w : \mathbb{R}^d \to [0, \infty) \) such that 0 < \( w < \infty \) a.e.

**Definition 2.1.** We say that a weight \( w \) belongs to the class \( A_p(\mathbb{R}^d) \), and we write \( w \in A_p(\mathbb{R}^d) \) if

\[
[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{p-1} < \infty,
\]

for 1 < \( p < \infty \), and

\[
[w]_{A_1} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \| w^{-1} \chi_Q \|_{\infty} < \infty,
\]

where the supremum is taken over the family of cubes \( Q \) with sides parallel to the coordinate axis. These quantities will be referred to as the \( A_p \)-constant of \( w \).

It is well known that, for 1 \( \leq p < \infty \) and \( w \in A_p(\mathbb{R}^d) \), \( S(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, w) \) and \( C_c^\infty(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d, w) \). We refer the reader to [10,11] for other properties and generalities of \( A_p \)-weights.

For any function \( f \), we shall denote by \( \hat{f} \) (resp. the inverse Fourier transform) of \( f \), whenever it is well defined.

**Definition 2.2.** Let 1 \( \leq p < \infty \). A function \( \mathbf{m} \in L^\infty(\mathbb{R}^d) \) is called a weak type multiplier on \( L^p(\mathbb{R}^d, w) \) (in symbols, \( \mathbf{m} \in M_p^{(w)}(\mathbb{R}^d) \)) if the mapping \( f \in S(\mathbb{R}^d) \mapsto (\mathbf{m} \hat{f})^\vee \) can be extended from \( S(\mathbb{R}^d) \) to a continuous linear mapping \( S_\mathbf{m} \) from \( L^p(\mathbb{R}^d, w) \) to \( L^{p, \infty}(\mathbb{R}^d, w) \). In this case we write

\[
\| \mathbf{m} \|_{M_p^{(w)}(\mathbb{R}^d)} = \| S_\mathbf{m} \|_{\mathfrak{B}(L^p(\mathbb{R}^d, w), L^{p, \infty}(\mathbb{R}^d, w))}.
\]

If \( S_\mathbf{m} \in \mathfrak{B} \left( L^p(\mathbb{R}^d, w) \right) \), we say that \( \mathbf{m} \) is a Fourier multiplier on \( L^p(\mathbb{R}^d, w) \) and write

\[
\| \mathbf{m} \|_{M_p,w(\mathbb{R}^d)} = \| S_\mathbf{m} \|_{\mathfrak{B}(L^p(\mathbb{R}^d, w))}.
\]

If \( \{ \mathbf{m}_j \}_j \) is a sequence in \( M_p^{(w)}(\mathbb{R}^d) \), we denote by \( \| \{ \mathbf{m}_j \}_j \|_{M_p^{(w)}(\mathbb{R}^d)} \) the norm of the operator defined for \( f \in S(\mathbb{R}^d) \) by

\[
S_{\{ \mathbf{m}_j \}_j} f(x) = \sup_j | S_{\mathbf{m}_j} f(x) |,
\]

provided it defines a continuous mapping from \( L^p(\mathbb{R}^d, w) \) to \( L^{p, \infty}(\mathbb{R}^d, w) \). If it extends to a bounded mapping on \( L^p(\mathbb{R}^d, w) \), we write its norm by \( \| \{ \mathbf{m}_j \}_j \|_{M_p,w(\mathbb{R}^d)} \).

We shall denote by \( \mathbb{T}^d \) the topological group \( \mathbb{R}^d/\mathbb{Z}^d \), which can be identified with the cube \([0, 1)^d \) or eventually with \([-1/2, 1/2)^d \) in \( \mathbb{R}^d \). Functions on \( \mathbb{T}^d \) will be identified with functions on \( \mathbb{R}^d \) which are 1-periodic in each variable. A function
f : \mathbb{T}^d \to \mathbb{C} such that for a finitely supported sequence \( \{a_k\}_{k \in \mathbb{Z}^d} \) of complex numbers written as

\[
f(x) = \sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k x}
\]
is called a trigonometric polynomial, and we write \( f \in P(\mathbb{T}^d) \). Let us recall that \( P(\mathbb{T}^d) \) is dense in \( L^p(\mathbb{T}^d, \mu) \) for any Radon measure \( \mu \) on \( \mathbb{T}^d \).

From now on, we work in the range

\[ 1 \leq p < \infty, \]

and \( w \) is a weight in \( \mathbb{R}^d \). Observe that if in addition \( w \) is 1-periodic, then \( w \in L^1(\mathbb{T}^d) \).

### 3. Restriction of Fourier multipliers from \( \mathbb{R}^d \) to \( \mathbb{T}^d \)

**Definition 3.1.** A function \( m \in \ell^\infty(\mathbb{Z}^d) \) is a weak type multiplier on \( L^p(\mathbb{T}^d, w) \) (in symbols, \( m \in M^{(w)}_{p,w}(\mathbb{T}^d) \)) if the mapping

\[
\sum_{k \in \mathbb{Z}^d} a_k e^{2\pi i k \theta} \in P(\mathbb{T}^d) \longrightarrow \sum_{k \in \mathbb{Z}^d} m(k) a_k e^{2\pi i k \theta}
\]
extends to a continuous operator \( T_m \in \mathcal{B}(L^p(\mathbb{T}^d, w), L^{p,\infty}(\mathbb{T}^d, w)) \). In this case,

\[
\|m\|_{M^{(w)}_{p,w}(\mathbb{T}^d)} = \|T_m\|_{\mathcal{B}(L^p(\mathbb{T}^d, w), L^{p,\infty}(\mathbb{T}^d, w))}.
\]

If \( T_m \in \mathcal{B}(L^p(\mathbb{T}^d, w)) \), \( m \) is said to be a multiplier on \( L^p(\mathbb{T}^d, w) \), we denote it by \( m \in M_{p,w}(\mathbb{T}^d) \) and

\[
\|m\|_{M_{p,w}(\mathbb{T}^d)} = \|T_m\|_{\mathcal{B}(L^p(\mathbb{T}^d, w))}.
\]

If \( \{m_j\}_j \) is a sequence in \( M^{(w)}_{p,w}(\mathbb{T}^d) \) we denote by \( \|\{m_j\}_j\|_{M^{(w)}_{p,w}(\mathbb{T}^d)} \) the norm of the operator defined for every \( f \in P(\mathbb{T}^d) \) by

\[
T_{\{m_j\}}^d f(x) = \sup_j |T_{m_j} f(x)|,
\]

provided it extends to a continuous mapping from \( L^p(\mathbb{T}^d, w) \) to \( L^{p,\infty}(\mathbb{T}^d, w) \). We shall write \( \|\{m_j\}_j\|_{M^{(w)}_{p,w}(\mathbb{T}^d)} \) in the case that it extends to a continuous operator on \( L^p(\mathbb{T}^d, w) \).

#### 3.1. Restriction results for weak type maximal multipliers.

**Theorem 3.2.** Let \( w \) be 1-periodic and let \( \{m_j\}_j \in M^{(w)}_{p,w}(\mathbb{R}^d) \) satisfying that, for each \( j \), there exists \( K_j \in L^1(\mathbb{R}^d) \) with compact support such that \( \hat{K}_j(x) = m_j(x) \) for every \( x \in \mathbb{R}^d \). Then \( \{m_j|_{\mathbb{Z}^d}\}_j \in M^{(w)}_{p,w}(\mathbb{T}^d) \) and

\[
\|\{m_j|_{\mathbb{Z}^d}\}_j\|_{M^{(w)}_{p,w}(\mathbb{T}^d)} \leq c_p \|\{m_j\}_j\|_{M^{(w)}_{p,w}(\mathbb{R}^d)},
\]

where \( c_p \) depends only on \( p \).

**Proof.** Let \( \mathcal{M} = \|\{m_j\}_j\|_{M^{(w)}_{p,w}(\mathbb{R}^d)} \). Since convolution operators commute with translations, it follows that for every \( \theta \in [0,1)^d \) and every \( N \in \mathbb{N} \),

\[
\left\| \sup_{1 \leq j \leq N} |K_j * g| \right\|_{L^{p,\infty}(\mathbb{R}^d, w(-\theta))} \leq \mathcal{M} \|g\|_{L^p(\mathbb{R}^d, w(-\theta))}.
\]

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Now, given \( f(\theta) = \sum_k a_k e^{2\pi i k \theta} \in P(T^d) \), let us consider
\[
\tilde{T}_{K_j} f(\theta) = \int_{\mathbb{R}^d} K_j(x) f(\theta - x) \, dx = \sum_k a_k \int_{\mathbb{R}^d} K_j(x) e^{2\pi i k(\theta - x)} \, dx
\]
that is, \( \tilde{T}_{K_j} \) coincides with the multiplier operator \( T_{m_{\pm}} \).

Let \( Q_r = (-r, r)^d \) with \( r > 0 \) such that \( \text{supp} \, K_j \subset Q_r \) for \( j = 1, \ldots, N \). Let \( q < p \), and for any measurable \( E \subset [0, 1)^d \), let \( \tilde{E} = \bigcup_{k \in \mathbb{Z}^d} E + k \) be its periodic extension. Set \( E_\theta = \{ x \in \mathbb{R}^d : x + \theta \in \tilde{E} \} \) with \( \theta \in \mathbb{T}^d \) and \( R_x f(\theta) = f(\theta + x) \).

Then, by translation invariance, we have that, for every \( x \in \mathbb{R}^d 
\]
\[
(2.2) \quad \left\| \sup_{1 \leq j \leq N} \left| \tilde{T}_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q = \int_{\mathbb{T}^d} \sup_{1 \leq j \leq N} \left| R_x \tilde{T}_{K_j} f(\theta) \right|^q w(x + \theta) \chi_E(x + \theta) \, d\theta.
\]

Therefore, for every \( s > 0 \),
\[
\left\| \sup_{1 \leq j \leq N} \left| \tilde{T}_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \leq \frac{1}{(2s)^d} \int_{\mathbb{T}^d} \int_{Q_s} \sup_{1 \leq j \leq N} \left| R_x \tilde{T}_{K_j} f(\theta) \right|^q w(x + \theta) \chi_E(x + \theta) \, d\theta \, dx.
\]

Now, using that \( \text{supp} \, K_j \subset Q_r \) for \( j = 1, \ldots, N \), one can easily see that, if \( x \in Q_s \),
\[
R_x \tilde{T}_{K_j} f(\theta) = B_{K_j} \left( R_{x-} f(\theta) \chi_{Q_r} \right)(x),
\]
where \( B_{K_j} (h)(x) = (K_j * h)(x) \), and hence,
\[
\left\| \sup_{1 \leq j \leq N} \left| \tilde{T}_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \leq \frac{1}{(2s)^d} \int_{\mathbb{T}^d} \int_{Q_s} \sup_{1 \leq j \leq N} \left| B_{K_j} \left( R_{x-} f(\theta) \chi_{Q_r} \right) \right|^q (x) \, d\theta \, dx.
\]

By (2.3) and (3.1), the term inside curly brackets is bounded by
\[
(c_{p,q} \mathcal{N})^q \left\{ \int_{Q_{r+s}} |R_x f(\theta)|^p w(x + \theta) \, dx \right\} \frac{2}{p} \left\{ \int_{E_\theta \cap Q_s} w(x + \theta) \, dx \right\}^{1 - \frac{2}{s}}.
\]

Also, using Hölder’s inequality, it follows that
\[
\left\| \sup_{1 \leq j \leq N} \left| \tilde{T}_{K_j} f \right| \chi_E \right\|_{L^q(\mathbb{T}^d, w)}^q \leq \frac{c_{p,q} \mathcal{N}^q}{(2s)^d} \left\{ \int_{\mathbb{T}^d} \int_{Q_{r+s}} |R_x f(\theta)|^p w(x + \theta) \, dt \, d\theta \right\} \frac{2}{p} \left\{ \int_{\mathbb{T}^d} \int_{Q_s \cap E_\theta} w(x + \theta) \, dx \, d\theta \right\}^{1 - \frac{2}{s}}
\]
\[
\leq \frac{c_{p,q} \mathcal{N}^q}{(2s)^d} (2(r + s))^{d(1 - \frac{2}{p})} w(E)^{1 - \frac{2}{s}} \| f \|_{L^p(\mathbb{T}^d, w)}^q
\]
\[
\leq \frac{c_{p,q} \mathcal{N}^q}{s} \left( \frac{r + s}{s} \right)^{\frac{d(1 - \frac{2}{p})}{p}} w(E)^{1 - \frac{2}{s}} \| f \|_{L^p(\mathbb{T}^d, w)}^q.
\]
Thus, taking $s \to +\infty$, and using Kolmogorov’s condition \[21\], we obtain that
\[
\left\| \sup_{1 \leq j \leq N} \left| \hat{T}_{K,j} f \right| \right\|_{L^{p,(\mathbb{T}^d,w)}} \leq c_{p,q} \| f \|_{L^p(\mathbb{T}^d,w)} .
\]
Now, considering $c_p = \inf_q c_{p,q}$, the result easily follows by Fatou’s Lemma and the density of $P(\mathbb{T}^d)$ in $L^p(\mathbb{T}^d,w)$. $\square$

The next step is to weaken the hypothesis assumed on $m_j$ in the previous theorem as is done both in [4] and [1]. As usually happens, this is the technical part of the work.

**Definition 3.3.** A bounded function $m$ defined in $\mathbb{R}^d$ is **normalized** if for any $x \in \mathbb{R}^d$,
\[
\lim_{n} \tilde{\varphi}_n * m(x) = m(x),
\]
where $\varphi_n(x) = \varphi(x/n)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\varphi \geq 0$ and $\| \varphi \|_1 = 1$.

It is easy to see that $\lim_n \tilde{\varphi}_n * m(x) = m(x)$ for every Lebesgue point $x$ of $m$. In particular, any continuous and bounded function is normalized.

In order to extend Theorem 3.2 to the class of normalized multipliers, we shall need some previous lemmas. The following one is a direct consequence of the proof of [15] Lemma 2.6] for $G = \mathbb{R}^d$.

**Lemma 3.4.** Let $J \in \mathbb{N}$ and let $\{ m_j \}_{j=1}^J$ be a family of $L^\infty(\mathbb{R}^d)$ functions. For $f \in S(\mathbb{R}^d)$, $j = 1, \ldots, J$ and $x \in \mathbb{R}^d$, let
\[
F_{j,x}(\xi) = S_{m_j}(e^{-2\pi i \xi \cdot f})(x), \quad \xi \in \mathbb{R}^d.
\]
Let $K$ be a compact set. Then, for each $k \in \mathbb{N} \setminus \{0\}$, there exists a finite family $\{ V_k \}_{l=1}^{L_k}$ of pairwise disjoint measurable sets in $\mathbb{R}^d$ such that

1. $K \subset \bigcup_{l=1}^{L_k} V_k^l$,
2. if $l = 1, \ldots, L_k$ and $\xi, \zeta \in V_k^l$, then
\[
|F_{j,x}(\xi) - F_{j,x}(\zeta)| \leq 1/k,
\]
uniformly on $j \in \{1, \ldots, J\}$ and $x \in \mathbb{R}$.

Another key ingredient is the following version of Marcinkiewicz-Zygmund’s inequality, whose proof is analogous to that given in [10] Theorem V.2.9 for $p = q = 1$ for linear operators.

**Theorem 3.5.** Let $\{ T_j \}_{j}$ be a countable family of linear operators such that
\[
\left\| \sup_j \left| T_j f \right| \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \leq \| \{ T_j \} \| \| f \|_{L^1(\mathbb{R}^d,w)} .
\]
Then
\[
\left\| \sup_j \left( \sum_l |T_j f_l|^2 \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R}^d,w)} \leq c_1 \| \{ T_j \} \| \left( \sum_l |f_l|^2 \right)^{1/2} \|_{L^1(\mathbb{R}^d,w)} ,
\]
where
\[
c_1 := \inf_{0 < r < 1} \frac{\sqrt{\pi}}{2 \left( (1 - r) \Gamma \left( 1 + \frac{r}{2} \right) \right)^{1/r} .}
\]

(3.3)
For $p > 1$, the next lemma is an immediate consequence of Minkowskii’s inequality, as $L^{p,\infty}$ is normable, but for $p = 1$ the convexity of the space $L^{1,\infty}$ fails. Similar results in the unweighted setting are given by [3, Lemma 2.1] and [4, Theorem 1.2].

**Lemma 3.6.** Let $\varphi \in L^1(\mathbb{R}^d)$ and $\{m_j\}_j \in M_{p,w}^{(m)}(\mathbb{R}^d)$. Then $\{\varphi \ast m_j\}_j \in M_{p,w}^{(m)}(\mathbb{R}^d)$ and

\[
\|\{\varphi \ast m_j\}_j\|_{M_{p,w}^{(m)}(\mathbb{R}^d)} \leq c_p \|\varphi\|_{L^1(\mathbb{R}^d)} \|\{m_j\}_j\|_{M_{p,w}^{(m)}(\mathbb{R}^d)},
\]

where $c_p = p'$ if $p > 1$ and $c_1$ is the constant given in (3.3).

**Proof.** We shall only prove the case $p = 1$. Without loss of generality, we can assume that $\{m_j\}_j$ is a finite family of multipliers of cardinality, say $J \in \mathbb{N}$. For $g \in C_0^\infty(\mathbb{R}^d)$,

\[
\int (\varphi \ast m_j)(\xi)\hat{g}(\xi)e^{2\pi i\xi x} \, d\xi = \int \varphi(y)e^{2\pi i xy}S_{m_j}(e^{-2\pi iy}g)(x) \, dy.
\]

Hence,

\[
|S_{\varphi \ast m_j}g(x)| \leq \int |\varphi(y)||S_{m_j}(e^{-2\pi iy}g)(x)| \, dy,
\]

and thus

\[
\sup_{1 \leq j \leq J} |S_{\varphi \ast m_j}g(x)| \leq \int |\varphi(y)| \sup_{1 \leq j \leq J} |S_{m_j}(e^{-2\pi iy}g)(x)| \, dy.
\]

Let us first assume that $\varphi \in L^1(\mathbb{R}^d)$ is supported on a compact set $K$. For each $k \geq 1$ let $\{V_k^l\}_{l=1}^{I_k}$ be the family of pairwise disjoint sets given by Lemma 3.3 and for each $l$, select $y_k^l \in V_k^l$. Then, for every $y \in K$ and any $k \geq 1$, there exists a unique $l \in \{1, \ldots, I_k\}$ such that $y \in V_k^l$, and hence

\[
|S_{m_j}(e^{-2\pi iy}g)(x) - S_{m_j}(e^{-2\pi iy}g)(x)| \leq \frac{1}{k},
\]

uniformly on $j = 1, \ldots, J$ and $x \in \mathbb{R}^d$. It follows that for every $x \in \mathbb{R}^d$, any $j \in \{1, \ldots, J\}$ and all $y \in K$,

\[
\lim_{k \to \infty} \sum_{l=1}^{I_k} S_{m_j}(e^{-2\pi iy_k^l}g)(x) \chi_{V_k^l}(y) = S_{m_j}(e^{-2\pi iy}g)(x).
\]

Then, by Fatou’s Lemma on (5.5),

\[
\sup_{1 \leq j \leq J} |S_{\varphi \ast m_j}g(x)| \leq \liminf_{k \to \infty} \sup_{1 \leq j \leq J} \left( \frac{1}{k} \sum_{l=1}^{I_k} \left| S_{m_j}(e^{-2\pi iy_k^l}g)(x) \right| \lambda_k^l \right),
\]

where $\lambda_k^l = \int_{V_k^l} |\varphi(y)| \, dy$. Observe that the term inside brackets is less than or equal to

\[
\|\varphi\|_{L^1(\mathbb{R}^d)}^{1/2} \left( \sum_{l=1}^{I_k} \left| S_{m_j}(\sqrt{\lambda_k^l}e^{-2\pi iy_k^l}g)(x) \right|^2 \right)^{1/2},
\]

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where we have used that $\sum_{k=1}^{n} \lambda_k = \int g_k \, dy = \| \varphi \|_{L^1(\mathbb{R}^d)}$. Then,

$$\left\| \sup_{1 \leq j \leq J} |S_{\varphi+\mu_j}g| \right\|_{L^1(\mathbb{R}^d)} \leq \| \varphi \|_{L^1(\mathbb{R}^d)}^2 \left( \sum_{j=1}^{J} \left| S_{\mu_j} \left( \sqrt{\lambda_k^2 e^{-2\pi iy_k^j} g} \right)(x) \right| \right)^{1/2} \left\| \sup_{1 \leq j \leq J} \left( \sum_{i=1}^{I_k} \left| S_{\mu_i} \left( \sqrt{\lambda_k^2 e^{-2\pi iy_k^j} g} \right)(x) \right| \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)}.$$ 

Applying Theorem 3.3 with the family of operators $\{ S_{\mu_i} \}_j$ to the functions $f_i = \sqrt{\lambda_k^2 e^{-2\pi iy_k^j} g}$, we obtain that

$$\left\| \sup_{1 \leq j \leq J} \left( \sum_{i=1}^{I_k} \left| S_{\mu_i} \left( \sqrt{\lambda_k^2 e^{-2\pi iy_k^j} g} \right)(x) \right| \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \leq c_1 \left\| \{ \mu_i \}_j \right\|_{M^{(m)}_1(\mathbb{R}^d)} \left\| \{ S_{\mu_i} \}_j \right\|_{L^1(\mathbb{R}^d)} \left\| g \right\|_{L^1(\mathbb{R}^d)}.$$ 

Therefore,

\begin{equation}
\left\| \sup_{1 \leq j \leq J} |S_{\varphi+\mu_j}g| \right\|_{L^1(\mathbb{R}^d)} \leq c_1 \left\| \varphi \right\|_{L^1(\mathbb{R}^d)} \left\| \{ \mu_i \}_j \right\|_{M^{(m)}_1(\mathbb{R}^d)} \left\| g \right\|_{L^1(\mathbb{R}^d)}.
\end{equation}

In the case that $\varphi$ is not compactly supported, considering $\varphi_n = \varphi \chi_{B(0,n)}$, we can write

$$\sup_{1 \leq j \leq J} |S_{\varphi+\mu_j}g(x)| \leq \lim_{n} \int |\varphi_n(y)| \sup_{1 \leq j \leq J} |S_{\mu_j} (e^{-2\pi iy_k^j} g)(x)| \, dy,$$

and using the previous argument we obtain that

$$\left\| \sup_{1 \leq j \leq J} |S_{\varphi+\mu_j}g| \right\|_{L^1(\mathbb{R}^d)} \leq c_1 \lim_{n} \left\| \varphi_n \right\|_{L^1(\mathbb{R}^d)} \left\| \{ \mu_i \}_j \right\|_{M^{(m)}_1(\mathbb{R}^d)} \left\| g \right\|_{L^1(\mathbb{R}^d)},$$

from where it follows that (3.6) holds for any $\varphi \in L^1(\mathbb{R}^d)$. The result now follows by the density of $C_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$.

**Lemma 3.7.** Let $w \in C(\mathbb{T}^d)$ such that $\inf_{x \in \mathbb{T}^d} w(x) > 0$. Consider $h \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq h \leq 1$ and $\int_{\mathbb{R}^d} h = 1$ and define $h_n(x) = n^d h(nx)$. Then,

1. There exists $n_0 = n_0(w) \in \mathbb{N}$ such that, for any $p \in [1, \infty)$,

   $$\sup_{n \geq n_0} \left\| \hat{h}_n \right\|_{M_{p,w}(\mathbb{R}^d)} \leq 2^{1/p}. $$

2. $\sup_n \left\| \hat{h}_n \right\|_{L^\infty(\mathbb{R}^d)} \leq 1$.

3. For every $\xi \in \mathbb{R}^d$, $\lim_n \hat{h}_n(\xi) = 1$.

**Proof.** Since $\left\| h_n \right\|_{L^1} = 1$, it follows that $\left\| \hat{h}_n \right\|_{\infty} \leq 1$. On the other hand, for every $\xi \in \mathbb{R}^d$ and for every $\epsilon > 0$ there exists $n_0$ such that for all $|x| < \frac{1}{n_0}$, $\left| 1 - e^{2\pi i x \xi} \right| < \epsilon$.
Hence, for every $n \geq n_0$,

$$|1 - \hat{h}_n(\xi)| \leq \int h_n(x) \left| 1 - e^{2\pi i x \xi} \right| \, dx.$$

Then, it follows that $\hat{h}_n \to 1$ pointwise. It remains to show that $\|\hat{h}_n\|_{L^p(w(\mathbb{R}^d))}$ are uniformly bounded on $n$.

Observe that $\|f\|_{L^\infty(w)} = \|f\|_{L^\infty}$ and hence, for any $n \geq 1$,

$$\|h_n * f\|_{L^\infty(\mathbb{R}^d,w)} \leq \|f\|_{L^\infty(\mathbb{R}^d,w)}.$$

Let $\delta = \inf_{x \in \mathbb{T}^d} w(x) > 0$. Since $w \in C(\mathbb{T}^d)$, there exists $n_0 = n_0(\delta)$ such that, for any $n \geq n_0$, for any $x$ and any $y \in \text{supp} h_n$,

$$|w(x) - w(x - y)| \leq \delta,$$

which implies that, for any $x \in \mathbb{T}^d$,

$$h_n * w(x) \leq \delta + w(x) \leq 2w(x).$$

Then, for $n \geq n_0$,

$$\|h_n * f\|_{L^1(\mathbb{R}^d,w)} \leq \int_{\mathbb{R}^d} |f(y)| h_n * w(y) \, dy \leq 2 \|f\|_{L^1(\mathbb{R}^d,w)}.$$

In other words, we have seen that for $n \geq n_0$ the linear operator defined by $T_n f = h_n * f$ is uniformly bounded on $L^1(\mathbb{R}^d,w)$ and $L^\infty(\mathbb{R}^d,w)$ with norm respectively bounded by 2 and 1. Riesz-Thorin’s Theorem implies the result. \hfill \Box

**Lemma 3.8.** Let $w$ be 1-periodic. If $g \in C_c(\mathbb{R}^d)$ is nonnegative, $\int_{\mathbb{R}^d} g = 1$ and $\text{supp} g \subset [-1/2,1/2]^d$, then $\inf_{x \in \mathbb{R}^d} g * w(x) > 0$.

**Proof.** Clearly $g * w \in C(\mathbb{T}^d)$, and hence there exists $x_0 \in [-1/2,1/2]^d$ such that $\inf_{x \in \mathbb{R}^d} g * w(x) = \min_{|x| < 1/2} g * w(x) = g * w(x_0)$. Since $g \in C_c(\mathbb{R}^d)$, there exists a set of positive Lebesgue measure $Q$ where $g(y) > 0$ for all $y \in Q$. Thus if $0 = g * w(x_0) = \int g(y) w(x_0 - y) \, dy$, then $g(y) w(x_0 - y) = 0$ a.e. $y \in Q$, which implies that $w(z) = 0$ a.e. $z \in x_0 - Q$, but this contradicts the fact that the set \{x : w(x) = 0\} is null. \hfill \Box

**Lemma 3.9.** Let $T$ be any bounded operator from $L^p(\mathbb{R}^d,w)$ to $L^{p,\infty}(\mathbb{R}^d,w)$ that commutes with translations. Then, for any nonnegative function $g \in C_c(\mathbb{R}^d)$, $T$ is bounded from $L^p(\mathbb{R}^d,g * w)$ to $L^{p,\infty}(\mathbb{R}^d,g * w)$ and

$$\|T\|_{B(L^p(\mathbb{R}^d,g * w),L^{p,\infty}(\mathbb{R}^d,g * w))} \leq c_p \|T\|_{B(L^p(\mathbb{R}^d,w),L^{p,\infty}(\mathbb{R}^d,w))},$$

where $c_p = \inf_{q < p} \left( \frac{p}{p-q} \right)^\frac{1}{2}$.\hfill \Box

**Proof.** Let $E$ be any measurable set in $\mathbb{R}^d$ such that $0 < g * w(E) < +\infty$. Then, for any $q < p$,

$$\|Tf \chi_E\|_{L^q(\mathbb{R}^d,g * w)}^q = \int_E |Tf(x)|^q g * w(x) \, dx = \int g(y) \int_E |Tf(x)|^q w(x - y) \, dx \, dy = \int g(y) \int_{E-y} |Tf_y(x)|^q w(x) \, dx \, dy,$$
with \( f_y(z) = f(z + y) \). Thus, by the boundedness hypothesis, Kolmogorov’s condition and Hölder’s inequality,

\[
\| T f \chi_E \|_{L^q(\mathbb{R}^d, g * w)} \leq \| T \|_p \left( \int |f_y(x)|^q w(x) \, dx \right)^{1/p} dy
\]

\[
\leq c_{p,q}^q \| T \|_p \| (g * w(E)) \|_{L^q(\mathbb{R}^d, g * w)},
\]

where \( c_{p,q}^q = p/(p-q) \). Then, the result follows by Kolmogorov’s condition and by taking the infimum for \( q < p \).

**Theorem 3.10.** Let \( w \) be a 1-periodic weight on \( \mathbb{R}^d \). Suppose that \( \{m_j\}_j \) are normalized functions and \( \{m_j\}_j \in M_{p,w}(\mathbb{R}^d) \). Then \( \{m_j\}^J \in M_{p,w}(\mathbb{T}^d) \) and

\[
\left\| \{m_j\}^J \right\|_{M_{p,w}(\mathbb{T}^d)} \leq c_p \left\| \{m_j\}_j \right\|_{M_{p,w}(\mathbb{R}^d)},
\]

where \( c_p \) depends only on \( p \).

**Proof.** Let \( \{g_t\}_t \) be a family of nonnegative functions in \( C^\infty_c(\mathbb{R}^d) \), supported in \([-1/2,1/2]^d \) such that it is an approximation of the identity in \( L^1(\mathbb{T}^d) \). We can also assume that \( \lim_{t \to 0} g_t * w(x) = w(x) \) a.e. \( x \in [-1/2,1/2]^d \).

For a fixed \( l \in \mathbb{N} \), by Lemma 3.9

\[
\left\| \{m_j\}_j \right\|_{M_{p_2,w}(\mathbb{R}^d)} \leq c_p \left\| \{m_j\}_j \right\|_{M_{p_2,w}(\mathbb{R}^d)}.
\]

By Lemma 3.8 it follows that for any \( h \in C^\infty_c(\mathbb{R}^d) \) such that \( 0 \leq h \leq 1 \) and that \( \int_{\mathbb{R}^d} h = 1 \), there exists an \( n_t \) such that, for any \( n \geq n_t \), the conclusions of Lemma 3.7 hold for the periodic weight \( g_t * w \).

Consider, for \( j, n \in \mathbb{N} \),

\[
m_{j,n}(\xi) = \widehat{K_{j,n}}(\xi) = (\varphi_n * m_j)(\xi) \widehat{h_n}(\xi),
\]

where \( \varphi_n \) are the functions given by the normalized condition. First observe that \( K_{j,n} \in \mathcal{S}(\mathbb{R}^d) \) and hence \( K_{j,n} \in \mathcal{S}(\mathbb{R}^d) \). Moreover, since

\[
K_{j,n}(x) = (\varphi_n m_j)(h_n(x - \cdot)) = m_j(\varphi_n(\cdot) h_n(x - \cdot)),
\]

and \( \varphi_n, h_n \) are compactly supported, it follows that \( K_{j,n} \in C^\infty_c(\mathbb{R}^d) \). On the other hand, since \( m_j \) is normalized and \( \widehat{h_n} \to 1 \), it holds that for every \( \xi \in \mathbb{R}^d \),

\[
\lim_n \widehat{K_{j,n}}(\xi) = m_j(\xi).
\]

Since \( \| \widehat{h_n} \|_{L^\infty(\mathbb{R}^d)} \leq 1 \) and \( \| \varphi_n \|_{L^1(\mathbb{R}^d)} \leq 1 \), then \( \| m_{j,n} \|_{L^\infty(\mathbb{R}^d)} \leq \| m_j \|_{L^\infty(\mathbb{R}^d)} \).

Let us fix \( J \in \mathbb{N} \). Since for any \( f \in C^\infty_c(\mathbb{R}) \)

\[
K_{j,n} * f = T_{\varphi_n * m_j}(h_n * f),
\]

it follows that for every \( n \geq n_t \),

\[
\left\| \sup_{1 \leq j \leq J} |K_{j,n} * f| \right\|_{L^p(\mathbb{R}^d, g_i * w)} \leq \left\| \{\varphi_n * m_j\}_J \right\|_{M_{p,w}(\mathbb{R}^d)} \| h_n * f \|_{L^p(\mathbb{R}^d, g_i * w)}
\]

\[
\leq c_p 2^{\frac{3p}{2}} \left\| \{m_j\}_J \right\|_{M_{p,w}(\mathbb{R}^d)} \| f \|_{L^p(\mathbb{R}^d, g_i * w)},
\]
where we have used that, by Lemma 3.6
\[
\left\| \{\hat{f}_n \ast m_j\}_j \right\|_{M^{(m)}_{p,q_i,w}(\mathbb{R}^d)} \leq c_p \left\| \{m_j\}_j \right\|_{M^{(m)}_{p,q_i,w}(\mathbb{R}^d)} \leq c_p^2 \left\| \{m_j\}_j \right\|_{M^{(m)}_{p,w}(\mathbb{R}^d)}.
\]
We can now apply Theorem 3.2 to deduce that for any \(n \geq n_1\),
\[
\left\| \{m_{j,n}\}_{j=1}^J \right\|_{M^{(m)}_{p,q_i,w}(\mathbb{T}^d)} \leq 2^{ \frac{1}{p} c_p^2 } \left\| \{m_j\}_j \right\|_{M^{(m)}_{p,w}(\mathbb{R}^d)}.
\]
Since, for any \(f \in P(\mathbb{T}^d)\),
\[
\lim_n S_{m_{j,n}} f(s) = \lim_n \sum_{k \in \mathbb{Z}^d} m_{j,n}(k) \hat{f}(k)e^{2\pi i k s} = \sum_{k \in \mathbb{Z}^d} m_j(k) \hat{f}(k)e^{2\pi i k s} = S_m f(s),
\]
by Fatou’s Lemma, the following inequality holds:
\[
\left\| \sup_{1 \leq j \leq J} \left| S_{m_j} f \right| \right\|_{L^p(\mathbb{T}^d, gl^{*}w)} \leq \liminf_n \left\| \sup_{1 \leq j \leq J} \left| S_{m_{j,n}} f \right| \right\|_{L^p(\mathbb{T}^d, gl^{*}w)} \leq 2^{ \frac{1}{p} c_p^2 } \left\| \{m_j\}_j \right\|_{M^{(m)}_{p,w}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{T}^d, gl^{*}w)},
\]
which implies
\[
\left\| \sup_{1 \leq j} \left| S_{m_j} f \right| \right\|_{L^p(\mathbb{T}^d, w)} \leq 2^{ \frac{1}{p} c_p^2 } \left\| \{m_j\}_j \right\|_{M^{(m)}_{p,w}(\mathbb{R}^d)} \liminf_{l \to \infty} \|f\|_{L^p(\mathbb{T}^d, gl^{*}w)}.
\]
Observe that
\[
\|f\|_{L^p(\mathbb{T}^d, gl^{*}w)} \leq \|f\|_{L^\infty(\mathbb{T}^d)} \|gl^{*}w - w\|_{L^1(\mathbb{T}^d)} + \|f\|_{L^p(\mathbb{T}^d, w)},
\]
and since \(\lim_l \|gl^{*}w - w\|_{L^1(\mathbb{T}^d)} = 0\), it follows that
\[
\left\| \sup_{1 \leq j} \left| S_{m_j} f \right| \right\|_{L^p(\mathbb{T}^d, w)} \leq 2^{ \frac{1}{p} c_p^2 } \|f\|_{L^p(\mathbb{T}^d, w)}.
\]
The result follows by the density of \(P(\mathbb{T}^d)\) in \(L^p(\mathbb{T}^d, w)\). \(\square\)

With minor modifications in the proofs, the analogous result for operators of strong type can be proved. In the particular case of a single multiplier, we recover K. Andersen and P. Mohanty’s [11, Theorem 1.1].

**Theorem 3.11.** Let \(w\) be 1-periodic. Suppose that \(\{m_j\}_j \subset M_{p,w}(\mathbb{R}^d)\) and that they are normalized functions. Then \(\{m_{j,z} \}_{j} \subset M_{p,w}(\mathbb{T}^d)\) and
\[
\left\| \{m_{j} \}_{j} \right\|_{M_{p,w}(\mathbb{T}^d)} \leq 2^{1/p} \left\| \{m_j\}_j \right\|_{M_{p,w}(\mathbb{R}^d)}.
\]

**3.2. An improvement for nonperiodic weights.** A similar approach to that in the previous section allows us to obtain a more general version of Theorem 3.11 (and also of Theorem 3.11) for a class of nonnecessarily periodic weights which includes those in \(A_p(\mathbb{T}^d)\).

**Definition 3.12.** We say that a weight \(v \in W(\mathbb{R}^d)\) if it satisfies the following conditions:

i) For every \(x \in \mathbb{R}^d, \theta \in [0,1)^d\),
\[
\frac{1}{\zeta} \leq \frac{v(x)}{v(x + \theta)} \leq \zeta.
\]
Theorem 3.13. Let $u$ be a periodic weight in $\mathbb{R}^d$, let $v \in W$ and set $w = uv$. Assume that $\{m_j\}_j \in M_p^{(m)}(\mathbb{R}^d)$ (respectively $M_{p,w}(\mathbb{R}^d)$) and that they are normalized functions in $\mathbb{R}^d$. Then $\{m_j(x)\}_j \in M_p^{(m)}(\mathbb{T}^d)$ (respectively $M_{p,w}(\mathbb{T}^d)$) and

$$
\left\| \left\{ m_j \right\}_{\mathbb{T}^d} \right\|_{M_p^{(m)}(\mathbb{T}^d)} \leq \|m\|_{M_p^{(m)}(\mathbb{R}^d)}.
$$

(respectively replacing $M_p^{(m)}$ with $M_{p,w}$ in the previous inequality), where $c_{p,w}$ and $c_p$ depend only on $p$.

Proof: We shall prove the weak case. The proof for the strong case is similar and we leave the details to the reader. Assume first that $\{m_j\}_j$ is a finite sequence. The argument is similar to that for Theorem 3.2 and we shall sketch the major changes to be done in the proof.

Let $\mathfrak{N} = \left\| \left\{ m_j \right\}_{\mathbb{R}^d} \right\|_{M_p^{(m)}(\mathbb{R}^d)}$. Since $v \in W$,

$$1 \frac{u(x + \theta)}{v(x + \theta)} \leq u(x + \theta)v(x) \leq \zeta(u(x + \theta)).$$

By (3.7), for every $f \in P(\mathbb{T})$ and every measurable set $E \subset \mathbb{T}$,

$$
\left\| \sup_{1 \leq j \leq N} |T_K f| \chi_E \right\|_{L^q(\mathbb{T}^d, u)}^q
= \frac{1}{v(Q_s)} \int_{Q_s} \left\{ \sup_{1 \leq j \leq N} |R_{x}T_K f| \right\}^q u(x + \theta)v(x) \chi_E(x + \theta) d\theta dx
\leq \frac{\zeta}{v(Q_s)} \int_{\mathbb{T}^d} \left\{ \sup_{1 \leq j \leq N} |B_K (R_{x}f \chi_{Q_s} \chi_{\tilde{E}})(x)| \right\}^q w(x + \theta) dx d\theta,
$$

where $E_0 = \{ x \in \mathbb{R}^d : x + \theta \in \tilde{E} \}$ and $\tilde{E}$ is the periodic extension of $E$. By (2.1) and (5.1), the term inside curly brackets is bounded by

$$(c_{p,q} \mathfrak{N})^q \left\{ \int_{Q_s} |R_{x}f| |P| w(x + \theta) dx \right\}^\frac{p}{q} \left\{ \int_{E_0 \cap Q_s} w(x + \theta) dx \right\}^{1 - \frac{p}{q}}.$$

Hence, by Hölder’s inequality, it follows that

$$
\left\| \sup_{1 \leq j \leq N} |T_K f| \chi_E \right\|_{L^q(\mathbb{T}^d, u)}^q
\leq c_{p,q} \mathfrak{N} \zeta \left[ \int_{\mathbb{T}^d} \int_{Q_s} |R_{x}f| |P| w(x + \theta) dtd\theta \right]^\frac{p}{q} \left[ \int_{\mathbb{T}^d} \int_{Q_s \cap E_0} w(x + \theta) dx d\theta \right]^{1 - \frac{p}{q}}.
$$

By (3.7), the first term is bounded by $[\zeta(u(Q_{r+s}))]^\frac{p}{q} \|f\|_{L^p(\mathbb{T}, u)}^q$, and the second one by $[\zeta(u(Q_s))u(E)]^{1 - \frac{p}{q}}$. Hence,

$$u(E)^{\frac{p}{q} - 1} \left\| \sup_{1 \leq j \leq N} |T_K f| \chi_E \right\|_{L^q(\mathbb{T}^d, u)}^q \leq c_{p,q} \mathfrak{N} \zeta^2 \left( \frac{v(Q_{r+s})}{v(Q_s)} \right)^\frac{p}{q} \|f\|_{L^p(\mathbb{T}^d, u)}^q.$$
Letting \( s \to \infty \) and using \( (2.1) \) we obtain that
\[
\left\| \sup_{1 \leq j \leq N} |T_{K_j}f| \right\|_{L^p(T^d, u)} \leq c_{p,v} \zeta^2 m f\|_{L^p(T^d, u)}.
\]

Considering \( c_{p,v} = \inf_{q < p} \zeta^{2/q} c_{p,q,v} \), the result easily follows by Fatou’s Lemma and the density of \( P(T^d) \) in \( L^p(T^d, u) \).

## 4. Restriction of Fourier multipliers to lower dimension

Restriction of Fourier multipliers of strong type to a lower dimensional space was studied in \([7, \text{Corollary 4.13}]\). Here we shall give a weak counterpart to that result.

We have to mention here that in this section we work with \( A_p(\mathbb{R}^d) \) weights mainly because, under this condition, we can prove the analogue to Lemma 3.7 (see Lemma 4.2 below). Other conditions that we can assume in \( w \) in order to have an approximation lemma are, for example, that \( w \) is uniformly continuous and \( \inf_{x \in \mathbb{R}^d} w(x) > 0 \). In this case the proof is a simple modification of the proof of Lemma 3.7.

### Lemma 4.1.
If \( w \in \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^d) \), then, for any \( s > 0 \), \( \lim_{s \to \infty} \frac{w(Q_{r+s})}{w(Q_r)} = 1 \).

**Proof.** By the \( A_\infty \)-condition \([10, \text{Theorem IV.2.9}]\), there exist \( \delta, C > 0 \) such that
\[
0 \leq 1 - \frac{w(Q_s)}{w(Q_{r+s})} = \frac{w(Q_{r+s} \setminus Q_s)}{w(Q_{r+s})} \leq C \left( 1 - \frac{s^d}{(r+s)^d} \right)^\delta,
\]
from where the result easily follows. \( \square \)

### Lemma 4.2.
If \( w \in A_p(\mathbb{R}^d) \), there exists \( \{h_n\}_n \subset C_c^\infty(\mathbb{R}^d) \), such that
1. \( \mathbf{g}_{p,w} := \sup_n |\hat{h}_n|_{M_{p,w}(\mathbb{R}^d)} < \infty \),
2. \( \sup_n \|\hat{h}_n\|_{L^\infty(\mathbb{R}^d)} \leq 1 \),
3. for every \( \xi \in \mathbb{R}^d \), \( \lim_n \hat{h}_n(\xi) = 1 \).

**Proof.** Properties (2) and (3) are proved as in Lemma 3.7. To prove (1), we first observe that clearly
\[
\sup_n |h_n * f|(x) \lesssim Mf(x)
\]
and hence the case \( p > 1 \) is trivial.

To prove the case \( p = 1 \), fix \( \beta \in \mathbb{N}, \beta > d \). Then,
\[
\|h_n * f\|_{L^1(w)} \leq \int |f(y)| \int h_n(x - y)w(x) \, dx \, dy.
\]

For a fixed \( y \in \mathbb{R}^d \) and \( n > 0 \), the inner integral can be split into
\[
\int_{|x-y| < n^{-1}} + \sum_{j \geq 0} \int_{2^j n^{-1} < |x-y| \leq 2^{j+1} n^{-1}} h_n(x - y) w(x) \, dx.
\]

The first term can be bounded by
\[
\|h\|_\infty n^d \int_{|x-y| < n^{-1}} w(x) \, dx \leq [w]_{A_1} 2^d w(y),
\]
as the ball $|x - y| < 1/n$ is included in $y + [-1/n, 1/n]^d$. On the other hand, if $p_{0,\beta}(h) = \sup_{x \in \mathbb{R}^d} |h(x)||x|^\beta$, each term on the sum can be bounded from above by

$$p_{0,\beta}(h)n^{d-\beta} \int_{2^{j-1} < |x - y| \leq 2^{j+1}} |x - y|^{-\beta} w(x) \, dx \leq p_{0,\beta}(h)4^{d(1-\beta)}[w]_{A_1} w(y).$$

Thus, the sum is bounded from above by $p_{0,\beta}(h)\frac{4^d}{1-2d}\|w\|_{A_1} w(y)$. Hence

$$\|h_n \ast f\|_{L^1(w)} \leq [w]_{A_1} c_{d,h} \|f\|_{L^1(w)},$$

where $c_{d,h} = 2^d \left(1 + \inf_{\beta > d} p_{0,\beta}(h) \frac{2^d}{1-2d}\right).$ 

The following result is the weighted version of \[Lemma 2\] and the weak type maximal counterpart of \[Proposition 4.10\].

**Proposition 4.3.** Let $w \in A_p(\mathbb{R}^d)$ and let $\{m_j\}_j \subset M_{p,w}^{(m)}(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$ that are normalized functions. Then, there exist $\{m_{j,n}\}_{j,n} \subset L^{\infty}(\mathbb{R}^d)$ satisfying:

1. For any $j$ and every $\xi \in \mathbb{R}^d$,

$$m_j(\xi) = \lim_n m_{j,n}(\xi).$$

2. $K_j = m_{j,n} \ast \check{\varphi}_n \in L^1(\mathbb{R}^d)$, and it is compactly supported.

3. $\sup_n \|m_{j,n}\|_{L^1(\mathbb{R}^d)} \leq \|m_j\|_{L^1(\mathbb{R}^d)}$.

4. $\sup_n \|\{m_{j,n}\}_j\|_{M_{p,w}^{(m)}(\mathbb{R}^d)} \leq d_{p,w} \|m_j\|_{M_{p,w}^{(m)}(\mathbb{R}^d)}$, where $d_{p,w}$ depends only on $p$, $d$ and the $A_p$-constant of $w$.

**Proof.** Let $\{h_n\}$ be the functions given by \[Lemma 4.2\] and $\varphi_n(x)$ as in Definition \[3.3\]. Consider, for $j, n \in \mathbb{N}$,

$$m_{j,n}(\xi) = K_{\check{\varphi}_n}(\xi) = (\varphi_n \ast m_j)(\xi) = \hat{h}_n(\xi),$$

and proceed as in the proof of Theorem \[4.10\].

**Theorem 4.4.** Let $d = d_1 + d_2$, $u \in A_p(\mathbb{R}^{d_1})$, $v \in A_p(\mathbb{R}^{d_2})$ and define $w(x,y) = u(x)v(y)$. Suppose that $\{m_j\}_j \subset M_{p,w}^{(m)}(\mathbb{R}^d)$ and are normalized functions. Then, for a fixed $\xi \in \mathbb{R}^{d_1}$, $\{m_j(\xi, \cdot)\}_j \subset M_{p,w}^{(m)}(\mathbb{R}^{d_2})$ and

$$\sup_{\xi \in \mathbb{R}^{d_1}} \left\|\left\{m_j(\xi, \cdot)\right\}_j\right\|_{M_{p,w}^{(m)}(\mathbb{R}^{d_2})} \leq c_{p,w} \left\|\left\{m_j\right\}_j\right\|_{M_{p,w}^{(m)}(\mathbb{R}^d)},$$

where $c_{p,w}$ depends only on $p$, $d$ and the $A_p$-constant of $w$.

**Proof.** Since $u \in A_p(\mathbb{R}^{d_1})$ and $v \in A_p(\mathbb{R}^{d_2})$ we have that $w \in A_p(\mathbb{R}^d)$ and $[w]_{A_p(\mathbb{R}^d)} \leq [u]_{A_p(\mathbb{R}^{d_1})}[v]_{A_p(\mathbb{R}^{d_2})}$. Then, by Proposition \[4.3\] we can assume that $\{m_j\}_{j=1}^\infty$ is a finite family such that $K_j = m_j \ast m_j \in L^1$ with compact support.

Let $N = \|\{m_j\}_j\|_{M_{p,w}^{(m)}(\mathbb{R}^d)}$. Since translations and convolution commute, it follows that for every $z \in \mathbb{R}^{d_2},$

$$\left\|\sup_{\xi \in \mathbb{R}^{d_1}} \left|B_{K_j} g\right|\right\|_{L^p(\mathbb{R}^{d_2}, u(\cdot)v(\cdot + z))} \leq N \|g\|_{L^p(\mathbb{R}^{d_2}, u(\cdot)v(\cdot + z))}. 

(4.2)$$

Fix $\xi \in \mathbb{R}^{d_1}$. For any $f \in C_c^{\infty}(\mathbb{R}^{d_2})$, write

$$R_{(x,y)} f(z) = e^{2\pi i \xi \xi} f(z + y), \quad (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. 

(4.3)$$
Observe that in this way,

\[ \hat{T}_{K_j} f(z) = \int_{\mathbb{R}^d} K_j(x, y) R_{-(x, y)} f(z) \, dx dy \]

\[ = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} K_j(x, y) e^{-2\pi i \xi \cdot x} \, dx \right) f(z - y) \, dy \]

\[ = \int_{\mathbb{R}^d} m_j(\xi, \eta) \hat{f}(\eta) e^{-2\pi i \xi \cdot \eta} \, d\eta, \]

Fix \( q < p \) and fix \( E \subset \mathbb{R}^{d_1} \) a set of finite measure. For any \( z \in \mathbb{R}^{d_2} \), let \( A_z = \{(x, y) \in \mathbb{R}^d : y + z \in E\} \). Let \( r > 0 \) such that \( \text{supp} \, K_j \subset (-r, r) = Q_r \) for \( j = 1, \ldots, J \).

Let \( s > 0 \). For any \((x, y) \in Q_s = (-s, s)^d\),

\[ \left\| \sup_{1 \leq j \leq J} |\hat{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{R}^{d_2}, w)}^q = \sup_{1 \leq j \leq J} \left| R_{(x, y)} \hat{T}_{K_j} f(z) \right|^q v(y + z) \chi_E(y + z) \, dz. \]

If we consider the weight \( \omega = u \otimes 1 \) on \( \mathbb{R}^d \), it follows that

\[ \left\| \sup_{1 \leq j \leq J} |\hat{T}_{K_j} f| \chi_E \right\|_{L^q(\mathbb{R}^d, u \otimes 1)}^q = \frac{1}{\omega(Q_s)} \int_{\mathbb{R}^d} \sup_{1 \leq j \leq J} \left| R_{(x, y)} \hat{T}_{K_j} f(z) \right|^q u(x) v(y + z) \chi_E(y + z) \, dx dy dz \]

\[ \leq \frac{1}{\omega(Q_s)} \int_{\mathbb{R}^d} \left\{ \int_{A_z \cap Q_s} \sup_{1 \leq j \leq J} |B_{K_j} (R(\cdot f(z) \chi_{Q_{r+s}})) (x, y)|^q u(x) v(y + z) \, dx dy \right\} dz. \]

By Kolmogorov’s condition \((2.1)\) and \((4.2)\), the term inside curly brackets is bounded by

\[ (c_{p,q})^q \left\{ \int_{Q_{r+s}} |R_{(x, y)} f(z)|^p u(x) v(y + z) \, dt \right\} \right\}^{\frac{q}{p}} \left\{ \int_{A_z \cap Q_s} u(x) v(y + z) \, dt \right\}^{1 - \frac{q}{p}}. \]

Then, by Hölder’s inequality, it follows that

\[ \left\| \sup_{1 \leq j \leq J} |T_{K_j} f| \chi_E \right\|_{L^q(\mathbb{R}^{d_2}, u \otimes 1)}^q \leq \frac{c_{p,q} \mathcal{M}}{\omega(Q_s)} \left\{ \int_{\mathbb{R}^d} \int_{Q_{r+s}} u(x) v(y + z) \, dt \, dz \right\}^{\frac{p}{q}} \]

\[ \times \left\{ \int_{\mathbb{R}^d} \int_{Q_{r+s}} |R_{(x, y)} f(z)|^p u(x) v(y + z) \, dx \, dy \, dz \right\}^{\frac{q}{p}} \]

\[ \leq c_{p,q} \mathcal{M} \left( \frac{\omega(Q_{r+s})}{\omega(Q_s)} \right)^{\frac{p}{q}} v(E)^{1 - \frac{q}{p}} \left\| f \right\|_{L^p(\mathbb{R}^{d_2}, u \otimes 1)}^q. \]

Since \( u \in A_p(\mathbb{R}^{d_1}), \omega \in A_p(\mathbb{R}^d) \). Then by Lemma \([1.1]\) and Kolmogorov’s condition \((2.1)\), it follows that

\[ \left\| \sup_{1 \leq j \leq J} |T_{K_j} f| \right\|_{L^p(\mathbb{R}^{d_2}, u \otimes 1)} \leq c_{p,q} \mathcal{M} \left\| f \right\|_{L^p(\mathbb{R}^{d_2}, u \otimes 1)} \]

Finally, considering \( c_p = \inf_{q < p} c_{p,q} \), the result easily follows by Fatou’s Lemma and the density of \( C_c^\infty(\mathbb{R}^{d_2}) \) in \( L^p(\mathbb{R}^{d_2}, u \otimes 1) \). \( \square \)
5. Consequences and Applications

5.1. Hörmander-Mihlin type multipliers. The first application involves multipliers satisfying a Hörmander-Mihlin type condition.

Definition 5.1 (see [14]). Let \( m \in L^\infty(\mathbb{R}^d) \cap C^d (\mathbb{R}^d \setminus \{0\}) \), \( l \in \mathbb{N} \) and \( s \geq 1 \). We say \( m \in M(s,l) \) if it satisfies

\[
(5.1) \quad c_{m,s,l} = \sup_{|\alpha| \leq l} \sup_{r > 0} \left( r^{|\alpha| - d} \int_{r < |x| < 2r} \left| \frac{\partial^{|\alpha|} m}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x) \right|^s \, dx \right)^{1/s} < \infty.
\]

In 1979, D. Kurtz and R. Wheeden proved the following result.

Theorem 5.2 ([14, Theorem 1]). Let \( 1 < s \leq 2 \), \( \frac{d}{s} < l \leq d \) and \( m \in M(s,l) \). If

1. \( d/l < p < \infty \) and \( w \in A_{pl/d}(\mathbb{R}^d) \) or
2. \( 1 < p < (d/l)' \) and \( w^{-1/(p-1)} \in A_{p'l/d}(\mathbb{R}^d) \),

then \( m \in M_{p,w}(\mathbb{R}^d) \). When \( l < d \) it can be taken \( p = d/l \) in (1) or \( p = (d/l)' \) in (2).

Moreover, if \( w^{d/l} \in A_1(\mathbb{R}^d) \), then \( m \in M_{1,w}(\mathbb{R}^d) \).

Corollary 5.3. Under the hypothesis of Theorem 5.2 and assuming that \( m \) is a normalized function, the following holds: If

1. \( d/l < p < \infty \) and \( w \in A_{pl/d}(\mathbb{T}^d) \) or
2. \( 1 < p < (d/l)' \) and \( w^{-1/(p-1)} \in A_{p'l/d}(\mathbb{T}^d) \),

then \( m|_{\mathbb{T}^d} \in M_{p,w}(\mathbb{T}^d) \). When \( l < d \) it can be taken \( p = d/l \) in (1) or \( p = (d/l)' \) in (2).

Moreover, if \( w^{d/l} \in A_1(\mathbb{T}^d) \), then \( m|_{\mathbb{T}^d} \in M_{1,w}(\mathbb{T}^d) \).

Proof. The result follows by applying Theorems 3.10 and 3.11 to \( m \).

5.2. Singular integral operators. Our second example involves the classical theory of Calderón-Zygmund singular integrals.

Definition 5.4 ([10, Definition II.5.17]). A function \( K \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \) is said to be a regular kernel if \( \tilde{K} \in L^\infty(\mathbb{R}^d) \) and it satisfies

\[
(5.2) \quad |K(x)| \leq C |x|^{-d}, \quad \forall x \in \mathbb{R}^d \setminus \{0\},
\]

\[
(5.3) \quad |K(x - y) - K(x)| \leq C |y| |x|^{-d-1}, \quad |x| > 2 |y|.
\]

Corollary 5.5. Let \( K \) be a regular kernel and consider for any \( 0 < r < s < \infty \), \( K_{r,s} = K_{r < |x| < s} \) and \( m_{r,s} = \tilde{K}_{r,s} \). If \( 1 < p < \infty \) and \( w \in A_p(\mathbb{T}^d) \) there exists a constant \( c \) such that

\[
\| \{ m_{r,s} \} \|_{M_{p,w}(\mathbb{T}^d)} \leq c.
\]

If \( w \in A_1(\mathbb{T}^d) \), then there exists a constant \( c \) such that

\[
\| \{ m_{r,s} \} \|_{M_{1,w}(\mathbb{T}^d)} \leq c.
\]

Proof. It is easy to see that \( T^d \{ m_{r,s} \}_{r<s} f(x) = T^d \{ m_{r,s} \}_{r<s} f(x) \) for every \( f \in P(\mathbb{T}^d) \). Then, the result follows by the known corresponding result for functions in \( \mathbb{R}^d \) (see [10, Theorem IV.3.6 and V.4.11]) by applying Theorem 5.2 and its corresponding strong version.
5.3. Bochner-Riesz partial sums. Our third application involves Bochner-Riesz partial sums. Let us recall that the Bochner-Riesz operators in \( \mathbb{R}^d \) are defined as

\[
\left( B_{\lambda}^r f \right)(\xi) = m_r(\xi) \hat{f}(\xi), \quad \text{where} \quad m_r(x) = \left( 1 - \frac{|x|^2}{r^2} \right)_+, 
\]

\( t_+ = \max(t, 0) \), and the associated maximal operator is defined by

\[
B_{\lambda}^r f(x) = \sup_{r > 0} |B_{\lambda}^r f(x)|
\]

for \( \lambda > 0 \). It is known that for \( \lambda > \frac{d-1}{2} \), \( B_{\lambda}^r f \) is pointwise majorized by the Hardy-Littlewood maximal operator; then it inherits its boundedness properties. For the critical index the following is known (S. Shi and Q. Sun [17, Theorem 1] and A. Vargas [16, Theorem 1]).

**Theorem 5.6.** Let \( \lambda = \frac{d-1}{2} \). If \( 1 < p < \infty \) and \( w \in A_p(\mathbb{R}^d) \), then

\[
\left\| B_{\lambda}^r f \right\|_{L^p(\mathbb{R}^d, w)} \leq C \left\| f \right\|_{L^p(\mathbb{R}^d, w)},
\]

and if \( w \in A_1(\mathbb{R}^d) \), there is a constant \( C \) such that for each \( r > 0 \),

\[
\left\| B_{\lambda}^r f \right\|_{L^1(\mathbb{R}^d, w)} \leq C \left\| f \right\|_{L^1(\mathbb{R}^d, w)},
\]

where the constants depend only on the \( A_p \)-constant of \( w \) and the dimension \( d \).

Let us observe that for \( \lambda = (d - 1)/2 \), the kernel of the operator \( B_{\lambda}^r \), say \( K \), satisfies the size condition \( |K(x)| \lesssim |x|^{-d} \), but it does not satisfy any Hörmander type condition such as (5.3) above. Then we can’t apply the result obtained in the previous example.

In the periodic case, for \( r > 0 \), the Bochner-Riesz partial sum of order \( \lambda > 0 \) is defined for every \( f \in P(\mathbb{T}^d) \) by

\[
S_{\lambda}^r f(\theta) = \sum_{|n| \leq r} \left( 1 - \frac{|n|^2}{r^2} \right)_+ \hat{f}(n) e^{2\pi i n \theta},
\]

and we denote by \( S_{\lambda}^r \) the associated maximal operator. Observe that since the function \( (1 - |x|^2)_+ \) is continuous, for every \( f \in P(\mathbb{T}^d) \),

\[
S_{\lambda}^r f(x) = \sup_{r \in Q_+} |S_{\lambda}^r f(x)|.
\]

Then, as a consequence of our results, the following counterpart to Theorem 5.6 is obtained.

**Corollary 5.7.** Let \( \lambda \geq \frac{d-1}{2} \). If \( 1 < p < \infty \) and \( w \in A_p(\mathbb{T}^d) \), then there exists \( C > 0 \) such that

\[
\left\| S_{\lambda}^r f \right\|_{L^p(\mathbb{T}^d, w)} \leq C \left\| f \right\|_{L^p(\mathbb{T}^d, w)}.
\]

If \( w \in A_1(\mathbb{T}^d) \) and \( \lambda = \frac{d-1}{2} \) there exists \( C > 0 \) such that for any \( r > 0 \),

\[
\left\| S_{\lambda}^r f \right\|_{L^1(\mathbb{T}^d, w)} \leq C \left\| f \right\|_{L^1(\mathbb{T}^d, w)}.
\]

and if \( \lambda > \frac{d-1}{2} \), there exists \( C > 0 \) such that

\[
\left\| S_{\lambda}^r f \right\|_{L^1(\mathbb{T}^d, w)} \leq C \left\| f \right\|_{L^1(\mathbb{T}^d, w)}.
\]
Corollary 5.8. Let $\lambda \geq \frac{d-1}{2}$, $1 \leq p < \infty$ and $w \in A_p(\mathbb{T}^d)$. For any $f \in L^p(\mathbb{T}^d, w)$,
\[
\lim_{r \to 0^+} S^r f = f,
\]
where the convergence is considered in measure for $p = 1$ and $\lambda = \frac{d-1}{2}$ and pointwise almost everywhere in the other cases.

For $\lambda$ below the critical index the study of the boundedness properties of the Bochner-Riesz operators constitutes an active area of research (see [9] for instance [9, Theorem 5.1] (taking $\lambda = \frac{d-1}{2}$ below the critical index). In this setting, the following is a direct consequence of Theorem 5.11 (taking $u_0 = 1$ with the notation therein).

Theorem 5.9. Let $0 < \lambda < (d-1)/2$. If $w(x) = v(x)^{2\lambda/d-1}$ with $v \in A_2(\mathbb{R}^d)$, then, for any $r > 0$, $B^r_\lambda$ is bounded in $L^2(\mathbb{R}^d, w)$ uniformly on $r$.

Theorem 5.10 leads to obtain the following periodic counterpart result.

Corollary 5.10. Let $0 < \lambda < (d-1)/2$. If $w(x) = v(x)^{2\lambda/d-1}$ with $v \in A_2(\mathbb{T}^d)$, then, for any $r > 0$, $S^r_\lambda$ is bounded in $L^2(\mathbb{T}^d, w)$ uniformly on $r$.

5.4. Extension of multipliers from $L^p(\mathbb{T})$ to $L^p(\mathbb{R}, w)$. In this section we are going to show how Theorem 5.11 allows us to see the strong ties between $M_p(\mathbb{T})$ and a subspace of $M_{p,w}(\mathbb{R})$ for a subclass of weights in $A_p(\mathbb{R})$ (see Corollary 5.12 below).

Following M. Jodeit’s ideas in [12], E. Berkson, M. Paluszyński and G. Weiss in [5] gave a way to extend multipliers from $L^p(\mathbb{T})$ to $L^p(\mathbb{R}, w)$ with $w \in A_p(\mathbb{R})$ satisfying that there exists a constant $\rho \geq 1$ such that for each $k \in \mathbb{Z}$
\[
\rho^{-1}w(k) \leq w(x) \leq \rho w(k), \quad \text{for all } x \in [k, k+1).
\]

These weights are said to be in $W_p$.

In this framework, E. Berkson, M. Paluszyński and G. Weiss proved the following result.

Theorem 5.11 ([5, Theorem 4.21]). Let $1 < p < \infty$, $w \in W_p$, $\Psi \in M_{p,w}(\mathbb{R})$ and the support of $\Psi$ is contained in $[-1/2, 1/2]$. Then, if $\{\phi_n\}_n \in M_p(\mathbb{T})$, we have that
\[
\mathcal{W}_{\phi_n, \Psi}(t) = \sum_{m \in \mathbb{Z}} \phi(m) \Psi(t-m) \in M_{p,w}(\mathbb{R})
\]
and
\[
\|\{\mathcal{W}_{\phi_n, \Psi}\}_n\|_{M_{p,w}(\mathbb{R})} \leq K_{p,w} \|\Psi\|_{M_{p,w}(\mathbb{R})} \|\{\phi_n\}_n\|_{M_p(\mathbb{T})}.
\]

Since $\mathcal{W}_{\phi_n, \Psi}|_{\mathbb{Z}} = \Psi(0)\hat{\phi}_n|_{\mathbb{Z}}$, a direct consequence of Theorem 5.13 with $u = 1$ and $v = w$ is that the converse of Theorem 5.11 also holds:

Corollary 5.12. Let $\{\phi_n\}_n \subset \ell^\infty(\mathbb{Z})$. Then, under the hypothesis of Theorem 5.11 we have that if $\Psi(0) \neq 0$,
\[
\|\{\mathcal{W}_{\phi_n, \Psi}\}_n\|_{M_{p,w}(\mathbb{R})} < +\infty \quad \text{if and only if} \quad \|\{\phi_n\}_n\|_{M_p(\mathbb{T})} < +\infty.
\]
Moreover,
\begin{equation}
C_{p,w} \frac{\|\{\phi_n\}_n\|_{M_p(T)}}{\|\Psi(0)\|} \leq \frac{\|\{W_{\phi_n,\Psi}\}_n\|_{M_{p,w}(\mathbb{R})}}{\|\Psi\|} \leq K_{p,w} \frac{\|\{\phi_n\}_n\|_{M_p(T)}}{\|\Psi\|_{M_{p,w}(\mathbb{R})}}.
\end{equation}

Observation 5.13. In the particular case of a single multiplier, inequality (5.5) yields that, for any $w \in W_p$, the map $\phi \mapsto W_{\phi, \Psi}$ induces an isomorphism between $M_p(T)$ and a subspace of $M_{p,w}(\mathbb{R})$. This result is a one dimensional weighted generalization of the unweighted result in [12, p. 225] for $\Psi$ the characteristic function of the interval $[-1/2, 1/2]$.

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